Hilbert's Monkey Saddle and other Curiosities

All 3-(Point-)Particle Riesz Equilibria on a Circle

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Experimental Mathematics Seminar
Rutgers University
December 07, 2017
Revision of Slides: Dec. 09, 2017

Optimal N-Particle Configurations on $\mathbb{S}^n$

- Pick N distinct points $\mathbf{p}_k \in \mathbb{R}^{n+1}$, $k \in \{1, ..., N\}$, with $|\mathbf{p}_k| = 1$
- Take any pair, say $\mathbf{p}_i$ and $\mathbf{p}_j$.
- Denote their Euclidean distance by $r_{ij} > 0$.
- Assign each ij pair a *Riesz s-energy* $V_s(r_{ij})$, with

$$V_s(r) := \frac{1}{s}\left(\frac{1}{r^s} - 1\right), \qquad s \in \mathbb{R}, \quad s \neq 0;$$

$$V_0(r) := \ln\frac{1}{r} \qquad \left(= \lim_{s\to 0} V_s(r)\right),$$

- Average pair energy of an N-pt. configuration $\omega^{(N)}$,

$$\langle V_s\rangle(\omega^{(N)}) := \frac{2}{N(N-1)} \sum\sum_{1\leq i<j\leq N} V_s(r_{ij})$$

- Optimal average s-Riesz pair energy of N point particles:

$$v_s(N) := \inf_{\omega^{(N)}} \langle V_s\rangle(\omega^{(N)})$$

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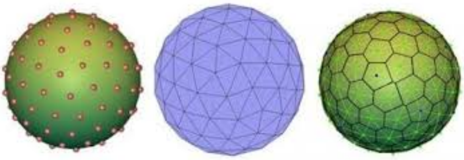

Optimal N-Particle Configurations on $\mathbb{S}^n$ when $s \geq -2$

- An optimizer $\omega_*^{(N)}$ exists whenever $s \geq -2$:

$$v_s(N) = \langle V_s \rangle (\omega_*^{(N)})$$

- Optimizers $\omega_*^{(N)}$ ($/O(n+1)$ and $/S_N$) generally not unique!
- Empirical: $\#$ of local minimizers $\propto e^{\gamma_n N}$ for some $\gamma_n > 0$.
- Asymptotic large-N expansion for $n = 2$ and $s = 0$:

$$\frac{N(N-1)}{2} v_0(N) = aN^2 + bN \ln N + cN + d \ln N + \mathcal{O}(1),$$

$$a = \tfrac{1}{4} \ln \tfrac{e}{4}, \quad b = -\tfrac{1}{4}, \quad c = \ln \left(2(2/3)^{1/4} \pi^{3/4} / \Gamma(1/3)^{3/2}\right), \quad d\ ?$$

- *Smale's 7th problem for the 21st century*

For each N, can one find $\omega_\clubsuit^{(N)}$ on $\mathbb{S}^2$ such that

$$\left| v_0(N) - \langle V_s \rangle (\omega_\clubsuit^{(N)}) \right| < D \frac{\ln N}{N(N-1)}$$

using not more than Poly(N) many steps?

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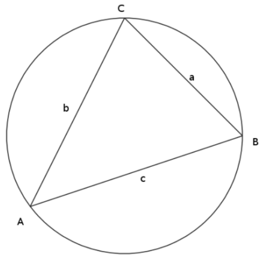
Optimal N-Particle Arrangements on $\mathbb{S}^n$ when $s < -2$

- An optimizer $\omega_*^{(N)}$ generally does not exist when $s < -2$.
  $\longrightarrow$ COAGULATION onto not more than 3 points! (Björk)



- 3 points inevitably on GREAT CIRCLE: $\mathbb{S}^1$.
- We speak of "*optimal N-particle arrangement*" in this case.
- $N = 2M$: optimizer $= M$ points each in two antipodal points
- $N = 2M + 1$: More COMPLICATED / INTERESTING!

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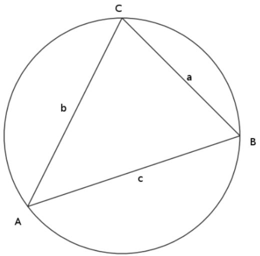

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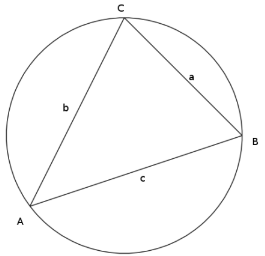

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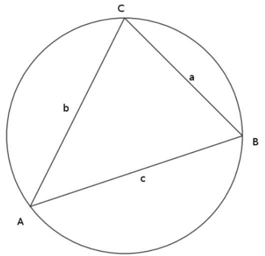

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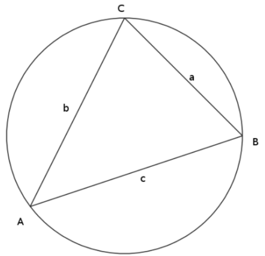

