Unexpected Connections Between Three Famous Old Formulas for Pi

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Vieta's product 1592

$$\frac{2}{\pi}=\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots$$

Wallis's product 1656

$$\frac{2}{\pi}=\frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdot\frac{7\cdot 9}{8\cdot 8}\cdots,$$

Lord Brouncker's continued fraction 1656

$$\frac{4}{\pi}=1+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\cdots}}}}.$$

Notation for continued fractions

$$\frac{4}{\pi} = 1 + \frac{1^2}{2\,+}\;\frac{3^2}{2\,+}\;\frac{5^2}{2\,+}\;\cdots$$

Morphing Wallis into Vieta

$$\frac{2}{\pi} = \prod_{k=1}^{n} \underbrace{\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \cdots + \frac{1}{2}\sqrt{\frac{1}{2}}}}}}_{(k\ radicals)} \; \prod_{k=1}^{\infty} \frac{2^{n+1}k - 1}{2^{n+1}k} \cdot \frac{2^{n+1}k + 1}{2^{n+1}k}.$$

$$n = 0, 1, 2, \cdots,$$

$n=0$:

$$\frac{2}{\pi}=\frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdot\frac{7\cdot 9}{8\cdot 8}\cdot\frac{9\cdot 11}{10\cdot 10}\cdot\frac{11\cdot 13}{12\cdot 12}\cdots$$

$n=1$:

$$\frac{2}{\pi}=\sqrt{\frac{1}{2}}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{7\cdot 9}{8\cdot 8}\cdot\frac{11\cdot 13}{12\cdot 12}\cdot\frac{15\cdot 17}{16\cdot 16}\cdot\frac{19\cdot 21}{20\cdot 20}\cdots$$

$n=2$:

$$\frac{2}{\pi}=\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\cdot\frac{7\cdot 9}{8\cdot 8}\cdot\frac{15\cdot 17}{16\cdot 16}\cdot\frac{23\cdot 25}{24\cdot 24}\cdot\frac{31\cdot 33}{32\cdot 32}\cdots$$

$n=3$:

$$\frac{2}{\pi}=\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdot\frac{15\cdot 17}{16\cdot 16}\cdot\frac{31\cdot 33}{32\cdot 32}\cdot\frac{47\cdot 49}{48\cdot 48}\cdots$$

$n\to\infty$:

$$\frac{2}{\pi}=\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots$$

Derivation of the Wallis product

$$\sin x = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{\pi^2 n^2}\right) = x\prod_{n=1}^{\infty}\left(\frac{\pi n - x}{\pi n}\cdot\frac{\pi n + x}{\pi n}\right).$$

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty}\left(\frac{\pi n - x}{\pi n}\cdot\frac{\pi n + x}{\pi n}\right)$$

set $x = \dfrac{\pi}{2}$

$$\frac{2}{\pi} = \prod_{n=1}^{\infty}\left(\frac{\pi n - \frac{\pi}{2}}{\pi n}\cdot\frac{\pi n + \frac{\pi}{2}}{\pi n}\right) = \prod_{n=1}^{\infty}\left(\frac{2n-1}{2n}\cdot\frac{2n+1}{2n}\right).$$

$$\frac{2}{\pi} = \frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdots$$

This last expression is the product of Wallis. The convergence is very slow. Multiplying 100,000 factors of the above product using Mathematica approximates π to only 4 decimal places!

Derivation of Vieta's product

$$\sin\theta = 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}$$

$$= 2^2\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\sin\frac{\theta}{2^2}$$

$$= 2^3\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\sin\frac{\theta}{2^3}$$

...

$$\sin\theta = 2^p\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\cdots\cos\frac{\theta}{2^p}\sin\frac{\theta}{2^p}$$

$$\cos\frac{\theta}{2} = \sqrt{\frac{1}{2}+\frac{1}{2}\cos\theta}$$

$$\cos\frac{\theta}{2^2} = \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos\theta}}$$

...

$$\cos\frac{\theta}{2^p} = \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos\theta}}}}$$

$$(p\ radicals)$$

$$\frac{\sin\theta}{2^p \sin\dfrac{\theta}{2^p}} = \prod_{n=1}^{p} \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos\theta}}}}$$

$$(n\ radicals)$$

Next we let $p \to \infty$ and since $2^p \sin\dfrac{\theta}{2^p} \to \theta$

$$\frac{\sin\theta}{\theta} = \prod_{n=1}^{\infty} \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos\theta}}}}$$

$$(n\ radicals)$$

If we set $\theta = \pi/2$ and simplify we obtain Vieta's product

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\ \cdots$$

.

