KAC-MOODY FIBONACCI SEQUENCES, HYPERBOLIC GOLDEN RATIOS, AND REAL QUADRATIC FIELDS

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ABSTRACT. Let A be the generalized Cartan matrix of rank 2 Kac-Moody algebra $\mathfrak{g}$. We write $\mathfrak{g} = \mathfrak{g}(a, b)$ when A has non-diagonal entries $-a$ and $-b$. To each such A, its Weyl group and corresponding root lattice, we associate a 'Fibonacci type' integer sequence. These sequences are derived from the coordinates of the real root vectors in the root space. Each element of each sequence can be expressed as a polynomial in the non-diagonal entries of the generalized Cartan matrix, whose coefficients are shallow diagonals of Pascal's triangle. Among the Fibonacci type sequences are the bisected Lucas and Fibonacci sequences, the Kekule numbers for the benzoids, the integers whose squares are triangular numbers, Chebyshev polynomials of the second kind, as well as some other known sequences. Each sequence has an associated 'hyperbolic golden ratio' Ψ which is a unit in a real quadratic field. We show that Ψ can be obtained as the limit of ratios of areas of hyperbolic triangles spanned by a triple of adjacent real roots in the root space. For the bisected Fibonacci sequence that occurs in our setting, Ψ coincides with the classical golden ratio $\psi = 1 + \phi$ for the bisection, where $\phi = 1.61803399\ldots$. It follows that ψ is an eigenvalue for the fundamental endomorphism w_1w_2 in the root system of $\mathfrak{g}(5, 1)$, and ψ^2 is an eigenvalue for w_1w_2 in $\mathfrak{g}(3, 3)$. This leads to an infinite family of identities involving continued fractions. We identify the generalized Lucas and Fibonacci sequences with Chebyshev polynomials of the 2nd kind, which in turn can be expressed as products. This leads us to an arithmetic proof that Kitchloo's 'generalized binomial coefficients' occuring in the cohomology of flag varieties of rank 2 Kac-Moody groups over $\mathbb{C}$ are integers.

1. INTRODUCTION

A *Kac-Moody Lie algebra* is the most natural generalization to infinite dimensions of the notion of a finite dimensional semisimple Lie algebra. The data for building a Kac-Moody algebra includes a square integer matrix A called a *generalized Cartan matrix* in which the diagonal entries equal 2, the other entries are nonpositive and A is symmetric in its zeros.

Let A be the 2×2 generalized Cartan matrix

$$A = (a_{ij})_{i,j=1,2} = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$$

with $a, b \in \mathbb{Z}_{>0}$, $ab \geq 4$, with Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$, root system $\Delta = \Delta(A)$, and Weyl group $W = W(A)$. When $ab = 4$, A is positive semi-definite but not positive definite, and we call A *affine*. When $ab > 4$, A is neither positive semi-definite nor positive definite, but every proper generalized Cartan submatrix is positive definite, and A is *hyperbolic*.

Let A be the generalized Cartan matrix of the rank 2 Kac-Moody algebra $\mathfrak{g}$. We write $\mathfrak{g} = \mathfrak{g}(a, b)$ when A has non-diagonal entries $-a$ and $-b$. To each such A, its Weyl group and corresponding root lattice, we associate a 'Fibonacci type' integer sequence (Section 3).

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These sequences are derived from the coordinates of the real root vectors in the root space. Each element of each sequence can be expressed as a polynomial in the non-diagonal entries of the generalized Cartan matrix, whose coefficients are shallow diagonals of Pascal's triangle. Among these sequences are the bisected Lucas and Fibonacci sequences, the Kekule numbers for the benzoids, the integers whose squares are triangular numbers, as well as some other known sequences. For each pair a, b there is a natural pairing of integer sequences. This gives rise to a relation between the entries generalizing the description of elements of the Lucas sequence as a sum of consecutive terms of the Fibonacci sequence. We obtain several other generalizations of Fibonacci type identities, including a generalization of Binet's formula (Lemma **??**). Each sequence has an associated 'hyperbolic golden ratio' Ψ which is an algebraic integer in a real quadratic field (Section 4). We show that Ψ can be obtained as the limit of ratios of areas of hyperbolic triangles spanned by a triple of adjacent real roots in the root space. This follows from the fact that the distribution of coordinates of adjacent real roots tends towards the 'golden' ratio Ψ. For the bisected Fibonacci sequence that occurs in our setting, Ψ coincides with the classical golden ratio $\psi = 1 + \phi$ for the bisection, where $\phi = 1.61803399\ldots$. It follows that ψ is an eigenvalue for the fundamental endomorphism w_1w_2 in the root system of $\mathfrak{g}(5,1)$, and ψ^2 is an eigenvalue for w_1w_2 in $\mathfrak{g}(3,3)$ (Section 5). This leads to an infinite family of identities involving continued fractions (Section 6).

We also identify the generalized Lucas and Fibonacci sequences with Chebyshev polynomials of the 2nd kind, which in turn can be expressed as products. We then show that each sequence element can be written as products of binomials of the form $k - 4\cos^2(\frac{j\pi}{n})$, where n is the index of the polynomial and j ranges from 1 to $\lfloor\frac{n-1}{2}\rfloor$. Since the resulting polynomials share like terms with our sequence polynomials, we identify the coefficients on both sides, hence indentifying binomial coefficients with elementary symmetric polynomials whose roots are $\cos^2(\frac{j\pi}{n})$ as above (Section 3).

It is interesting to ask if certain ratios of products of consecutive sequence elements are integers. Using topological methods, N. Kitchloo answered this in the affirmative, showing that these integers appear in the cohomology of flag varieties associated to rank 2 Kac-Moody groups over $\mathbb{C}$. Kitchloo also asked if an arithmetic proof was possible. Using properties of Chebyshev polynomials of the second kind, we provide an arithmetic proof (Section 7). We also give a detailed computation of the p-adic valuation of our sequence entries (Section 8).

In Section 10, we give a brief indication of how our methods may be used to study the Weyl group orbits of roots in certain rank 3 hyperbolic root systems.

At the time of submission, there is no description of the infinite family of sequences in the *Online Encyclopedia of Integer Sequences*, though some of the sequences do appear there, as indicated in Table 1 and Table 2.

We remark that our work is within the root system corresponding to a 2×2 generalized Cartan matrix A. In this case, the Cartan subalgebra can be mapped into the Lorentzian space $\mathbb{R}^{(2,1)}$, a *finite dimensional* space. Hence our results do not use the full structure of the Kac-Moody algebra. Furthermore, since we deal only with real roots, the phenomena we describe occur mostly in the 'Euclidean part' of $\mathbb{R}^{(2,1)}$. Hence much of what we describe can be characterized in terms of linear dynamical systems in $\mathbb{R}^2$, though we did not find a suitable reference.

Some of the proofs of our results are elementary, and will be omitted if they are obvious or straightforward. Alex Feingold first discovered the appearance of Fibonacci numbers in a rank 2 Kac-Moody root system when $a = b = 3$ ([4]). This was also observed by Kang and Melville ([7]). Some of the ideas here are implicitly included in the paper of Lepowsky and Moody ([11]) though they do not appear explicitly there. We also suspect that some of the results included here are well known as isolated facts and that our main contribution is to describe them in the framework of rank 2 Kac-Moody root systems.

Omission of relevant references in parts of this work is unintentional. We are grateful to Nitu Kitchloo who was a driving force behind Sections 7 and 8.

2. Rank 2 Kac-Moody root systems

Let A be the generalized Cartan matrix

$$A = (a_{ij})_{i,j=1,2} = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$$

with $a, b \in \mathbb{Z}_{>0}$, $ab \geq 4$, $a \geq b$, with Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$, root system $\Delta = \Delta(A)$, and Weyl group $W = W(A)$. Let $\Pi = \{\alpha_1, \alpha_2\}$ be the simple roots and let w_1, w_2 be the simple reflections. We have $W = \langle w_1 \rangle * \langle w_2 \rangle \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, so every element of W is of the form $w_1^{\epsilon_1}(w_2w_1)^n w_2^{\epsilon_2}$, where $\epsilon_i \in \{0, 1\}$, $i = 1, 2$ and $n \in \mathbb{Z}_{\geq 0}$. Since W is an infinite group, the number of translates of Π by W is infinite, and $\mathfrak{g}(A)$ is infinite dimensional. However, not all roots Δ are of the form $W\Pi$. We will refer to roots of the form $W\Pi$ as *real roots*. Let

$$B = (b_{ij})_{i,j=1,2} = \begin{pmatrix} 2 & -a \\ -a & 2a/b \end{pmatrix}$$

be a symmetrization of A, so that we have a short root (say α_1) of squared length 2, and a long root (say α_2) of squared length $2a/b$. Let

$$Q(x, y) = 2x^2 - 2axy + (2a/b)y^2$$

be the associated quadratic form. We wish to determine the integer lattice points such that $Q(x, y) = 2$ and $Q(x, y) = 2a/b$ on the corresponding conic section. These correspond to a subset of the root system, namely the roots of positive squared length (real roots). Let $P(x, y)$ be the quadratic form

$$P(x, y) = bx^2 - abxy + ay^2.$$

We have

$$\{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid Q(x, y) = 2\} = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid P(x, y) = b\}$$

$$\{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid Q(x, y) = 2a/b\} = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid P(x, y) = a\},$$

so we may characterize the integer solutions of $P(x, y)$ rather than $Q(x, y)$.

We have

$$W\{\alpha_1\} \subseteq \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid P(x, y) = b\}$$

$$W\{\alpha_2\} \subseteq \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid P(x, y) = a\}.$$

We refer the reader to [6, Exercise 5.25] for a more precise statement of about the relationship between real roots and the quadratic form $P(x, y)$. The real roots then correspond to the set $\Phi = W\{\alpha_1\} \sqcup W\{\alpha_2\}$, and Φ is a proper subset of Δ. When $a = b$, α_1 and α_2 have the same length, as do all translates $W\Pi$.

It is known that we can map Δ into a 2-dimensional Euclidean space E which can be endowed with a Lorentzian metric of signature (1,1) when $ab > 4$. The real roots $\Phi = W\{\alpha_1\} \sqcup W\{\alpha_2\}$ lie on the root lattice $\mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \subset E$. Each root $\beta \in \Phi$ is a linear combination of simple roots with coefficients that are either all positive or all negative. We call β positive or negative accordingly.

3. KAC-MOODY FIBONACCI SEQUENCES

In this section, we demonstrate that the Weyl group translates of the simple roots α_1 and α_2 give rise to a sequence of polynomials in a and b, where $-a$ and $-b$ are the off diagonal entries of the generalized Cartan matrix. We first tabulate the coefficients of $W\{\alpha_1\} \sqcup W\{\alpha_2\}$ in the root lattice, and then in Lemma 3.1 below, we rewrite these in the language of sequences.

To set up the appropriate notation, we identify the simple roots α_1, α_2 with the vectors $(1,0)$ and $(0,1)$, respectively. For every root $\beta \in \Phi$ we write

$$\beta = x(\beta)\alpha_1 + y(\beta)\alpha_2,$$

for $x(\beta), y(\beta) \in \mathbb{Z}$ to give an expression for β in

$$\Phi \cap (\mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2).$$

Then

$$x(\alpha_1) = 1,\ y(\alpha_1) = 0$$
$$x(\alpha_2) = 0,\ y(\alpha_2) = 1.$$

For $i = 1, 2$, we write $W^+\{\alpha_i\} = W\{\alpha_i\} \cap \Phi^+$, where Φ denotes the real roots. Then

$$W^+\{\alpha_1\} = \{(w_1w_2)^n\alpha_1,\ n \geq 0\} \sqcup \{w_2(w_1w_2)^n\alpha_1,\ n \geq 0\},$$

$$W^+\{\alpha_2\} = \{(w_2w_1)^n\alpha_2,\ n \geq 0\} \sqcup \{w_1(w_2w_1)^n\alpha_2,\ n \geq 0\}.$$

Note that for each real root $\alpha \in \Phi$, if we write $\alpha = w\alpha_i$, the word w is reduced and ends in w_{3-i}.

We tabulate $x(\alpha)$ and $y(\alpha)$ for all $\alpha \in \Phi$, where $l(w) = n$ for some $n \geq 0$, and $l(\cdot)$ is the standard length function on W:

$n = l(w)$	$\alpha \in \{(w_1w_2)^{\lfloor n/2 \rfloor}\alpha_1 \mid n \geq 0\}$	$x(\alpha)$	$y(\alpha)$
0	α_1	1	0
2	$w_1w_2\alpha_1$	$ab-1$	b
4	$w_1w_2w_1w_2\alpha_1$	$a^2b^2-3ab+1$	ab^2-2b
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$n = l(w)$	$\alpha \in \{w_2(w_1w_2)^{\lfloor n/2 \rfloor}\alpha_1 \mid n \geq 0\}$	$x(\alpha)$	$y(\alpha)$
1	$w_2\alpha_1$	1	b
3	$w_2w_1w_2\alpha_1$	$ab-1$	ab^2-2b
5	$w_2(w_1w_2)^2\alpha_1$	$a^2b^2-3ab+1$	$a^2b^3-4ab^2+3b$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$n = l(w)$	$\alpha \in \{(w_2w_1)^{\lfloor n/2 \rfloor}\alpha_2 \mid n \geq 0\}$	$x(\alpha)$	$y(\alpha)$
0	α_2	0	1
2	$w_2w_1\alpha_2$	a	$ab-1$
4	$w_2w_1w_2w_1\alpha_2$	a^2b-2a	$a^2b^2-3ab+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$n = l(w)$	$\alpha \in \{w_1(w_2w_1)^{\lfloor n/2 \rfloor}\alpha_2 \mid n \geq 0\}$	$x(\alpha)$	$y(\alpha)$
1	$w_1\alpha_2$	a	1
3	$w_1w_2w_1\alpha_2$	$a^2b - 2a$	$ab - 1$
5	$w_1(w_2w_1)^2\alpha_2$	$a^3b^2 - 4a^2b + 3a$	$a^2b^2 - 3ab + 1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

