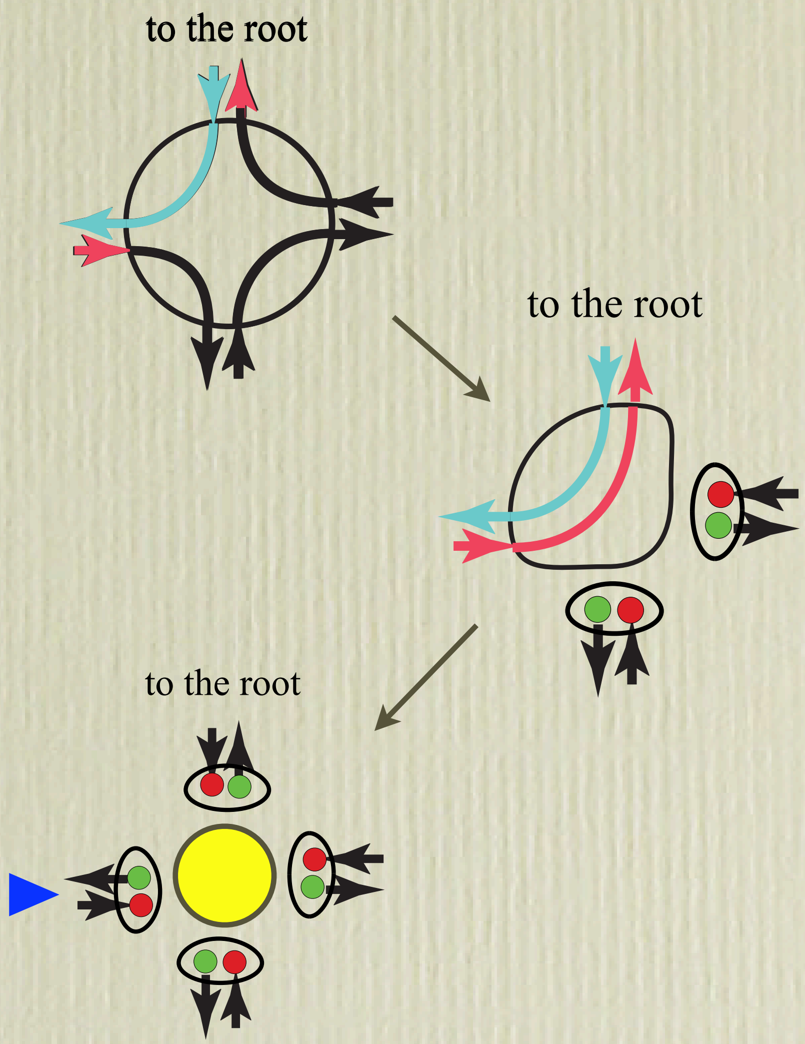
On the minors of the paths matrix in a tree

Pierre Lalonde,
LaCIM &
Collège de Maisonneuve

With the support of NSERC (Canada)



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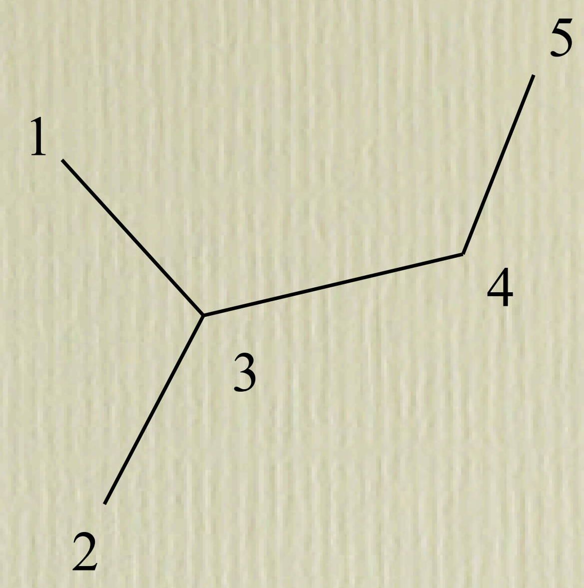

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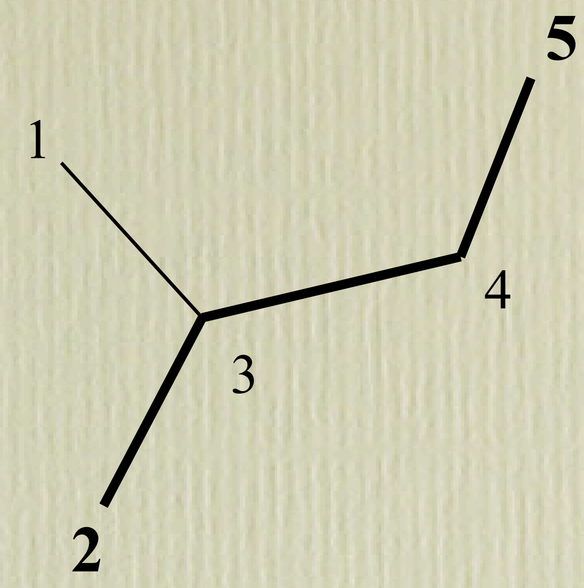

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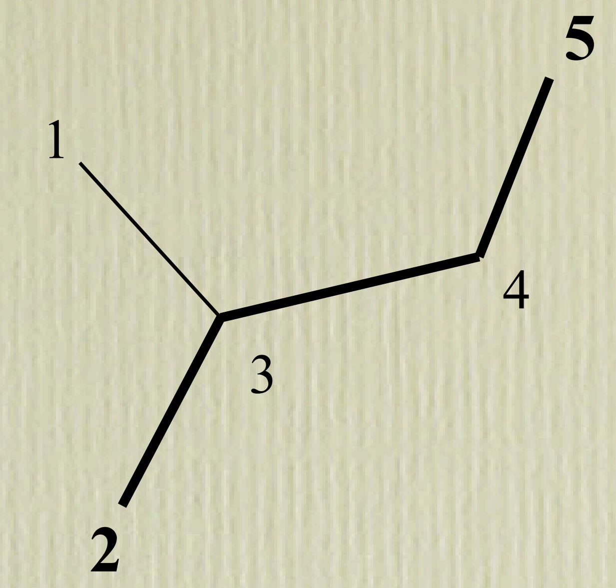

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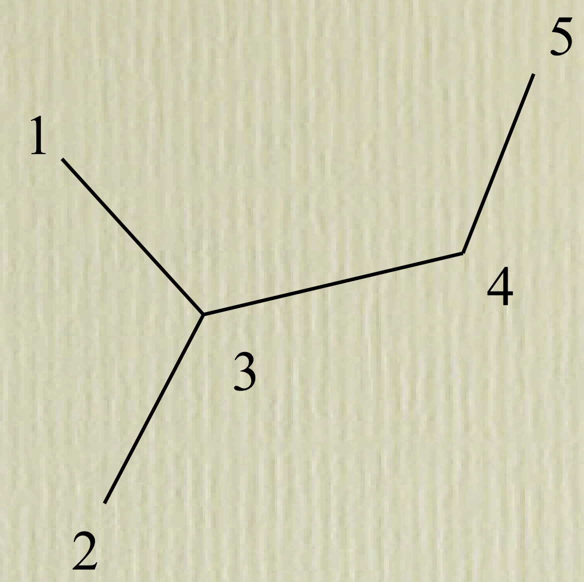

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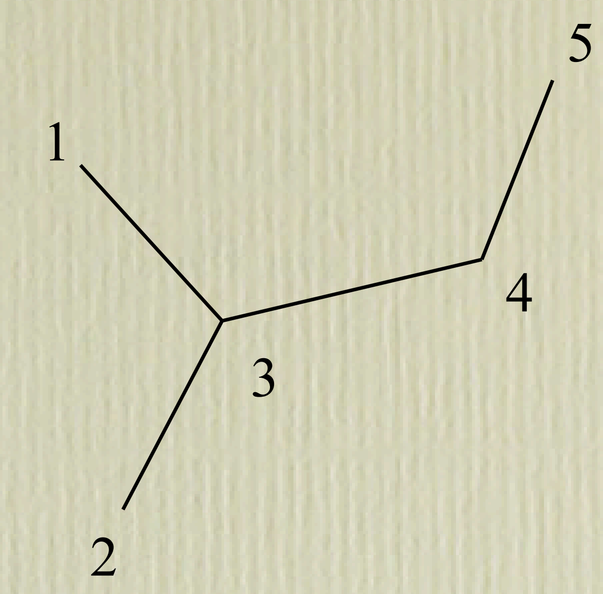

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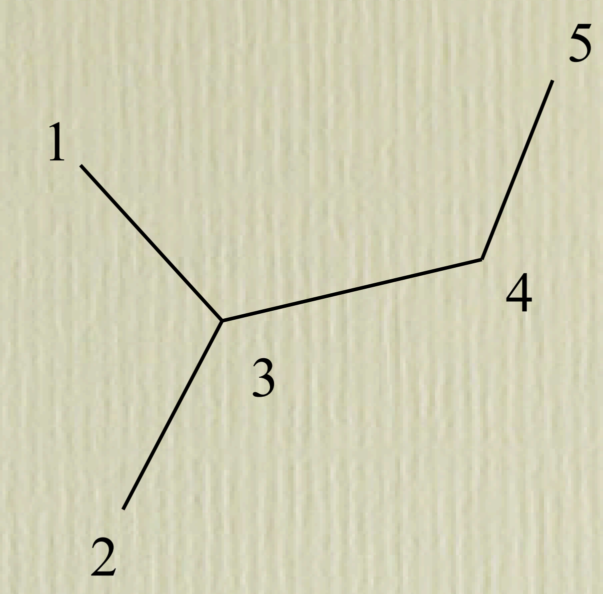

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- (Graham-Lovász, 78): det(D - x I)

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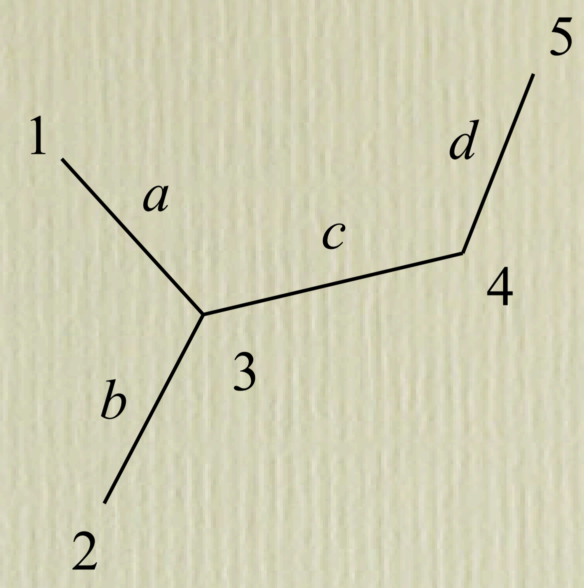

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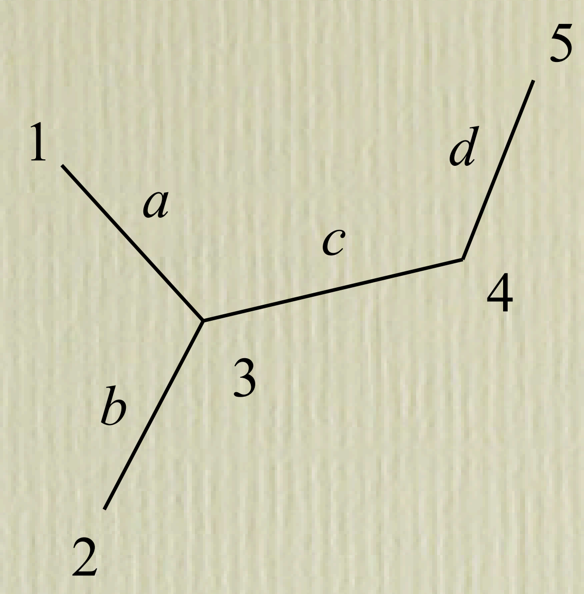

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$$D = \begin{pmatrix} 0 & a+b & a & a+c & a+c+d \\ a+b & 0 & b & b+c & b+c+d \\ a & b & 0 & c & c+d \\ a+c & b+c & c & 0 & d \\ a+c+d & b+c+d & c+d & d & 0 \end{pmatrix}$$

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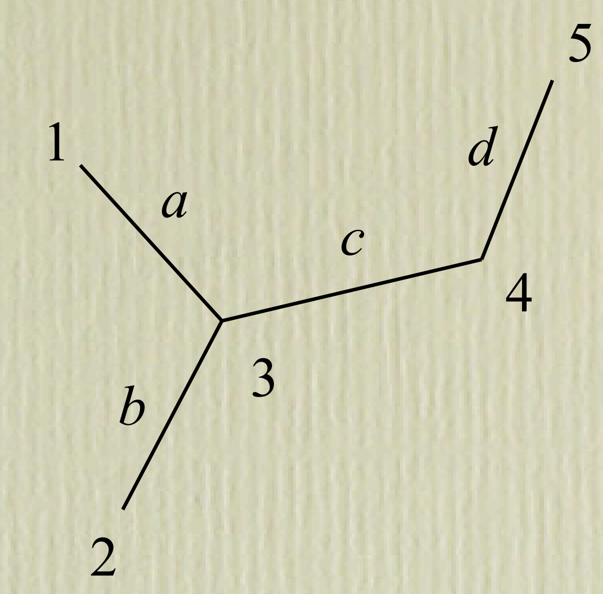

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- **Thm** (Bapat-Kirkland-Neumann, 05)

$$\det(D + xJ) = (-1)^{n-1} ab\cdots c\,(2x + a + b + \cdots + c)\,2^{n-2}$$

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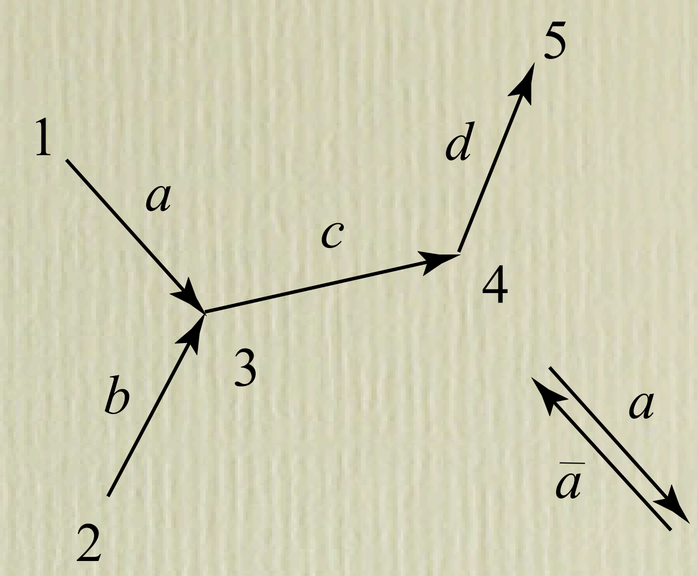

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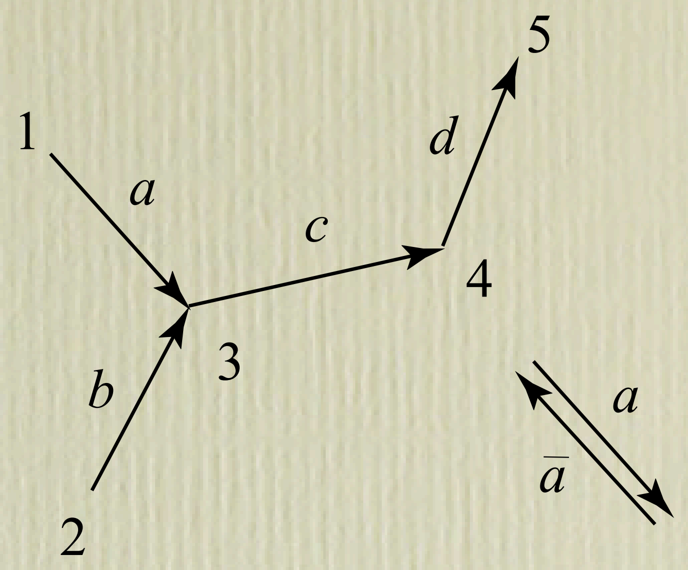

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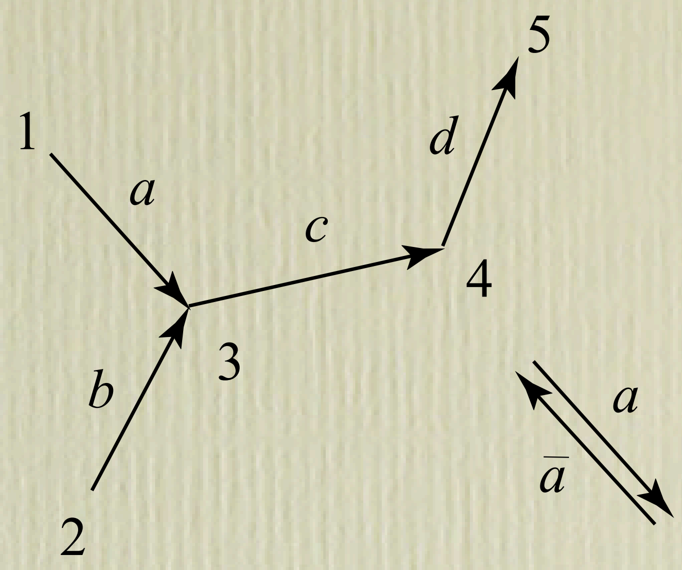

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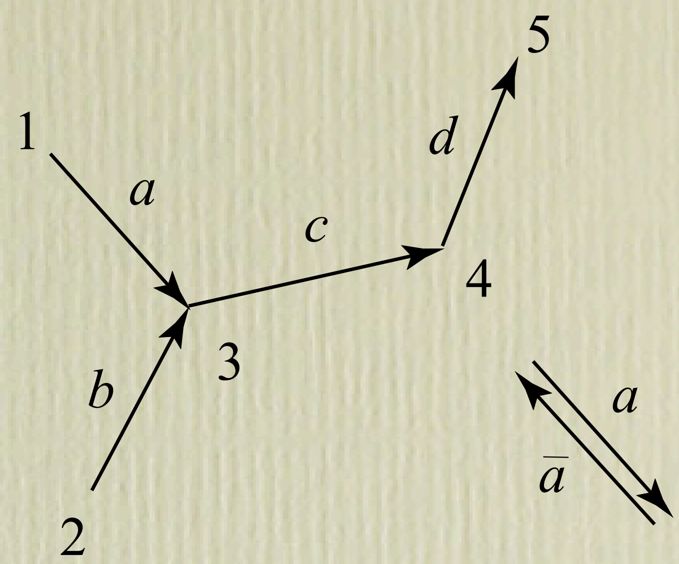

- Paths matrix $P = (p_{ij})_{n\times n}$, where p_{ij} is the product of the weights from i to j

$$P = \begin{pmatrix} 1 & a\bar{b} & a & ac & acd \\ \bar{a}b & 1 & b & bc & bcd \\ \bar{a} & \bar{b} & 1 & c & cd \\ \bar{a}\bar{c} & \bar{b}\bar{c} & \bar{c} & 1 & d \\ \bar{a}\bar{c}\bar{d} & \bar{b}\bar{c}\bar{d} & \bar{c}\bar{d} & \bar{d} & 1 \end{pmatrix}$$

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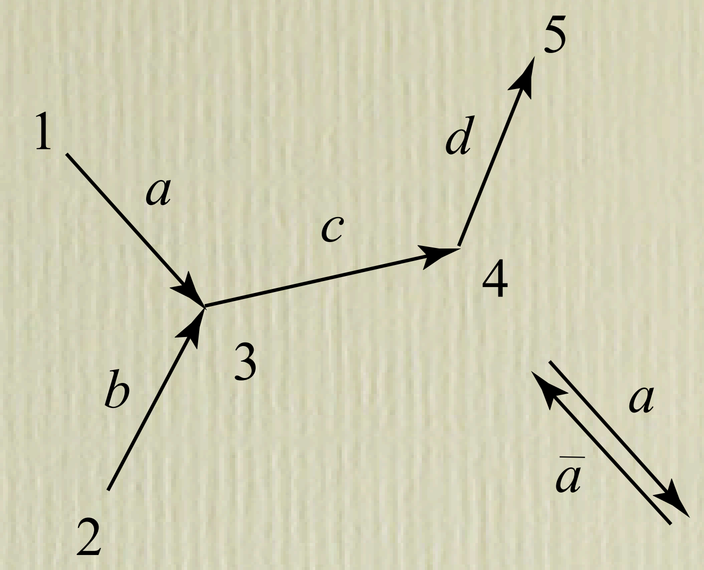

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$$\det(P) = (1 - a\bar{a})(1 - b\bar{b}) \cdots (1 - c\bar{c})$$

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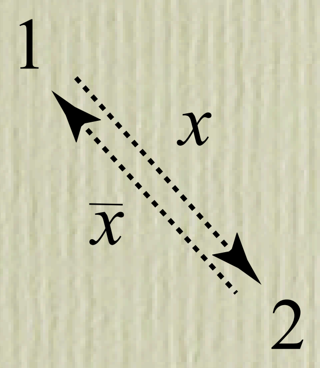

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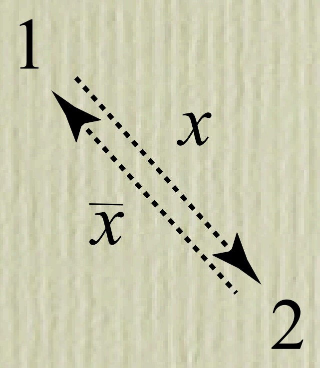