Multiplying 35 factors of the above product will approximate π to 20 decimal places.

LORD BROUNCKER'S FORGOTTEN SEQUENCE OF CONTINUED FRACTIONS FOR PI

$$1+\frac{1^2}{2\,+}\;\frac{3^2}{2\,+}\;\frac{5^2}{2\,+}\cdots=\frac{4}{\pi}$$

$$3+\frac{1^2}{6\,+}\;\frac{3^2}{6\,+}\;\frac{5^2}{6\,+}\cdots=\pi$$

$$5+\frac{1^2}{10+}\;\frac{3^2}{10+}\;\frac{5^2}{10+}\cdots=3\cdot\frac{2\cdot 2}{1\cdot 3}\cdot\frac{4}{\pi}$$

$$7+\frac{1^2}{14+}\;\frac{3^2}{14+}\;\frac{5^2}{14+}\cdots=3\cdot\frac{1\cdot 3}{2\cdot 2}\pi$$

$$9+\frac{1^2}{18+}\frac{3^2}{18+}\frac{5^2}{18+}\cdots=5\cdot\frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{4}{\pi}$$

$$11+\frac{1^2}{22+}\frac{3^2}{22+}\frac{5^2}{22+}\cdots=5\cdot\frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\pi$$

$$13+\frac{1^2}{26+}\frac{3^2}{26+}\frac{5^2}{26+}\cdots=7\cdot\frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{6\cdot 6}{5\cdot 5}\cdot\frac{4}{\pi}$$

$$15+\frac{1^2}{30+}\frac{3^2}{30+}\frac{5^2}{30+}\cdots=7\cdot\frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\pi$$

Table from Wallis's *Arithmetica*

Q2	Q4	Q6	Q8	Q10	Q12	Q14	Q16	
□	B	C	D	E	F	G	H	I
$1\ \frac{1}{2}\ \frac{9}{2}$ +	$3\ \frac{1}{6}\ \frac{9}{6}$ +	$5\ \frac{1}{10}\ \frac{9}{10}$ +	$7\ \frac{1}{14}\ \frac{9}{14}$ +	$9\ \frac{1}{18}\ \frac{9}{18}$ +	$11\ \frac{1}{22}\ \frac{9}{22}$ +	$13\ \frac{1}{26}\ \frac{9}{26}$ +	$15\ \frac{1}{30}\ \frac{9}{30}$ +	$17\ \frac{1}{34}\ \frac{9}{34}$ +
		$\frac{4}{1}$ □	$\frac{9}{4}$ B	$\frac{16}{9}$ C	$\frac{25}{16}$ D	$\frac{36}{25}$ E	$\frac{49}{36}$ F	$\frac{64}{49}$ G
				$\frac{4}{1}\times\frac{16}{9}$ □	$\frac{9}{4}\times\frac{25}{16}$ B	$\frac{16}{9}\times\frac{36}{25}$ C	$\frac{25}{16}\times\frac{49}{36}$ D	$\frac{36}{25}\times\frac{64}{49}$ E
						$\frac{4}{1}\times\frac{16}{9}\times\frac{36}{25}$ □	$\frac{9}{4}\times\frac{25}{16}\times\frac{49}{36}$ B	$\frac{16}{9}\times\frac{36}{25}\times\frac{64}{49}$ C
								$\frac{4}{1}\times\frac{16}{9}\times\frac{36}{25}\times\frac{64}{49}$ □

$$1+\cfrac{\frac{1}{2}}{\frac{9}{2}+},\quad 3+\cfrac{\frac{1}{6}}{\frac{9}{6}+},\quad 5+\cfrac{\frac{1}{10}}{\frac{9}{10}+},\quad 7+\cfrac{\frac{1}{14}}{\frac{9}{14}+},\ \text{etc.}$$