We now rewrite the coefficients $x(\alpha)$ and $y(\alpha)$ as tabulated above in the language of sequences. For $i = 1, 2$ and for $\alpha \in W^+\{\alpha_i\}$, we have $\alpha = w\alpha_i$ for some $w \in W$, where $l(w) = n$ for some $n \geq 0$, and $l(\cdot)$ is the standard length function on W. For $i = 1, 2$, we introduce the following notation for the polynomials $x(\alpha)$ and $y(\alpha)$. We set

$$x_{i,n} := x_{i,n}(a, b) = x(\alpha) \text{ where } \alpha = w\alpha_i,\ l(w) = n,$$
$$y_{i,n} := y_{i,n}(a, b) = y(\alpha) \text{ where } \alpha = w\alpha_i,\ l(w) = n.$$

Lemma 3.1. *We have*

$$x_{1,n} = x_{1,n+1},\ n \text{ even, } n \geq 0,$$
$$y_{1,n} = y_{1,n-1},\ n \text{ even, } n \geq 2.$$

$$x_{2,n} = x_{2,n-1},\ n \text{ even, } n \geq 2,$$
$$y_{2,n} = y_{2,n+1},\ n \text{ even, } n \geq 0.$$

Proof. Let A be the generalized Cartan matrix defined in Section 1 with $ab \geq 4$. Let $\beta = w\alpha_1$ and $\gamma = w'\alpha_2$ where $l(w) = l(w') = n$ for some $n \in 2\mathbb{Z}_{\geq 0}$. That is to say, $w = (w_1w_2)^{n/2}$ and $w' = (w_2w_1)^{n/2}$. The Weyl group generators w_1 and w_2 act on α_1 and α_2 by the general formula:

$$w_i\alpha_j = \alpha_j - A_{ij}\alpha_i.$$

Thus, writing $\beta = x(\beta)\alpha_1 + y(\beta)\alpha_2$ and $\gamma = x(\gamma)\alpha_1 + y(\gamma)\alpha_2$, we obtain:

(1)
$$w_2\beta = x(\beta)\alpha_1 + [b \cdot x(\beta) - y(\beta)]\,\alpha_2$$

(2)
$$w_1\gamma = [a \cdot y(\gamma) - x(\gamma)]\,\alpha_1 + y(\gamma)\alpha_2.$$

Note that the Weyl words w_2w and w_1w' associated to $w_2\beta$ and $w_1\gamma$ have lengths $l(w_2(w_1w_2)^{n/2}) = l(w_1(w_2w_1)^{n/2}) = n+1$. Thus we observe, $y(w_1\gamma) = y(\gamma)$ which implies $y_{2,n+1} = y_{2,n}$ and $x(w_2\beta) = x(\beta)$ proving $x_{1,n+1} = x_{1,n}$.

The two remaining identities in Lemma 3.1 are obtained by the actions of w_1 on β and w_2 on γ, where β and γ are as before, but now $n \geq 2$. Here, we note that the Weyl words associated to the new roots $w_1\beta$ and $w_2\gamma$ begin with w_1^2 and w_2^2, respectively, and so they have length $n - 1$:

$$l(w_1w) = l(w_1(w_1w_2)^{n/2}) = l(w_2(w_1w_2)^{\frac{n}{2}-1}) = n - 1,$$
$$l(w_2w') = l(w_2(w_2w_1)^{n/2}) = l(w_1(w_2w_1)^{\frac{n}{2}-1}) = n - 1.$$

Thus we obtain:

(3)
$$w_1\beta = [a \cdot y(\beta) - x(\beta)]\,\alpha_1 + y(\beta)\alpha_2$$

(4)
$$w_2\gamma = x(\gamma)\alpha_1 + [b \cdot x(\gamma) - y(\gamma)]\,\alpha_2,$$

and we observe that $y(w_1\beta) = y(\beta)$ and $x(w_2\gamma) = x(\gamma)$ so that $y_{1,n-1} = y_{1,n}$ and $x_{2,n-1} = x_{2,n}$ completing the proof. $\square$

Lemma 3.2. *We have the following recursion relations*

$$\text{(i)}\ \ x_{1,2m} = \begin{cases} 1, & m = 0 \\ ay_{1,2m} - x_{1,2m-2}, & m \geq 1, \end{cases}$$

$$\text{(ii)}\ \ y_{1,2m} = \begin{cases} 0, & m = 0 \\ bx_{1,2m-2} - y_{1,2m-2}, & m \geq 1, \end{cases}$$

$$\text{(iii)}\ \ x_{2,2m} = \begin{cases} 0, & m = 0 \\ ay_{2,2m-2} - x_{2,2m-2}, & m \geq 1, \end{cases}$$

$$\text{(iv)}\ \ y_{2,2m} = \begin{cases} 1, & m = 0 \\ bx_{2,2m} - y_{2,2m-2}, & m \geq 1. \end{cases}$$

(cf. [6, Exercise 5.25]*).*

Proof. The case $m = 0$ in each recursion follows from the vector coordinates of the simple roots α_1 and α_2, since the trivial Weyl word of length 0 is associated to them.

For the case $m \geq 1$, we use Lemma 3.1 and equation (3) with $n = 2m$ in the last proof to obtain the following identity:

$$x_{1,2m-2} = x_{1,2m-1} = a \cdot y_{1,2m} - x_{1,2m},$$

which by re-arranging terms gives us (i). Similarly, we use Lemma 3.1 and equation (1) with $n = 2m - 2$ to obtain:

$$y_{1,2m} = y_{1,2m-1} = b \cdot x_{1,2m-2} - y_{1,2m-2},$$

which gives us (ii). Finally, Lemma 3.1 and equations (2) with $n = 2m - 2$ and (4) with $n = 2m$ give us

$$x_{2,2m} = x_{2,2m-1} = a \cdot y_{2,2m-2} - x_{2,2m-2}$$

and

$$y_{2,2m-2} = y_{2,2m-1} = b \cdot x_{2,2m} - y_{2,2m},$$

respectively. The first equation is (iii) and the second gives us (iv). □

By Lemma 3.1, it suffices to restrict our attention to even indices ($n = 2m$) from this point forward, unless otherwise indicated. It follows that for $i = 1, 2$, $x_{i,2m}$ and $y_{i,2m}$ are subject to the following recursion relations.

Lemma 3.3. *We have the following recursion relations*

$$\text{(i)}\ \ x_{1,2m} = \begin{cases} 1, & m = 0 \\ ab - 1, & m = 1 \\ (ab-2)x_{1,2m-2} - x_{1,2m-4}, & m \geq 2, \end{cases}$$

$$\text{(ii)}\ \ y_{1,2m} = \begin{cases} 0, & m = 0 \\ b, & m = 1 \\ (ab-2)y_{1,2m-2} - y_{1,2m-4}, & m \geq 2, \end{cases}$$

$$\text{(iii)}\ \ x_{2,2m} = \begin{cases} 0, & m = 0 \\ a, & m = 1 \\ (ab-2)x_{2,2m-2} - x_{2,2m-4}, & m \geq 2. \end{cases}$$

$$\text{(iv)}\ \ y_{2,2m} = \begin{cases} 1, & m = 0 \\ ab - 1, & m = 1 \\ (ab-2)y_{2,2m-2} - y_{2,2m-4}, & m \geq 2, \end{cases}$$

Proof. As in Lemma 3.2, the base case $m = 0$ is trivial, and the values for $m = 1$ can be read from the tables. For the case $m \geq 2$, substituting Lemma 3.2(ii) into Lemma 3.2(i) gives

$$x_{1,2m} = a \cdot y_{1,2m} - x_{1,2m-2} = ab \cdot x_{1,2m-2} - a \cdot y_{1,2m-2} - x_{1,2m-2},$$

and since Lemma 3.2(ii) shows that $a \cdot y_{1,2m-2} = x_{1,2m-2} + x_{1,2m-4}$, we obtain:

$$x_{1,2m} = (ab - 2)x_{1,2m-2} - x_{1,2m-4}.$$

The proofs for (ii), (iii), and (iv) are similar, and we leave them to the reader. □

Similar recursions for rank 2 Kac-Moody root systems are given in [6, Exercises in Ch. 5]. We also refer the reader to
http://www.tanyakhovanova.com/RecursiveSequences/RecursiveSequences.html
for a detailed discussion on recursions $a(n) = d \cdot a(n-1) - a(n-2)$.

We adopt the convention $y_{1,-1} = x_{2,-1} = 0$.

For each pair a, b, where $ab \geq 4$, we obtain the following integer sequences

$X_1(a, b) := (x_{1,2m})_{m \in \mathbb{Z}_{\geq 0}} = \{1, ab - 1, a^2b^2 - 3ab + 1, \dots\}$,
$Y_1(a, b) := (y_{1,2m})_{m \in \mathbb{Z}_{\geq 0}} = \{0, b, ab^2 - 2b, a^2b^3 - 4ab^2 + 3b, \dots\}$,
$X_2(a, b) := (x_{2,2m})_{m \in \mathbb{Z}_{\geq 0}} = \{0, a, a^2b - 2a, a^3b^2 - 4a^2b + 3a, \dots\}$,
$Y_2(a, b) := (y_{2,2m})_{m \in \mathbb{Z}_{\geq 0}} = \{1, ab - 1, a^2b^2 - 3ab + 1, \dots\}$.

In Theorem 3.4(i) and (iii) below we prove that $Y_2(a, b) = X_1(a, b)$ and that $X_2(a, a) = Y_1(a, a)$. In particular we can define the sequence:

$$\begin{aligned} X(a) &= \{x_{2,2m}(a, a), y_{2,2m}(a, a)\}_{m \in \mathbb{Z}_{\geq 0}} = \{y_{1,2m}(a, a), x_{1,2m}(a, a)\}_{m \in \mathbb{Z}_{\geq 0}} \\ &= \{0, 1, a, a^2 - 1, a^3 - 2a, a^4 - 3a^2 + 1, \dots\}. \end{aligned}$$

Note that $X(a)$ is distinct from $X_i(a, a)$ and $Y_i(a, a)$, $i = 1, 2$. In fact for each $a \geq 2$, $X_i(a, a)$ and $Y_i(a, a)$ are each bisections of $X(a)$. It can also be shown that $X(a) = X_2(1, a + 2)$.

Theorem 3.4. *For each pair a, b, $ab \geq 4$, we have*

(i) $X_1(a, b) = X_1(b, a) = Y_2(a, b) = Y_2(b, a)$.
(ii) *Let $k \geq 4$. If $k = a_ib_i$ for some $a_i, b_i \in \mathbb{Z}_{\geq 1}$, $i = 1, \dots, t$, then*

$$X_1(a_1, b_1) = X_1(a_2, b_2) = \cdots = X_1(a_t, b_t),$$

and in particular $X_1(a, b) = X_1(b, a) = X_1(ab, 1) = X_1(1, ab)$.
(iii) $bX_2(a, b) = aY_1(a, b)$ *and* $X_2(a, b) = Y_1(b, a)$. *In particular* $X_2(a, a) = Y_1(a, a)$.
(iv) $X_2(1, ab) = Y_1(ab, 1)$.

Proof. Part (i) follows from Lemma 3.3: Observe that the recursion relations for $x_{1,2m}$ and $y_{2,2m}$ share the same initial values $\{1, ab-1\}$ and recursive formula, namely $d(n) = (ab-2)d(n-1)-d(n-2)$. Since this data completely determines any recursive sequence, we have $X_1(a, b) = Y_2(a, b)$. Moreover, since $ab = ba$, Lemma 3.3 parts (i) and (iv) are invariant under permuting a and b. Thus, $x_{1,2m}(a, b) = x_{1,2m}(b, a)$ and $y_{2,2m}(a, b) = y_{2,2m}(b, a)$, or to summarize in our sequence notation, $X_1(a, b) = X_1(b, a) = Y_2(a, b) = Y_2(b, a)$. Since the polynomial $x_{1,2m}(a, b)$ depends only on the product ab, part (ii) also follows. For part (iii) we use Lemma 3.3 and do induction on m. For $m = 0$ and $m = 1$ we have

$$b \cdot x_{2,0} = b \cdot 0 = 0 = a \cdot 0 = a \cdot y_{1,0}$$

and

$$b \cdot x_{2,2} = ba = ab = a \cdot y_{1,2}.$$

For $m \geq 2$ assume that $b \cdot x_{2,2m} = a \cdot y_{1,2m}$ for all $m < M$ for some $M \in \mathbb{Z}_{\geq 2}$. Then by the induction hypothesis,

$$b \cdot x_{2,2M} = (ab-2)b \cdot x_{2,2M-2} - b \cdot x_{2,2M-4} = (ab-2)a \cdot y_{1,2M-2} - a \cdot y_{1,2M-4} = a \cdot y_{1,2M},$$

and so $bX_2(a,b) = aY_1(a,b)$. The second equality $X_2(a,b) = Y_1(b,a)$ follows from observing that the recursion relations in Lemma 3.3(ii) and (iii) are identical when interchanging a and b. Lastly, (iv) follows from (iii). □

Remark 3.5. *The sequences above yield many known sequences as well as some new ones. We highlight the following:*

(i) $X_1(1,5) = Y_2(5,1)$ *is the bisected Lucas sequence* $(L_{2s+1})_{s\geq 0} = \{1,4,11,\dots\}$.
(ii) $X_2(1,5) = Y_1(5,1) = X(3)$ *is the bisected Fibonacci sequence* $(F_{2s})_{s\geq 0} = \{0,1,3,8,\dots\}$.
(iii) $X(6)^2$ *is a sequence of triangular numbers.*

Additional known sequences are given in the tables in Section 9. The following theorem reveals that for each pair a, b, $ab \geq 4$, our sequences are a generalization of the Lucas and Fibonacci sequences and the relationship between them.

Theorem 3.6. *For each pair a, b, $ab \geq 4$ and $n > 0$ we have*

(i) $x_{1,n}(1,ab) \;=\; x_{2,n-1}(1,ab) + x_{2,n+1}(1,ab)$,
(ii) $y_{1,n}(1,ab) \;=\; x_{1,n-1}(1,ab) + x_{1,n+1}(1,ab)$,
(iii) *When $a = 1$, $b = 5$, the relation in (i) encodes the relationship $L_{2s+1} = F_{2s} + F_{2s+2}$ that gives the Lucas number L_{2s+1} as the sum of consecutive terms in the bisected Fibonacci sequence F_{2s}.*

Proof. (i) Let $n > 0$, n even. By Lemma 3.2(iii),

$$a \cdot y_{2,n} = x_{2,n+2} + x_{2,n}.$$

If $a = 1$, Theorem 3.4(i) and Lemma 3.1 allow us to rewrite this equation as $x_{1,n} = x_{2,n-1} + x_{2,n+1}$. To address the case of n odd, note that we could also rewrite as $x_{1,n+1} = x_{2,n} + x_{2,n+2}$, by Lemma 3.1. This does not however address the case $n = 1$, so to complete the proof, Lemma 3.1 gives us $x_{1,1} = x_{1,0} = 1 = 0 + 1 = x_{2,0} + x_{2,2}$, since for $a = 1$, $x_{2,2} = 1$.

