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INTERACTIVE COMPUTATION OF HOMOLOGY OF FINITE PARTIALLY ORDERED SETS*

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Abstract. We outline a method for practical use of an interactive system (APL) to compute the homology of finite partially ordered sets.

1. Prerequisites. All partially ordered sets (posets) are assumed finite.

Given a poset $\langle P, \leqq \rangle$, we say that b *covers* a if $b > a$ and $a \leqq c \leqq b$ implies $a = c$ or $b = c$. Since we deal with finite posets, the order relation can be obtained as the reflexive, transitive closure of the cover relation. Our programs allow the user to describe posets by the cover relation considered as a list of ordered pairs (more precisely, as an $N \times 2$ matrix). For its own convenience, the program only accepts cover relations which are subsets of the usual order on the natural numbers. There is no difficulty representing any poset in this fashion, e.g., the cover relation of a "labeled Hasse diagram" [1].

In order to calculate the homology of a poset, we define a functor, $C : \mathscr{P}_0 \to \mathscr{A}b^Z$, from the category of finite posets to the category of finite chain complexes of abelian groups. If P is a poset, then the group of n-chains, $C_n(P)$, is the free abelian group generated by symbols $a_0 < a_1 < \cdots < a_n$ in P. The *boundary operator* ∂ is defined on each generator $a_0 < a_1 < \cdots < a_n$ by the formula

$$\partial(a_0 < \cdots < a_n) = \sum_{0 \leqq i \leqq n} (-1)^i a_0 < \cdots < \hat{a}_i < \cdots < a_n,$$

where $a_0 < \cdots < \hat{a}_i < \cdots < a_n$ is the generator of $C_{n-1}(P)$ obtained from $a_0 < \cdots < a_i < \cdots < a_n$ by deleting the element a_i. The *n-th homology group* of P, $H_n(P)$, is defined to be the nth homology group of the complex $C(P)$. For the category of small categories, $\mathscr{C}$at, which includes $\mathscr{P}_0$, homology is usually defined as the homology of the simplicial set nerve of P, $N(P)$. It is well known [3], [6], that these homology theories are isomorphic.

2. Method. We begin by describing some of the functions in our APL-workspace:

PO: PO computes the graph of the $<$ relation in poset P and represents it as an $N \times N$ matrix called POMAT.

CHAIN: CHAIN computes the list of $K - 1$-chains in the poset P from the list of K-chains and POMAT.

BD: BD computes the matrix representing the boundary homomorphism. Input to this function consists of the list of K-chains and the list of $K + 1$-chains.

Following the sketch for computation of the homology of finite chain complexes found in Eilenberg and Steenrod [2, p. 138], we diagonalize the matrix giving the boundary map while saving the left transition matrix. For this diagonalization, we use a method of Nijenhuis [5] for determining the Smith canonical form of an integral matrix [4]. The functions actually used in the workspace are:

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NIJ1: NIJ1 reduces an integer matrix to a matrix whose nonzero entries are confined to main diagonal and an adjacent diagonal. Only these diagonals need to be stored for the remainder of the computation.

NIJ2: NIJ2 reduces output of NIJ1 to diagonal form.

Communication with user is accomplished via the function HOM. After loading the workspace, entering the single command HOM causes the response "ENTER POSET", followed by the request for input (□:). The user then enters the $N \times 2$ matrix of the cover relation, either directly or from a stored array. After computing the list of 1-chains, $H_0(P)$ is computed and displayed in the form

HO:RANK(#),

where (#) is the rank of $H_0(P)$. From here HOM enters a loop, which computes the chains of next highest length, computes the structure of the next homology group and displays it in a format similar to that used for $H_0(P)$ (see examples). If there are any elements of finite order in $H_k(P)$, the display includes the word "TORSION" followed by the orders of the factors in a direct sum decomposition.

3. Miscellaneous comments. An outline of an algorithm for performing this computation was developed by the second author. Actual programming was done by the first and third authors.

An early version of the workspace was produced fairly quickly, but proved too wasteful of space in the diagonalization routine. The appearance of Nijenhuis' abstract [5], while we were attempting to avoid WS-FULL errors, encouraged us to rewrite the workspace in the present form. In addition, this allowed a certain saving of time by not computing the Smith canonical form, but rather stopping as soon as the matrix was diagonalized. The workspace includes all functions necessary for the computation of the Smith canonical form of any given integer matrix.

Computation of integral cohomology via the dual chain complex, $C^*(P)$, can be computed similarly, and is included in the workspace.