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Optimal 3-Particle Arrangements on $\mathbb{S}^1$

- Average pair energy of 3-particle arrangement $\omega^{(3)} \equiv (\alpha, \beta, \gamma)$

$$\langle V_s \rangle(\alpha, \beta, \gamma) := \tfrac{1}{3}\big(V_s(|a|) + V_s(|b|) + V_s(|c|)\big);$$

$$|a| = 2\sin\alpha, \quad |b| = 2\sin\beta, \quad |c| = 2\sin\gamma$$

- *Minimal average s-Riesz pair-energy* of $N = 3$ particles:

$$v_s(3) := \inf_{\alpha+\beta+\gamma=\pi} \langle V_s \rangle(\alpha, \beta, \gamma).$$

- Compactify via:

$$V_s(0) := -s^{-1}, \qquad s < 0;$$
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and find all extremal / critical points of $\langle V_s \rangle(\alpha, \beta, \gamma)$!

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The absolute minimizers (Nerattini-Brauchart-K.):

Theorem

Set $s_3 := \ln(4/9)/\ln(4/3)$.

- *Then for* $s \neq s_3$ *the optimal minimal Riesz s-energy* $N = 3$ *arrangement on* $\mathbb{S}^1$ *is unique (up to rotation/permutation).*
- *For* $s < s_3$ *it is given by the antipodal arrangement* $(\alpha, \beta, \gamma) = (\frac{\pi}{2}, \frac{\pi}{2}, 0)$.
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The relative minimzers of $\langle V_s \rangle(\alpha, \beta, \gamma)$ which are not absolute:

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Recall that $s_3 := \ln(4/9)/\ln(4/3)$. *The following list exhausts all the relative minimizers which are not absolute.*

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The relative maximizers of $\langle V_s \rangle(\alpha, \beta, \gamma)$ which are not absolute:

Theorem

For $s < -4$ the equilateral configuration $(\alpha, \beta, \gamma) = (\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$ is a relative maximizer which is not absolute. This exhausts all relative maximers of $\langle V_s \rangle(\alpha, \beta, \gamma)$ which are not absolute.

Optimal 3-Particle Arrangements on $\mathbb{S}^1$

The absolute maximizers of $\langle V_s \rangle(\alpha, \beta, \gamma)$:

Theorem

For all $s < 0$ the completely degenerate triangular configuration (i.e. the one-point arrangement) given by $(\alpha, \beta, \gamma) = (0, 0, \pi)$ (up to permutation) is the unique (up to rotation) absolute maximizer of $\langle V_s \rangle(\alpha, \beta, \gamma)$. Contracting and compactifying all energies to $[-1, 1]$, the one-pt arrangement is the unique (up to rotation) absolute maximizer for all s.

Optimal 3-Particle Arrangements on $\mathbb{S}^1$

The saddle points of $\langle V_s \rangle(\alpha, \beta, \gamma)$:

Theorem

The following list exhausts all the saddle points:

- *The equilateral configuration* $(\alpha, \beta, \gamma) = (\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$ *is a saddle point at* $s = -4$.
- *The antipodal arrangement* $(\alpha, \beta, \gamma) = (\frac{\pi}{2}, \frac{\pi}{2}, 0)$ *(up to permutation) is a saddle for* $-2 \leq s < 0$.
  Contracting and compactifying all energies to $[-1, 1]$, *the antipodal arrangement is a saddle for all* $s \geq 0$, *too.*
- *For* $\{s < -2\} \cap \{s \neq -4\}$ *there are two families (disjoint open sets) of non-universal isosceles triangular equilibrium configurations, and all these are saddle points.*

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Optimal 3-Particle Arrangements on $\mathbb{S}^1$

Non-proper pseudo equilibria:

Theorem

The following are the only non-proper pseudo Riesz s-force equilibria of $N = 3$ point particles on $\mathbb{S}^1$.

- *The completely degenerate triangular configuration given by $(\alpha, \beta, \gamma) = (0, 0, \pi)$ (up to permutation) is a non-proper pseudo Riesz s-force equilibrium for all $s \geq -1$.*
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Optimal 3-Particle Arrangements on $\mathbb{S}^1$

The proper Riesz *s*-force equilibria:

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This list exhausts all proper Riesz s-force 3-particle equilibria:

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- *For each $s < -2$ except $s = -4$, there exists a non-universal isosceles triangular proper Riesz s-force equilibrium, i.e. its shape depends on s.*

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The two isosceles families of non-universal equilibria:

Theorem

- *The family of isosceles triangular Riesz s-force equilibria for* $s \in (-\infty, -4)$ *interpolates continuously and monotonically between a right triangular configuration* $(\gamma = \pi/2)$*, to which it converges when* $s \downarrow -\infty$*, and the equilateral configuration* $(\gamma = \pi/3)$*, to which it converges when* $s \uparrow -4$*.*
- *The family of isosceles triangular Riesz s-force equilibria for* $s \in (-4, -2)$ *interpolates continuously and monotonically between the equilateral configuration* $(\gamma = \pi/3)$*, to which it converges when* $s \uparrow -4$*, and the antipodal arrangement* $(\gamma = 0)$*, to which it converges when* $s \uparrow -2$*.*