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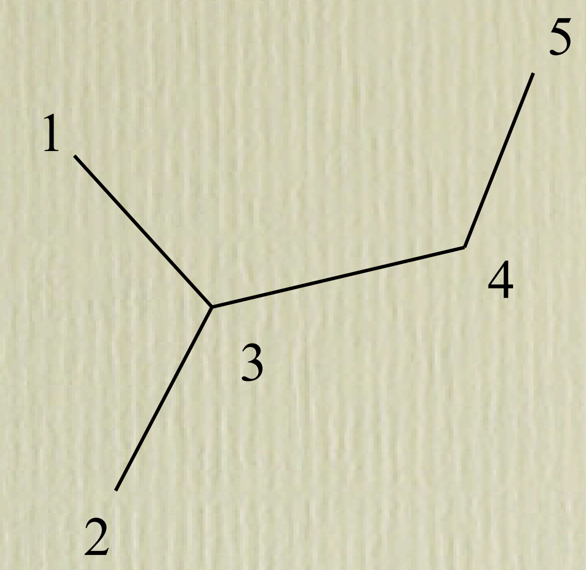

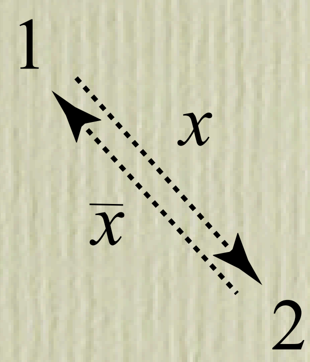

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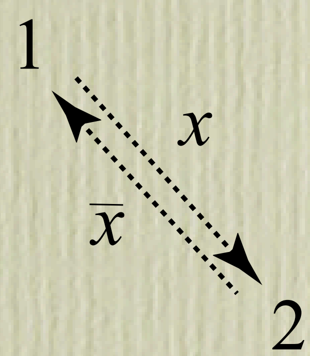

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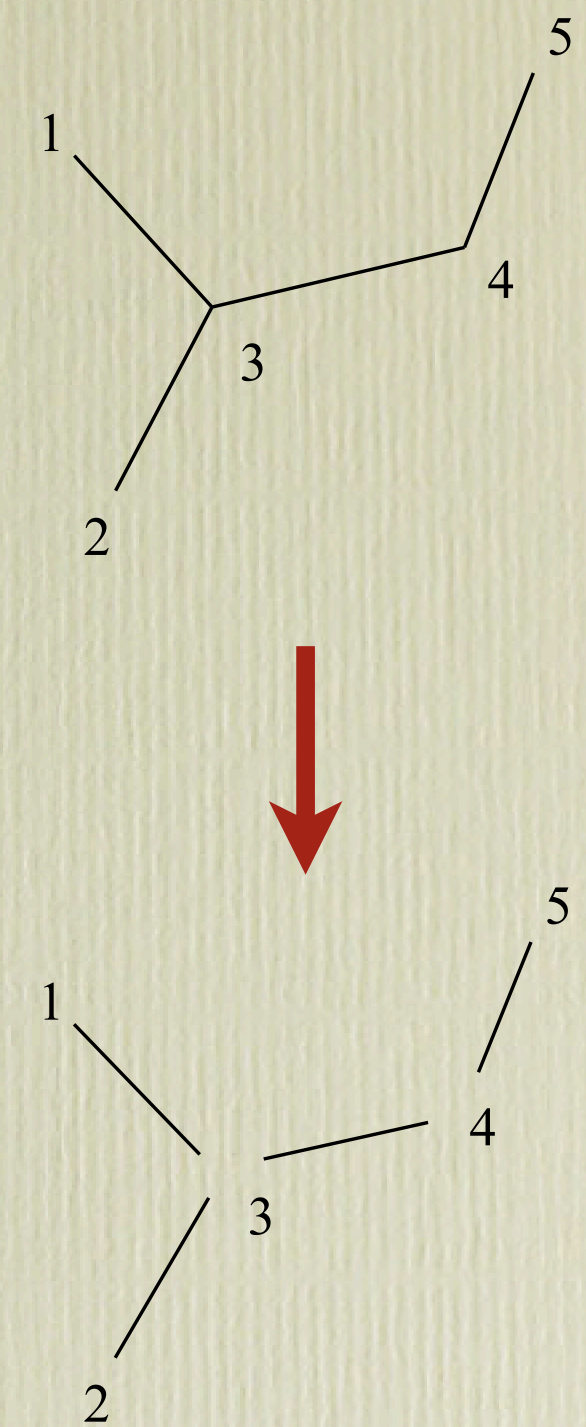

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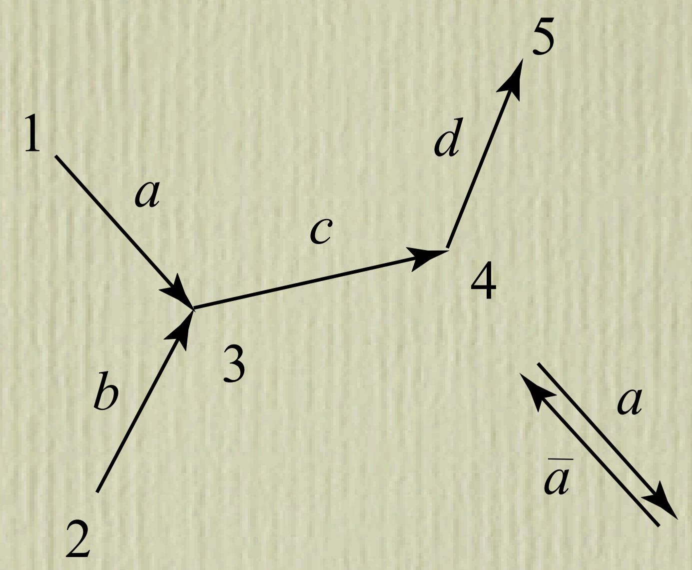

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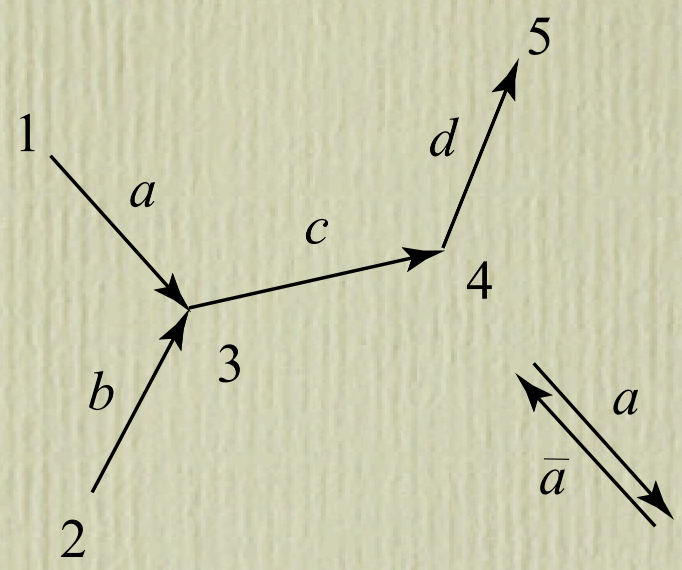

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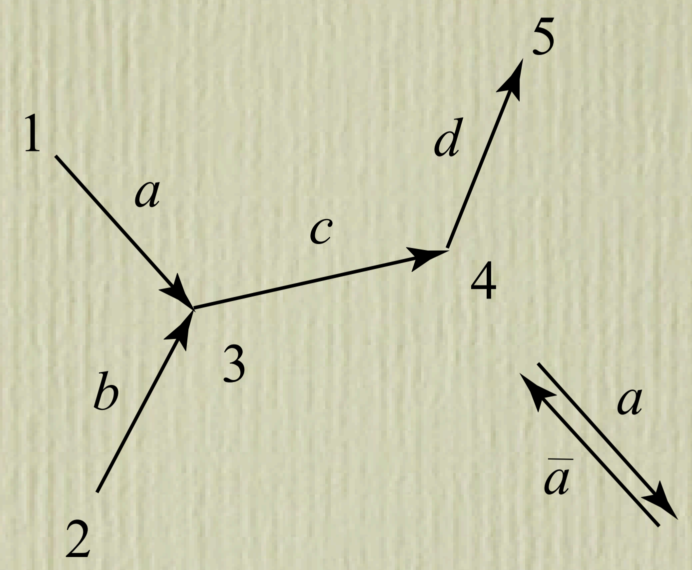

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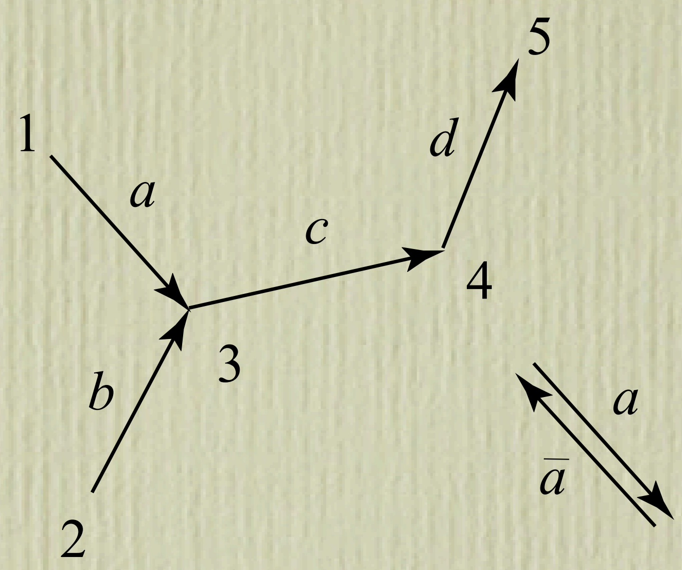

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$$P^{-1} = \frac{1}{1-q^2} \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & -1 & 1+2q^2 & -1 & 0 \\ 0 & 0 & -1 & 1+q^2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

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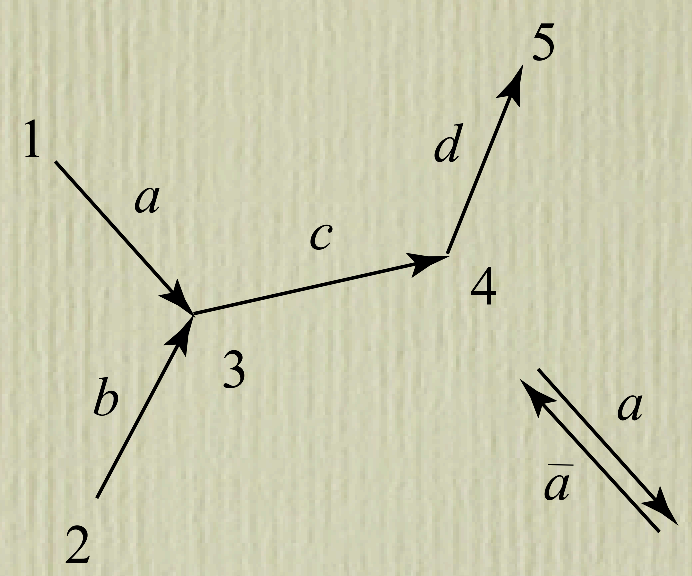

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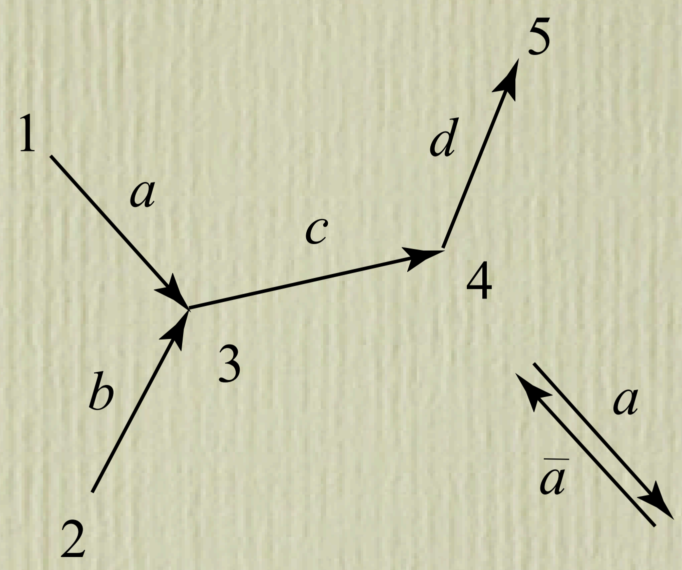

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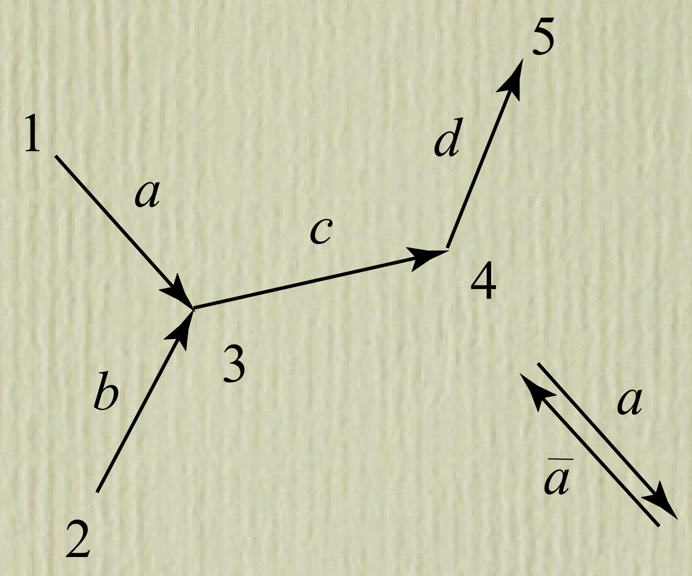
- **Notation**: let

  $$S = \{s_1 < s_2 < \cdots < s_k\} \quad \text{and} \quad T = \{t_1 < t_2 < \cdots < t_k\}$$

  be sets of vertices,

  then

  $$P(S,T) = (p_{s_i\,t_j})_{i,j\in[k]}$$



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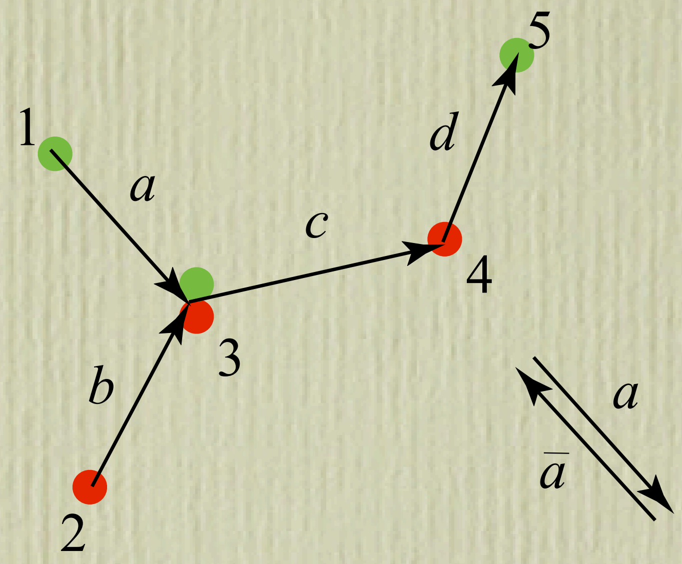

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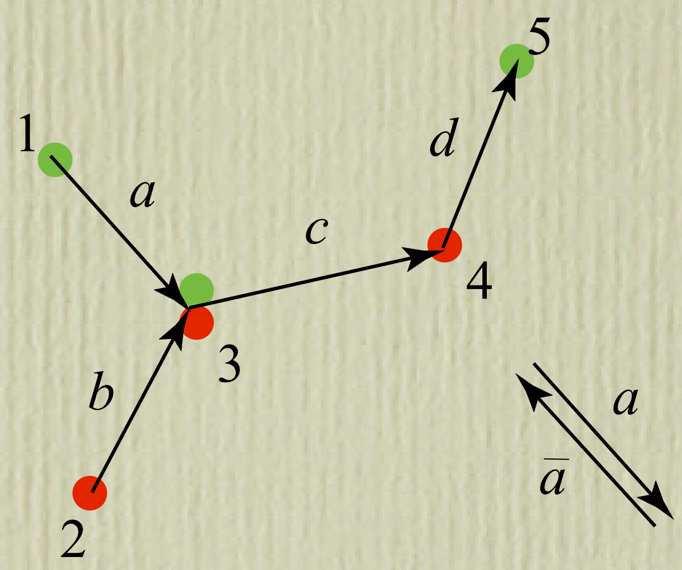

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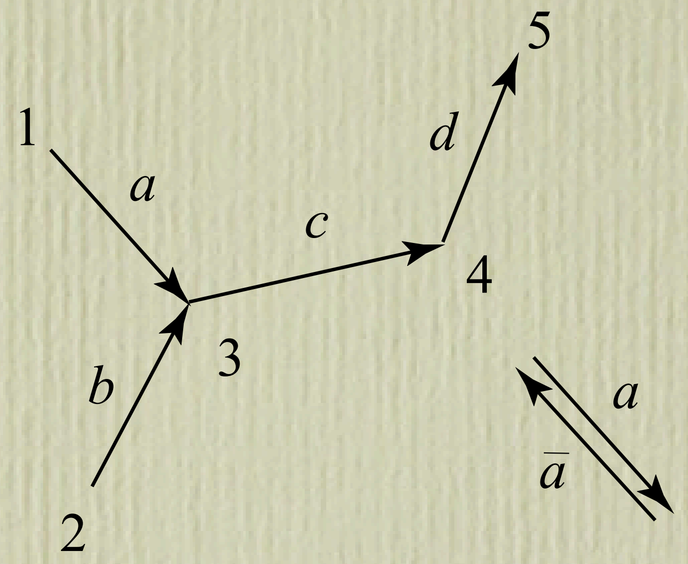
- For instance:

$$\det(P(\{1,3,5\},\{2,3,4\})) = \begin{vmatrix} a\bar{b} & a & ac \\ \bar{b} & 1 & c \\ \bar{b}\bar{c}\bar{d} & \bar{c}\bar{d} & \bar{d} \end{vmatrix} = 0$$



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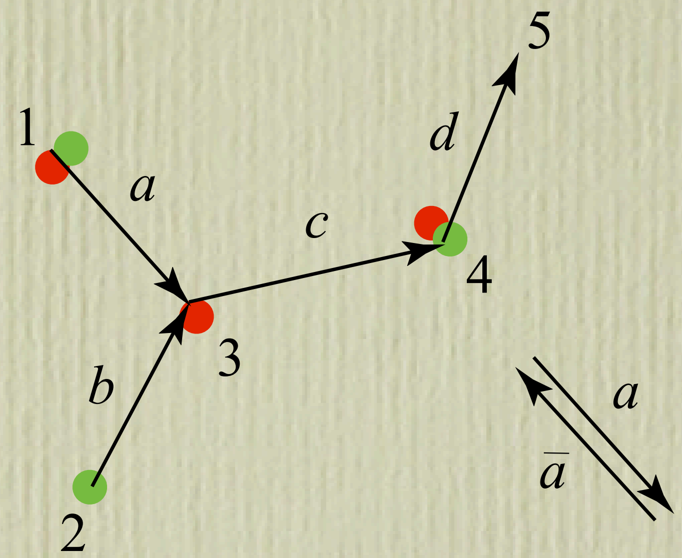

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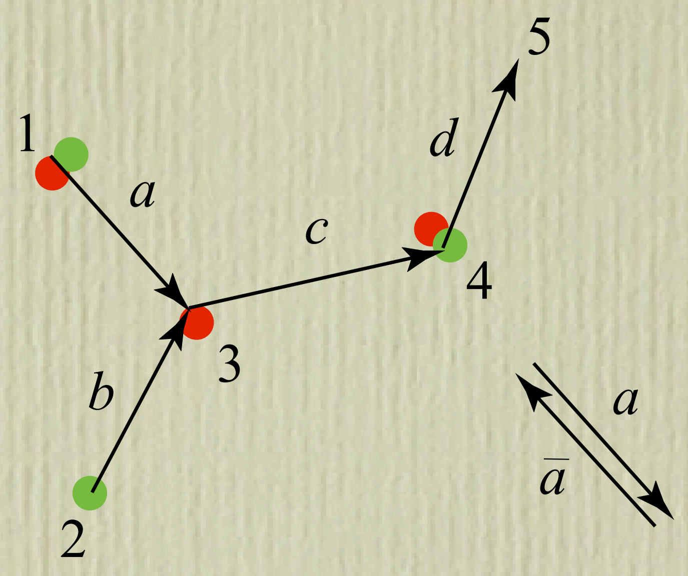