(ii) Again let $n > 0$, n even. By Lemma 3.2(i),

$$a \cdot y_{1,n} = x_{1,n-2} + x_{1,n}.$$

As before, if $a = 1$, Lemma 3.1 allows us to rewrite this equation as either $y_{1,n} = x_{1,n-1} + x_{1,n+1}$ or as $y_{1,n-1} = x_{1,n-2} + x_{1,n}$.

(iii) Part (i) and Remark 3.5 parts (i) and (ii) give us:

$$L_{2s+1} = x_{1,2s}(1,5) = x_{1,2s+1}(1,5) = x_{2,2s}(1,5) + x_{2,2s+2}(1,5) = F_{2s} + F_{2s+2}.$$

□

Remark 3.7. *Since $\beta = x_{1,n}\alpha_1 + y_{1,n}\alpha_2$ is a positive root, we have $x_{1,n} \geq 0$ and $y_{1,n} \geq 0$ for all n. Moreover we have $x_{1,n} = 0$ if and only if $y_{1,n}\alpha_2$ is a positive root, and since α_2 is a real root this is equivalent to $y_{1,n} = 1$. We conclude that $x_{1,n} > 0$ for $n \geq 0$. Similarly we get $y_{1,n} = 0$ if and only if $\beta = \alpha_1$, so $y_{1,n} > 0$ for $n \geq 1$. The same arguments show that $x_{2,n} > 0$ for $n \geq 1$ and $y_{2,n} > 0$ for $n \geq 0$.*

For later use, we also prove that the four sequences are increasing. Consider the sequence $\{x_{1,n}\}_{n\geq 0}$. Since $x_{1,2m} = x_{1,2m+1}$ for $m \geq 0$ by Lemma 3.1 it suffices to prove that $x_{1,2m+2} \geq x_{1,2m}$ for $m \geq 0$. By Lemma 3.3(i) we have $x_{1,0} = 1$, $x_{1,2} = ab - 1$ and $x_{1,2m+2} = (ab-2)x_{1,2m} - x_{1,2m-2}$ for $m \geq 1$. Hence the inequality holds for $m = 0$ (since $ab \geq 4$) and if $x_{1,2m} \geq x_{1,2m-2}$ then

$$x_{1,2m+2} = (ab-3)x_{1,2m} + (x_{1,2m} - x_{1,2m-2}) \geq 0,$$

since $ab \geq 4$ and $x_{1,2m} \geq 0$. This proves that $\{x_{1,n}\}_{n\geq 0}$ is increasing. The proof for the three other sequences is identical (again using Lemmas 3.1 and 3.3).

We adopt the terminology used in the literature that Chebyshev polynomials of the first and second kind are denoted by T- and U-polynomials, respectively. These are given by the following recurrence relations:

$$T_n(x) = \begin{cases} 1, & n = 0 \\ x, & n = 1 \\ 2xT_{n-1}(x) - T_{n-2}(x), & n \geq 2, \end{cases}$$

$$U_n(x) = \begin{cases} 1, & n = 0 \\ 2x, & n = 1 \\ 2xU_{n-1}(x) - U_{n-2}(x), & n \geq 2, \end{cases}$$

We also adopt the convention that $U_{-1}(x) = 0$. It is not hard to show from the above recursions that $T_n(\cos(\theta)) = \cos(n\theta)$ and $U_n(\cos(\theta)) = \frac{\sin((n+1)\theta)}{\sin(\theta)}$ for all n and θ (with $\sin(\theta) \neq 0$ in the last case).

Viewing the sequence entries as polynomials in a and b, we show that they are in fact scalar multiples of Chebyshev U-polynomials.

Proposition 3.8. *For each $m \in \mathbb{Z}_{\geq 0}$, $a, b \in \mathbb{Z}_{\geq 1}$, $ab \geq 4$ we have*

$$x_{1,2m}(a,b) = y_{2,2m}(a,b) = U_{2m}\left(\frac{\sqrt{ab}}{2}\right),$$

$$y_{1,2m-1}(a,b) = \sqrt{\frac{b}{a}} \cdot U_{2m-1}\left(\frac{\sqrt{ab}}{2}\right),$$

and

$$x_{2,2m-1}(a,b) = \sqrt{\frac{a}{b}} \cdot U_{2m-1}\left(\frac{\sqrt{ab}}{2}\right),$$

where $U_n\left(\frac{\sqrt{ab}}{2}\right)$ is the Chebyshev U-polynomial $U_n(x)$ evaluated at $x = \frac{\sqrt{ab}}{2}$.

Proof. We first determine the recurrence relations for both the even and odd bisections of the Chebyshev U-polynomials. For initial values, we note that $U_1(x) = 2x$ and $U_2(x) = (2x)(2x) - 1 = 4x^2 - 1$. To find the recurrence formula for $n \geq 3$, we apply the convention $U_{-1}(x) = 0$ to find

$$\begin{aligned} U_n(x) &= (2x)U_{n-1}(x) - U_{n-2}(x) = (2x)[(2x)U_{n-2}(x) - U_{n-3}(x)] - U_{n-2}(x) \\ &= (4x^2-1)U_{n-2}(x) - (2x)U_{n-3}(x) = (4x^2-1)U_{n-2}(x) - [U_{n-2}(x) + U_{n-4}(x)] \\ &= (4x^2-2)U_{n-2}(x) - U_{n-4}(x). \end{aligned}$$

Thus, evaluating both the even and odd bisections at $x = \frac{\sqrt{ab}}{2}$ gives us the recursion relations

$$U_{2m}\left(\frac{\sqrt{ab}}{2}\right) = \begin{cases} 1, & m = 0 \\ ab - 1, & m = 1 \\ (ab-2)U_{2m-2}(\frac{\sqrt{ab}}{2}) - U_{2m-4}(\frac{\sqrt{ab}}{2}), & m \geq 2, \end{cases}$$

which is $x_{1,2m}$ by Lemma 3.3(i), and

$$U_{2m-1}\left(\frac{\sqrt{ab}}{2}\right) = \begin{cases} 0, & m = 0 \\ \sqrt{ab}, & m = 1 \\ (ab-2)U_{2m-3}(\frac{\sqrt{ab}}{2}) - U_{2m-5}(\frac{\sqrt{ab}}{2}), & m \geq 2, \end{cases}$$

which is $\sqrt{\frac{a}{b}} \cdot y_{1,2m-1}$ by Lemma 3.3(ii), and $\sqrt{\frac{b}{a}} \cdot x_{2,2m-1}$ by Lemma 3.3(iii) (switching to odd indices by Lemma 3.1). $\square$

Corollaries 3.9 and 3.10 below follow immediately from the familiar properties of Chebyshev polynomials, which are found in [14].

Corollary 3.9. *For each $m \geq 1$, $a, b \in \mathbb{Z}_{\geq 1}$, $ab \geq 4$, $x_{1,2m}$ is given by the $2m \times 2m$ determinant*

$$x_{1,2m}(a,b) = \begin{vmatrix} \sqrt{ab} & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & \sqrt{ab} & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \sqrt{ab} & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & \sqrt{ab} & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \sqrt{ab} & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & \sqrt{ab} \end{vmatrix}.$$

In the following, $\mathbb{Z}[ab]$ denotes polynomials in ab with $\mathbb{Z}$-coefficients.

Corollary 3.10. *Let $a, b \in \mathbb{Z}$, $ab \geq 4$. Then, we have:*

(i) *For $s \geq 0$,* $\quad x_{1,2s+1}(a,b) = y_{2,2s+1}(a,b) = \prod_{j=1}^{s} \left(ab - 4 \cos^2 \left(\frac{j\pi}{2s+1} \right) \right),$

(ii) *For $s > 0$,* $\quad y_{1,2s}(a,b) = b \cdot \prod_{j=1}^{s-1} \left(ab - 4 \cos^2 \left(\frac{j\pi}{2s} \right) \right),$

(iii) *For $s > 0$,* $\quad x_{2,2s}(a,b) = a \cdot \prod_{j=1}^{s-1} \left(ab - 4 \cos^2 \left(\frac{j\pi}{2s} \right) \right).$

Moreover, since the sequence entries are polynomials in a and b, the right-hand sides of (i), (ii), and (iii) are polynomials in $\mathbb{Z}[ab]$, $b \cdot \mathbb{Z}[ab]$, and $a \cdot \mathbb{Z}[ab]$, respectively.

Proof. By Proposition 3.8, for n even, $x_{1,n}$ can be written as a Chebyshev U-polynomial. As shown in [14], for $n \geq 0$, the n^{th} Chebyshev U-polynomial can be factored as

$$U_n(x) = 2^n \prod_{j=1}^{n} \left(x - \cos \left(\frac{j\pi}{n+1} \right) \right).$$

Setting $n = 2s$, and bringing 2^{2s} into the product, using that $\cos(\pi - x) = -\cos(x)$ gives

$$x_{1,2s} = x_{1,2s+1} = \prod_{j=1}^{s} \left(\sqrt{ab} - 2 \cos \left(\frac{j\pi}{2s+1} \right) \right) \left(\sqrt{ab} + 2 \cos \left(\frac{j\pi}{2s+1} \right) \right).$$

We now obtain (i) by pairing up conjugates:

$$x_{1,2s+1}(a,b) = \prod_{j=1}^{s} \left(ab - 4 \cos^2 \left(\frac{j\pi}{2s+1} \right) \right).$$

Similarly, setting $n = 2s - 1$ ($s > 0$) gives us the following identity for $y_{1,2s-1}$:

$$y_{1,2s-1} = y_{1,2s} = \sqrt{\frac{b}{a}} \cdot \prod_{j=1}^{2s-1} \left(\sqrt{ab} - 2 \cos \left(\frac{j\pi}{2s} \right) \right)$$

When $j = s$ we have the special factor $\left(\sqrt{ab} - 2\ \cos(\frac{\pi}{2})\right) = \sqrt{ab}$. Taking this out of the product allows us to pair up conjugates as before:

$$y_{1,2s} = \sqrt{\frac{b}{a}} \cdot \sqrt{ab} \prod_{j=1}^{\lfloor \frac{2s-1}{2} \rfloor} \left(\sqrt{ab} - 2\cos\left(\frac{j\pi}{2s}\right)\right)\left(\sqrt{ab} + 2\cos\left(\frac{j\pi}{2s}\right)\right)$$

$$y_{1,2s}(a,b) = b \cdot \prod_{j=1}^{s-1} \left(ab - 4\ \cos^2\left(\frac{j\pi}{2s}\right)\right).$$

Lastly, (iii) follows immediately from (ii) by Theorem 3.4(i). $\square$

It is remarkable that the products of $\mathbb{R}$-linear factors shown in Corollary 3.10 will yield polynomials in $\mathbb{Z}[ab]$. Indeed, roots of the type shown in Corollary 3.10, namely $4\cos^2(\frac{j\pi}{n}) \in \mathbb{R}$, have shown similar striking properties in the past. Studies by [13] and [8] revealed that suitable products of these roots yield integers.

We now view the sequence entries as polynomials in a and b. An immediate consequence of Corollary 3.10 is that two such polynomials will share some of the same linear factors if their indices are not coprime. For example, $x_{1,21}$ and $x_{1,7}$ share the linear factor $(ab - 4\cos^2(\frac{\pi}{7}))$, and $y_{1,10}$ and $x_{1,15}$ share the linear factors $(ab - 4\cos^2(\frac{\pi}{5}))$ and $(ab - 4\cos^2(\frac{2\pi}{5}))$. This motivates the question of cancellation: will a ratio of such polynomials sharing a common $\mathbb{R}$-linear factor yield a polynomial in $\mathbb{Z}[ab]$? Indeed, [12] has shown that this is true for Chebyshev U-polynomials. In particular, [12] found that if two U polynomials divide with remainder zero (as in the case of $\frac{x_{1,21}}{x_{1,7}}$ or $\frac{y_{1,18}}{x_{1,9}}$), the resulting product of $\mathbb{R}$-linear factors is a composition of Chebyshev polynomials. It should be noted that striking properties such as this are not uncommon for Chebyshev polynomials, which is why they have found use in so many applications.

The proof of the following theorem uses Chebyshev T-polynomials (defined earlier), which when evaluated at $x = \sqrt{ab}/2$ give us the recurrence relation:

$$T_n(\sqrt{ab}/2) = \begin{cases} 1, & n = 0 \\ \sqrt{ab}/2, & n = 1 \\ \sqrt{ab} \cdot T_{n-1}(\sqrt{ab}/2) - T_{n-2}(\sqrt{ab}/2), & n \geq 2, \end{cases}$$

Theorem 3.11. *We have*

(i) *For all $n \geq 0$ and $s > 0$,* $\dfrac{y_{1,2ns}}{y_{1,2s}}, \dfrac{x_{2,2ns}}{x_{2,2s}} \in \mathbb{Z}[\sqrt{ab}/2]$,

(ii) *For odd $n > 0$ and $s \geq 0$,* $\dfrac{x_{1,n(2s+1)}}{x_{1,2s+1}} \in \mathbb{Z}[\sqrt{ab}/2]$,

(iii) *For even $n \geq 0$ and all $s \geq 0$,* $\dfrac{y_{1,n(2s+1)}}{x_{1,2s+1}}, \dfrac{x_{2,n(2s+1)}}{x_{1,2s+1}} \in \sqrt{\dfrac{b}{a}} \cdot \mathbb{Z}[\sqrt{ab}/2]$.