In the second row Wallis uses a box π to stand for our $\frac{4}{\pi}$ and the remaining letters B, C, D, etc., to stand for fractions beneath them. Continuing down the columns we find values for the fractions. For example, at the bottom of the third column we find

$\frac{4}{1}\Pi$ which is $\frac{4}{1}\cdot\frac{4}{\pi}$ the correct value for the third fraction in Brouncker’s list. We see in this table that Wallis and Brouncker have written the equivalent of the value of these fractions in terms of rational numbers and pi.

$$CF(x,0) = x + \frac{1^2}{2x+} \frac{3^2}{2x+} \frac{5^2}{2x+} \cdots$$

Morphing Brounckner into Wallis

$$\frac{4}{\pi} = W(n)\frac{1}{2n+1}\left[(4n+1) + \frac{1^2}{2(4n+1) + \frac{3^2}{2(4n+1) + \frac{5^2}{2(4n+1) + \cdots}}}\right]$$

,

$$W(n) = \frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdots\frac{(2n-1)(2n+1)}{2n\cdot 2n}$$

$n = 0$:

$$\frac{4}{\pi} = 1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \cdots}}}$$

(

$n = 1$:

$$\frac{4}{\pi} = \frac{1\cdot 3}{2\cdot 2}\times\frac{1}{3}\left[5 + \frac{1^2}{10 + \frac{3^2}{10 + \frac{5^2}{10 + \cdots}}}\right]$$

$n = 2$

$$\frac{4}{\pi} = \frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\times\frac{1}{5}\left[9 + \frac{1^2}{18 + \frac{3^2}{18 + \frac{5^2}{18 + \cdots}}}\right]$$

$$n = 3$$

$$\frac{4}{\pi} = \frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\times\frac{1}{7}\left[13+\frac{1^2}{26+\frac{3^2}{26+\frac{5^2}{26+\cdots}}}\right]$$

…

$n \to \infty$:

$$\frac{4}{\pi} = 2\times\frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdot\frac{7\cdot 9}{8\cdot 8}\cdots$$

Another Morphing of Brouncker into Wallis

$$\pi = \frac{1}{W(n)(2n+1)}\left[(4n+3) + \frac{1^2}{2(4n+3)+}\ \frac{3^2}{2(4n+3)+}\ \frac{5^2}{2(4n+3)+}\ \cdots\right]$$

,

$n = 0$:

$$\pi = 3 + \frac{1^2}{6\ +}\ \frac{3^2}{6\ +}\ \frac{5^2}{6\ +}\ \cdots.$$

$n = 1$:

$$\pi = \frac{2\cdot 2}{1\cdot 3}\times\frac{1}{3}\left[7 + \frac{1^2}{14+}\ \frac{3^2}{14+}\ \frac{5^2}{14+}\ \cdots\right].$$

$n = 2$:

$$\pi = \frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\times\frac{1}{5}\left[11 + \frac{1^2}{22+}\ \frac{3^2}{22+}\ \frac{5^2}{22+}\ \cdots\right].$$

$\cdots$

$n \to \infty$:

$$\pi = 2\times\frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{6\cdot 6}{5\cdot 7}\cdots.$$

Derivation of the results

$$\frac{4\Gamma\left(\frac{x+y+3}{4}\right)\Gamma\left(\frac{x-y+3}{4}\right)}{\Gamma\left(\frac{x+y+1}{4}\right)\Gamma\left(\frac{x-y+1}{4}\right)}=x+\frac{1^2-y^2}{2x+}\frac{3^2-y^2}{2x+}\frac{5^2-y^2}{2x+}\cdots,$$

$$=CF(x,y)$$

valid for either y an odd integer and x any complex number or y any complex number and $\mathrm{Re}(x)>0$.

The names of Euler, Stieltjes, and Ramanujan have been associated with this result. Use

$$\Gamma(n+1)=n!,$$

$$\Gamma(x)x=\Gamma(x+1)$$

$$\Gamma(1/2)=\sqrt{\pi}$$

$$\Gamma\left(\frac{2k+1}{2}\right)=\frac{1\cdot 3\cdot 5\cdots(2k-1)}{2^k}\sqrt{\pi},$$

to get our results.