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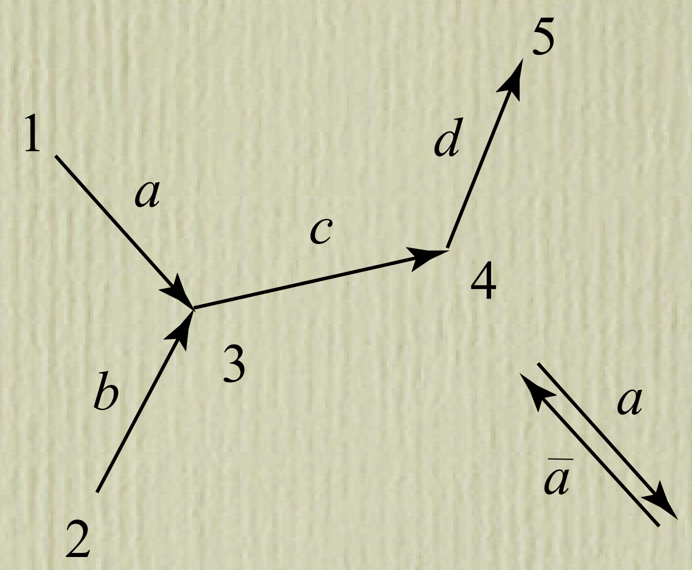
- For instance:

$$\det(P(\{1,2,4\},\{1,3,4\})) = \begin{vmatrix} 1 & a & ac \\ \bar{a}b & b & bc \\ \bar{a}\bar{c} & \bar{c} & 1 \end{vmatrix} = b\,(1-a\bar{a})(1-c\bar{c})$$



$$P = \begin{pmatrix} 1 & a\bar{b} & a & ac & acd \\ \bar{a}b & 1 & b & bc & bcd \\ \bar{a} & \bar{b} & 1 & c & cd \\ \bar{a}\bar{c} & \bar{b}\bar{c} & \bar{c} & 1 & d \\ \bar{a}\bar{c}\bar{d} & \bar{b}\bar{c}\bar{d} & \bar{c}\bar{d} & \bar{d} & 1 \end{pmatrix}$$

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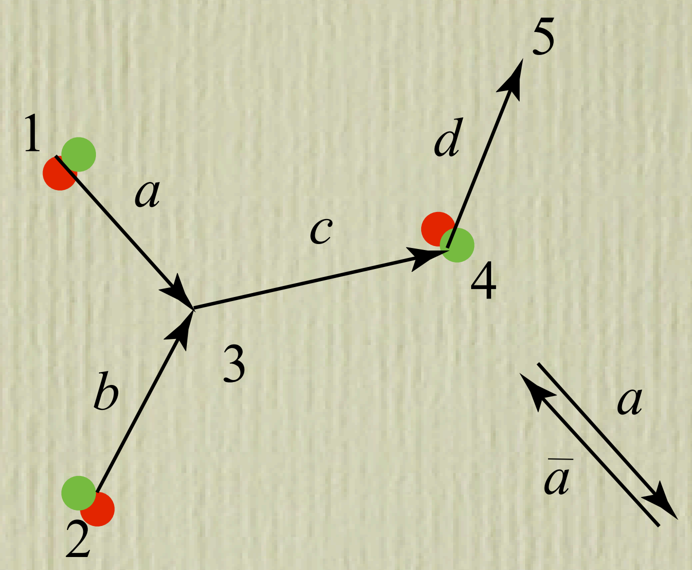
On the minors ... : Introduction



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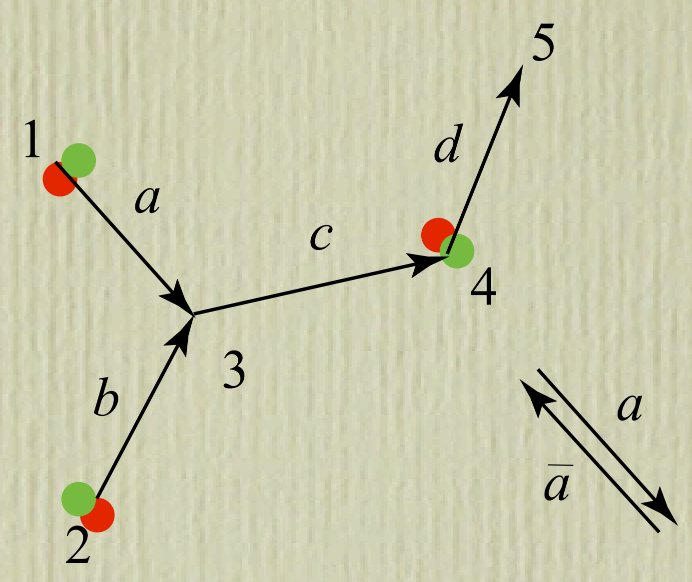

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On the minors ... : Interpreting the minors

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- Given a forest with 2 sets S, T of vertices (such that $|S| = |T|$), we have

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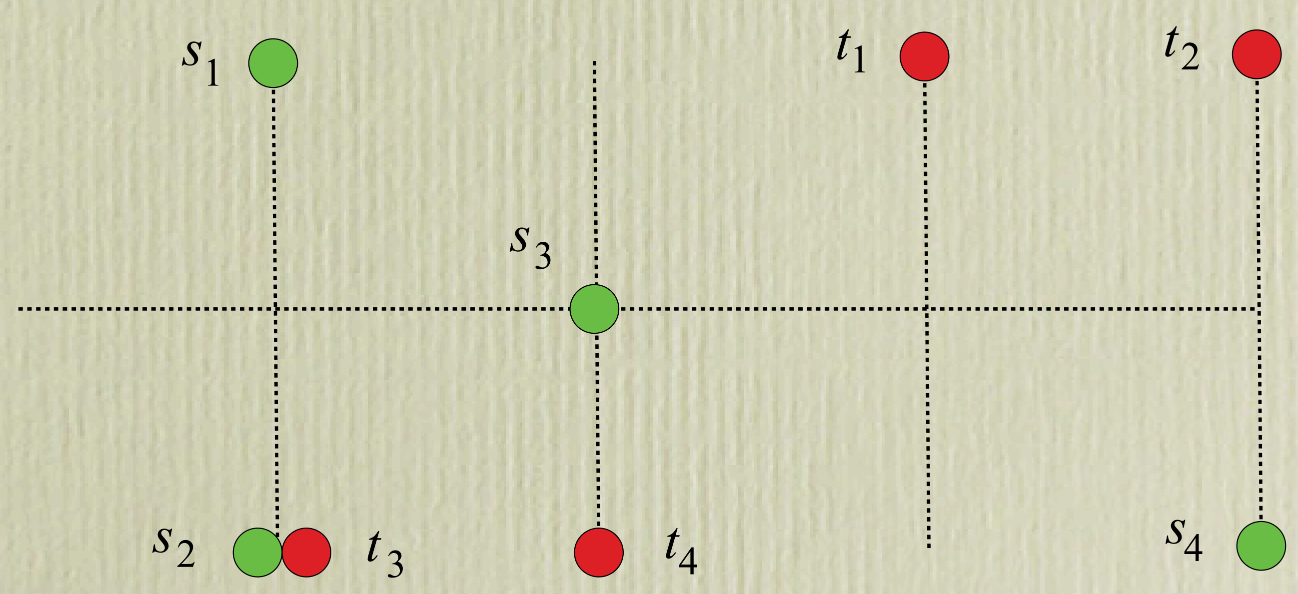

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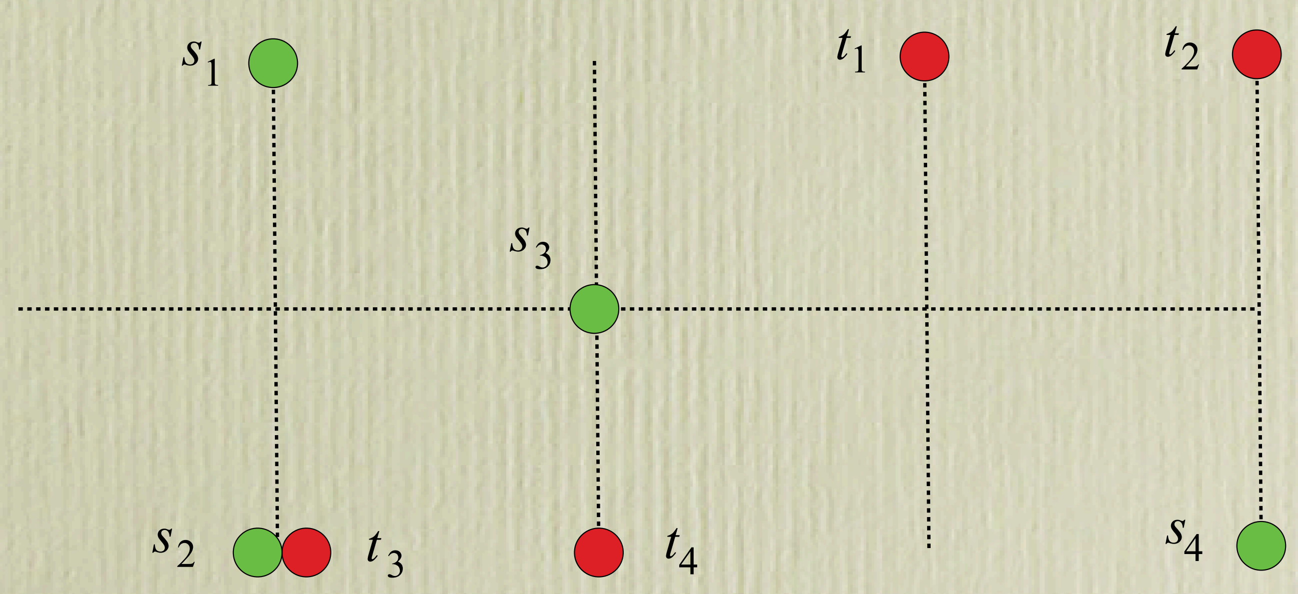

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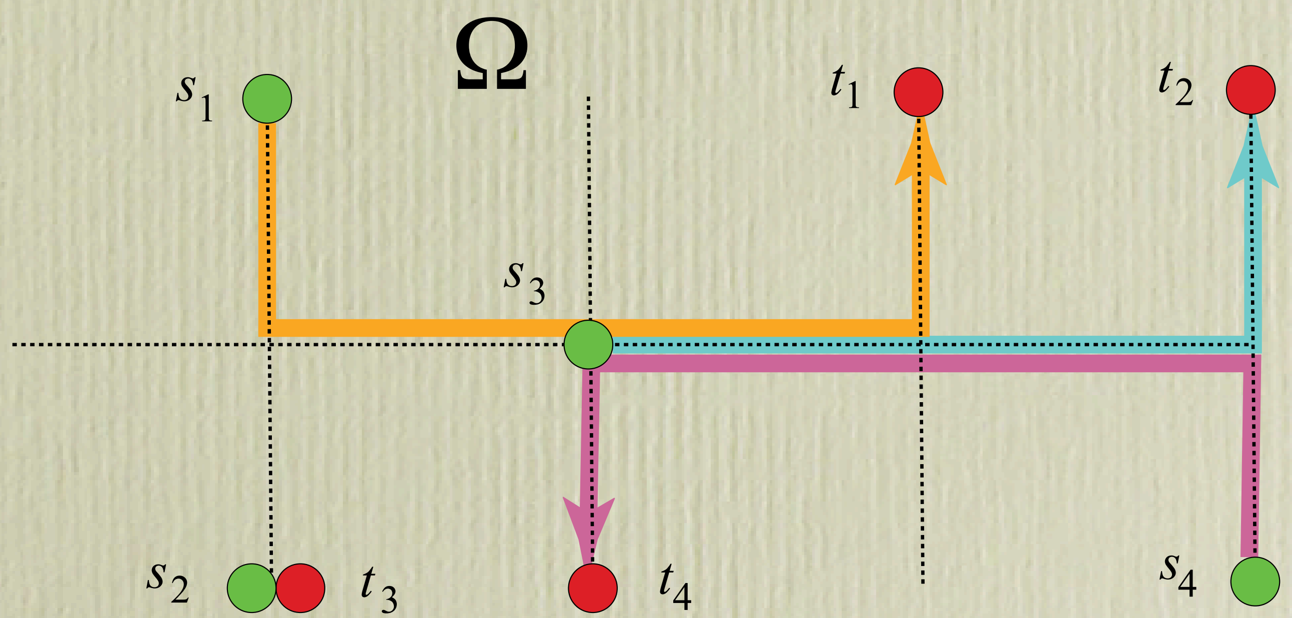

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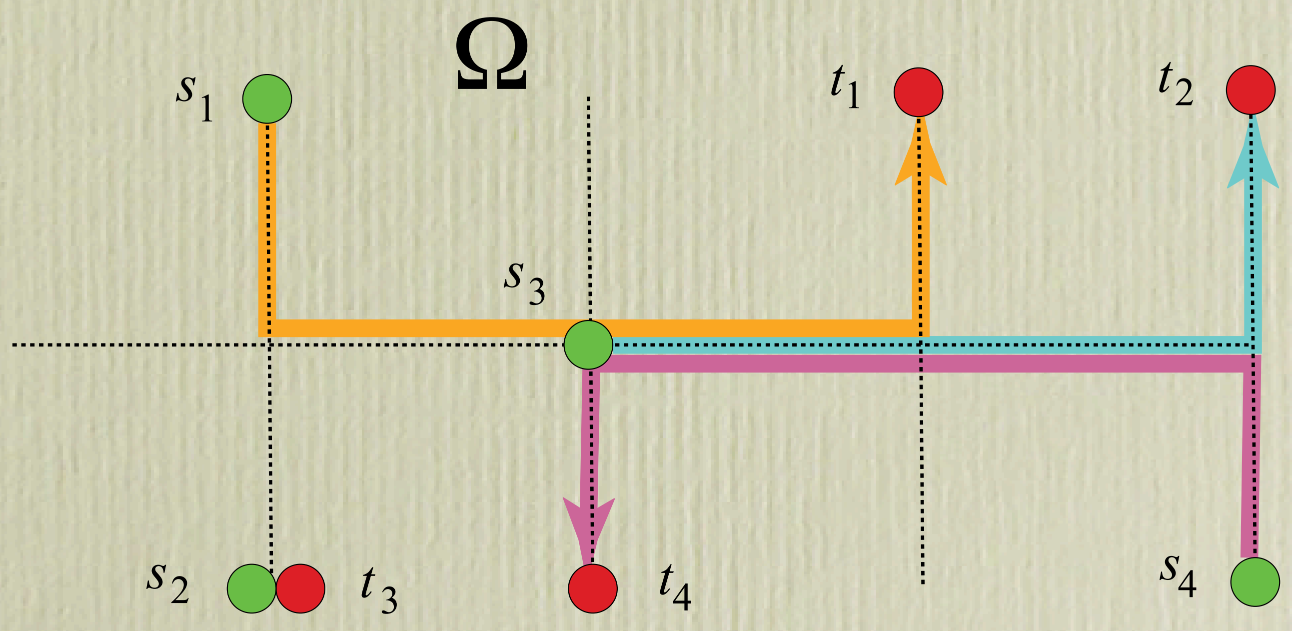

$\sigma = 1324$

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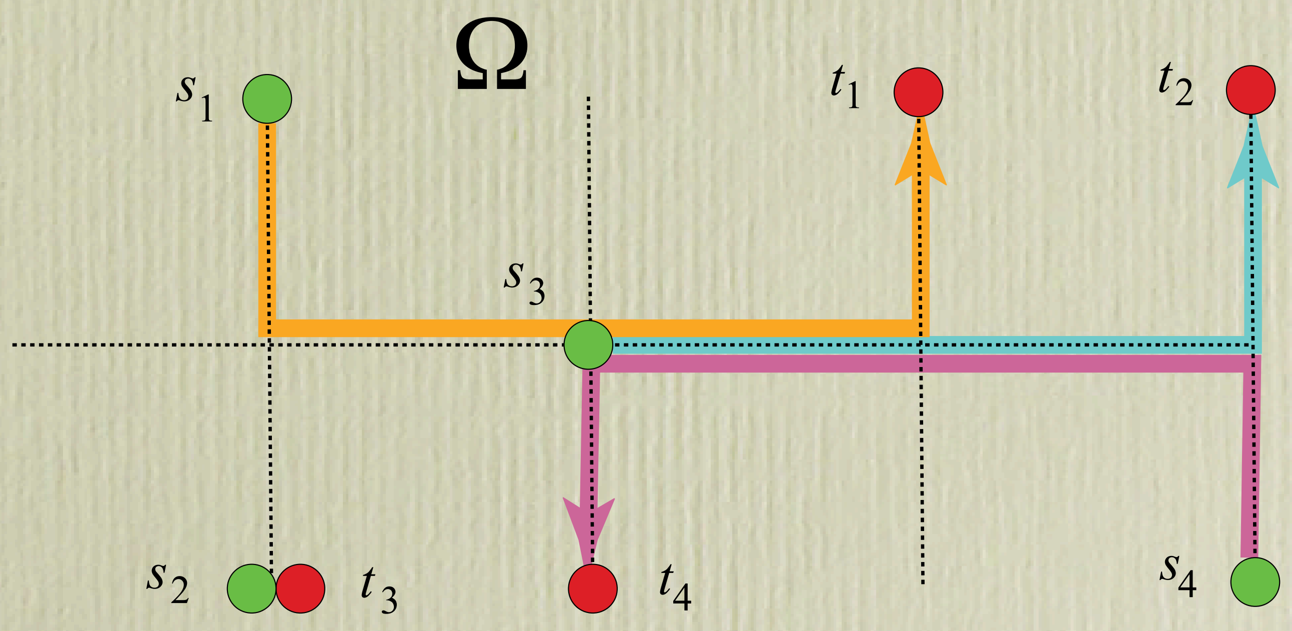

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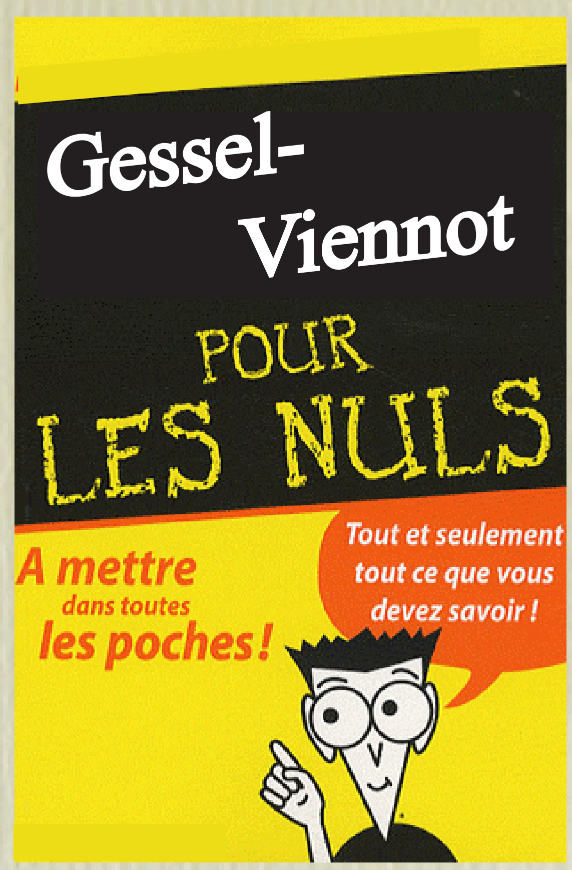
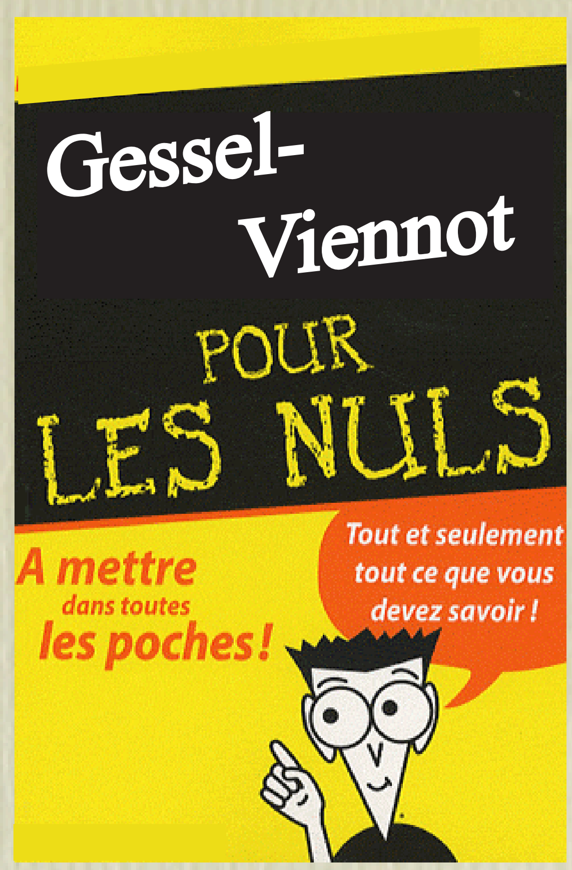

Gessel-
Viennot
POUR
LES NULS
A mettre
dans toutes
les poches!
Tout et seulement
tout ce que vous
devez savoir !