Proof. (i) Let $s, n \in \mathbb{Z}_{>0}$. By Proposition 3.8, $y_{1,2ns} = \sqrt{\frac{b}{a}} U_{2ns-1}$ and $y_{1,2s} = \sqrt{\frac{b}{a}} U_{2s-1}$. The paper [12] gives the divisibility criteria for two U polynomials $U_q(x)$ and $U_p(x)$ thus: $U_q(x)$ divides $U_p(x)$ if there exists an integer $n > 0$ such that $p = n(q+1) - 1$. Moreover, the U polynomials decompose as follows:

$$\frac{U_p(x)}{U_q(x)} = \frac{U_{n(q+1)-1}(x)}{U_{(q+1)-1}(x)} = U_{n-1}(T_{q+1}(x)).$$

Since the polynomial ring $\mathbb{Z}[x]$ is also a composition ring, composing any two polynomials in $\mathbb{Z}[x]$ will yield another polynomial in $\mathbb{Z}[x]$. Thus we have

$$\frac{y_{1,2ns}}{y_{1,2s}} = \frac{\sqrt{\frac{b}{a}} \cdot U_{n(2s)-1}(\sqrt{ab}/2)}{\sqrt{\frac{b}{a}} \cdot U_{(2s)-1}(\sqrt{ab}/2)} = U_{n-1}\left(T_{2s}\left(\frac{\sqrt{ab}}{2}\right)\right) \in \mathbb{Z}[\sqrt{ab}/2].$$

Similarly, we find that for n odd,

$$\frac{x_{1,n(2s+1)}}{x_{1,2s+1}} = \frac{U_{n(2s+1)-1}(\sqrt{ab}/2)}{U_{(2s)}(\sqrt{ab}/2)} = U_{n-1}\left(T_{2s+1}\left(\frac{\sqrt{ab}}{2}\right)\right) \in \mathbb{Z}[\sqrt{ab}/2],$$

and for n even,

$$\frac{y_{1,n(2s+1)}}{x_{1,2s+1}} = \frac{\sqrt{\frac{b}{a}} \cdot U_{n(2s+1)-1}(\sqrt{ab}/2)}{U_{(2s)}(\sqrt{ab}/2)} = \sqrt{\frac{b}{a}} \cdot U_{n-1}\left(T_{2s+1}\left(\frac{\sqrt{ab}}{2}\right)\right) \in \sqrt{\frac{b}{a}} \cdot \mathbb{Z}[\sqrt{ab}/2].$$

Lastly, Theorem 3.4(iii) gives us that the remaining ratios are also in $\sqrt{\frac{b}{a}} \cdot \mathbb{Z}[\sqrt{ab}/2]$. □

It turns out that the expressions in the statement of Theorem 3.11 are integers. Indeed, we will prove that all ratios in Theorem 3.11 are polynomials in ab with coefficients in $\mathbb{Z}$, $a\mathbb{Z}$, or $b\mathbb{Z}$ (see Corollary 3.15).

We will use the following Lemmas (3.12, 3.13) to prove our Theorem.

Lemma 3.12. *For $n > 0$ and $0 \le i \le \frac{n}{2}$ we have $\frac{n}{n-i}\binom{n-i}{i} \in \mathbb{Z}$.*

Proof. This is clear for $i = 0$. For $i \ge 1$ we have

$$\begin{aligned}\frac{n}{n-i}\binom{n-i}{i} - \binom{n-i}{i} &= \frac{i}{n-i}\binom{n-i}{i} = \frac{i}{n-i} \cdot \frac{(n-i)!}{i! \cdot (n-2i)!} \\ &= \frac{(n-i-1)!}{(i-1)! \cdot (n-2i)!} = \binom{n-i-1}{i-1}.\end{aligned}$$

It follows that $\frac{n}{n-i}\binom{n-i}{i} = \binom{n-i-1}{i-1} + \binom{n-i}{i} \in \mathbb{Z}$. □

Lemma 3.13. *For $m \ge 0$ we have $T_{2m}\left(\frac{x}{2}\right) \in \frac{1}{2} \cdot \mathbb{Z}[x^2]$, $T_{2m+1}\left(\frac{x}{2}\right) \in \frac{x}{2} \cdot \mathbb{Z}[x^2]$, $U_{2m}\left(\frac{x}{2}\right) \in \mathbb{Z}[x^2]$ and $U_{2m+1}\left(\frac{x}{2}\right) \in x \cdot \mathbb{Z}[x^2]$. In particular $U_n\left(\frac{x}{2}\right) \in \mathbb{Z}[x]$ for $n \ge 0$.*

Proof. The claim about T_0 is clear since $T_0(x) = 1$. For $n \ge 1$ we have

$$T_n(x) = \frac{n}{2}\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^i}{n-i}\binom{n-i}{i} \cdot (2x)^{n-2i}$$

cf. [1, Formula 22.3.6]. This immediately implies that

$$T_n\left(\frac{x}{2}\right) = \frac{1}{2}\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^i \cdot \frac{n}{n-i}\binom{n-i}{i} \cdot x^{n-2i}.$$

By Lemma 3.12 this sum belongs to $\mathbb{Z}[x^2]$ for n even and to $x \cdot \mathbb{Z}[x^2]$ for n odd. The result concerning the Chebyshev U-polynomials follows similarly from the formula

$$U_n(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^i\binom{n-i}{i} \cdot (2x)^{n-2i} \tag{5}$$

for $n \ge 0$, cf. [1, Formula 22.3.7], except that one does not need Lemma 3.12 in this case. □

Theorem 3.14. *Let $n, s \geq 0$ be integers. Then*

(i) *For $s > 0$ we have $\frac{y_{1,2ns}}{y_{1,2s}} = \frac{x_{2,2ns}}{x_{2,2s}} \in \mathbb{Z}[ab]$.*

(ii) *For n odd, $\frac{x_{1,n(2s+1)}}{x_{1,2s+1}} \in \mathbb{Z}[ab]$.*

(iii) *For n even, $\frac{y_{1,n(2s+1)}}{x_{1,2s+1}} \in b \cdot \mathbb{Z}[ab]$ and $\frac{x_{2,n(2s+1)}}{x_{1,2s+1}} \in a \cdot \mathbb{Z}[ab]$.*

Proof. In the first case we have $\frac{y_{1,2ns}}{y_{1,2s}} = \frac{x_{2,2ns}}{x_{2,2s}} = U_{n-1}\left(T_{2s}\left(\sqrt{ab}/2\right)\right)$. Lemma 3.13 shows that $T_{2s}\left(\sqrt{ab}/2\right) = \frac{p}{2}$ with $p \in \mathbb{Z}[ab]$. Applying the last part of Lemma 3.13 shows that $U_{n-1}\left(\frac{p}{2}\right) \in \mathbb{Z}[p] \subseteq \mathbb{Z}[ab]$.

In the second case $\frac{x_{1,n(2s+1)}}{x_{1,2s+1}} = U_{n-1}\left(T_{2s+1}\left(\sqrt{ab}/2\right)\right)$. Lemma 3.13 implies that $T_{2s+1}\left(\sqrt{ab}/2\right) = \frac{\sqrt{ab}\cdot p}{2}$ with $p \in \mathbb{Z}[ab]$. Using Lemma 3.13 again implies that $U_{n-1}\left(\frac{\sqrt{ab}\cdot p}{2}\right) \in \mathbb{Z}[ab \cdot p^2] \subseteq \mathbb{Z}[ab]$.

The third claim is clear for $n = 0$. For $n > 0$ we have $\frac{y_{1,n(2s+1)}}{x_{1,2s+1}} = \sqrt{\frac{b}{a}} \cdot U_{n-1}\left(T_{2s+1}\left(\sqrt{ab}/2\right)\right)$. As above Lemma 3.13 shows that $T_{2s+1}\left(\sqrt{ab}/2\right) = \frac{\sqrt{ab}\cdot p}{2}$ with $p \in \mathbb{Z}[ab]$. Since n is even, Lemma 3.13 implies that $U_{n-1}\left(\frac{\sqrt{ab}\cdot p}{2}\right) \in \sqrt{ab} \cdot p \cdot \mathbb{Z}[ab \cdot p^2] \subseteq \sqrt{ab} \cdot \mathbb{Z}[ab]$. The last claim follows from the formula $x_{2,2k} = \frac{a}{b} \cdot y_{1,2k}$. □

Corollary 3.15. *All ratios in Theorem 3.11 are polynomials in ab with coefficients in $\mathbb{Z}$, $a\mathbb{Z}$, or $b\mathbb{Z}$.*

Theorem 3.16. *For $s \geq 0$ we have closed form expressions*

(i) $x_{1,2s} = y_{2,2s} = x_{1,2s+1} = y_{2,2s+1} = \sum_{i=0}^{s} (-1)^i \binom{2s-i}{i} (ab)^{s-i}$,

and the coefficients in this summation (modulo sign) are the shallow diagonals

$$\binom{2s}{0}, \binom{2s-1}{1}, \binom{2s-2}{2}, \ldots, \binom{s}{s}$$

of Pascal's triangle.

(ii) $y_{1,2s} = y_{1,2s-1} = \sum_{i=0}^{s-1} (-1)^i \binom{2s-1-i}{i} a^{s-1-i} b^{s-i}$, *and*

$$x_{2,2s} = x_{2,2s-1} = \sum_{i=0}^{s-1} (-1)^i \binom{2s-1-i}{i} a^{s-i} b^{s-1-i},$$

and the coefficients in both summations (modulo sign) are the shallow diagonals

$$\binom{2s-1}{0}, \binom{2s-2}{1}, \binom{2s-3}{2}, \ldots, \binom{s}{s-1}$$

of Pascal's triangle.

(iii) *When $a = b$, $s \geq 0$, we have*

$$x_{2,2s} = x_{2,2s-1} = y_{1,2s} = y_{1,2s-1} = \sum_{i=0}^{s-1} (-1)^i \binom{2s-1-i}{i} a^{2s-1-2i},$$

$$y_{2,2s} = y_{2,2s+1} = x_{1,2s} = x_{1,2s+1} = \sum_{i=0}^{s} (-1)^i \binom{2s-i}{i} a^{2s-2i}.$$

Proof. By Proposition 3.8 combined with (5) we have

$$x_{1,2s} = U_{2s}(\sqrt{ab}/2) = \sum_{i=0}^{\lfloor 2s/2 \rfloor} (-1)^i \binom{2s-i}{i} \left(2 \cdot \sqrt{ab}/2\right)^{2s-2i} = \sum_{i=0}^{s} (-1)^i \binom{2s-i}{i} \cdot (ab)^{s-i}$$

proving (i). The proof of (ii) is similar, and part (iii) follows from (i) and (ii). □

We remark that we do not recognize the expressions involving shallow diagonals in Pascal's triangle, except for the familiar formula for Fibonacci numbers that arises in $x_{1,2s}$ when $ab = 5$. We expect that this sum can be found in the literature, though we did not find a reference.

As we have shown that our sequence polynomials can be written as both products and sums, we now explore the consequence of relating Corollary 3.10 with Theorem 3.16. Denote the root $4\cos^2\left(\frac{j\pi}{n}\right)$ by $4r_j$, and let $k = ab$. Expanding the $x_{1,2s+1}$ product as in Corollary 3.10, we get (by the binomial theorem),

$$k^s - 4\left(\sum_{f=1}^{s} r_f\right)k^{s-1} + 4^2\left(\sum_{f<g}^{s} r_f r_g\right)k^{s-2} - 4^3\left(\sum_{f<g<h}^{s} r_f r_g r_h\right)k^{s-3} + \cdots \pm 4^s \prod_{j=1}^{s} r_j.$$

We may write the polynomials for $y_{1,2s}$ (resp. $x_{2,2s}$) in the same fashion, remembering the extra factor of b (resp. a).

We recognize the i^{th} coefficient as 4^i times the i^{th} elementary symmetric polynomial, $e_i(r_1, r_2, \ldots, r_s)$.

By Theorem 3.16, all x_j and y_j can be written in closed form as polynomials in a and b with coefficients from shallow diagonals of Pascal's Triangle. Since $k = ab$, these closed forms and the polynomials above are identical. Hence, the coefficients above are in fact binomial coefficients, and are therefore integers. This gives us the new identity (in the notation of elementary symmetric polynomials):

Lemma 3.17. $\binom{u}{v} = 4^v e_v\left(\cos^2\left(\frac{\pi}{u+v+1}\right), \cos^2\left(\frac{2\pi}{u+v+1}\right), \ldots, \cos^2\left(\frac{\lfloor\frac{u+v}{2}\rfloor\pi}{u+v+1}\right)\right),$ *where* $u, v \in \mathbb{Z}_{\geq 0}$.

(See [8], [13] where the number of tilings of a square with dominos is given by a similar expression).

Theorem 3.18. *Suppose that* $a, b \in \mathbb{Z}_{>0}$ *with* $ab \geq 4$. *Then*

(i) $aX_2(1, ab) = X_2(a, b)$.

(ii) *More generally, let* $q \in \mathbb{Q}_{>0}$. *Then if* $qa, b/q \in \mathbb{Z}$, *we have* $qX_2(a, b) = X_2(qa, b/q)$.

Furthermore, (ii) *allows us to generalize* (i) *as:*

$(i)'$ $qX_2(1, ab) = X_2(u, v)$ *for some* $u, v \in \mathbb{Z}_{\geq 1}$, $uv \geq 4$, *if and only if* q *is a positive integer dividing* ab.

Proof. Statements (i) and (ii) follow immediately from Proposition 3.8. We now prove the biconditional. $(\Leftarrow)$ Assume q is a positive integer dividing ab. Then by (ii),

$$qX_2(1, ab) = X_2(q, ab/q) = X_2(u, v)$$

where we have $u = q \in \mathbb{Z}_{\geq 1}$ and $v = ab/q \in \mathbb{Z}_{\geq 1}$ by assumption.
$(\Rightarrow)$ Assume $a, b \geq 1$, $ab \geq 4$, and let $q \in \mathbb{Q}$ satisfy $qX_2(1, ab) = X_2(u, v)$ for some positive integers u and v. By comparing first and second terms of the sequences we get $q = u$ and $q(ab - 2) = u(uv - 2)$. Hence $q = u$ is a positive integer and $ab = qv$. Hence q divides ab. $\square$

4. Hyperbolic golden ratios and real quadratic fields

There are well-documented identities relating Fibonacci numbers to the classical golden ratio, $\phi = \frac{1+\sqrt{5}}{2} = 1.61803399\ldots$, such as Binet's formula which expresses the n-th Fibonacci number, $F(n)$, as the linear combination

$$F(n) = \frac{\phi^n - \phi'^n}{\sqrt{5}},$$

where ϕ' is the conjugate of ϕ. In Theorem 3.6 we showed that, for each $k = ab \geq 4$, sequences $X_1(1, k)$ and $X_2(1, k)$ are generalizations of the bisected Lucas and Fibonacci sequences, respectively. In this section, we investigate our sequences further to determine if there are other generalizations of Fibonacci-type identities. We begin by recognizing that the ratios of consecutive bisected Fibonacci and Lucas numbers both tend to the square of the golden ratio,

$$\lim_{m\longrightarrow\infty}\frac{F(2m+2)}{F(2m)}=\lim_{m\longrightarrow\infty}\frac{L(2m+2)}{L(2m)}=\phi^2=2.61803399\ldots$$

This classical number, sometimes referred to by ψ and which we will call the "hyperbolic golden ratio", can indeed be generalized.