Euler has shown

$$A=(f-b)\frac{\displaystyle\int_0^1 \frac{x^{f+a-b-1}dx}{\sqrt{1-x^{2a}}}}{\displaystyle\int_0^1 \frac{x^{f+b-1}dx}{\sqrt{1-x^{2a}}}}$$

$$2A=2f-a+\cfrac{(a+2b)(a-2b)}{4f-2a+\cfrac{(3a+2b)(3a-2b)}{4f-2a+\cfrac{(5a+2b)(5a-2b)}{4f-2a+\cfrac{(7a+2b)(7a-2b)}{4f-2a+etc...}}}}$$

.

$$A=\frac{2a\Gamma\left(\frac{f+a+b}{2a}\right)\Gamma\left(\frac{f+a-b}{2a}\right)}{\Gamma\left(\frac{f+b}{2a}\right)\Gamma\left(\frac{f-b}{2a}\right)}.$$

The Tables of John Wallis and the Discovery of his Product for Pi

$$\frac{2}{\pi} = \frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdots.$$

Wallis knows

$$\int_0^c x^p\,dx = \frac{c^{p+1}}{p+1},$$

Wallis wants

$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}.$$

How Wallis did it in modern notation

$$1/\int_0^1 \left(1-x^{1/Q}\right)^P dx$$

We will use the symbol $\{Q, P\}$ to denote the entry in a cell of the table where Q is the row and P is the column. Notice that $\{1/2, 1/2\} = 4/\pi$ since we know that $\int_0^1 \sqrt{1-x^2}\,dx = \dfrac{\pi}{4}$.

We also notice that the values obtained are symmetric. The number $\{_{P,Q}\}$ is the same as the number $\{_{Q,P}\}$.

Table 1: The reciprocal integral

$$1/\int_0^1 \left(1-x^{1/Q}\right)^P dx$$ **for integer *P* and *Q*.**

	$P=0$	$P=1/2$	$P=1$	$P=3/2$	$P=2$	$P=5/2$	$P=3$
$Q=0$	$\binom{0}{0}$		$\binom{1}{0}$		$\binom{2}{0}$		$\binom{3}{0}$
$Q=1/2$		$\frac{4}{\pi}$					
$Q=1$	$\binom{1}{1}$		$\binom{2}{1}$		$\binom{3}{1}$		$\binom{4}{1}$
$Q=3/2$							
$Q=2$	$\binom{2}{2}$		$\binom{3}{2}$		$\binom{4}{2}$		$\binom{5}{2}$
$Q=5/2$							
$Q=3$	$\binom{3}{3}$		$\binom{4}{3}$		$\binom{5}{3}$		$\binom{6}{3}$

Table 2: Extending the table first by integration to fractional *Q* then to fractional *P* using symmetry.

	$P=0$	$P=1/2$	$P=1$	$P=3/2$	$P=2$	$P=5/2$	$P=3$
$Q=0$	1	1	1	1	1	1	1
$Q=1/2$	1	$\frac{4}{\pi}$	$\frac{3}{2}$		$\frac{15}{8}$		$\frac{105}{48}$
$Q=1$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4
$Q=3/2$	1		$\frac{5}{2}$		$\frac{35}{8}$		$\frac{315}{48}$
$Q=2$	1	$\frac{15}{8}$	3	$\frac{35}{8}$	6	$\frac{63}{8}$	10
$Q=5/2$	1		$\frac{7}{2}$		$\frac{63}{8}$		$\frac{693}{48}$
$Q=3$	1	$\frac{105}{48}$	4	$\frac{315}{48}$	10	$\frac{693}{48}$	20

Table 3: Replacing cells with integer P by "growth revealing expressions"