On the minors ... : Interpreting the minors



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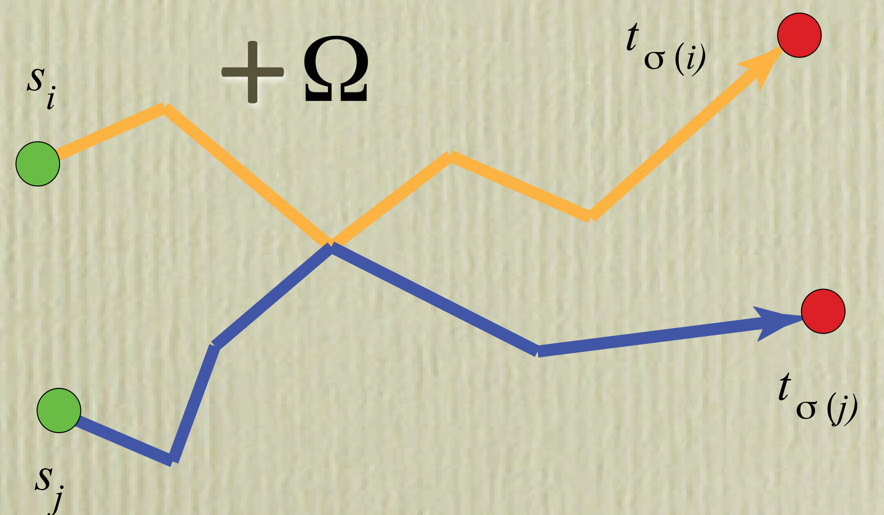
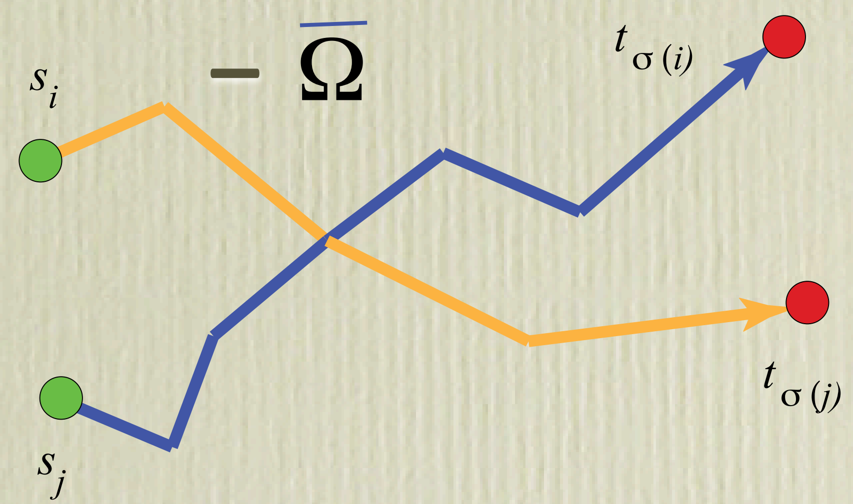


permutation σ



permutation $\sigma \circ (i, j)$

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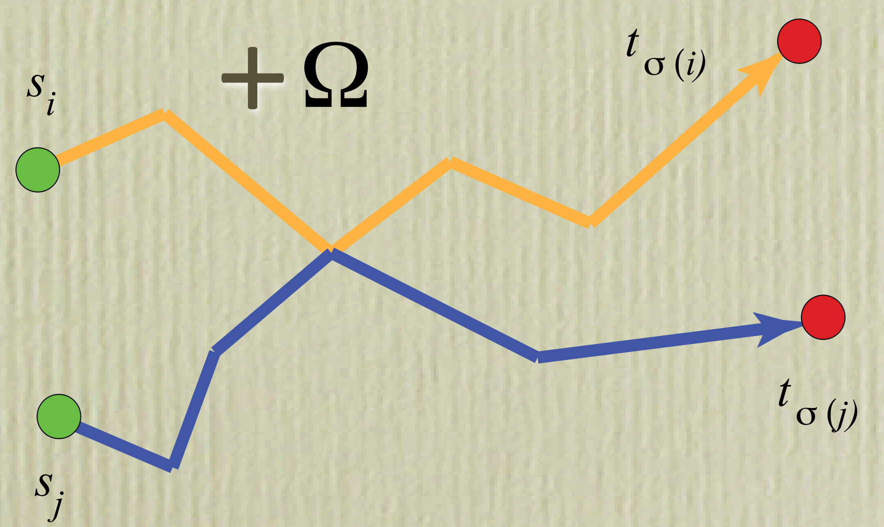

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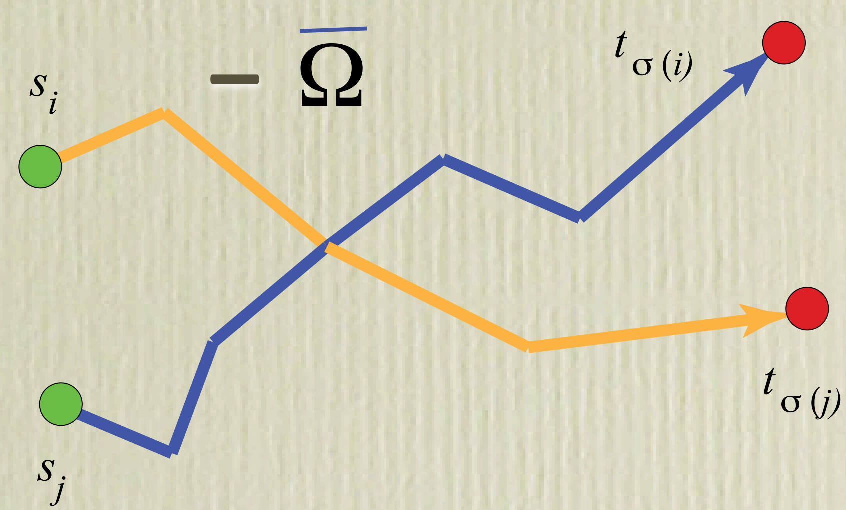

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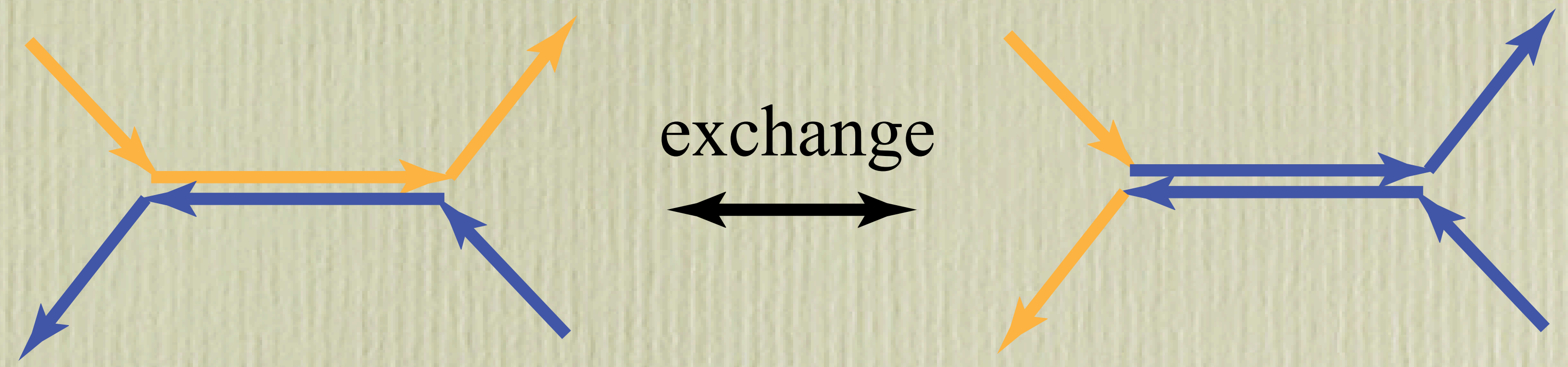

Not a shortest path

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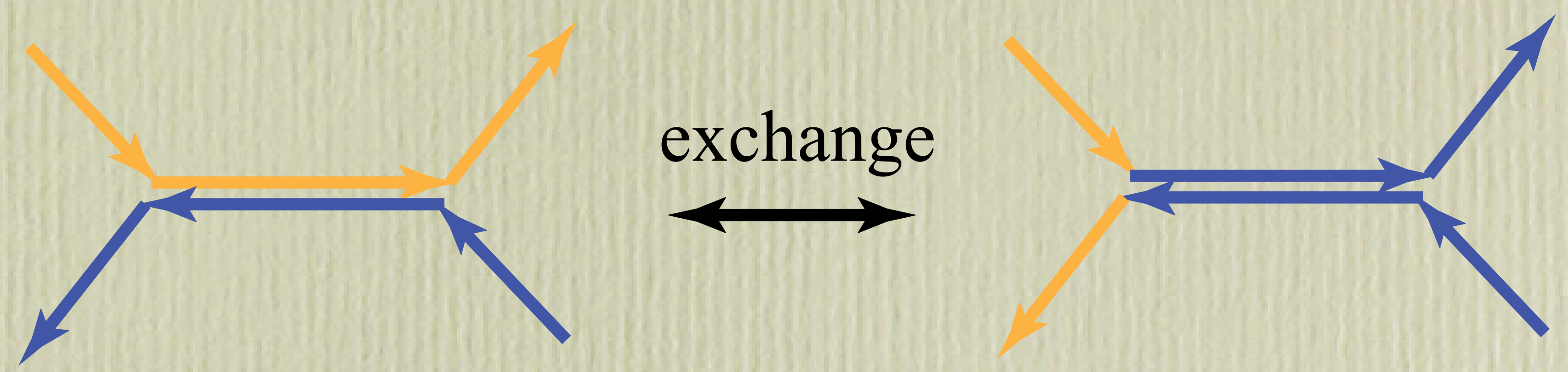

Not a shortest path

  - **Problem 2:** What are the paths to be transformed ?

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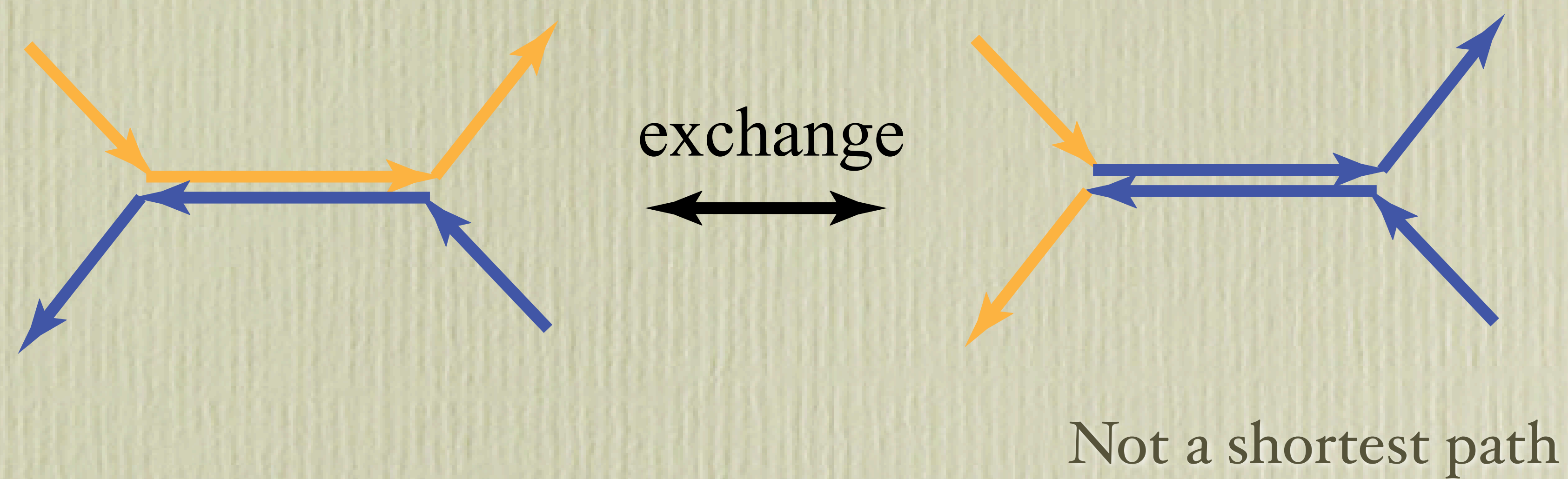

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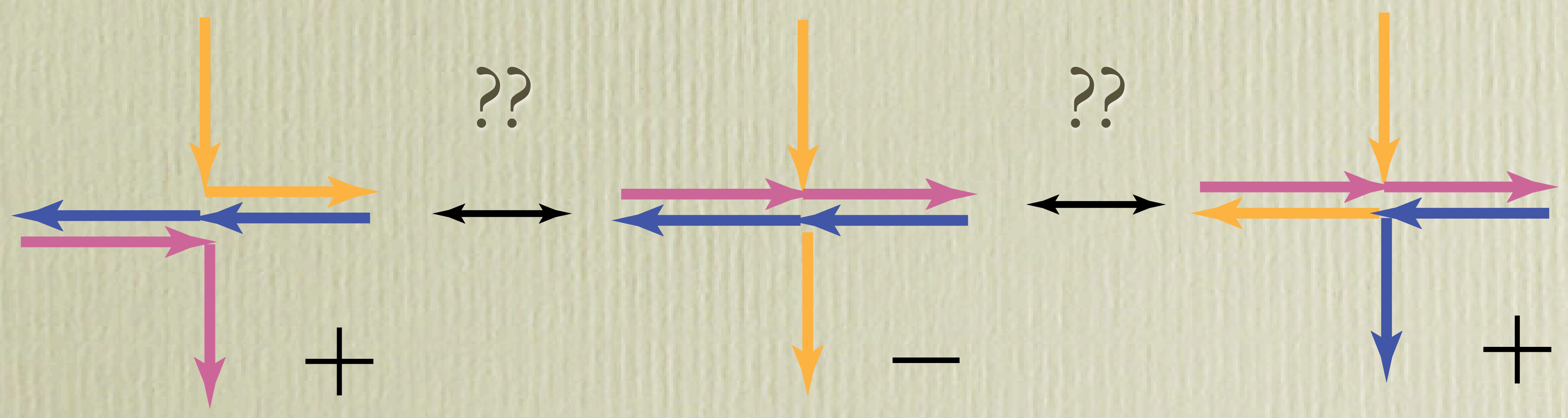

- **Partial solution:**

  The exchange works if there is a common **arrow**,
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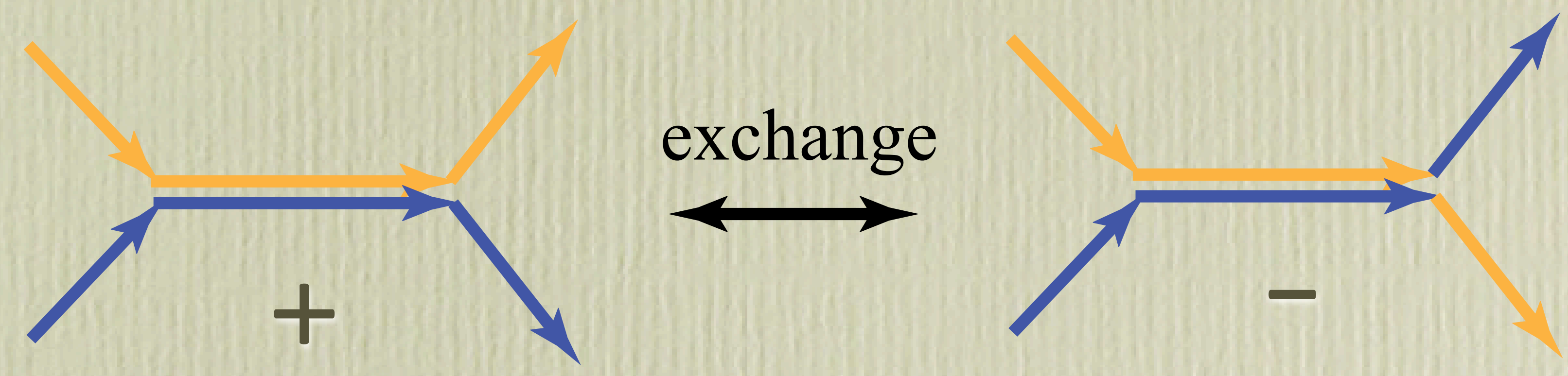

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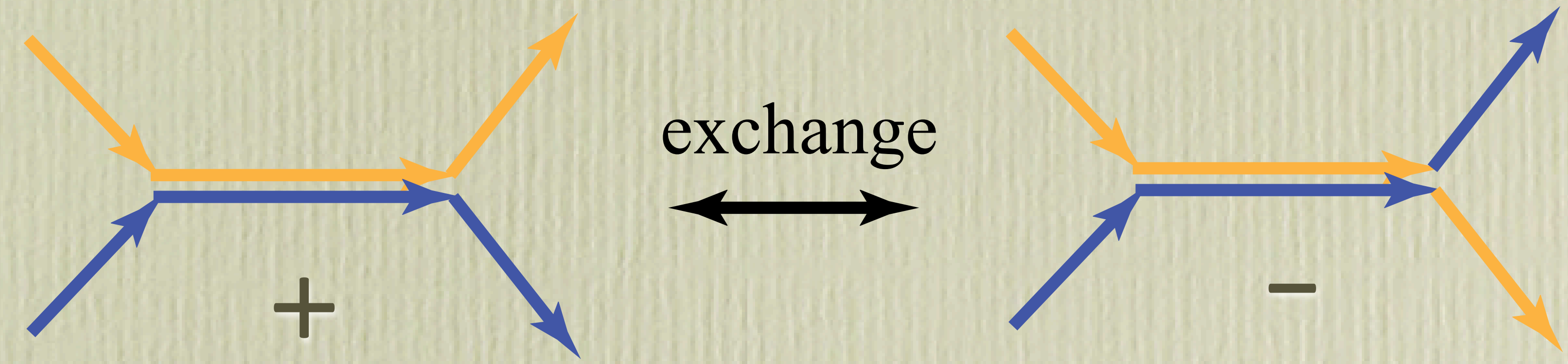

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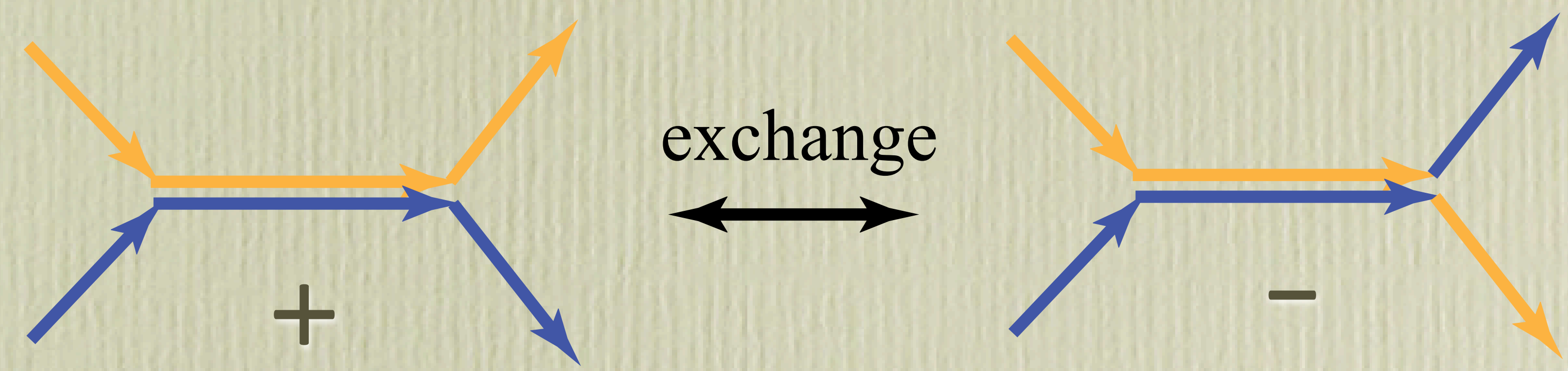

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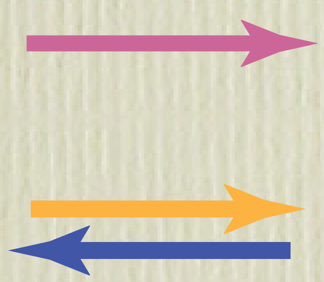
- The **arrows** in the surviving configurations are either **single** or come in **pairs of opposite**.