Proposition 4.1. *Let $a,b\in\mathbb{Z}_{>0}$ with $k=ab\geq 4$. Then the sequences $\left\{\frac{x_{1,n+2}(a,b)}{x_{1,n}(a,b)}\right\}_{n\geq 0}$, $\left\{\frac{x_{2,n+2}(a,b)}{x_{2,n}(a,b)}\right\}_{n\geq 1}$, $\left\{\frac{y_{1,n+2}(a,b)}{y_{1,n}(a,b)}\right\}_{n\geq 1}$ and $\left\{\frac{y_{2,n+2}(a,b)}{y_{2,n}(a,b)}\right\}_{n\geq 0}$ all converge to $\Psi(k)=\frac{1}{2}[(k-2)+\sqrt{k(k-4)}]$.*

Proof. By Remark 3.7, all numerators and denominators in the fractions above are positive and in particular the fractions are well defined. We first prove that the sequence $\left\{\frac{x_{1,n+2}}{x_{1,n}}\right\}_{n\geq 0}$ is decreasing. Since all entries are positive, this implies that it is convergent. Thus we have to show that for $n\geq 0$ we have $\frac{x_{1,n+2}}{x_{1,n}}\geq\frac{x_{1,n+3}}{x_{1,n+1}}$. For n even the numerators and denominators agree, so it suffices to consider the case $n=2m-1$, $m\geq 1$. Since $\frac{x_{1,n+2}}{x_{1,n}}=\frac{x_{1,2m}}{x_{1,2m-2}}$ we have to prove the inequality $\frac{x_{1,2m}}{x_{1,2m-2}}\geq\frac{x_{1,2m+2}}{x_{1,2m}}$ for all $m\geq 1$. We prove the inequality true by induction on m. Since k is positive, we have

$$\frac{x_{1,2}}{x_{1,0}}=\frac{k-1}{1}=\frac{k^2-2k+1}{k-1}\geq\frac{k^2-3k+1}{k-1}=\frac{x_{1,4}}{x_{1,2}},$$

and so the base case holds true. Assume the inequality holds for all $m\leq M$ for some $M\in\mathbb{Z}_{>0}$. Using Lemma 3.3 part (i), we see that

$$\frac{x_{1,2M+2}}{x_{1,2M}}=\frac{(k-2)x_{1,2M}-x_{1,2M-2}}{x_{1,2M}}=k-2-\frac{x_{1,2M-2}}{x_{1,2M}}.$$

Multiplying the induction inequality by $-\frac{x_{1,2m-2}}{x_{1,2m+2}}$ (recalling positivity of both numerator and denominator), we see that $-\frac{x_{1,2m}}{x_{1,2m+2}}\leq-\frac{x_{1,2m-2}}{x_{1,2m}}$, which gives us

$$\frac{x_{1,2M+2}}{x_{1,2M}}\geq k-2-\frac{x_{1,2M}}{x_{1,2M+2}}=\frac{(k-2)x_{1,2M+2}-x_{1,2M}}{x_{1,2M+2}}=\frac{x_{1,2M+4}}{x_{1,2M+2}},$$

and so the inequality holds true for all $m\in\mathbb{Z}_{>0}$.

This shows that $\left\{\frac{x_{1,n+2}}{x_{1,n}}\right\}_{n\geq 0}$ is convergent. Similarly one proves (using the relevant parts of Lemma 3.3) that the three other sequences are all decreasing and hence convergent.

By Lemma 3.3(i) we have $\frac{x_{1,2m}}{x_{1,2m-2}}=ab-2-\left(\frac{x_{1,2m-2}}{x_{1,2m-4}}\right)^{-1}$ and hence $\lambda=\lim_{n\to\infty}\frac{x_{1,n+2}}{x_{1,n}}$ satisfies $\lambda=ab-2-\lambda^{-1}$. Since $x_{1,n+2}\geq x_{1,n}$ by Remark 3.7 we get $\lambda\geq 1$ so $\lambda=\Psi(k)=\frac{1}{2}[(k-2)+\sqrt{k(k-4)}]$ as claimed. The proof that the remaining quotients also converge to $\Psi(k)$ is identical. $\square$

We now give two continued fraction expansions of $\Psi(k)$.

Proposition 4.2. *For $k > 4$ we have*

$$\Psi(k) = (k-2) - \cfrac{1}{(k-2) - \cfrac{1}{(k-2) - \cfrac{1}{(k-2) - \cdots}}} = (k-3) + \cfrac{1}{1 + \cfrac{1}{(k-4) + \cfrac{1}{1 + \cfrac{1}{(k-4) + \cfrac{1}{1+\cdots}}}}}$$

Proof. This follows directly from [5, Theorem 3.1]. □

The classical hyperbolic golden ratio ψ then corresponds to the special case $k = 5$.

Further generalizations of identities involving ψ and ϕ are also found. Set

$$\Psi'(k) = \frac{1}{2}[(k-2) - \sqrt{k(k-4)}].$$

then $\Psi(k)$ and $\Psi'(k)$ are the solutions of the quadratic equation

$$x^2 - (k-2)x + 1 = 0.$$

Note that for $k > 4$ we have $(k-3)^2 < k(k-4) < (k-2)^2$, so $\Psi(k)$ and $\Psi'(k)$ are algebraic integers in the real quadratic field $\mathbb{Q}(\sqrt{d})$, where d denotes the squarefree part of $k(k-4)$. Furthermore $\Psi(k)$ satisfies

$$\Psi(k) = (k-2) - \frac{1}{\Psi(k)}.$$

Multiplying this equation by $\Psi(k)^n$ for any $n \in \mathbb{Z}$ gives the identity

$$\Psi(k)^{n+1} = (k-2)\Psi(k)^n - \Psi(k)^{n-1}. \tag{6}$$

Since $\Psi(k)\Psi'(k) = 1$, $\Psi(k)$ and $\Psi'(k)$ are units in the ring of algebraic integers in $\mathbb{Q}(\sqrt{d})$. Now let $\Phi(k) = \Psi(k) - 1$ (a clear analog of the golden ratio ϕ). Then

$$\Phi(k) = \frac{1}{2}[(k-4) + \sqrt{k(k-4)}].$$

If we set

$$\Phi'(k) = \frac{1}{2}[(k-4) - \sqrt{k(k-4)}].$$

then $\Phi'(k)\Phi(k) = -(k-4)$, $\Psi(k) - \Psi'(k) = \Phi(k) - \Phi'(k) = \sqrt{k(k-4)}$ and $\Phi(k)$ and $\Phi'(k)$ are the solutions of the quadratic equation

$$x^2 - (k-4)x - (k-4) = 0,$$

so $\Phi(k)$ and $\Phi'(k)$ are algebraic integers in the real quadratic field $\mathbb{Q}(\sqrt{d})$ and $\Phi(k)$ is the continued fraction

$$\Phi(k) = (k-4)[1 + \frac{1}{\Phi(k)}].$$

As before, multiplying this equation by $\Phi(k)^n$ for any $n \in \mathbb{Z}$ gives the identity

$$\Phi(k)^{n+1} = (k-4)[\Phi(k)^n + \Phi(k)^{n-1}]. \tag{7}$$

Substituting $k = 5$ recovers the well-known identity relating powers of the classical golden ratio, ϕ.

The following is a generalization of Binet's formula for the Fibonacci numbers.

Proposition 4.3. *Let $a, b \in \mathbb{Z}_{>0}$ with $k = ab > 4$. For $m \geq 1$ we have*

$$x_{1,2m-1}(a,b) = y_{2,2m-1}(a,b) = \frac{1}{\Phi(k)}\Psi(k)^m + \frac{1}{\Phi'(k)}\Psi'(k)^m = \frac{\Phi(k)^{2m-1} + \Phi'(k)^{2m-1}}{(k-4)^m}, \tag{8}$$

and for $m \geq 0$ we have

$$y_{1,2m}(a,b) = \sqrt{\frac{b}{a(ab-4)}} \cdot (\Psi(k)^m - \Psi'(k)^m) = \sqrt{\frac{b}{a(ab-4)}} \cdot \frac{\Phi(k)^{2m} - \Phi'(k)^{2m}}{(k-4)^m}, \tag{9}$$

$$x_{2,2m}(a,b) = \sqrt{\frac{a}{b(ab-4)}} \cdot (\Psi(k)^m - \Psi'(k)^m) = \sqrt{\frac{a}{b(ab-4)}} \cdot \frac{\Phi(k)^{2m} - \Phi'(k)^{2m}}{(k-4)^m}. \tag{10}$$

Proof. From Lemma 3.1 and Lemma 3.3 it follows that the sequence $\{x_{1,2m-1}\}_{m\geq 1} = \{y_{2,2m-1}\}_{m\geq 1}$ satisfies the recurrence $X(m) = (ab-2)X(m-1) - X(m-2)$, $m \geq 3$. The set of solutions to this recurrence is clearly a two dimensional vector space, and if $\lambda^2 = (ab-2)\lambda - 1$, then the sequence $\{\lambda^m\}_{m\geq 1}$ is a solution. Since the solutions to this equation are given by $\lambda = \Psi(k)$ and $\lambda = \Psi'(k)$, it follows that $\{\Psi(k)^m\}_{m\geq 1}$ and $\{\Psi'(k)^m\}_{m\geq 1}$ form a basis for the set of solutions to the recurrence relation. Hence there exists unique constants c_1 and c_2 such that $x_{1,2m-1} = y_{2,2m-1} = c_1\Psi(k)^m + c_2\Psi'(k)^m$ for $m \geq 1$. The values of the constants are easily computed from the values for $m = 1$ and $m = 2$. The second formula in (8) follows from the first using the relation $\Psi(k) = \Phi(k)^2/(k-4)$ and its conjugate $\Psi'(k) = \Phi'(k)^2/(k-4)$. The identities (9) and (10) are proved similarly. □

Now let $a = 1$ and $b = 5$. Then $k = 5$ and

$$\Phi(5) = \frac{1}{2}[1 + \sqrt{5}]$$

is the classical golden ratio $\phi = 1.61803399\ldots$, and as we have already seen, $\Psi(5)$ is the classical golden ratio $\psi = 1 + \phi$ for the bisected Fibonacci sequence. Proposition 4.3 and equations (6), (7), and (10) are generalizations of identities and recurrences of these classical objects.

The following lemma follows directly from Proposition 4.1.

Lemma 4.4. *For $i = 1, 2$, Ψ is the limit of the difference of x coordinates for adjacent pairs of real roots:*

$$\Psi(k) = \lim_{n\longrightarrow\infty} \frac{x_{i,n+2} - x_{i,n}}{x_{i,n} - x_{i,n-2}}.$$

A similar identity holds for $y_{1,n}$ and $y_{2,n}$. This lemma shows that the limit of the ratio of the differences of x-coordinates of adjacent pairs of real roots is the golden ratio $\Psi(k)$. This can also be used to show that for each $k = ab \geq 4$, $\Psi(k)$ is the limit of ratios of areas of hyperbolic triangles spanned by a triple of adjacent roots in the root system for a rank 2 Kac-Moody algebra. This configuration of hyperbolic triangles is depicted in Fig. 1.

5. Eigenvalues of Weyl transformations

Let $w \in W$ with $n = l(w)$. If

$$w = \begin{cases} (w_1w_2)^m & n = 2m, m \in \mathbb{Z}_{>0} \\ w_2(w_1w_2)^m & n = 2m+1, m \in \mathbb{Z}_{>0} \end{cases}$$

then w has matrix

$$M_w = \begin{pmatrix} x_{1,n} & -x_{2,n-1} \\ y_{1,n} & -x_{1,n-1} \end{pmatrix}.$$

If

$$w = \begin{cases} (w_2w_1)^m & n = 2m, m \in \mathbb{Z}_{>0} \\ w_1(w_2w_1)^m & n = 2m+1, m \in \mathbb{Z}_{>0} \end{cases}$$

then w has matrix

$$M_w = \begin{pmatrix} -x_{1,n-1} & x_{2,n} \\ -y_{1,n-1} & x_{1,n} \end{pmatrix}.$$

In both cases, the characteristic polynomial is

$$\lambda(n)^2 - (x_{1,n} - x_{1,n-1})\lambda(n) + \det(M_w).$$

Since w is a product of the simple Weyl reflections w_1 and w_2 we obtain

$$\det(M_w) = (-1)^{l(w)} = (-1)^n.$$

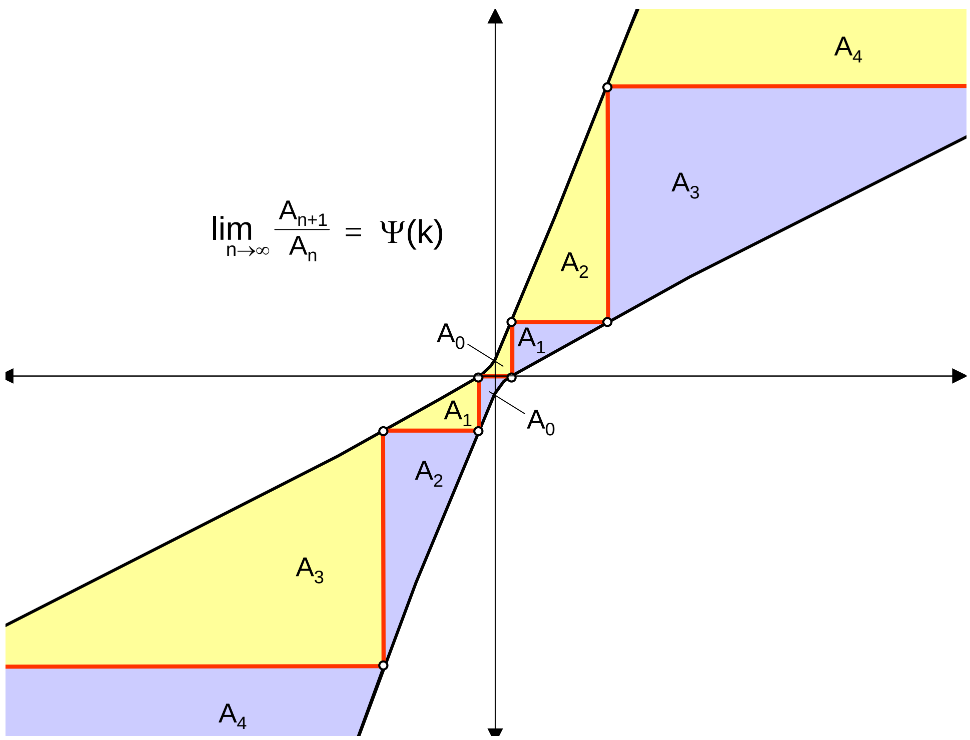

Fig 1.

