Computational Lists and Challenges in Mathematics

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Three Scientific Quotations. Kurt Gödel overturned the mathematical apple cart entirely deductively, but he could hold quite different ideas about legitimate forms of mathematical reasoning:

> *If mathematics describes an objective world just like physics, there is no reason why inductive methods should not be applied in mathematics just the same as in physics.* (Kurt Gödel[1], 1951)

Greg Chaitin takes this much further:

> *Over the past few decades, Gregory Chaitin, a mathematician at IBM's T.J. Watson Research Center in Yorktown Heights, N.Y., has been uncovering the distressing reality that much of higher math may be riddled with unprovable truths–that it's really a collection of random facts that are true for no particular reason. And rather than deducing those facts from simple principles,* "I'm making the suggestion that mathematics is done more like physics in that you come about things experimentally," *he says.* "This will still be controversial when I'm dead. It's a major change in how you do mathematics." (Time Magazine, Sept 4, 2005)

And Christoph Koch accurately captures a great scientific distaste for philosophizing:

> *Whether we scientists are inspired, bored, or infuriated by philosophy, all our theorizing and experimentation depends on particular philosophical background assumptions. This hidden influence is an acute embarrassment to many researchers, and it is therefore not often acknowledged.* (Christof Koch[2], 2004)

My Intentions in this Lecture. I, like Gödel, Chaitin, and surprisingly many others, suggest that both modes should be openly entertained in mathematical discourse, [5]. I aim to discuss Experimental Mathodology, its *philosophy*, *history*, *current practice* and *proximate future*, and using concrete accessible—entertaining I hope—examples, to explore implications for mathematics and for mathematical philosophy. *Thereby, to persuade you both of the power of mathematical experiment and that the traditional accounting of mathematical learning and research is largely an ahistorical caricature.*

I shall do so with a sample of material largely from the 2005 *Clifford Lectures* which I gave at Tulane University in New Orleans in April 2005.

1. **Plausible Reasoning in the 21st Century, I** is a general introduction to *Experimental Mathematics, its Practice and its Philosophy*. It reprises the 'Experimental methodology' that David Bailey and I—among many others—have practiced over the past two decades [6, 7].[3]

[1]Taken from an until then unpublished manuscript in his *Collected Works*, Volume III.
[2]In "Thinking About the Conscious Mind," a review of John R. Searle's *Mind. A Brief Introduction*, OUP 2004.
[3]All resources are available at www.experimentalmath.info.

2. **Plausible Reasoning in the 21st Century, II** focusses on the differences between *Determining Truths and Proving Theorems.* It explores various of the tools available for deciding what to believe in mathematics, and—using accessible examples—illustrates the rich experimental tool-box mathematicians can now have access to.

3. **Ten Computational Challenge Problems** is a more advanced analysis of the themes developed in Lectures 1 and 2. It discusses examples in [3], including

$$\int_0^\infty \cos(2x) \prod_{n=1}^\infty \cos\left(\frac{x}{n}\right) dx \stackrel{?}{=} \frac{\pi}{8}.$$

   This problem set was stimulated by Nick Trefethen's recent more numerical *SIAM 100 Digit, 100 Dollar Challenge* [4].

4. **Apéry-Like Identities for** $\zeta(n)$. The final lecture comprises a research level case study of generating functions for zeta functions. One example is

$$3\sum_{k=1}^\infty \frac{1}{\binom{2k}{k}(k^2-x^2)} \prod_{n=1}^{k-1} \frac{4x^2-n^2}{x^2-n^2} = \sum_{n=1}^\infty \frac{1}{n^2-x^2} \qquad \left[= \sum_{k=0}^\infty \zeta(2k+2)\, x^{2k} \;=\; \frac{1-\pi x\cot(\pi x)}{2x^2}\right].$$

   With $x=0$ this recovers the well known identity $3\sum_{k=1}^\infty 1/(\binom{2k}{k}k^2) = \sum_{n=1}^\infty 1/n^2 = \zeta(2)$.

References

[1] D.H. Bailey and J.M Borwein, "Experimental Mathematics: Examples, Methods and Implications," *Notices Amer. Math. Soc.*, **52** No. 5 (2005), 502–514. [CoLab Preprint 269].

[2] David Bailey, Jonathan Borwein and David Bradley, "Experimental Determination of Apéry-type Formulae for $\zeta(2n+2)$," preprint, 2005. [D-drive Preprint 295].

[3] D. Bailey, J. Borwein, V. Kapoor and E. Weisstein, "Ten Problems in Experimental Mathematics," *MAA Monthly,* in press, 2005. [CoLab Preprint 270].

[4] J.M. Borwein, "The SIAM 100 Digits Challenge," Extended review for the *Mathematical Intelligencer*, **27** (4) (2005). [D-drive preprint 285].

[5] J.M. Borwein, "Implications of Experimental Mathematics for the Philosophy of Mathematics," to appear in *Current Issues in the Philosophy of Mathematics From the Viewpoint of Mathematicians and Teachers of Mathematics,* Bonnie Gold, editor, 2006. [D-drive Preprint 280].

[6] Jonathan M. Borwein and David H. Bailey, *Mathematics by Experiment: Plausible Reasoning in the 21st Century*, A.K. Peters, Natick, MA, 2004.

[7] Jonathan M. Borwein, David H. Bailey and Roland Girgensohn, *Experimentation in Mathematics: Computational Paths to Discovery*, A.K. Peters, Natick, MA, 2004.