On the minors ... : Single arrows

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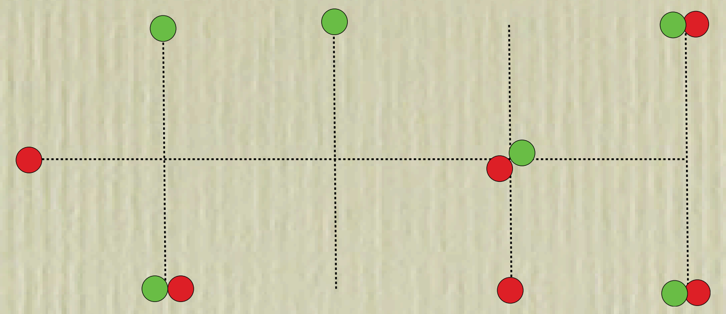
- Some configurations $\Omega : S \to T$



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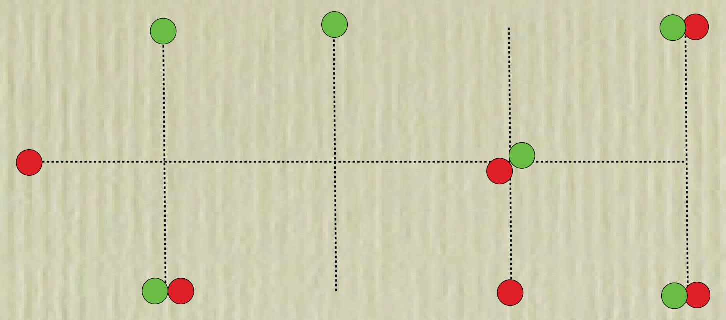

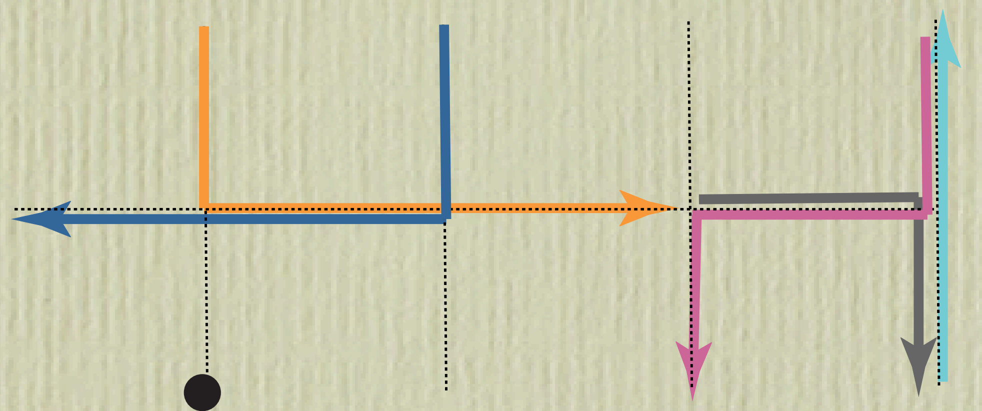

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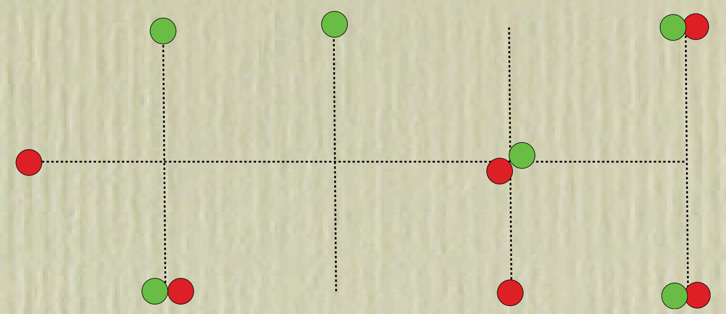

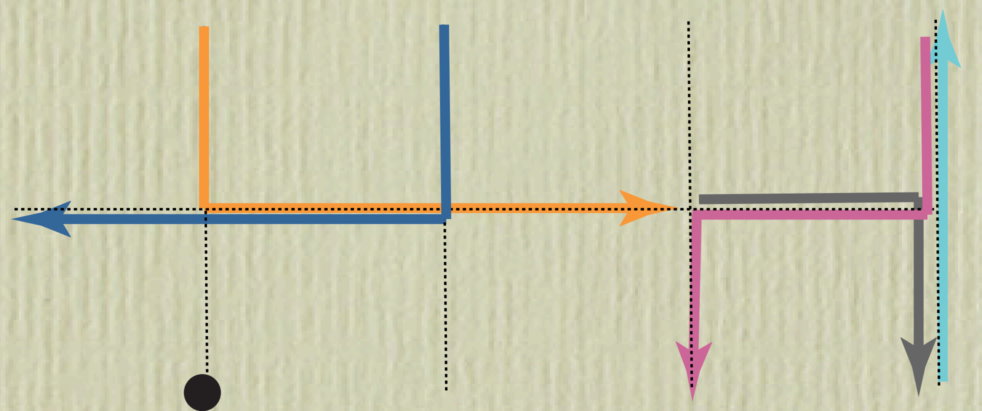

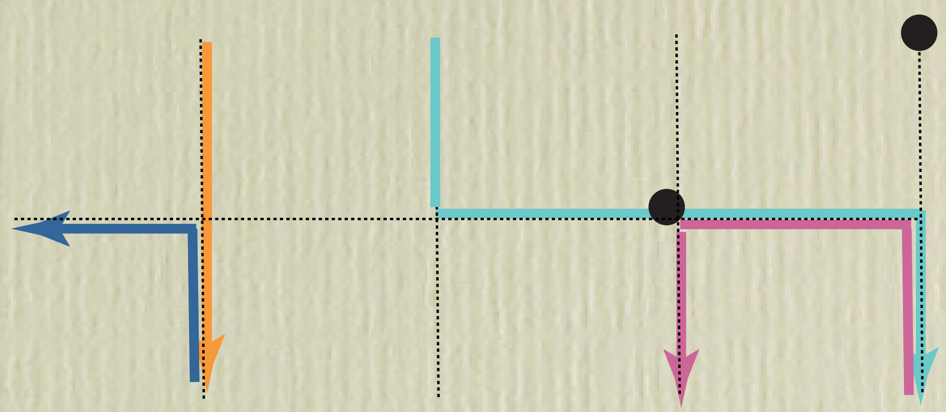

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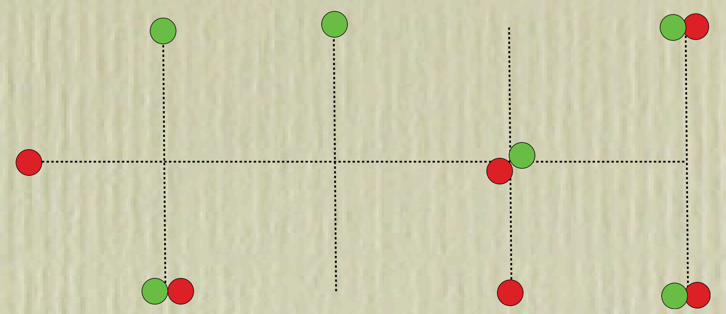

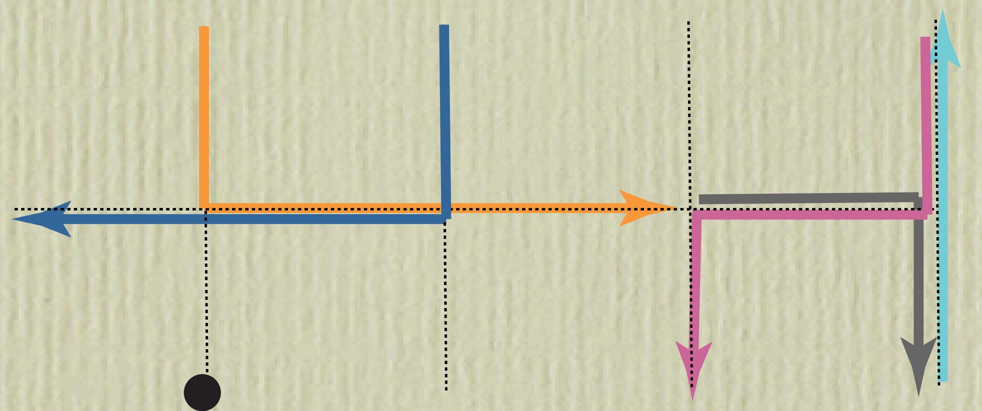

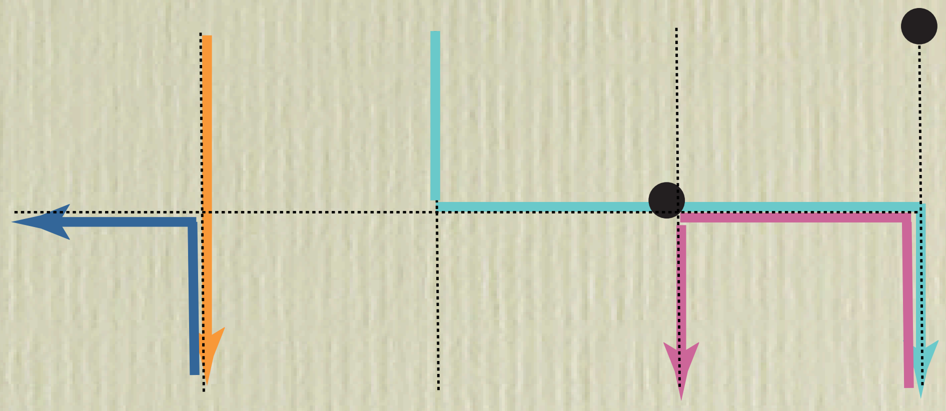

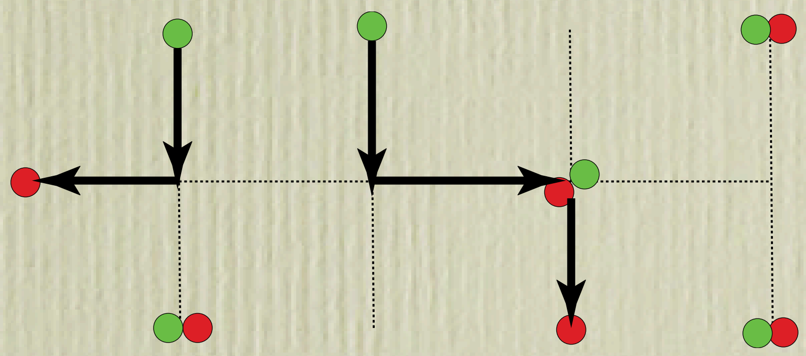

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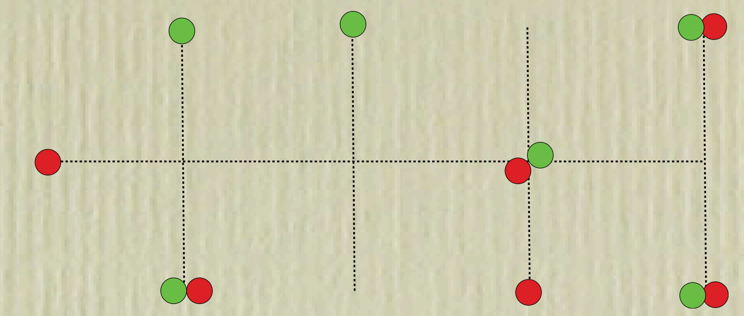
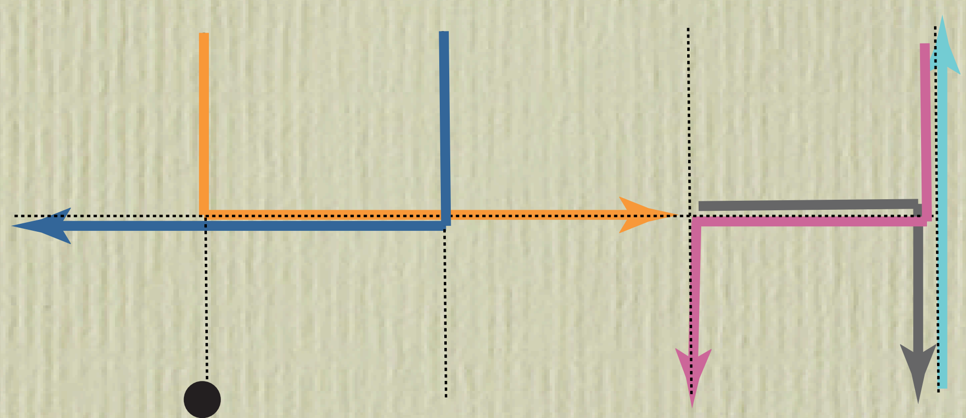
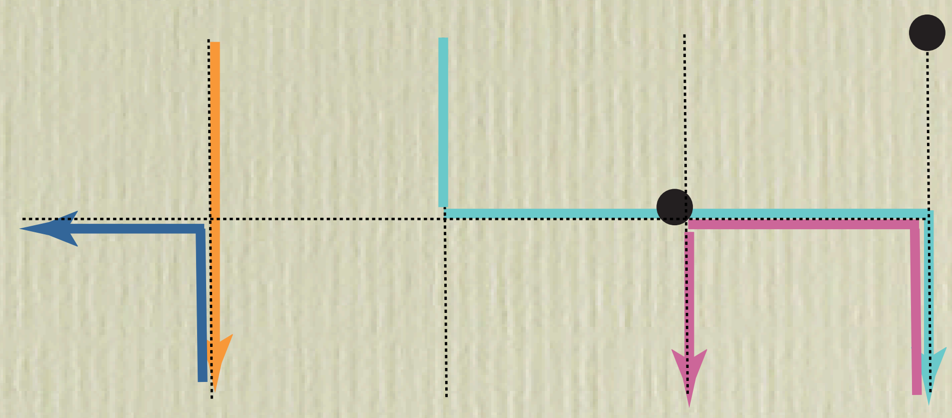
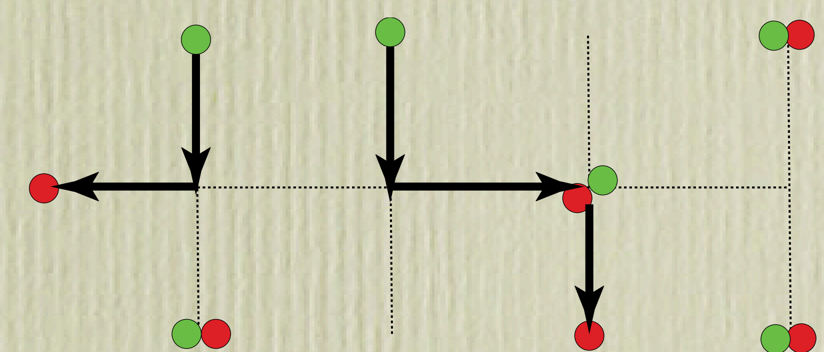




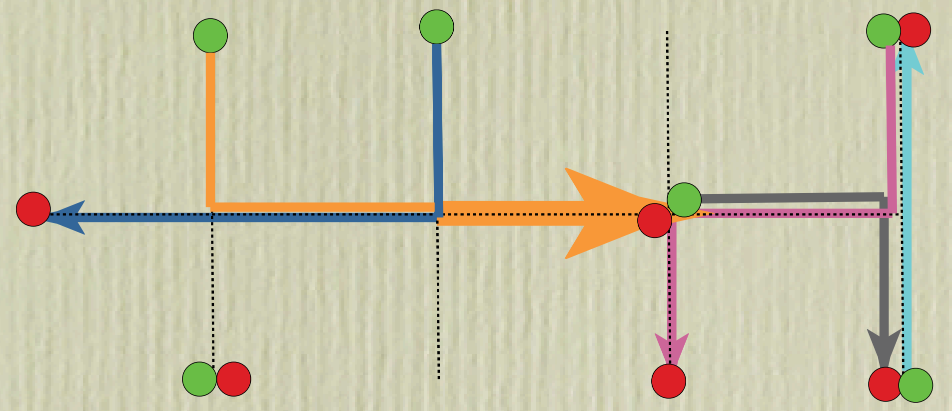




- **Remarks:**
    - Single arrows are the same for all configurations (?)
    - Pairs of opposites may vary

On the minors ... : Single arrows

- Consider a single arrow in a configuration



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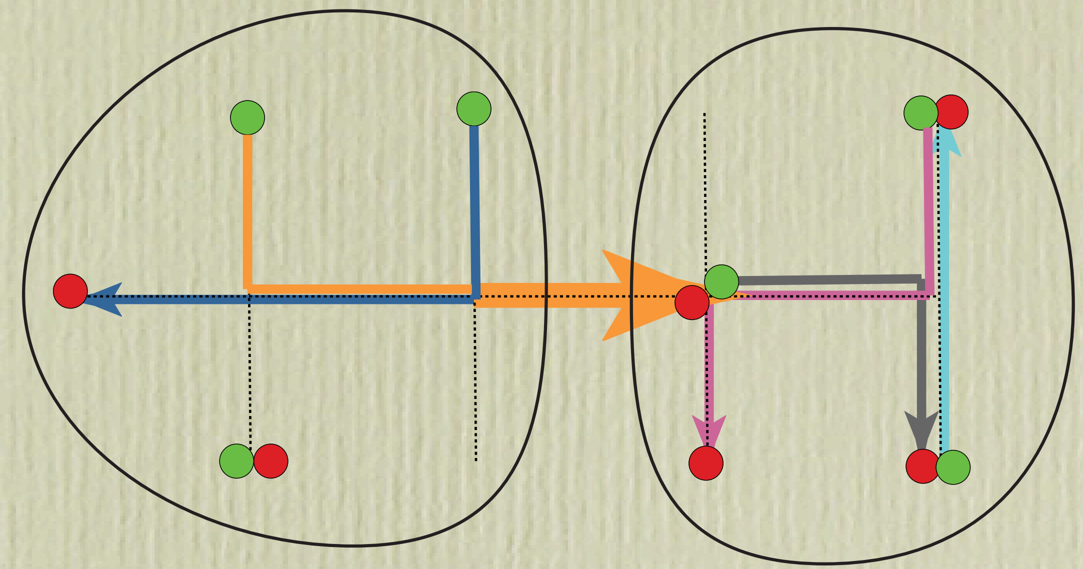

- In this subtree, | S | - | T| = 1 In this subtree, | S | - | T| = -1

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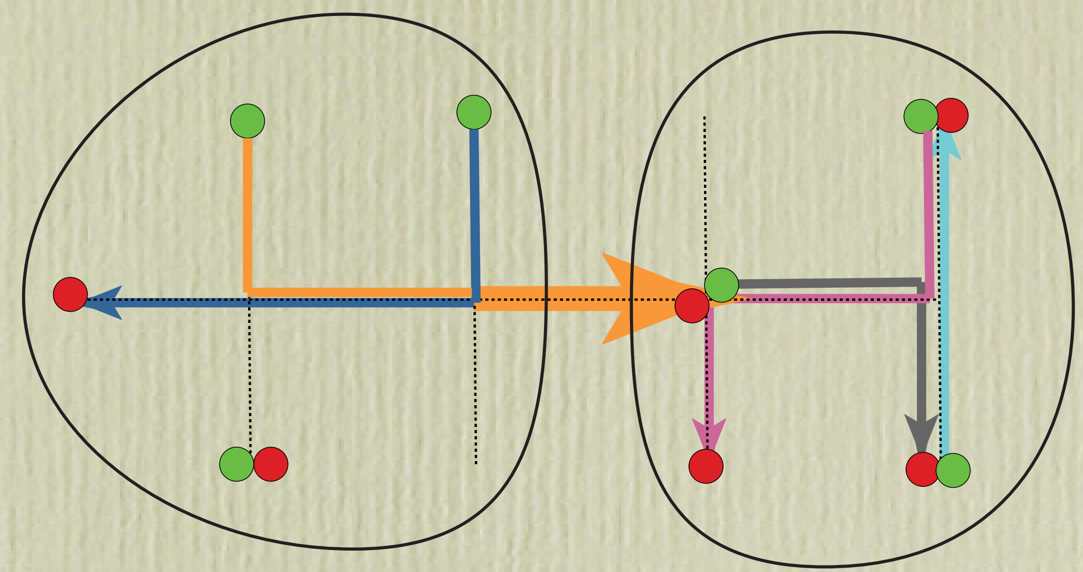

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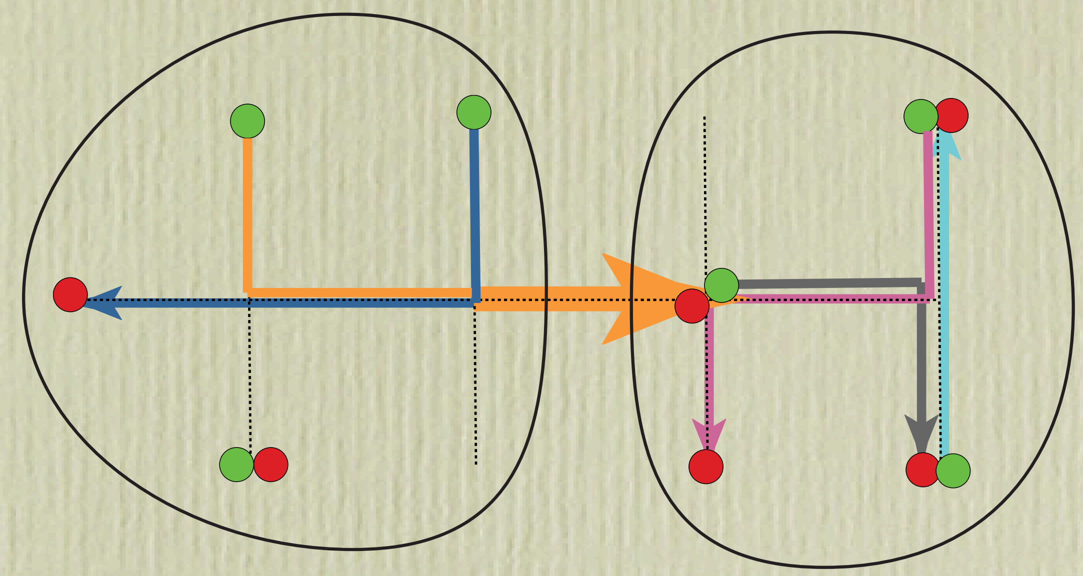

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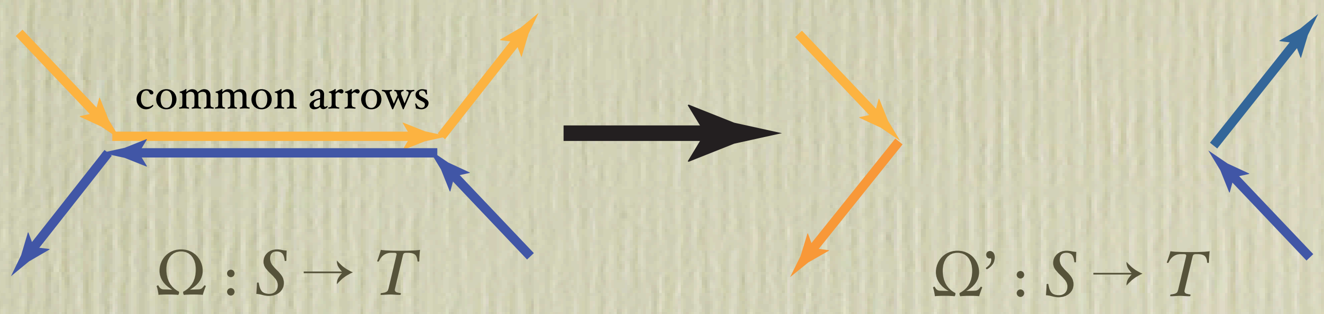
- N.B. $||S| - |T|| > 1$ would implies double arrows in all configurations