Moreover $x_{1,n} - x_{1,n-1} = 0$ when n is odd, so the characteristic polynomial becomes

$$\begin{cases} \lambda(n)^2 - 1 & n \text{ odd} \\ \lambda(n)^2 - (x_{1,n} - x_{1,n-2})\lambda(n) + 1 & n \text{ even.} \end{cases}$$

It follows that $\lambda(n) = \pm 1$ when n is odd, and when $n = 2$

$$\lambda(2) = \frac{1}{2}[(ab-2) \pm \sqrt{ab(ab-4)}] = \Psi(ab) \text{ or } \Psi'(ab).$$

A basic result from linear algebra shows that for $n = 2m$, $m \in \mathbb{Z}_{>0}$,

$$\lambda(2m) = \Psi(ab)^m \text{ or } \Psi'(ab)^m.$$

Hence for each $k = ab \geq 4$, $m \in \mathbb{Z}_{>0}$, we obtain:

$\Psi(k)^m$ and $\Psi'(k)^m$ are the eigenvalues for $(w_1 w_2)^m$ and $(w_2 w_1)^m$, respectively.

We believe that for a linear recursion in $\mathbb{R}^2$, it is known that the eigenvalues are quadratic units when the matrix of the recursion has determinant 1 or -1, though we did not find a suitable reference.

6. Identities with continued fractions

We now fix a real quadratic field $\mathbb{Q}(\sqrt{d})$, where $d \in \mathbb{Z}_{\geq 2}$ is squarefree. Let $\mathcal{O}$ be the ring of algebraic integers in $\mathbb{Q}(\sqrt{d})$. For $k > 4$, let $\Psi(k) = \frac{1}{2}[(k-2) + \sqrt{k(k-4)}]$ as in Section 4. If $\Psi(k) \in \mathbb{Q}(\sqrt{d})$, we showed that $\Psi(k)$ is a unit in $\mathcal{O}$. By Dirichlet's unit theorem, $\Psi(k)$ must be a power η^n of the fundamental unit η of $\mathbb{Q}(\sqrt{d})$, cf. [3]. The following theorem shows that the only possible values for n are $n = 1$ and 2.

Theorem 6.1. *Let $d \in \mathbb{Z}_{\geq 2}$ be squarefree. Then there exists an integer $k > 4$ such that $\Psi(k) \in \mathbb{Q}(\sqrt{d})$, where $\Psi(k) = \frac{1}{2}[(k-2) + \sqrt{k(k-4)}]$ as above. If k is the smallest such integer, then $\Psi(k) = \eta$ or η^2, where η is the fundamental unit in $\mathbb{Q}(\sqrt{d})$.*

Proof. The fundamental unit of $\mathbb{Q}(\sqrt{d})$ has the form $\eta = x + y\omega(d)$ where $x, y \in \mathbb{Z}_{\geq 0}$ and

$$\omega(d) = \begin{cases} \frac{1}{2}(1+\sqrt{d}) & \text{if } d \equiv 1 \pmod 4 \\ \sqrt{d} & \text{otherwise.} \end{cases}$$

Clearly the norm of η, $N(\eta) = \eta\bar{\eta}$ equals ± 1. Note that $N(\eta) = x^2 - dy^2$ when $d \not\equiv 1 \pmod 4$ and that $N(\eta) = \frac{1}{4}\left((2x+y)^2 - dy^2\right)$ for $d \equiv 1 \pmod 4$. We consider the following cases:

- $N(\eta) = 1$ and $d \not\equiv 1 \pmod 4$: With $k' = 2x + 2$ we have

  $$\Psi(k') = \frac{1}{2}\left(2x + \sqrt{(2x+2)(2x-2)}\right) = x + \sqrt{x^2 - 1} = x + \sqrt{dy^2} = \eta$$

  since $x^2 - dy^2 = 1$.
- $N(\eta) = 1$ and $d \equiv 1 \pmod 4$: A similar computation shows that with $k' = 2x + y + 2$ we have $\Psi(k') = \eta$.
- $N(\eta) = -1$ and $d \not\equiv 1 \pmod 4$: Taking $k' = 4x^2 + 4$ we get

  $$\begin{aligned}\Psi(k') &= \frac{1}{2}\left(4x^2 + 2 + \sqrt{(4x^2+4)\cdot 4x^2}\right) = 2x^2 + 1 + 2x\sqrt{x^2+1} = (x^2 + dy^2) + 2x\sqrt{dy^2} \\ &= (x^2 + dy^2) + 2xy\sqrt{d} = \eta^2,\end{aligned}$$

  using that $x^2 - dy^2 = -1$.
- $N(\eta) = -1$ and $d \equiv 1 \pmod 4$: It is easy to verify that with $k' = (2x+y)^2 + 4$ we have $\Psi(k') = \eta^2$.

To sum up we have shown that there exists an integer $k' > 4$ such that

$$\Psi(k') = \begin{cases} \eta & \text{for } N(\eta) = 1 \\ \eta^2 & \text{for } N(\eta) = -1. \end{cases}$$

Now let $k > 4$ be the smallest integer such that $\Psi(k) \in \mathbb{Q}(\sqrt{d})$. Since $\Psi(k)$ is a unit in $\mathcal{O}$ we must have $\Psi(k) = \eta^n$ for some $n \geq 1$, and clearly n must be even if $N(\eta) = -1$. It follows that $\Psi(k) = \Psi(k')^m$ for some $m \geq 1$. By minimality of k, we must have $m = 1$ proving the result. $\square$

Let d be squarefree and let $k_1 > 4$ be the smallest positive integer for which $\Psi(k_1) \in \mathbb{Q}(\sqrt{d})$. Let $S = \{\dots, \pm\Psi(k_1)^{-3}, \pm\Psi(k_1)^{-2}, \pm\Psi(k_1)^{-1}, \pm 1, \pm\Psi(k_1), \pm\Psi(k_1)^2, \dots\}$. Then by the proof of Theorem 6.1, S coincides with the set of norm 1 units of $\mathcal{O}$. Moreover, for each $j \geq 1$,

$$\Psi(k_1)^j = \Psi(k_j),$$

where

$$k_1 < k_2 < k_3 < \cdots < k_j < \dots$$

are the integers $k_j > 4$ such that $\Psi(k_j) \in \mathbb{Q}(\sqrt{d})$.

In $\mathbb{Q}(\sqrt{5})$, the sequence

$$\{\Psi(5),\ \Psi(9), \Psi(20), \Psi(49), \dots\}$$

is the sequence

$$\{\psi,\ \psi^2,\ \psi^3,\ \psi^4, \dots\},$$

for $\psi = 1 + \phi = 2.61803399\dots$, which are eigenvalues for

$$(w_1w_2),\ (w_1w_2)^2,\ (w_1w_2)^3,\ (w_1w_2)^4, \dots$$

in $\mathfrak{g}(1,5)$. Note that in this case $k_j = L_{2j} + 2$ is given in terms of the Lucas numbers L_{2j}. It can also be shown that the bisection

$$\{\Psi(9),\ \Psi(49), \dots\}$$

are

$$\{\psi^2,\ \psi^4, \dots\},$$

eigenvalues for

$$(w_1w_2),\ (w_1w_2)^2,\ \dots$$

in $\mathfrak{g}(3,3)$.

Corollary 6.2. *The relation $\Psi(k_s) = \Psi(k_1)^s$ gives an infinite family of identities expressing the continued fraction for $\Psi(k_s)$ as the s-th power of the continued fraction for $\Psi(k_1)$:*

$$\Psi(k_s) = (k_s - 2) - \cfrac{1}{(k_s - 2) - \cfrac{1}{(k_s - 2) - \cdots}} = \left[(k_1 - 2) - \cfrac{1}{(k_1 - 2) - \cfrac{1}{(k_1 - 2) - \cdots}} \right]^s$$

where $4 < k_1 < k_2 < \dots$ are the integers such that $\Psi(k_s) \in \mathbb{Q}(\sqrt{d})$.

7. Generalized binomial coefficients

In this section we define a new sequence $(z_n)_{n\geq 0}$ in terms of the sequences $(x_{2,2n})_{n\geq 0}$ and $(x_{1,2n+1})_{n\geq 0}$. We use the sequence $(z_n)_{n\geq 0}$ to define generalized binomial coefficients, which remarkably turn out to be integers. We provide an arithmetic proof of this, established previously by Kitchloo using topological methods ([9]). In [9], Kitchloo also showed that these integers appear in the cohomology of flag varieties associated to rank 2 Kac-Moody groups over $\mathbb{C}$.

Definition 7.1. *For natural numbers a and b with $ab \geq 4$ we define the sequence $(z_n)_{n\geq 0}$ by $z_{2n} = x_{2,2n}/a$ and $z_{2n+1} = x_{1,2n+1}$.*

Note that by Theorem 3.4 we have $x_{1,2n} = y_{2,2n} = z_{2n+1}$, $x_{2,2n} = az_{2n}$ and $y_{1,2n} = bz_{2n}$ so the sequence $(z_n)_{n\geq 0}$ encodes all the previous sequences.

Proposition 7.2. *The sequence $(z_n)_{n\geq 0}$ is determined by the recursion $z_0 = 0$, $z_1 = 1$, $z_2 = 1$, $z_3 = ab - 1$ and $z_n = (ab - 2)z_{n-2} - z_{n-4}$ for $n \geq 4$. In particular $z_n \in \mathbb{Z}[ab]$ for $n \geq 0$. Moreover $z_n > 0$ for $n \geq 1$.*

Proof. The values of z_n, $0 \le n \le 3$, are easily checked. For n even, $n \ge 4$, the definition of z_n shows that

$$z_n = \frac{x_{2,n}}{a} = \frac{(ab-2)x_{2,n-2} - x_{2,n-4}}{a} = (ab-2)z_{n-2} - z_{n-4}$$

by Lemma 3.3(iii). Similarly, by definition and Lemma 3.1, $z_{2n+1} = x_{1,2n+1} = x_{1,2n}$, so the recursion formula for n odd, $n \ge 5$, follows from Lemma 3.3(i). This proves the recursion and from this it follows directly that $z_n \in \mathbb{Z}[ab]$ for $n \ge 0$. Finally, Remark 3.7 shows that $z_n > 0$ for $n \ge 1$. ☐

Note that it follows from Proposition 3.8 and Lemma 3.1 that we have the following formula for z_n in terms of the Chebyshev U-polynomial

$$z_n = \begin{cases} \frac{1}{\sqrt{ab}} \cdot U_{n-1}\left(\frac{\sqrt{ab}}{2}\right) & \text{for } n \text{ even,} \\ U_{n-1}\left(\frac{\sqrt{ab}}{2}\right) & \text{for } n \text{ odd.} \end{cases} \tag{11}$$

Proposition 7.3. *For integers $m \ge 1$ and $n \ge 0$ we have*

$$z_{m+n} = \begin{cases} z_{n+1}z_m - z_{m-1}z_n & \text{if } m \equiv 0 \pmod 2,\ n \equiv 0 \pmod 2, \\ ab \cdot z_{n+1}z_m - z_{m-1}z_n & \text{if } m \equiv 0 \pmod 2,\ n \equiv 1 \pmod 2, \\ z_{n+1}z_m - ab \cdot z_{m-1}z_n & \text{if } m \equiv 1 \pmod 2,\ n \equiv 0 \pmod 2, \\ z_{n+1}z_m - z_{m-1}z_n & \text{if } m \equiv 1 \pmod 2,\ n \equiv 1 \pmod 2. \end{cases}$$

Proof. Note that we have the following trigonometric identity

$$\frac{\sin((m+n)\theta)}{\sin(\theta)} = \frac{\sin((n+1)\theta)}{\sin(\theta)} \cdot \frac{\sin(m\theta)}{\sin(\theta)} - \frac{\sin((m-1)\theta)}{\sin(\theta)} \cdot \frac{\sin(n\theta)}{\sin(\theta)}$$

for $\theta \ne p\pi$, $p \in \mathbb{Z}$. Using the definition of the Chebyshev U-polynomials this translates into the identity

$$U_{m+n-1}(x) = U_n(x) \cdot U_{m-1}(x) - U_{m-2}(x) \cdot U_{n-1}(x).$$

Letting $x = \frac{\sqrt{ab}}{2}$ and using that for $k \ge -1$ we have

$$U_k\left(\frac{\sqrt{ab}}{2}\right) = \begin{cases} z_{k+1} & \text{for } k \text{ even,} \\ \sqrt{ab} \cdot z_{k+1} & \text{for } k \text{ odd,} \end{cases}$$

(cf. (11)) now proves the result. ☐

Corollary 7.4. *For $m, n \ge 0$ we have $z_{m+n} \in z_m\mathbb{Z}[ab] + z_n\mathbb{Z}[ab]$.*

Proof. This is clear for $m = 0$, and for $m \ge 1$ the claim follows directly from Proposition 7.3 and Proposition 7.2. ☐

Definition 7.5. *Let $m, n \ge 1$. The* generalized binomial coefficient *$Z(m,n)$ is defined by*

$$Z(m,n) = \frac{z_{m+n} \cdot z_{m+n-1} \cdot \ldots \cdot z_1}{(z_m \cdot z_{m-1} \cdot \ldots \cdot z_1) \cdot (z_n \cdot z_{n-1} \cdot \ldots \cdot z_1)}$$

Note that this expression makes sense by Proposition 7.2.