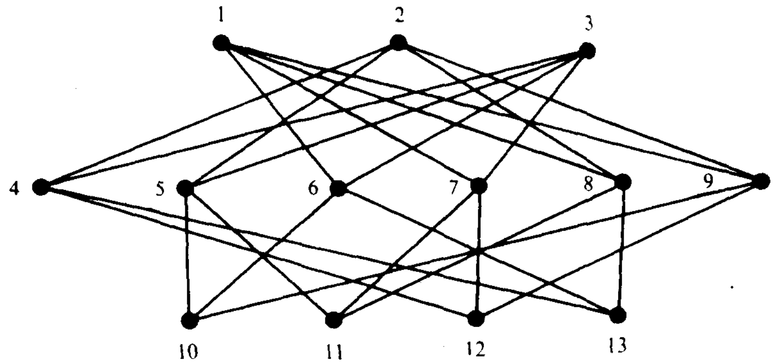
The workspace is currently being used on the Rutgers University System. A listing of the contents of the workspace will be furnished upon request from Professor R. Bumby.

4. Examples. The cover relation A was derived from a triangulation of the projective plane. The arrays EL 291, etc., are cover relations of posets on at most 6 points named according to their occurrence in the list obtained by Ellis Cooper [1].

```
        HOM
ENTER POSET
□:
        A
H0:     RANK 1
H1:     RANK 0   TORSION 2
H2:     RANK 0
```

A

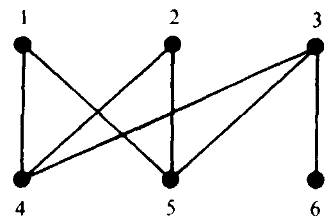
1	6
1	7
1	8
1	9
2	4
2	5
2	8
2	9
3	4
3	5
3	6
3	7
4	12
4	13
5	10
5	11
6	10
6	13
7	11
7	12
8	11
8	13
9	10
9	12



HOM
ENTER POSET
□:
*EL*291
*H*0: *RANK* 1
*H*1: *RANK* 2

*EL*291

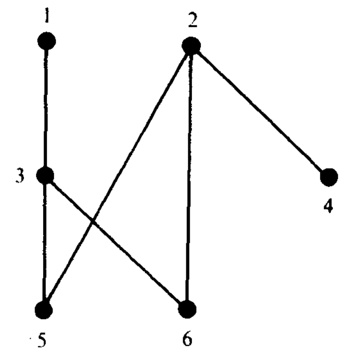
1	4
1	5
2	4
2	5
3	4
3	5
3	6



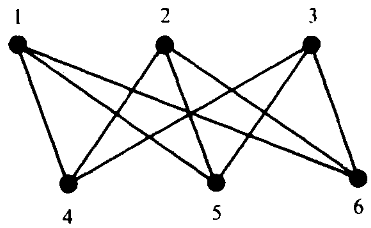
*EL*137

1	3
2	4
2	5
2	6
3	5
3	6

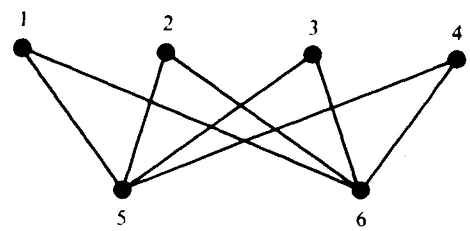
```
HOM
 ENTER POSET
 □:
           EL137
 H0:   RANK 1
 H1:   RANK 1
 H2:   RANK 0
```



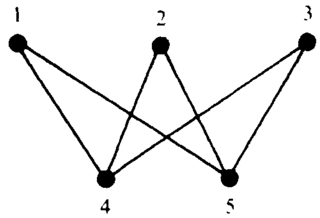

```
          EL316
  1   4
  1   5
  1   6
  2   4
  2   5
  2   6
  3   4
  3   5
  3   6
          HOM
ENTER POSET
□:
          EL316
H0:   RANK 1
H1:   RANK 4
```



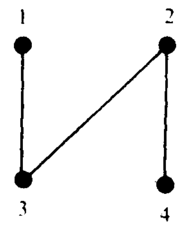

```
          HOM
ENTER POSET
□:
       □←EL315
  1   6
  1   5
  2   5
  2   6
  3   5
  3   6
  4   5
  4   6
H0:   RANK 1
H1:   RANK 3
          HOM
ENTER POSET
□:
```



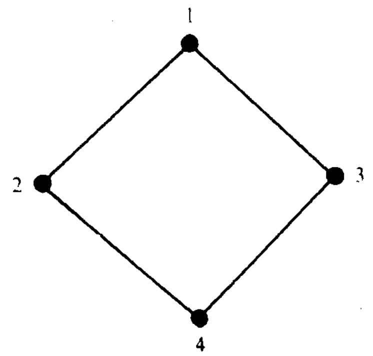
```
    ⎕←EL 61
 1  4
 1  5
 2  4
 2  5
 3  4
 3  5
H0:  RANK 1
H1:  RANK 2
        HOM
ENTER POSET
⎕:
```




```
    ⎕←EL 11
 1  3
 2  3
 2  4
H0:  RANK 1
H1:  RANK 0
        HOM
ENTER POSET
⎕:
```




```
    ⎕←EL 13
 1  2
 1  3
 2  4
 3  4
H0:  RANK 1
H1:  RANK 0
H2:  RANK 0
```



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