On the minors ... : Single arrows

On the minors ... : Single arrows

- Any configuration $\Omega : S \rightarrow T$ contains the forced (single) arrows



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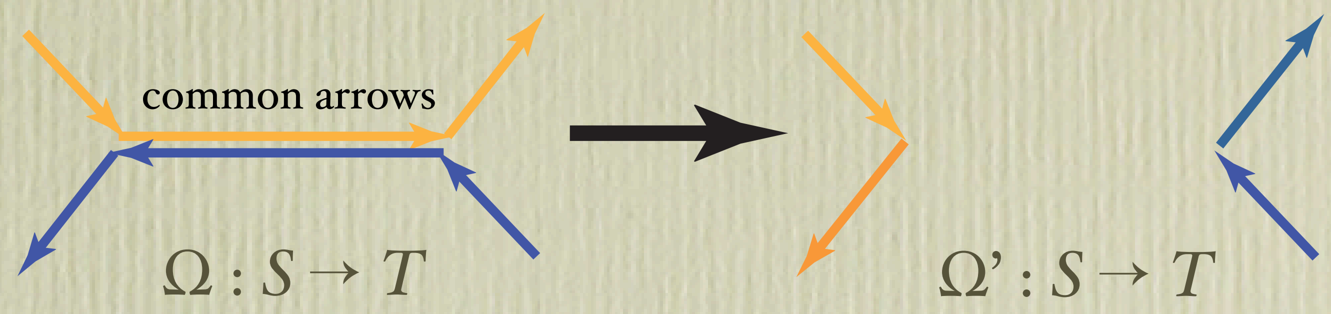

- Forced arrows can be connected to one another to form *minimal* configurations
$$\Omega_0 : S \to T$$
(not necessarily unique)

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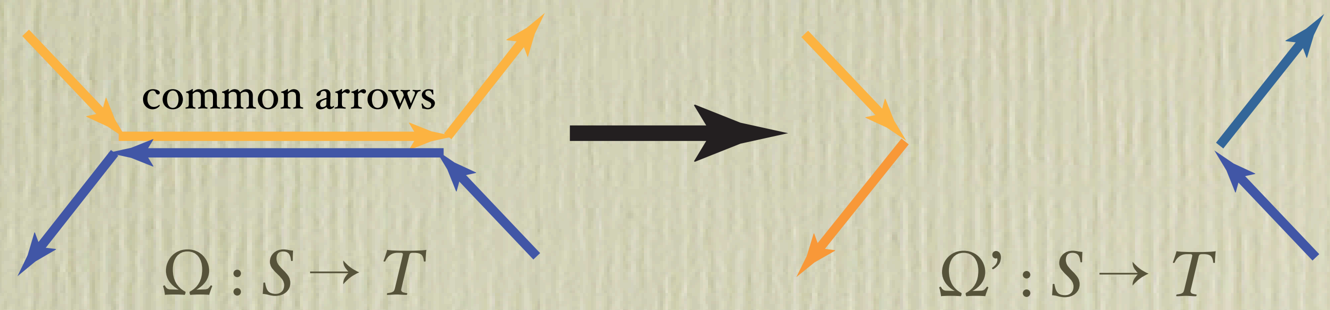

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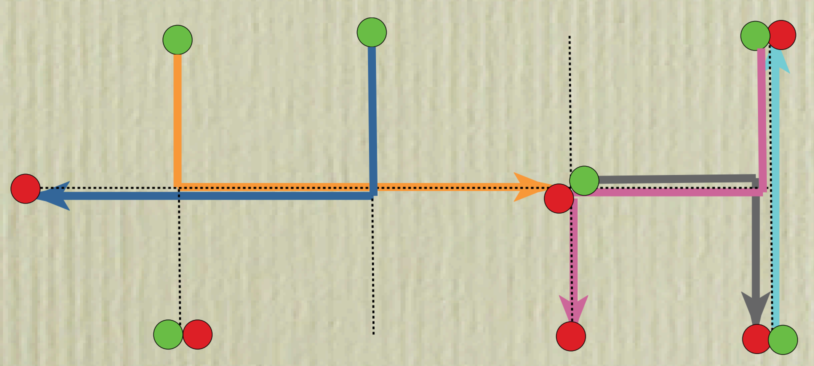

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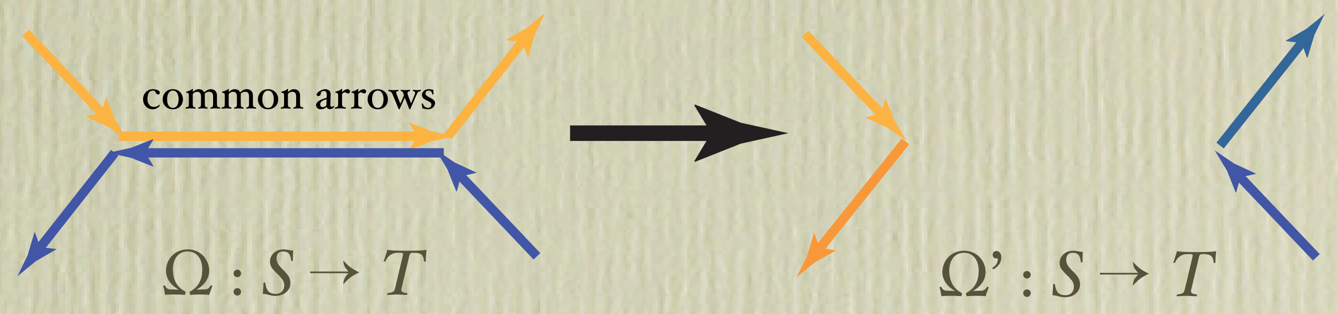

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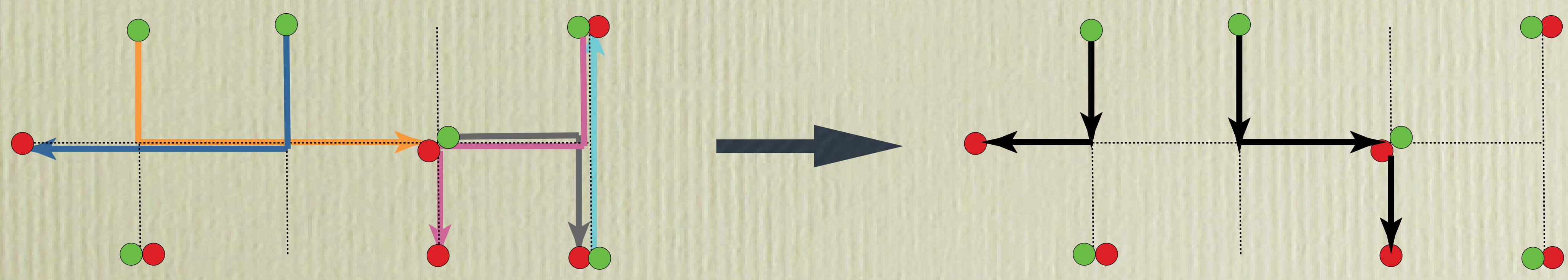

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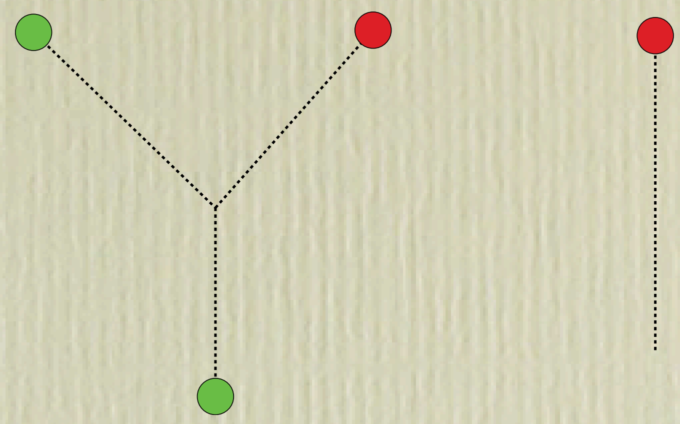

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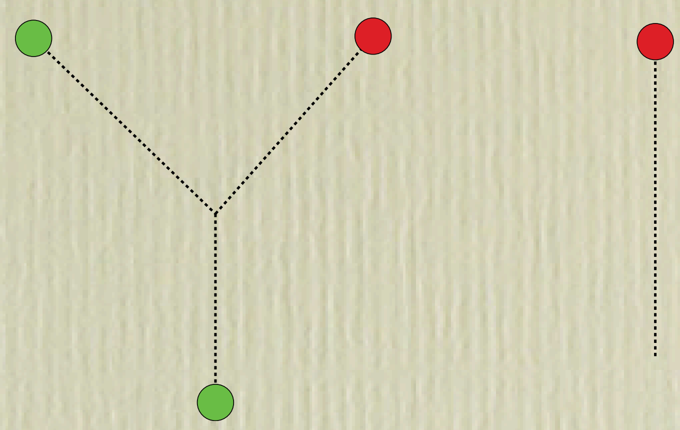

No configuration $\Omega : S \to T$ exists

$$\det(P[S,T]) = 0$$

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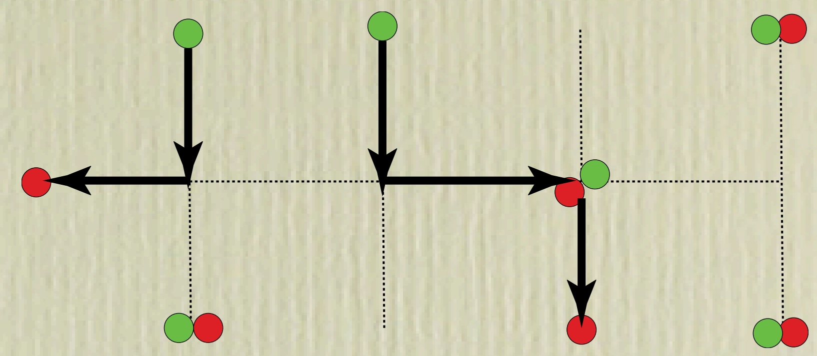

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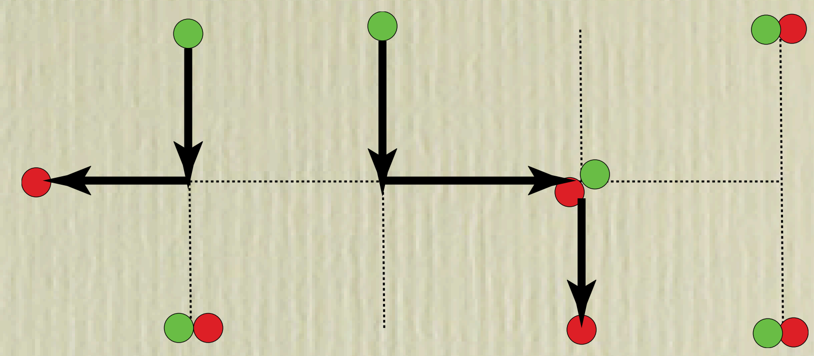
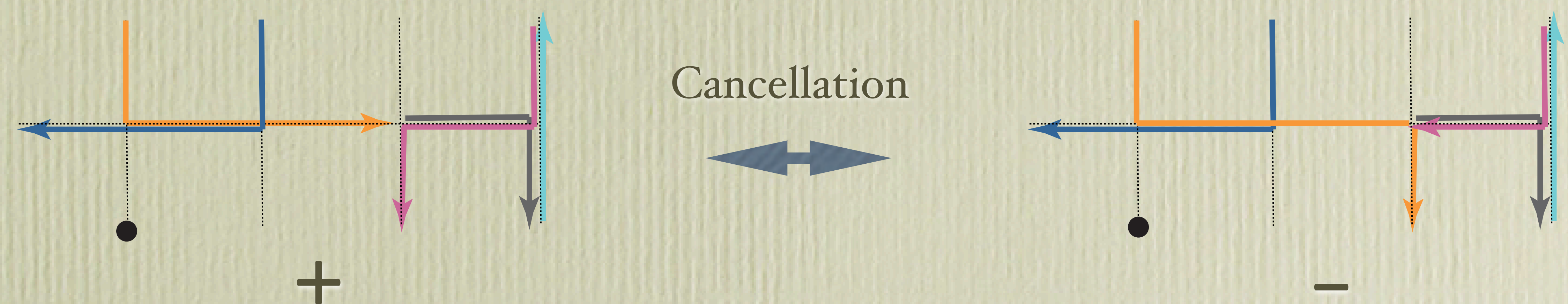


Minimal configurations not unique



$$\det(P[S,T]) = 0$$

On the minors ... : When does $|P[S,T]| \neq 0$?

- Necessary conditions for $\det(P[S,T]) \neq 0$:
  - There must exist at least one minimal configuration.
  - The minimal configuration must be unique
- Sufficient conditions

  We have:

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- Remark: If no two paths of a minimal configuration have a common vertex, the minimal configuration is unique

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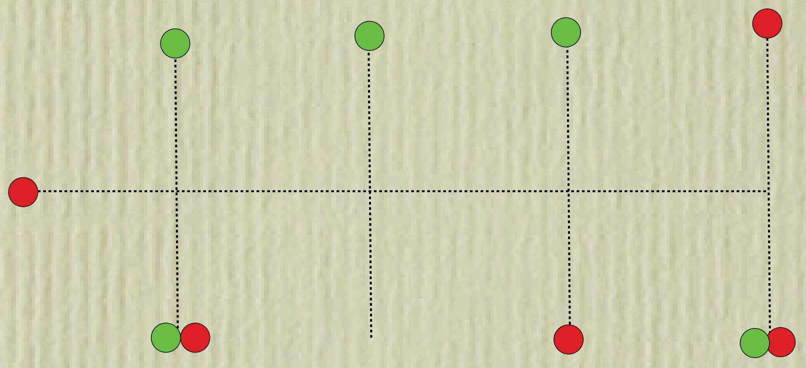

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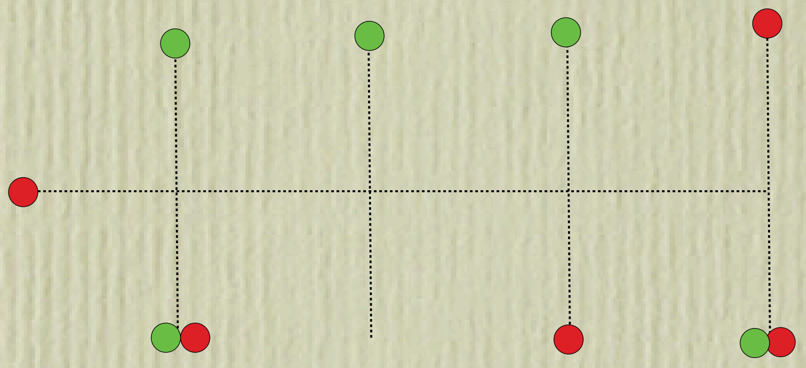

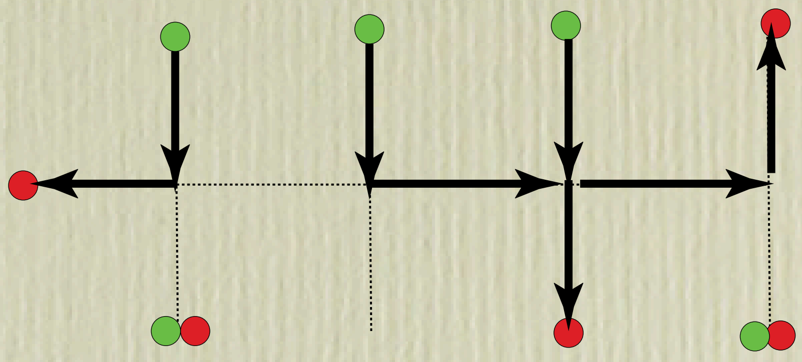

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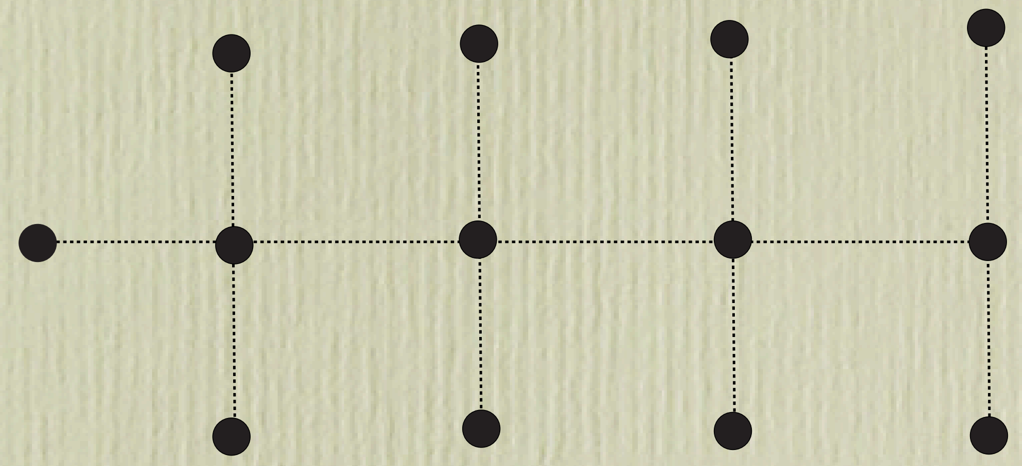

Unique minimal configuration

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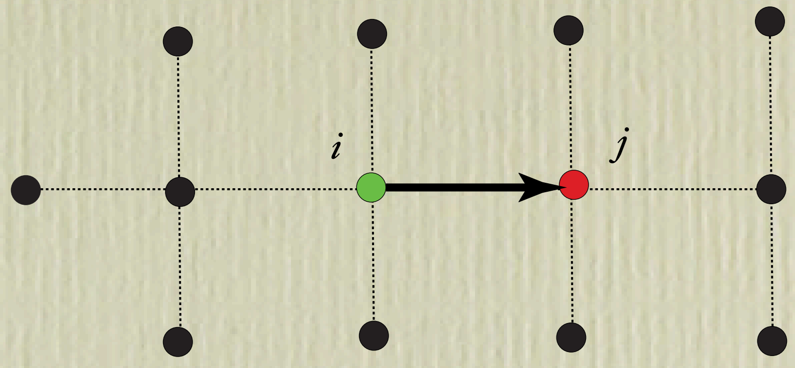

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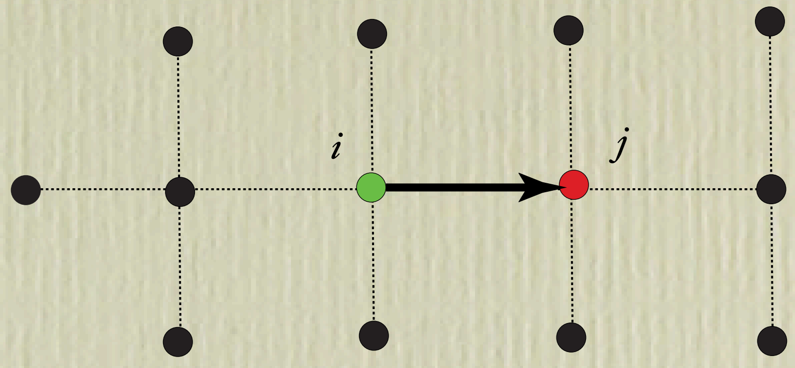

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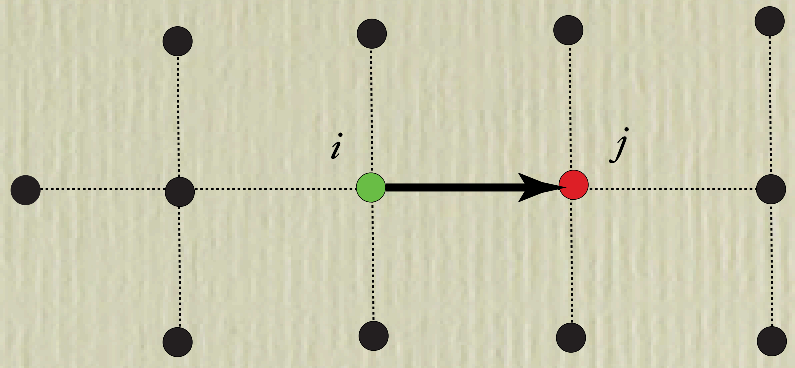