Lemma 7.6. *For $m, n \ge 2$ we have the Pascal triangle type relation*

$$Z(m,n) \in Z(m-1,n)\mathbb{Z}[ab] + Z(m,n-1)\mathbb{Z}[ab].$$

Proof. After cancelling the common factor $z_{m+n-1} \cdot \ldots \cdot z_1$ and multiplying by $(z_m \cdot \ldots \cdot z_1) \cdot (z_n \cdot \ldots \cdot z_1)$ on both sides, we get the equivalent statement $z_{m+n} \in z_m\mathbb{Z}[ab] + z_n\mathbb{Z}[ab]$. This follows directly from Corollary 7.4. ☐

Theorem 7.7. *For $m, n \ge 1$ we have $Z(m,n) \in \mathbb{Z}[ab]$.*

Proof. Note first that by definition $Z(1,n) = \frac{z_{n+1}}{z_1} = z_{n+1} \in \mathbb{Z}[ab]$. Similarly $Z(m,1) = z_{m+1} \in \mathbb{Z}[ab]$. This proves the result if $m = 1$ or $n = 1$. For $m, n \geq 2$ Lemma 7.6 shows that the result holds for (m,n) if it holds for $(m-1,n)$ and for $(m,n-1)$. Induction on $m+n$ combined with the above now shows that the result holds for all (m,n). $\square$

Remark 7.8. *In* [9, §10] *Kitchloo considers the sequences* $(c_n)_{n\geq 0}$ *and* $(d_n)_{n\geq 0}$ *given by* $c_0 = d_0 = 0$, $c_1 = d_1 = 1$ *and* $c_n = ad_{n-1} - c_{n-2}$, $d_n = bc_{n-1} - d_{n-2}$ *for* $n \geq 2$. *By induction it is easy to see that* $c_n = az_n$, $d_n = bz_n$ *for* n *even and* $c_n = d_n = z_n$ *for* n *odd. Using this it follows that the numbers*

$$C(m,n) = \frac{c_{m+n} \cdot \ldots \cdot c_1}{(c_m \cdot \ldots \cdot c_1) \cdot (c_n \cdot \ldots \cdot c_1)}, \qquad D(m,n) = \frac{d_{m+n} \cdot \ldots \cdot d_1}{(d_m \cdot \ldots \cdot d_1) \cdot (d_n \cdot \ldots \cdot d_1)}$$

are given by $C(m,n) = aZ(m,n)$ *and* $D(m,n) = bZ(m,n)$ *if* $m \equiv n \equiv 1 \pmod 2$ *and* $C(m,n) = D(m,n) = Z(m,n)$ *otherwise. In particular these numbers are integers. In* [9, Thm. 10.3] *this is proved by showing that these numbers occur as structure coefficients in the cohomology ring* $H^*(G/P;\mathbb{Z})$ *where* G *is the Kac-Moody group corresponding to the generalized Cartan matrix* $\begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$ *and* P *is a maximal parabolic subgroup of* K.

8. Aritmetic properties of the sequence $(z_n)_{n\geq 0}$

In this section we compute the p-adic valuation of the numbers z_n, see Theorem 8.5. We first need a number of auxiliary results.

Lemma 8.1. *We have* $\gcd(z_n, z_{n-1}) = 1$ *for* $n \geq 1$.

Proof. For $m \geq 1$, Proposition 7.3 shows that

$$z_{m+1} = \begin{cases} ab \cdot z_m - z_{m-1} & \text{for } m \text{ even, } m \geq 2 \\ z_m - z_{m-1} & \text{for } m \text{ odd, } m \geq 1. \end{cases} \tag{12}$$

It follows that $\gcd(z_{m+1}, z_m) = \gcd(z_m, z_{m-1})$ for all $m \geq 1$. By induction we now get $\gcd(z_n, z_{n-1}) = \ldots = \gcd(z_1, z_0) = \gcd(1, 0) = 1$ for all $n \geq 1$. $\square$

Lemma 8.2. *If* p *is a prime number with* $p \mid ab$ *and* $p \mid z_n$ *then* n *is even.*

Proof. For $s \geq 0$ Theorem 3.16(i) shows that

$$z_{2s+1} = \sum_{i=0}^{s} (-1)^i \binom{2s-i}{i} (ab)^{s-i} \equiv (-1)^s \binom{2s-s}{s} = (-1)^s \pmod p,$$

since $p \mid ab$. Hence n must be even. $\square$

Proposition 8.3. *Let* p *be a prime number. Then there exists a natural number* k *such that* $p \mid z_n$ *if and only if* $k \mid n$.

Proof. Let $X = \{n \geq 0 \mid z_n \equiv 0 \pmod p\}$. Since $z_0 = 0$ we have $0 \in X$. It is clear from Proposition 7.3 that $i, j \in X \implies i + j \in X$. Now suppose $i, j \in X$ with $1 \leq i \leq j$. Setting $m = i$ and $n = j - i$ in Proposition 7.3 shows that $p \mid ab \cdot z_{i-1} z_{j-i}$ if $i \equiv 1 \pmod 2$ and $j - i \equiv 0 \pmod 2$ and that $p \mid z_{i-1} z_{j-i}$ otherwise. However by Lemma 8.2, $p \mid ab$ and $p \mid z_i$ implies that i is even so the first case does not occur. Hence $p \mid z_{i-1} z_{j-i}$. By Lemma 8.1 $\gcd(z_{i-1}, z_i) = 1$ so $p \nmid z_{i-1}$. It follows that $p \mid z_{j-i}$ so $j - i \in X$. The above properties show that $X = k\mathbb{Z}_{\geq 0}$ for a unique $k \geq 0$. It remains to show that $k \neq 0$, that is $X \neq \{0\}$. Since there are only p^4 possibilities for $(z_n, z_{n+1}, z_{n+2}, z_{n+3})$ modulo p we can find $0 \leq i < j$ such that $z_j \equiv z_i \pmod p$, $z_{j+1} \equiv z_{i+1} \pmod p$, $z_{j+2} \equiv z_{i+2} \pmod p$ and $z_{j+3} \equiv z_{i+3} \pmod p$. Since $z_{n-4} = (ab-2)z_{n-2} - z_n$ for $n \geq 4$ we see that

$$z_{j-1} \equiv z_{i-1} \pmod p, \qquad z_{j-2} \equiv z_{i-2} \pmod p, \qquad \ldots, \qquad z_{j-i} \equiv z_0 = 0 \pmod p.$$

Hence $X \neq \{0\}$ as desired. $\square$

Remark 8.4. *In the language of* [10] *the preceding proposition says that the sequence* $(z_n)_{n\geq 0}$ *is* regularly divisible*. By* [10, Cor. 2] *this implies that the numbers* $Z(m,n)$ *are integers.*

Theorem 8.5. *Let* p *be a prime number and let* k *be the smallest natural number with* $p\,|\,z_k$ *(cf. Proposition 8.3). For* $n\geq 1$*, the* p*-adic valuation of* z_n *is given by*

$$\nu_2(z_n) = \begin{cases} 0 & \text{if } k\nmid n, \\ \nu_2(z_k) & \text{if } k\,|\,n \text{ and } 2k\nmid n, \\ \nu_2(z_{2k}) + \nu_2(\frac{n}{2k}) & \text{if } 2k\,|\,n, \end{cases} \tag{13}$$

for $p=2$ *and*

$$\nu_p(z_n) = \begin{cases} 0 & \text{if } k\nmid n, \\ \nu_p(z_k) + \nu_p(\frac{n}{k}) & \text{if } k\,|\,n. \end{cases} \tag{14}$$

for p *odd.*

Remark 8.6. *This is proved for* p *odd in* [9, Thm. 10.10] *by considering* $H^*(K;\mathbb{F}_p)$ *where* K *is the Kac-Moody group associated to the generalized Cartan matrix* $\begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$.

Definition 8.7. *We define the sequence* $(y_n)_{n\geq 0}$ *of integers by* $y_0=2$ *and* $y_n = z_{n+1}-z_{n-1}$ *for* $n\geq 1$.

Lemma 8.8. *We have* $y_n>0$ *for all* $n\geq 0$.

Proof. By definition we have $y_0=2$, $y_1=1$, $y_2=ab-2$ and $y_3=ab-3$. Moreover from the recursion relation for $(z_n)_{n\geq 0}$ in Proposition 7.2 it follows that $y_n=(ab-2)y_{n-2}-y_{n-4}$ for $n\geq 4$. We now prove by induction that $y_n\geq y_{n-2}$ for $n\geq 2$. This is clear for $n=2$ and $n=3$ by the above computations. Suppose that the inequality holds for all n with $2\leq n\leq N$ for some $N\geq 3$. Then

$$y_{N+1}-y_{N-1} = (ab-3)y_{N-1}-y_{N-3} \geq y_{N-1}-y_{N-3} \geq 0$$

proving the claim. The inequality combined with induction shows immediately that $y_n\geq y_0=2$ for n even and $y_n\geq y_1=1$ for n odd. This proves the result. □

Proposition 8.9. *The sequence* $(y_n)_{n\geq 0}$ *may be expressed in terms of Chebyshev* T*-polynomials as follows*

$$y_n = \begin{cases} 2\cdot T_n(\sqrt{ab}/2) & \text{for } n \text{ even}, \\ \frac{2}{\sqrt{ab}}\cdot T_n(\sqrt{ab}/2) & \text{for } n \text{ odd}. \end{cases} \tag{15}$$

Proof. For $n=0$ this is clear since $T_0(x)=1$. Assume now that $n\geq 1$. We have the trigonometric identity

$$\frac{\sin((n+1)\theta)}{\sin(\theta)} - \frac{\sin((n-1)\theta)}{\sin(\theta)} = 2\cos(n\theta)$$

for $\theta\neq p\pi$, $p\in\mathbb{Z}$ which translates into the identity $U_n(x)-U_{n-2}(x)=2\cdot T_n(x)$. Formula (15) follows from this identity combined with the definition $y_n=z_{n+1}-z_{n-1}$ and (11). □

Lemma 8.10. *We have* $\gcd(z_n,y_n)\in\{1,2\}$ *for* $n\geq 0$.

Proof. The claim is clear for $n=0$ since $z_0=0$ and $y_0=2$. For $n\geq 1$, (12) shows that

$$y_n = z_{n+1}-z_{n-1} = \begin{cases} z_n - 2z_{n-1} & \text{for } n \text{ odd}, n\geq 1, \\ ab\cdot z_n - 2z_{n-1} & \text{for } n \text{ even}, n\geq 2. \end{cases}$$

Hence $\gcd(z_n,y_n)=\gcd(z_n,2z_{n-1})$, so the result follows from Lemma 8.1. □

Proposition 8.11. *For integers $m \geq 1$ and $n \geq 0$ we have*

$$z_{mn} = \frac{1}{2^{m-1}} \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2j+1} z_n^{2j+1} y_n^{m-1-2j} \cdot (ab)^j \cdot (ab-4)^j \qquad \textit{for } n \textit{ even,}$$

$$z_{mn} = \frac{1}{2^{m-1}} \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2j+1} z_n^{2j+1} y_n^{m-1-2j} \cdot (ab)^{\lfloor \frac{m-1}{2} \rfloor - j} \cdot (ab-4)^j \qquad \textit{for } n \textit{ odd.}$$

Proof. Note that for $\sin(\theta) \neq 0$ we have

$$\begin{aligned}
\sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2j+1} \cdot \left(\frac{\sin(n\theta)}{\sin(\theta)}\right)^{2j+1} &\cdot \cos(n\theta)^{m-1-2j} \cdot (-\sin(\theta)^2)^j \\
&= \frac{1}{i\sin(\theta)} \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2j+1} \cdot (i\sin(n\theta))^{2j+1} \cdot \cos(n\theta)^{m-(2j+1)} \\
&= \frac{1}{i\sin(\theta)} \cdot \frac{1}{2} \cdot \big((\cos(n\theta) + i\sin(n\theta))^m - (\cos(n\theta) - i\sin(n\theta))^m\big) \\
&= \frac{1}{2i\sin(\theta)} \cdot (e^{imn\theta} - e^{-imn\theta}) = \frac{\sin(mn\theta)}{\sin(\theta)}.
\end{aligned}$$

This translates into the following identity

$$U_{mn-1}(x) = \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2j+1} \cdot U_{n-1}(x)^{2j+1} \cdot T_n(x)^{m-1-2j} \cdot (x^2-1)^j \tag{16}$$

between Chebyshev polynomials. The formulas of the proposition follow easily from this identity by setting $x = \sqrt{ab}/2$ and using that

$$U_{k-1}\left(\frac{\sqrt{ab}}{2}\right) = \begin{cases} \sqrt{ab} \cdot z_k & \text{for } k \text{ even,} \\ z_k & \text{for } k \text{ odd,} \end{cases} \qquad T_k\left(\frac{\sqrt{ab}}{2}\right) = \begin{cases} \frac{1}{2} \cdot y_k & \text{for } k \text{ even,} \\ \frac{\sqrt{ab}}{2} \cdot y_k & \text{for } k \text{ odd,} \end{cases}$$

for $k \geq 0$ (cf. (11) and Proposition 8.9). ☐

Proposition 8.12. *Let $n \geq 1$ and $p \neq 2$ be a prime number with $p \mid z_n$. Then $\nu_p(z_{pn}) = \nu_p(z_n) + 1$ and $\nu_p(z_{mn}) = \nu_p(z_n)$ for $m \geq 1$ with $p \nmid m$.*

Proof. Note first that $z_n \neq 0$ since $n \geq 1$. Set $e = \nu_p(z_n) \geq 1$. Since $\gcd(z_n, y_n) \mid 2$ by Lemma 8.10 we must have $p \nmid y_n$. Recall that if $p \mid ab$ then n is even by Lemma 8.2. Now let $m \geq 1$ and consider the formula for z_{mn} from Proposition 8.11. From the above it follows that the term for $j = 0$ in the sum for z_{mn} has p-adic valuation equal to $\nu_p(m) + e$. This equals $e+1$ for $m = p$ and e for $p \nmid m$. The remaining terms all have p-adic valuations which are $\geq 3e > e+1$, so $\nu_p(z_{mn})$ is given by $e+1$ for $m = p$ and by e for $p \nmid m$ as claimed. ☐