	$P=0$	$P=1/2$	$P=1$	$P=3/2$	$P=2$	$P=5/2$	$P=3$
$Q=0$	1	1	$1\cdot\frac{1}{1}$	1	$1\cdot\frac{1}{1}\cdot\frac{2}{2}$	1	$1\cdot\frac{1}{1}\cdot\frac{2}{2}\cdot\frac{3}{3}$
$Q=1/2$	1	$\frac{4}{\pi}$	$\frac{3}{2}$		$\frac{15}{8}$		$\frac{105}{48}$
$Q=1$	1	$\frac{3}{2}$	$1\cdot\frac{2}{1}$	$\frac{5}{2}$	$1\cdot\frac{2}{1}\cdot\frac{3}{2}$	$\frac{7}{2}$	$1\cdot\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}$
$Q=3/2$	1		$\frac{5}{2}$		$\frac{35}{8}$		$\frac{315}{48}$
$Q=2$	1	$\frac{15}{8}$	$1\cdot\frac{3}{1}$	$\frac{35}{8}$	$1\cdot\frac{3}{1}\cdot\frac{4}{2}$	$\frac{63}{8}$	$1\cdot\frac{3}{1}\cdot\frac{4}{2}\cdot\frac{5}{3}$
$Q=5/2$	1		$\frac{7}{2}$		$\frac{63}{8}$		$\frac{693}{48}$
$Q=3$	1	$\frac{105}{48}$	$1\cdot\frac{4}{1}$	$\frac{315}{48}$	$1\cdot\frac{4}{1}\cdot\frac{5}{2}$	$\frac{693}{48}$	$1\cdot\frac{4}{1}\cdot\frac{5}{2}\cdot\frac{6}{3}$

Table 4: Replacing cells by improved "growth revealing expressions"

	$P=0$	$P=1/2$	$P=1$	$P=3/2$	$P=2$	$P=5/2$	$P=3$
$Q=0$	1	1	$1\cdot\frac{2}{2}$	1	$1\cdot\frac{2}{2}\cdot\frac{4}{4}$	1	$1\cdot\frac{2}{2}\cdot\frac{4}{4}\cdot\frac{6}{6}$
$Q=1/2$	1	$\frac{4}{\pi}$	$\frac{3}{2}$		$\frac{15}{8}$		$\frac{105}{48}$
$Q=1$	1	$\frac{3}{2}$	$1\cdot\frac{4}{2}$	$\frac{5}{2}$	$1\cdot\frac{4}{2}\cdot\frac{6}{4}$	$\frac{7}{2}$	$1\cdot\frac{4}{2}\cdot\frac{6}{4}\cdot\frac{8}{6}$
$Q=3/2$	1		$\frac{5}{2}$		$\frac{35}{8}$		$\frac{315}{48}$
$Q=2$	1	$\frac{15}{8}$	$1\cdot\frac{6}{2}$	$\frac{35}{8}$	$1\cdot\frac{6}{2}\cdot\frac{8}{4}$	$\frac{63}{8}$	$1\cdot\frac{6}{2}\cdot\frac{8}{4}\cdot\frac{10}{6}$
$Q=5/2$	1		$\frac{7}{2}$		$\frac{63}{8}$		$\frac{693}{48}$
$Q=3$	1	$\frac{105}{48}$	$1\cdot\frac{8}{2}$	$\frac{315}{48}$	$1\cdot\frac{8}{2}\cdot\frac{10}{4}$	$\frac{693}{48}$	$1\cdot\frac{8}{2}\cdot\frac{10}{4}\cdot\frac{12}{6}$

Table 5: More growth revealing expressions in rows where *Q* is a fraction followed by new growth revealing expressions in columns where *P* is a fraction.

