Rational generating functions and compositions

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A *composition* of the nonnegative interger n is a way of writing n as an ordered sum. So the compositons of 3 are $1+1+1$, $1+2$, $2+1$, and 3 itself. It is well-known and easy to prove that if c_n is the number of compositions of n then $c_n = 2^{n-1}$ for $n \geq 1$ and $c_0 = 1$. Equivalently, we have the generating function

$$\sum_{n\geq 0} c_n x^n = \frac{1-x}{1-2x}$$

which is a rational function. We show that this is a special case of a more general family of rational functions associated with compositions. Our techniques include the use of formal languages. Surprisingly, identities from the theory of hypergeometric series are needed to do some of the computations.