Lemma 8.13. *The values of z_n and y_n modulo 4 are periodic and are given as follows.*

(i) *For $ab \equiv 0 \pmod 4$ we have*

n mod 8	0	1	2	3	4	5	6	7
z_n mod 4	0	1	1	3	2	1	3	3
y_n mod 4	2	1	2	1	2	1	2	1

(ii) *For* $ab \equiv 1 \pmod 4$ *we have*

$n \bmod 6$	0	1	2	3	4	5
$z_n \bmod 4$	0	1	1	0	3	3
$y_n \bmod 4$	2	1	3	2	3	1

(iii) *For* $ab \equiv 2 \pmod 4$ *we have*

$n \bmod 8$	0	1	2	3	4	5	6	7
$z_n \bmod 4$	0	1	1	1	0	3	3	3
$y_n \bmod 4$	2	1	0	3	2	3	0	1

(iv) *For* $ab \equiv 3 \pmod 4$ *we have*

$n \bmod 12$	0	1	2	3	4	5	6	7	8	9	10	11
$z_n \bmod 4$	0	1	1	2	1	1	0	3	3	2	3	3
$y_n \bmod 4$	2	1	1	0	3	3	2	3	3	0	1	1

Proof. This follows easily by direct computation using the values $z_0 = 0$, $z_1 = 1$, $z_2 = 1$, $z_3 = ab - 1$, $y_0 = 2$, $y_1 = 1$, $y_2 = ab - 2$ and $y_3 = ab - 3$ combined with the recursions $z_n = (ab-2)z_{n-2} - z_{n-4}$ and $y_n = (ab-2)y_{n-2} - y_{n-4}$ for $n \geq 4$. $\square$

Proposition 8.14. *Let $n \geq 1$ and assume $2 \,|\, z_n$. Then $\nu_2(z_{mn}) = \nu_2(z_n)$ for m odd.*

Proof. Set $e = \nu_2(z_n) \geq 1$ and $f = \nu_2(y_n) \geq 0$ (using that $y_n \neq 0$ by Lemma 8.8). Now let $m \geq 1$ be odd and consider the formula for z_{mn} from Proposition 8.11.

If $2 \,|\, ab$, Lemma 8.13 shows that $n \equiv 0 \pmod 4$ and $f = 1$. The term for $j = 0$ in the sum for z_{mn} has 2-adic valuation e, and for $1 \leq j \leq \frac{m-1}{2}$ the j'th term in the sum has 2-adic valuation

$$\geq -(m-1) + (2j+1)e + (m-1-2j) + 2j = (2j+1)e \geq 3e > e.$$

It follows that $\nu_2(z_{mn}) = e$ if $2 \,|\, ab$.

Now consider the case $2 \nmid ab$. By Lemma 8.13 it follows that $n \equiv 0 \pmod 3$ and that either $e \geq 2$ and $f = 1$ or $e = 1$ and $f \geq 2$. For $0 \leq j \leq \frac{m-1}{2}$, the j'th term in the sum for z_{mn} has 2-adic valuation equal to

$$\nu_j = -(m-1) + \nu_2\left(\binom{m}{2j+1}\right) + (2j+1)e + (m-1-2j)f. \tag{17}$$

If $e \geq 2$ and $f = 1$ we have $\nu_0 = e$ and $\nu_j \geq (2j+1)e - 2j > e$ for $1 \leq j \leq \frac{m-1}{2}$, so $\nu_2(z_{mn}) = e$ in this case. If $e = 1$ and $f \geq 2$, then $\nu_{(m-1)/2} = 1$ and $\nu_j \geq 2f - 1 > 1$ for $0 \leq j < \frac{m-1}{2}$. It follows that $\nu_2(z_{mn}) = 1 = e$ in this case as well. $\square$

Proposition 8.15. *Let $k \geq 1$ be the least integer such that $2 \,|\, z_k$ (cf. Proposition 8.3). If $2k \,|\, n$ then $\nu_2(z_{2n}) = \nu_2(z_n) + 1$.*

Proof. By Proposition 8.11 we have $z_{2n} = z_n y_n$ for all $n \geq 0$. If ab is even it follows from Lemma 8.13 that $k = 4$, so $n \equiv 0 \pmod 8$ and $\nu_2(y_n) = 1$. If ab is odd, then Lemma 8.13 shows that $k = 3$, so $n \equiv 0 \pmod 6$ and $\nu_2(y_n) = 1$. Thus $\nu_2(z_{2n}) = \nu_2(z_n) + \nu_2(y_n) = \nu_2(z_n) + 1$ as claimed. $\square$

Proof of Theorem 8.5. The case $p = 2$ follows from Propositions 8.14 and 8.15 and the case p odd follows from Proposition 8.12. $\square$

9. Tables of integer sequences

Unless otherwise noted, the sequences are not in the Online Encyclopedia of Integer Sequences at the time of publication. In Table 1 we exhibit sequences $X_1(1,k)$, $k = 4, \ldots, 20$. In Table 2, we give a choice of sequence $X_2(a,b)$ for each $k = ab = 4, \ldots, 12$, and in Table 3 we exhibit sequences $X(a)$, $a = 2, \ldots, 12$. We refer the reader to Theorem 3.4 for the properties of these sequences.

$X_1(1,4)$	$\{1,3,5,7,9,\ldots\}$	OEIS sequence A005408
$X_1(1,5)$	$\{1,4,11,29,76,\ldots\}$	OEIS sequence A002878 Bisection of the Lucas sequence
$X_1(1,6)$	$\{1,5,19,71,265,\ldots\}$	OEIS sequence A001834
$X_1(1,7)$	$\{1,6,29,139,666,\ldots\}$	OEIS sequence A030221 Chebyshev even indexed U-polynomials evaluated at $\sqrt{7}/2$
$X_1(1,8)$	$\{1,7,41,239,1393,\ldots\}$	OEIS sequence A002315 the Newman-Shanks-Williams numbers
$X_1(1,9)$	$\{1,8,55,377,2584,\ldots\}$	OEIS sequence A033890 Fibonacci(4n+2)
$X_1(1,10)$	$\{1,9,71,559,4401,\ldots\}$	OEIS sequence A057080 Chebyshev even indexed U-polynomials evaluated at $\sqrt{10}/2$ with $\lim_{n\longrightarrow\infty} a(n)/a(n-1) = 4+\sqrt{15}$
$X_1(1,11)$	$\{1,10,89,791,7030,\ldots\}$	OEIS sequence A057081 Chebyshev even indexed U-polynomials evaluated at $\sqrt{11}/2$
$X_1(1,12)$	$\{1,11,109,1079,10681,\ldots\}$	OEIS sequence A054320 Chebyshev even indexed U-polynomials evaluated at $\sqrt{3}$ Squares of entries are star numbers
$X_1(1,13)$	$\{1,12,131,1429,15588,\ldots\}$	OEIS sequence A097783 Chebyshev polynomials with diophantine property
$X_1(1,14)$	$\{1,13,155,1847,22009,\ldots\}$	OEIS sequence A077416 Chebyshev S-sequence with diophantine property
$X_1(1,15)$	$\{1,14,181,2339,30226,\ldots\}$	OEIS sequence A126866
$X_1(1,16)$	$\{1,15,209,2911,40545,\ldots\}$	OEIS sequence A028230 Indices of square numbers which are also octagonal
$X_1(1,17)$	$\{1,16,239,3569,53296,\ldots\}$	
$X_1(1,18)$	$\{1,17,271,4319,68833,\ldots\}$	
$X_1(1,19)$	$\{1,18,305,5167,87534,\ldots\}$	
$X_1(1,20)$	$\{1,19,341,6119,109801,\ldots\}$	OEIS sequence A049629 $a(n) = 1/4[F(6n+4)+F(6n+2)]$ where $F(n)$ is the Fibonacci sequence

Table 1. $X_1(1,k)$, $k = 4, \ldots, 20$

$X_2(2,2)$	$\{0,2,4,6,8,\dots\}$	OEIS sequence A005843
$X_2(1,5)$	$\{0,1,3,8,21,\dots\}$	OEIS sequence A001906 Even bisection of the Fibonacci sequence
$X_2(2,3)$	$\{0,2,8,30,112,\dots\}$	OEIS sequence A052530
$X_2(1,7)$	$\{0,1,5,24,115,\dots\}$	OEIS sequence A004254
$X_2(4,2)$	$\{0,4,24,140,816,\dots\}$	OEIS sequence A005319 For any term n, $2n^2+4$ is a perfect square
$X_2(1,9)$	$\{0,1,7,48,329,\dots\}$	OEIS sequence A004187
$X_2(1,10)$	$\{0,1,8,63,496,\dots\}$	OEIS sequence A001090
$X_2(1,11)$	$\{0,1,9,80,711,\dots\}$	OEIS sequence A018913
$X_2(4,3)$	$\{0,4,40,396,3920,\dots\}$	OEIS sequence A122652 The Kekule numbers for the benzenoids

TABLE 2. $X_2(a,b)$, $k = ab = 4,\dots,12$

$X(2)$	$\{0,1,2,3,4,5,\dots\}$	OEIS sequence A001477
$X(3)$	$\{0,1,3,8,21,\dots\}$	OEIS sequence A001906 The bisection of the Fibonacci sequence
$X(4)$	$\{0,1,4,15,56,209,\dots\}$	OEIS sequence A001353 The number of spanning trees in a $2\times n$ grid
$X(5)$	$\{0,1,5,24,115,551,\dots\}$	OEIS sequence A004254
$X(6)$	$\{0,1,6,35,204,1189,\dots\}$	OEIS sequence A001109 The squares of entries are triangular numbers
$X(7)$	$\{0,1,7,48,329,2255,\dots\}$	OEIS sequence A004187
$X(8)$	$\{0,1,8,63,496,3905,\dots\}$	OEIS sequence A001090
$X(9)$	$\{0,1,9,80,711,6319,\dots\}$	OEIS sequence A018913
$X(10)$	$\{0,1,10,99,980,\dots\}$	OEIS sequence A004189 Indices of square numbers which are also generalized pentagonal numbers
$X(11)$	$\{0,1,11,120,\dots\}$	OEIS sequence A004190
$X(12)$	$\{0,1,12,143,\dots\}$	OEIS sequence A004191

TABLE 3. $X(a)$, $a = 2,\dots,12$

10. FURTHER DIRECTIONS - ARITHMETIC IN RANK 3 HYPERBOLIC ROOT SYSTEMS

In this section we give a brief indication of how our methods may extend to certain rank 3 hyperbolic root systems.

Let

$$A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \qquad A_0 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix},$$

so that A_0 is an affine submatrix of A. It follows that A has noncompact hyperbolic type.

We let W denote the Weyl group of A and let $W_0 \subset W$ denote the Weyl group of A_0. Then $W \cong PGL_2(\mathbb{Z})$. Let Φ denote the full real root system of A, let $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$ denote the simple roots and let

$$\Phi \cap (\mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\alpha_3)$$

denote the points on the root lattice corresponding to real roots.

The group W_0 acts on $\Phi \cap (\mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\alpha_3)$. Since W_0 is generated by two simple root reflections w_1 and w_2 (as in Section 3), the coefficient of α_3 in each orbit under the action of W_0 on $\Phi \cap (\mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\alpha_3)$ is fixed. We denote this coefficient by $z \in \mathbb{Z}$.

The orbit space for the action of W_0 on $\Phi \cap (\mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\alpha_3)$ consists of an infinite family of (lattice points on) curves indexed by $\mathbb{Z}$. These lie on an infinite family of 'slices' of the hyperboloid of one sheet supporting the real roots Φ, thus each orbit lives in a unique 2 dimensional 'sheet'

$$S_z = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus z\alpha_3$$

for a fixed $z \in \mathbb{Z}$. Conversely, for each $z \in \mathbb{Z}$, the sheet S_z contains at least one orbit of W_0.

As in Section 3, every element of W_0 is of the form $w_1^{\epsilon_1}(w_2w_1)^n w_2^{\epsilon_2}$, where $\epsilon_i \in \{0,1\}$, $i = 1,2$. We consider the action of $(w_2w_1)^n$ on points $(x,y,z) \in \Phi \cap (\mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus z\alpha_3)$, where z is fixed and we obtain the formula

$$(w_2w_1)^n(x,y,z) = (a_n, b_n, z),$$

where

$$a_0 = x, \qquad a_n = 2T_{n-1}z + 2ny - (2n-1)x, \qquad b_n = n^2z + (2n+1)y - 2nx,$$

and

$$T_n = \sum_{i=1}^{n} i = \frac{1}{2}(n^2+n) = \binom{n+1}{2} \text{ is the } n^{\text{th}} \text{ triangular number.}$$

To prove this formula (which was first shown to us by Alex Conway) note that the Jordan decomposition of w_2w_1 is given by $w_2w_1 = SJS^{-1}$, where

$$S = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since

$$J^n = \begin{pmatrix} 1 & n & \binom{n}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$$

by induction, it follows that

$$(w_2w_1)^n = SJ^nS^{-1} = \begin{pmatrix} -2n+1 & 2n & n^2-n \\ -2n & 2n+1 & n^2 \\ 0 & 0 & 1 \end{pmatrix}$$

which proves the claim.

It is not surprising to find symmetries here relating to Pascal's triangle, though we have not yet understood the full Weyl group action on roots.

We mention the following result which is also of interest. In [2], the authors consider the rank 3 generalized Cartan matrix

$$A = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}$$

which is also of noncompact hyperbolic type and contains A_0 as an affine submatrix. The Weyl group W of A is $W \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$.

The authors show that the Weyl group images of twice the fundamental weights give rise to Pythagorean triples, and these in turn correspond to vertices of ideal triangles of the W-tessellation on the perimeter of the Poincaré disk.

We suspect that a detailed investigation of the Weyl group orbits of these rank 3 hyperbolic root systems on their roots and weights will reveal a rich arithmetic structure. We hope to take this up elsewhere.

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