	$P=0$	$P=1/2$	$P=1$	$P=3/2$	$P=2$	$P=5/2$	$P=3$
$Q=0$	1	$1\cdot\frac{1}{1}$	$1\cdot\frac{2}{2}$	$1\cdot\frac{1}{1}\cdot\frac{3}{3}$	$1\cdot\frac{2}{2}\cdot\frac{4}{4}$	$1\cdot\frac{1}{1}\cdot\frac{3}{3}\cdot\frac{5}{5}$	$1\cdot\frac{2}{2}\cdot\frac{4}{4}\cdot\frac{6}{6}$
$Q=1/2$	1	$\frac{4}{\pi}$	$1\cdot\frac{3}{2}$		$1\cdot\frac{3}{2}\cdot\frac{5}{4}$		$1\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}$
$Q=1$	1	$\frac{1}{2}\cdot\frac{3}{1}$	$1\cdot\frac{4}{2}$	$\frac{1}{2}\cdot\frac{3}{1}\cdot\frac{5}{3}$	$1\cdot\frac{4}{2}\cdot\frac{6}{4}$	$\frac{1}{2}\cdot\frac{3}{1}\cdot\frac{5}{3}\cdot\frac{7}{5}$	$1\cdot\frac{4}{2}\cdot\frac{6}{4}\cdot\frac{8}{6}$
$Q=3/2$	1		$1\cdot\frac{5}{2}$		$1\cdot\frac{5}{2}\cdot\frac{7}{4}$		$1\cdot\frac{5}{2}\cdot\frac{7}{4}\cdot\frac{9}{6}$
$Q=2$	1	$\frac{3}{8}\cdot\frac{5}{1}$	$1\cdot\frac{6}{2}$	$\frac{3}{8}\cdot\frac{5}{1}\cdot\frac{7}{3}$	$1\cdot\frac{6}{2}\cdot\frac{8}{4}$	$\frac{3}{8}\cdot\frac{5}{1}\cdot\frac{7}{3}\cdot\frac{9}{5}$	$1\cdot\frac{6}{2}\cdot\frac{8}{4}\cdot\frac{10}{6}$
$Q=5/2$	1		$1\cdot\frac{7}{2}$		$1\cdot\frac{7}{2}\cdot\frac{9}{4}$		$1\cdot\frac{7}{2}\cdot\frac{9}{4}\cdot\frac{11}{6}$
$Q=3$	1	$\frac{5}{8}\cdot\frac{7}{1}$	$1\cdot\frac{8}{2}$	$\frac{5}{8}\cdot\frac{7}{1}\cdot\frac{9}{3}$	$1\cdot\frac{8}{2}\cdot\frac{10}{4}$	$\frac{5}{8}\cdot\frac{7}{1}\cdot\frac{9}{3}\cdot\frac{11}{5}$	$1\cdot\frac{8}{2}\cdot\frac{10}{4}\cdot\frac{12}{6}$

Looking at the column where $P=1/2$ It seems clear from other expressions just discovered that the value $\frac{4}{\pi}$ in cell $\{1/2, 1/2\}$ should be written as $\frac{2}{\pi}\cdot\frac{2}{1}$, and the remaining empty cells in that row with $\frac{2}{\pi}\cdot\frac{2}{1}\cdot\frac{4}{3}$ and $\frac{2}{\pi}\cdot\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}$.

Table 6: Completing the critical row with *Q* = ½

	$P=0$	$P=1/2$	$P=1$	$P=3/2$	$P=2$	$P=5/2$	$P=3$
$Q=0$	1	$1\cdot\frac{1}{1}$	$1\cdot\frac{2}{2}$	$1\cdot\frac{1}{1}\cdot\frac{3}{3}$	$1\cdot\frac{2}{2}\cdot\frac{4}{4}$	$1\cdot\frac{1}{1}\cdot\frac{3}{3}\cdot\frac{5}{5}$	$1\cdot\frac{2}{2}\cdot\frac{4}{4}\cdot\frac{6}{6}$
$Q=1/2$	1	$\frac{2}{\pi}\cdot\frac{2}{1}$	$1\cdot\frac{3}{2}$	$\frac{2}{\pi}\cdot\frac{2}{1}\cdot\frac{4}{3}$	$1\cdot\frac{3}{2}\cdot\frac{5}{4}$	$\frac{2}{\pi}\cdot\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}$	$1\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}$
$Q=1$	1	$\frac{1}{2}\cdot\frac{3}{1}$	$1\cdot\frac{4}{2}$	$\frac{1}{2}\cdot\frac{3}{1}\cdot\frac{5}{3}$	$1\cdot\frac{4}{2}\cdot\frac{6}{4}$	$\frac{1}{2}\cdot\frac{3}{1}\cdot\frac{5}{3}\cdot\frac{7}{5}$	$1\cdot\frac{4}{2}\cdot\frac{6}{4}\cdot\frac{8}{6}$
$Q=3/2$	1		$1\cdot\frac{5}{2}$		$1\cdot\frac{5}{2}\cdot\frac{7}{4}$		$1\cdot\frac{5}{2}\cdot\frac{7}{4}\cdot\frac{9}{6}$
$Q=2$	1	$\frac{3}{8}\cdot\frac{5}{1}$	$1\cdot\frac{6}{2}$	$\frac{3}{8}\cdot\frac{5}{1}\cdot\frac{7}{3}$	$1\cdot\frac{6}{2}\cdot\frac{8}{4}$	$\frac{3}{8}\cdot\frac{5}{1}\cdot\frac{7}{3}\cdot\frac{9}{5}$	$1\cdot\frac{6}{2}\cdot\frac{8}{4}\cdot\frac{10}{6}$
$Q=5/2$	1		$1\cdot\frac{7}{2}$		$1\cdot\frac{7}{2}\cdot\frac{9}{4}$		$1\cdot\frac{7}{2}\cdot\frac{9}{4}\cdot\frac{11}{6}$
$Q=3$	1	$\frac{5}{8}\cdot\frac{7}{1}$	$1\cdot\frac{8}{2}$	$\frac{5}{8}\cdot\frac{7}{1}\cdot\frac{9}{3}$	$1\cdot\frac{8}{2}\cdot\frac{10}{4}$	$\frac{5}{8}\cdot\frac{7}{1}\cdot\frac{9}{3}\cdot\frac{11}{5}$	$1\cdot\frac{8}{2}\cdot\frac{10}{4}\cdot\frac{12}{6}$