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The two isosceles families of non-universal equilibria (cont.d):

Theorem

The asymptotics of γ as function of s for the isosceles triangles is given by the following:
(a) in a left neighborhood of $\gamma = \pi/2$ (as $s \downarrow -\infty$),

$$\gamma(s) \asymp \tfrac{\pi}{2} - \sqrt{2}^{1+s},$$

(b) in a neighborhood of $\gamma = \pi/3$ (for $s \approx -4$),

$$\gamma(s) = \tfrac{\pi}{3} - \tfrac{1}{2\sqrt{3}}(4+s) + \mathcal{O}((s+4)^2)$$

(c) in a right neighborhood of $\gamma = 0$ (as $s \uparrow -2$),

$$\gamma(s) \asymp 0 + 2^{\frac{1}{2+s}}.$$

Equilibrium BIFURCATION diagram:

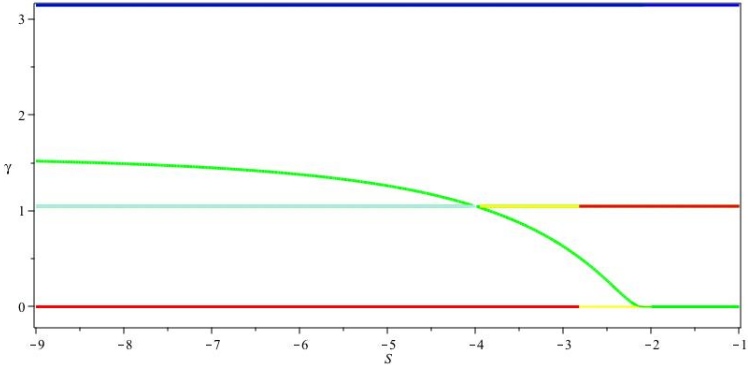


Figure: Bifurcation diagram (γ vs. *s*) of the Riesz *s*-force equilibria.
Color code: MINIMUM, minimum, saddle, maximum, MAXIMUM.

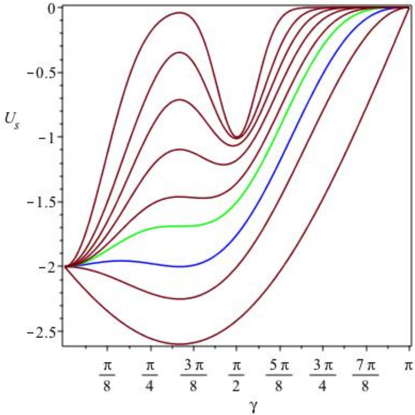
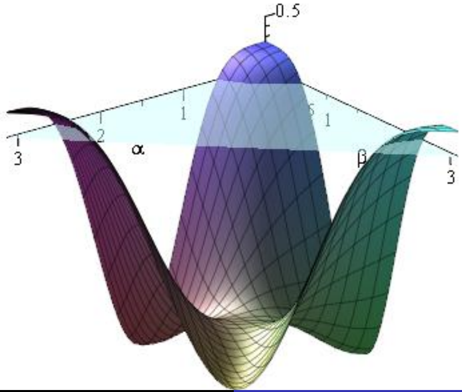
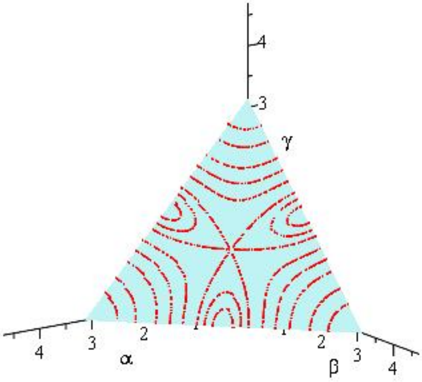


Figure: $U_s(\gamma)$: $s \in \{-30, -15, -10, -7, -5, -4, \frac{\ln(4/9)}{\ln(4/3)}, -2, -1\}$.
The graphs are monotonically ordered with s, decreasing with s

The Figure shows the graph of $\langle V_{-4}\rangle(\alpha, \beta, \pi - \alpha - \beta)$ over the isosceles triangle $(\alpha, \beta) \in [0, \pi]^2 \cap \{\alpha + \beta \leq \pi\}$ ($\leftarrow$ this is the projection of the fundamental triangle into the (α, β) plane. Note that this projected illustration somewhat distorts the three-fold symmetry of Hilbert's Monkey Saddle.)



The Figure shows the contour lines of $\langle V_{-4}\rangle(\alpha, \beta, \gamma)$ in the fundamental triangle in (α, β, γ) space.



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