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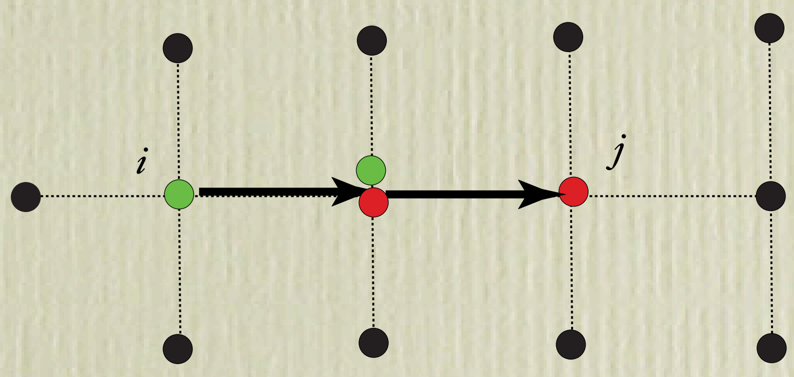

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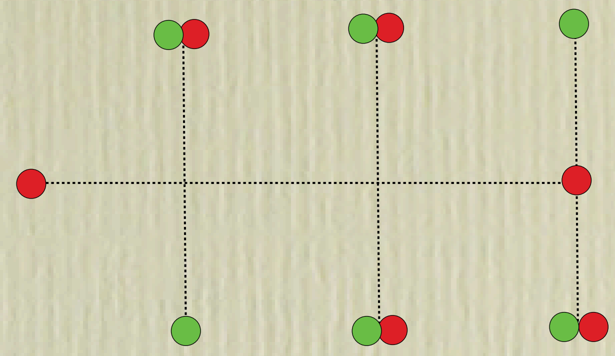
On the minors ... : Enumeration



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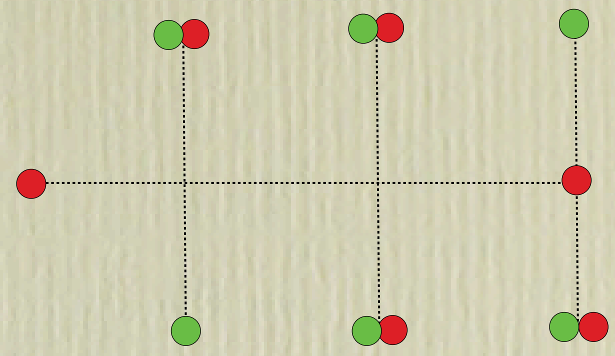

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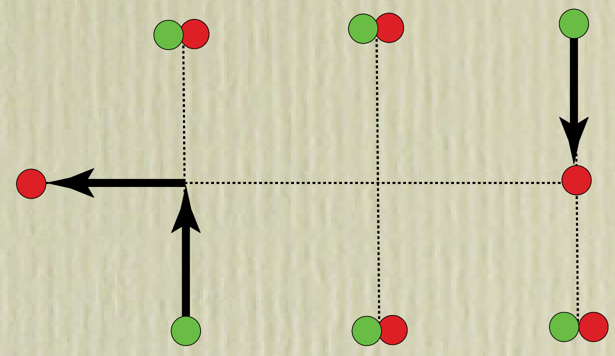

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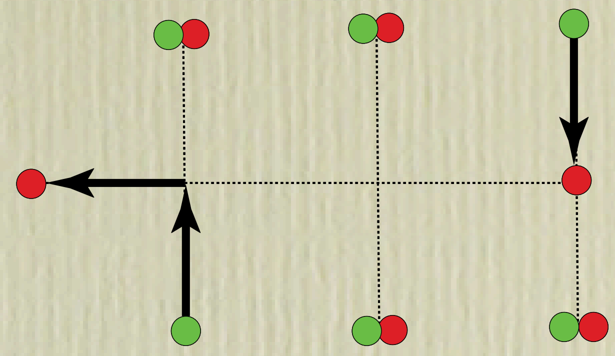

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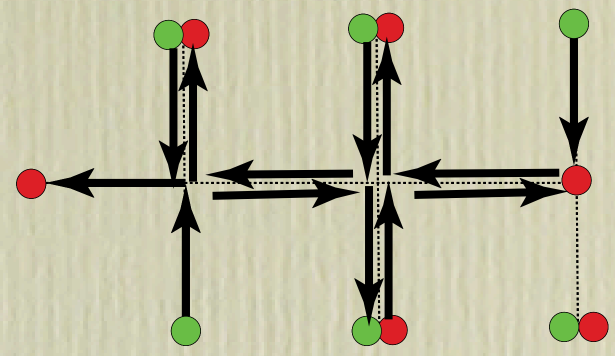

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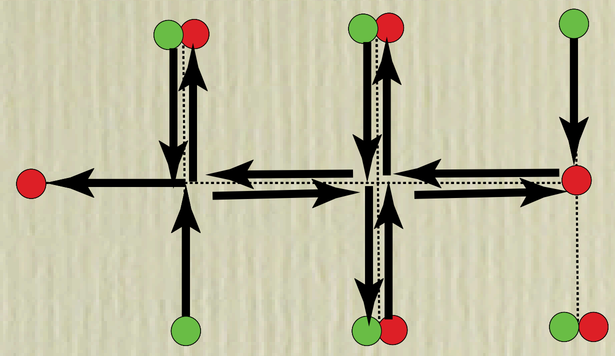

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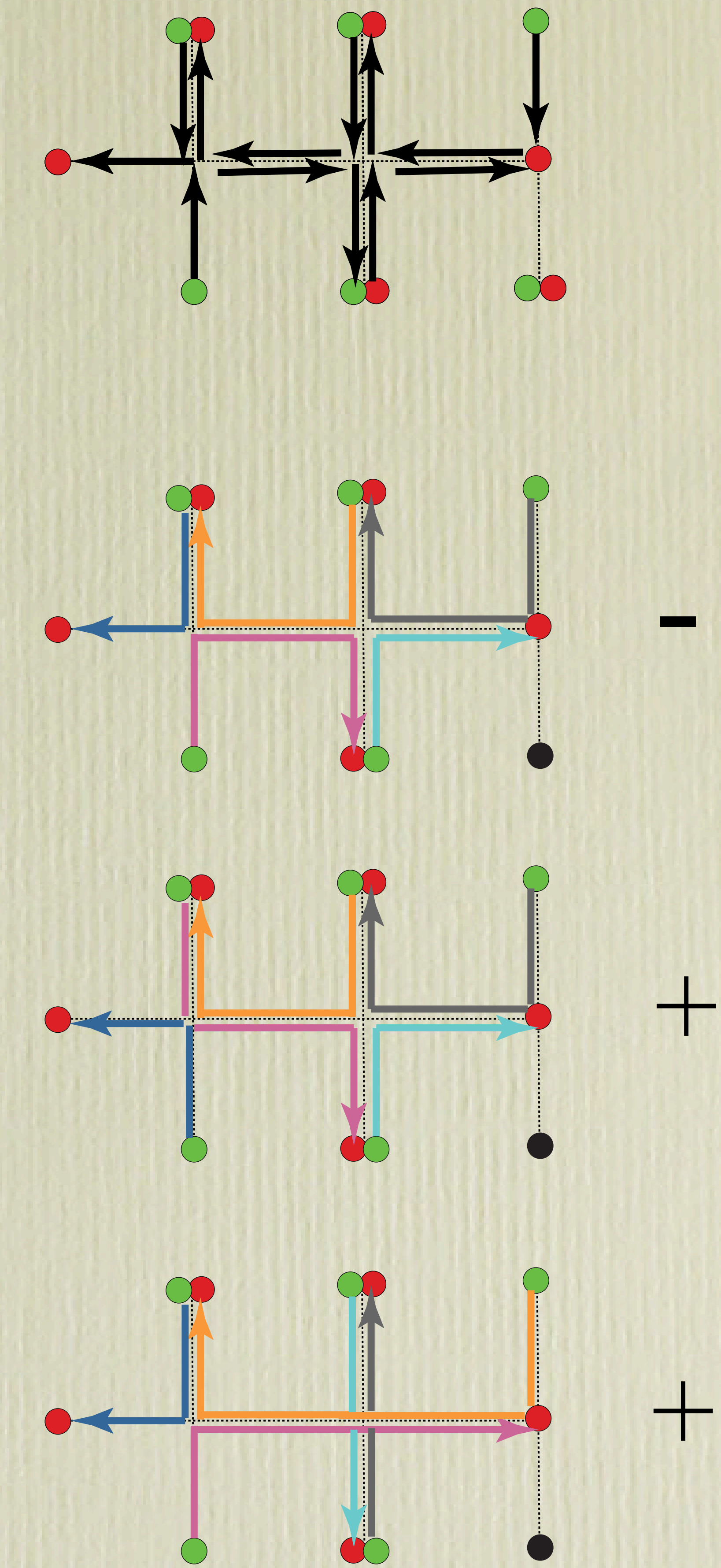

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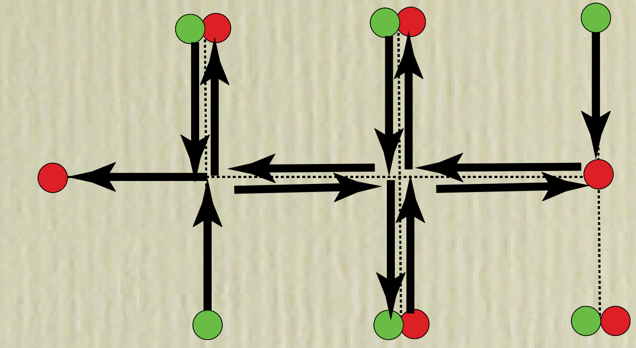

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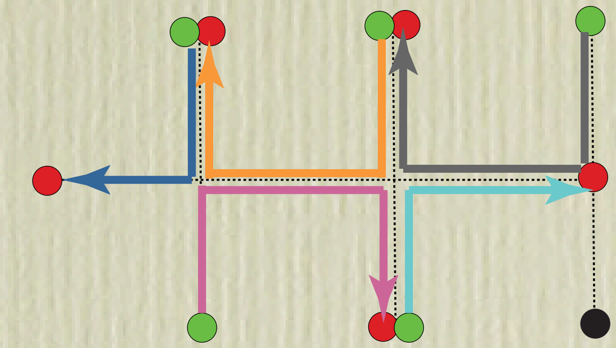

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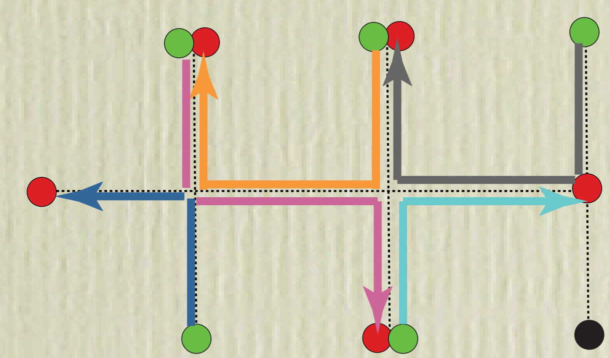

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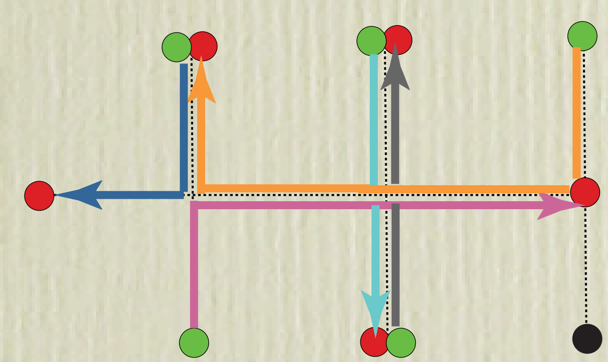
- **Wanted:**



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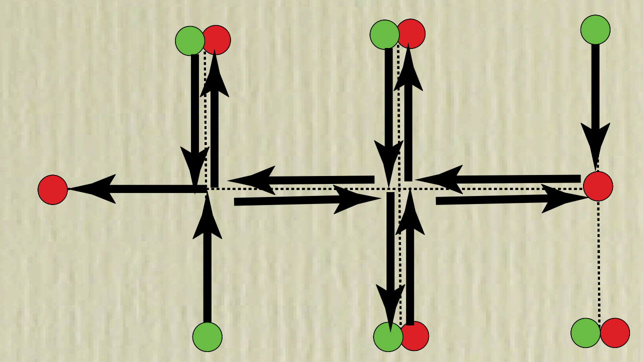

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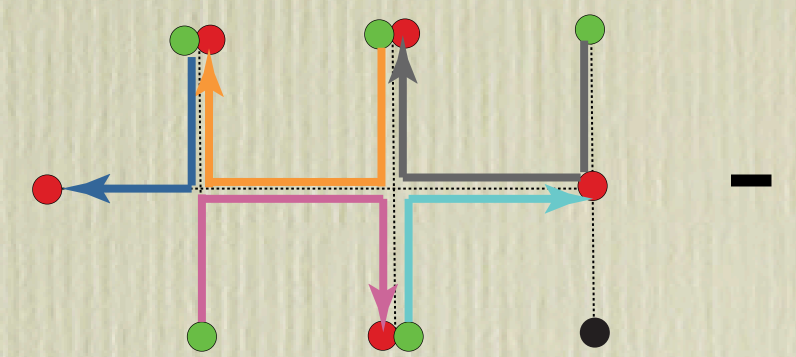

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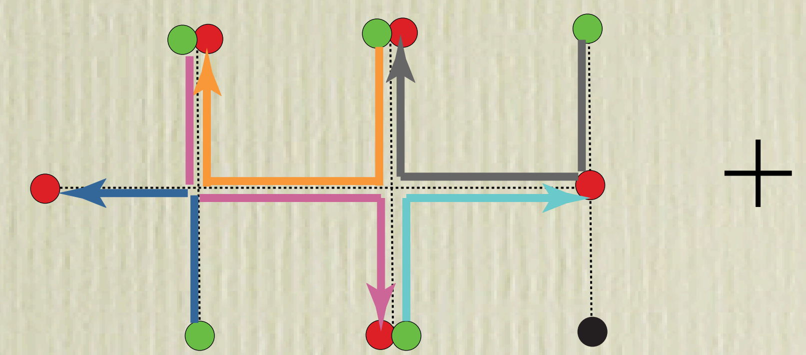

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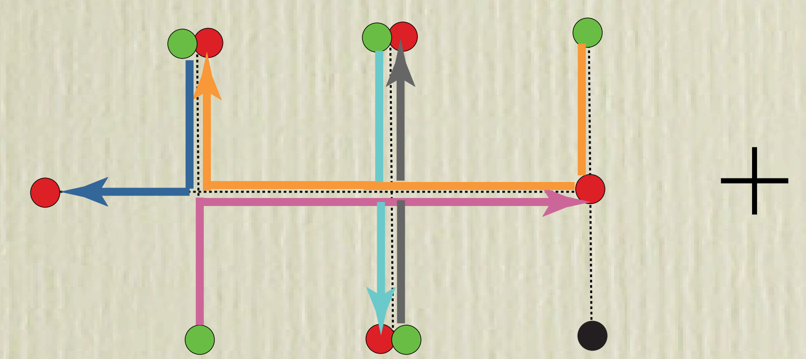

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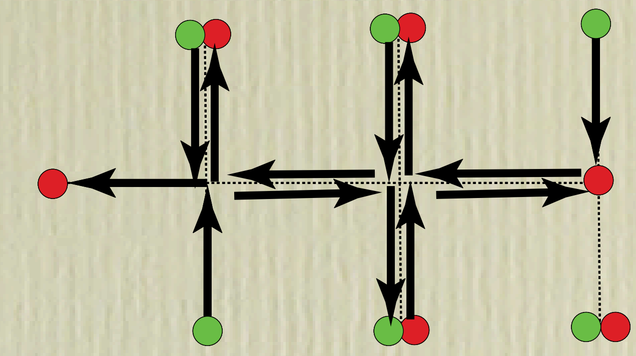

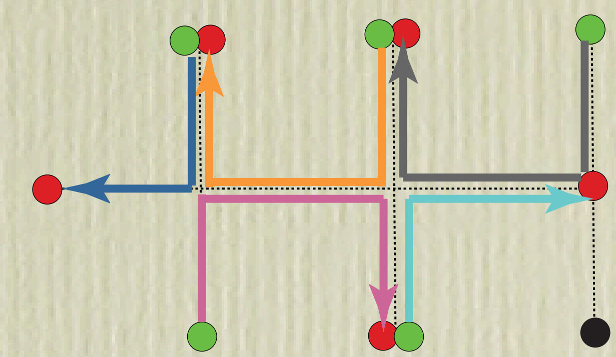

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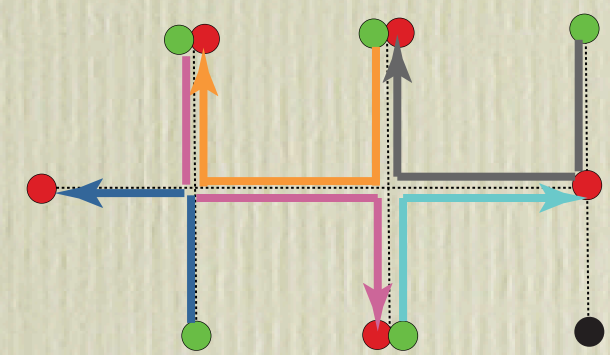

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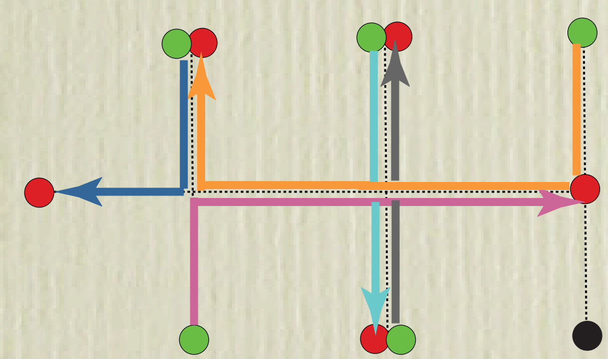

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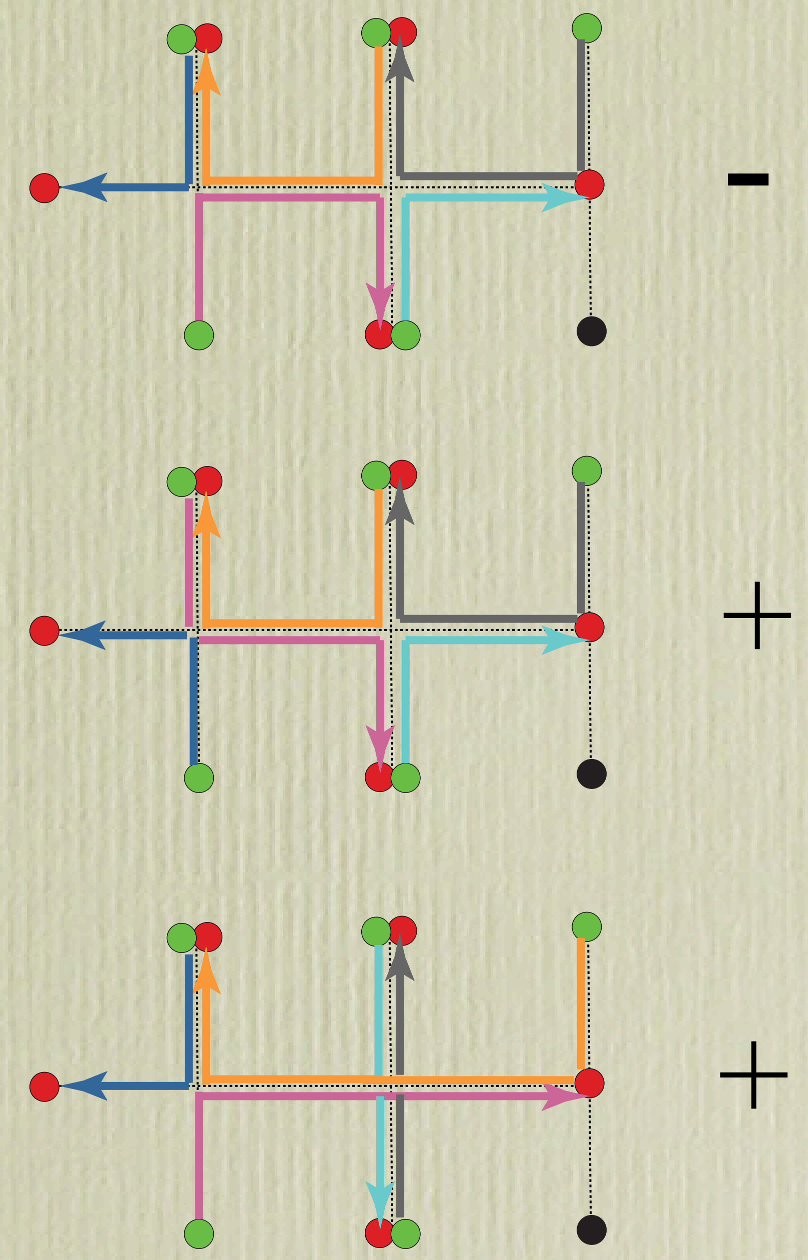