Final argument

Call the ratio of two successive entries in the row where

$$Q=1/2\,,$$

$$R(n)=\frac{\{1/2,\ n+1/2\}}{\{1/2,\ n+1\}}$$

and notice that

$$R(0) = \frac{\frac{2}{\pi} \cdot \frac{2}{1}}{1 \cdot \frac{3}{2}} = \frac{2}{\pi} \cdot \frac{2 \cdot 2}{1 \cdot 3},$$

$$R(1) = \frac{\frac{2}{\pi} \cdot \frac{2}{1} \cdot \frac{4}{3}}{1 \cdot \frac{3}{2} \cdot \frac{5}{4}} = \frac{2}{\pi} \cdot \frac{2 \cdot 2}{1 \cdot 3} \cdot \frac{4 \cdot 4}{3 \cdot 5},$$

$$R(2) = \frac{\frac{2}{\pi} \cdot \frac{2}{1} \cdot \frac{4}{3} \cdot \frac{6}{5}}{1 \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6}} = \frac{2}{\pi} \cdot \frac{2 \cdot 2}{1 \cdot 3} \cdot \frac{4 \cdot 4}{3 \cdot 5} \cdot \frac{6 \cdot 6}{5 \cdot 7},$$

and in general we have

$$R(n) = \frac{2}{\pi} \cdot \frac{2 \cdot 2}{1 \cdot 3} \cdot \frac{4 \cdot 4}{3 \cdot 5} \cdot \frac{6 \cdot 6}{5 \cdot 7} \cdots \frac{(2n+2)(2n+2)}{(2n+1)(2n+3)}$$

.

Thus it is clear that

(1) $$\frac{2}{\pi} = \frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{3 \cdot 5}{4 \cdot 4} \cdot \frac{5 \cdot 7}{6 \cdot 6} \cdots \frac{(2n+1)(2n+3)}{(2n+2)(2n+2)} R(n) .$$

What can we say about $R(n)$ as *n* grows large? To answer this question Wallis examines another ratio in this row where $Q = 1/2$. He looks at the ratio of two entries with integer values of *P*. He observes

$$\frac{\{1/2,\, 1\}}{\{1/2,\, 2\}} = \frac{4}{5},\ \frac{\{1/2,\, 2\}}{\{1/2,\, 3\}} = \frac{6}{7},\ \frac{\{1/2,\, 3\}}{\{1/2,\, 4\}} = \frac{8}{9}, \text{ etc. },$$

and it is clear that

$$\lim_{n\to\infty} \frac{\{1/2,\, n\}}{\{1/2,\, n+1\}} = 1.$$

Now assume that also

$$\lim_{n\to\infty} \frac{\{1/2,\, n+1/2\}}{\{1/2,\, n+1\}} = \lim_{n\to\infty} R(n) = 1.$$

Returning to (1) and letting *n* tend to infinity, Wallis now has his product

$$\frac{2}{\pi} = \frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdots.$$

This completes our explanation of how Wallis conjectured his product.

THE LEMNISCATE CONSTANT AND THE MISSING FRACTIONS IN BROUNCKER'S SEQUENCE OF CONTINUED FRACTIONS FOR PI

Lemniscates curve $r^2 = \cos 2\theta$.

Lemniscate constant is ½ the perimeter

$$L = \frac{B\left(\frac{1}{4}, \frac{1}{4}\right)}{2\sqrt{2}} = \frac{\Gamma\left(\frac{1}{4}\right)^2}{2\sqrt{2\pi}} = 2.6220575542\ldots$$

New list of items missing from Brouncker's sequence:

$$2+\frac{1^2}{4+}\frac{3^2}{4+}\frac{5^2}{4+}\cdots=\frac{L^2}{\pi}=\frac{\Gamma(1/4)^4}{8\pi^2}$$

$$4+\frac{1^2}{8+}\frac{3^2}{8+}\frac{5^2}{8+}\cdots=\frac{9\pi}{L^2}=\frac{72\pi^2}{\Gamma(1/4)^4}$$

$$6+\frac{1^2}{12+}\frac{3^2}{12+}\frac{5^2}{12+}\cdots=\frac{5^2L^2}{3^2\pi}=\frac{5^2\Gamma(1/4)^4}{8\cdot 3^2\pi^2}$$

$$8+\frac{1^2}{16+}\frac{3^2}{16+}\frac{5^2}{16+}\cdots=\frac{7^2}{5^2}\cdot\frac{9\cdot\pi}{L^2}=\frac{7^2}{5^2}\cdot\frac{72\pi^2}{\Gamma(1/4)^4}$$

$$10+\frac{1^2}{20+}\frac{3^2}{20+}\frac{5^2}{20+}\cdots=\frac{5^2\cdot 9^2L^2}{3^2\cdot 7^2\pi}=\frac{5^2\cdot 9^2\Gamma(1/4)^4}{8\cdot 3^2\cdot 7^2\pi^2}2$$

$$12+\frac{1^2}{24+}\frac{3^2}{24+}\frac{5^2}{24+}\cdots=\frac{7^2\cdot 11^2}{5^2\cdot 9^2}\cdot\frac{9\cdot\pi}{L^2}=\frac{7^2\cdot 11^2}{5^2\cdot 9^2}\cdot\frac{72\pi^2}{\Gamma(1/4)^4}$$

These can be expressed in general by:

$$CF(x,0) = x + \frac{1^2}{2x+}\,\frac{3^2}{2x+}\,\frac{5^2}{2x+}\cdots$$

$$CF(4n+2,0) = \frac{5^2\cdot 9^2\cdot 13^2\cdots(4n+1)^2 L^2}{3^2\cdot 7^2\cdot 11^2\cdots(4n-1)^2 \pi}$$

$$CF(4n,0) = \frac{7^2\cdot 11^2\cdot 15^2\cdots(4n-1)^2}{5^2\cdot 9^2\cdot 13^2\cdots(4n-3)^2}\cdot\frac{9\cdot\pi}{L^2}$$

for $n = 1,2,3,\cdots$.

Notice also that

$$CF(2n-1)CF(2n+1) = (2n)^2.$$

Continued Fractions of Brouncker and Wallis

$$\frac{2}{\pi} = \frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{3 \cdot 5}{4 \cdot 4} \cdot \frac{5 \cdot 7}{6 \cdot 6} \cdots$$

Double each factor

$$\frac{2}{\pi} = \frac{2 \cdot 6}{4 \cdot 4} \cdot \frac{6 \cdot 10}{8 \cdot 8} \cdot \frac{10 \cdot 14}{12 \cdot 12} \cdots$$

Multiply by 2

$$\frac{4}{\pi} = \underset{BC}{\overset{AB}{\frac{2\cdot 2}{4\cdot 4}}} \cdot \underset{DE}{\overset{CD}{\frac{6\cdot 6}{8\cdot 8}}} \cdot \underset{FG}{\overset{EF}{\frac{10\cdot 10}{12\cdot 12}}} \cdots = A$$

$$AB = 2^2$$

$$BC = 4^2$$

$$CD = 6^2$$

$$DE = 8^2$$

This leads to

$$A = \frac{4}{\pi} = 1 + \frac{1^2}{2+}\,\frac{3^2}{2+}\,\frac{5^2}{2+}\cdots$$

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