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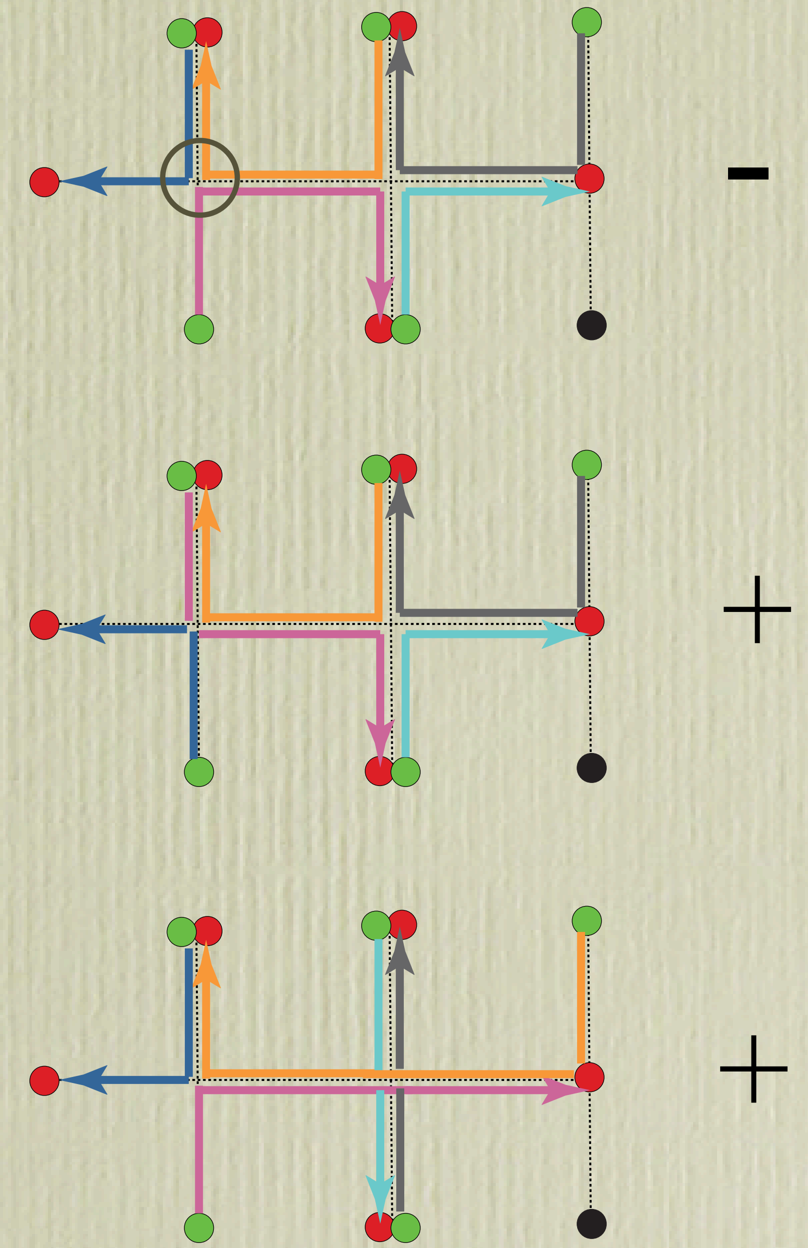

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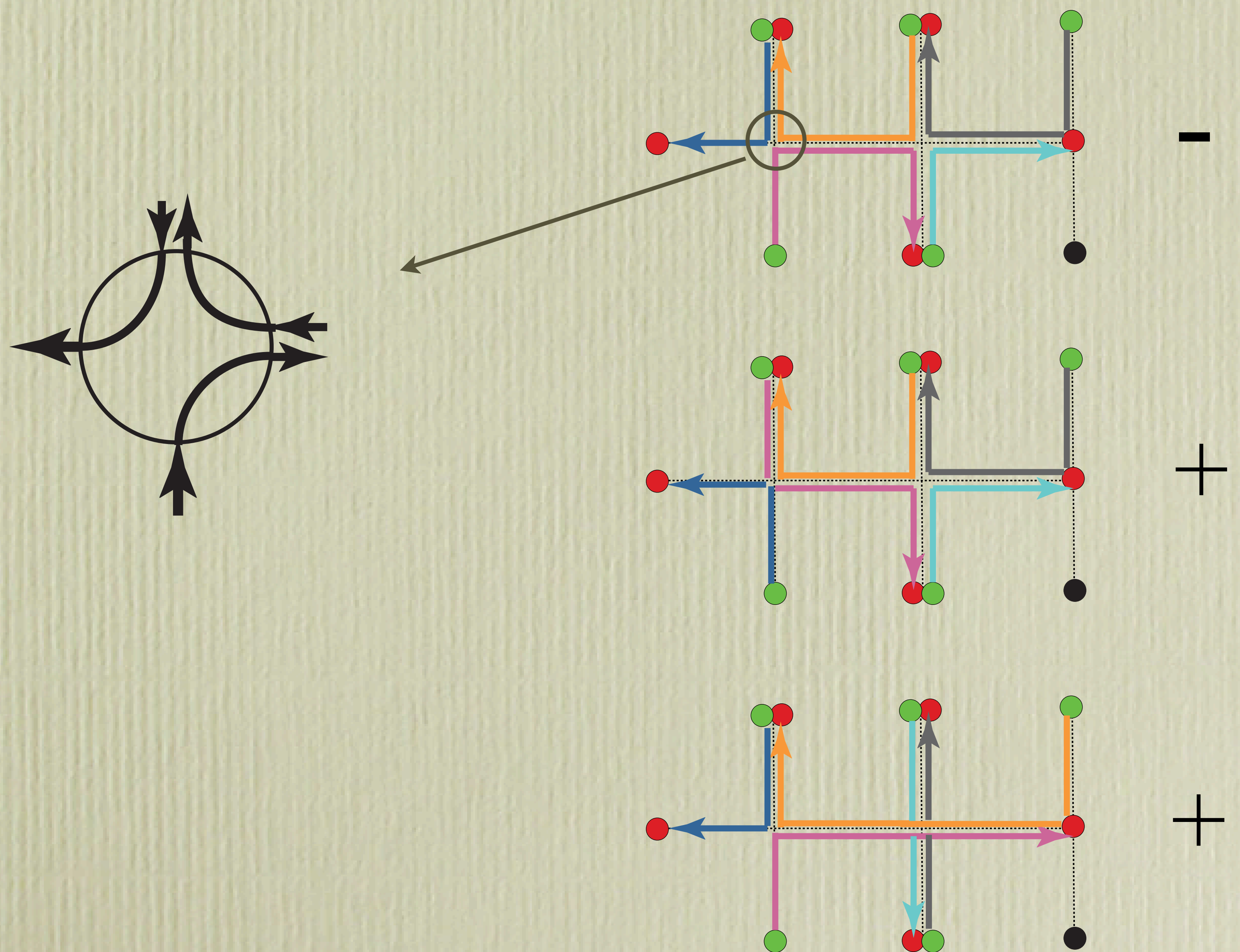

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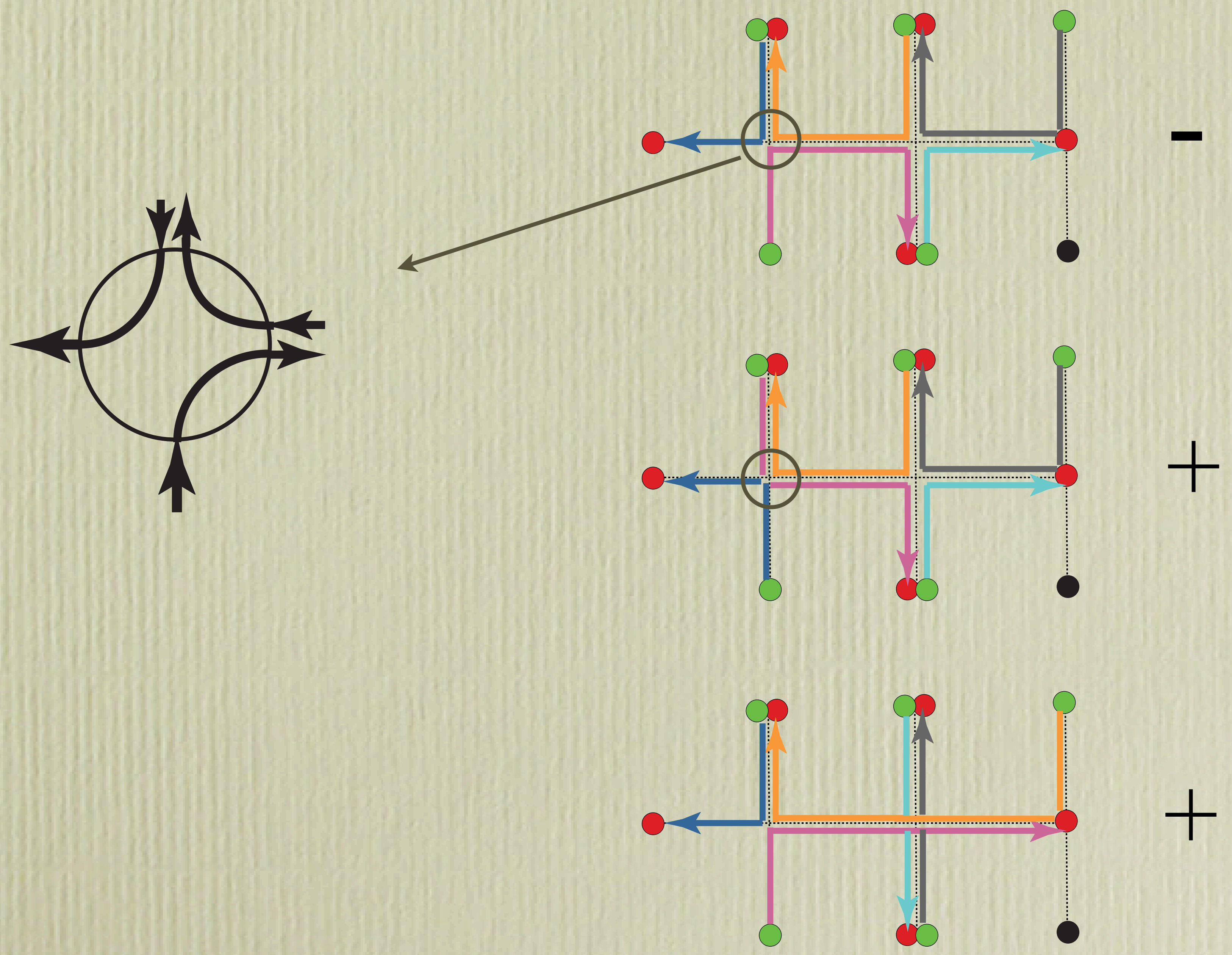

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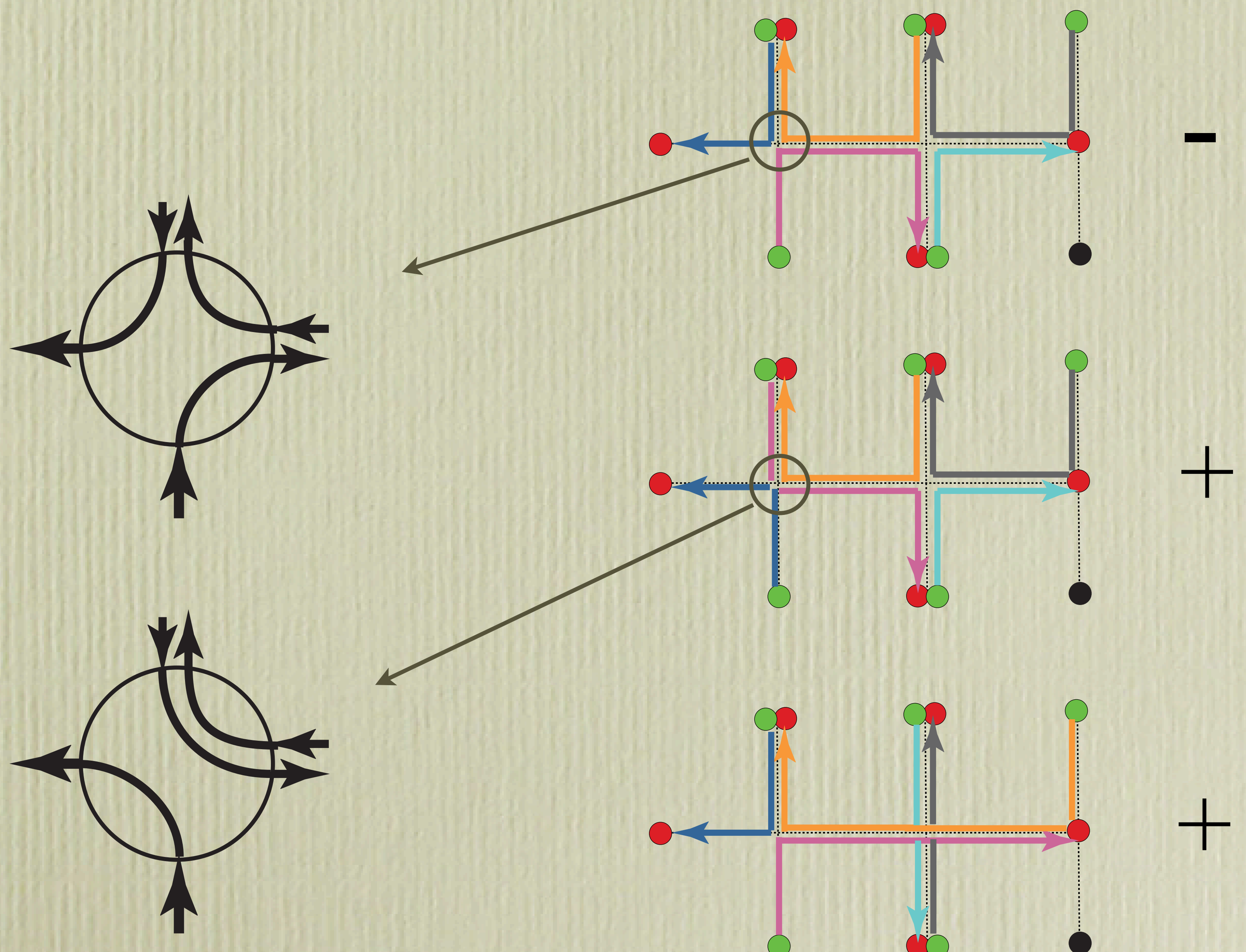

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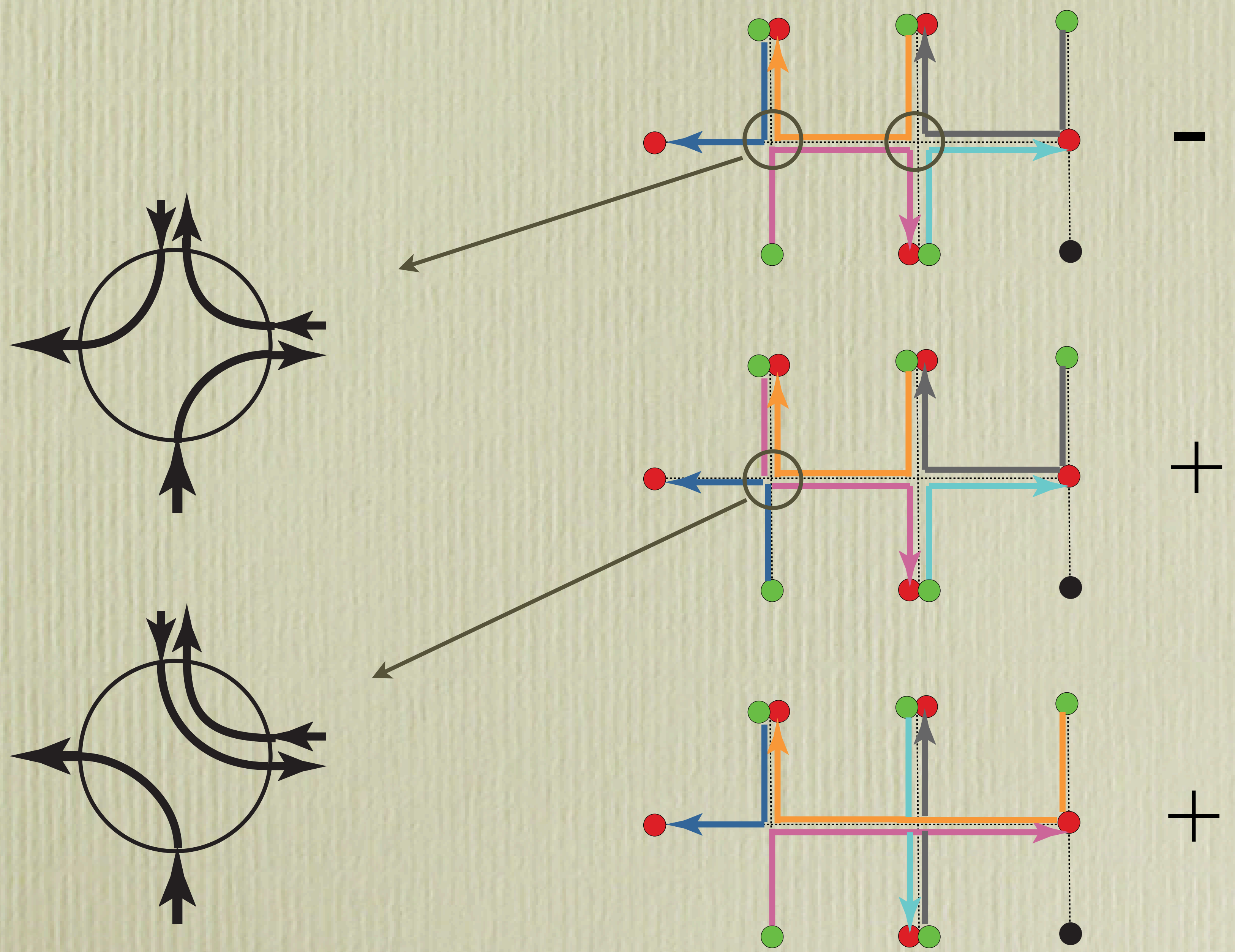

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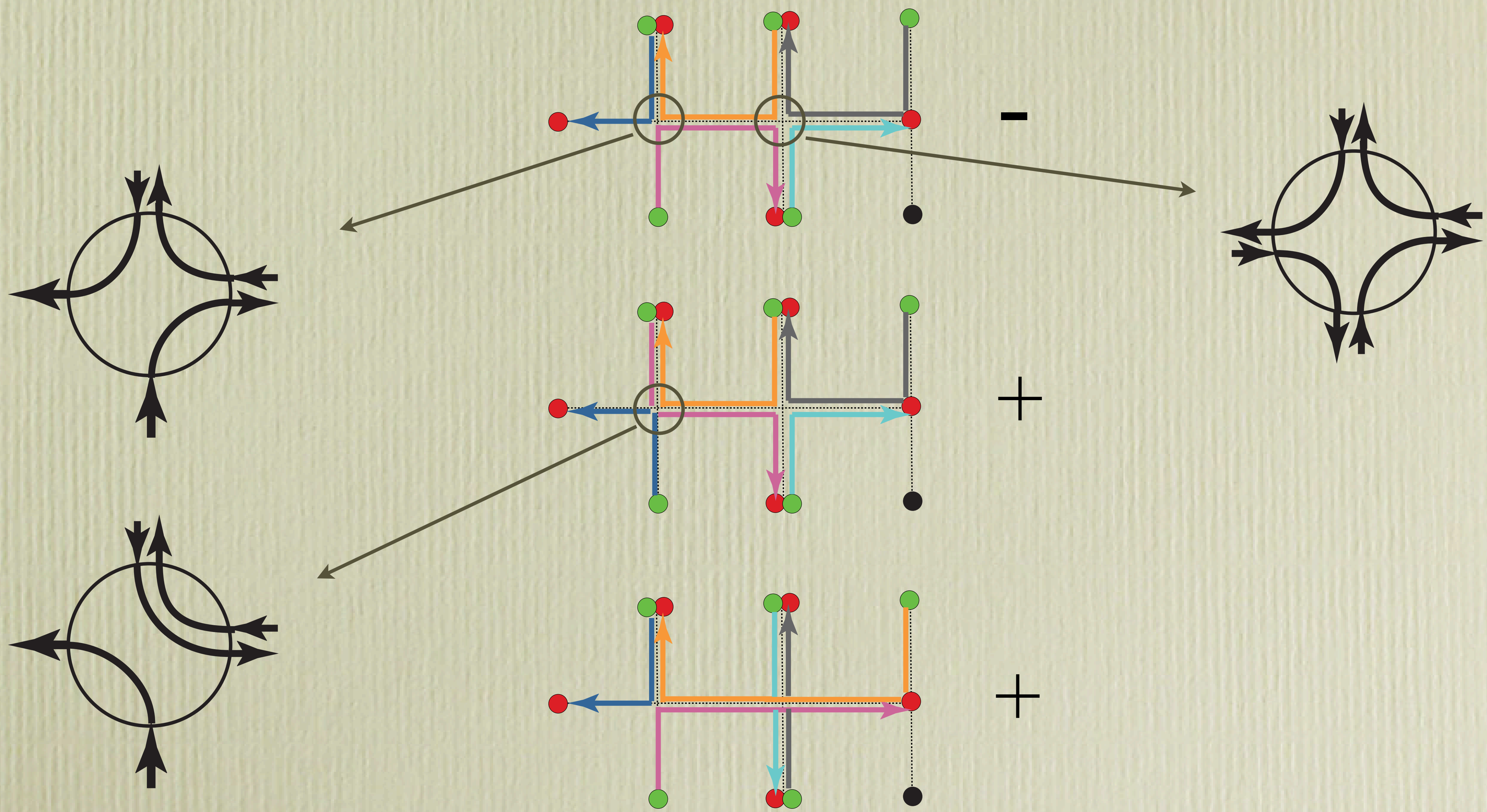

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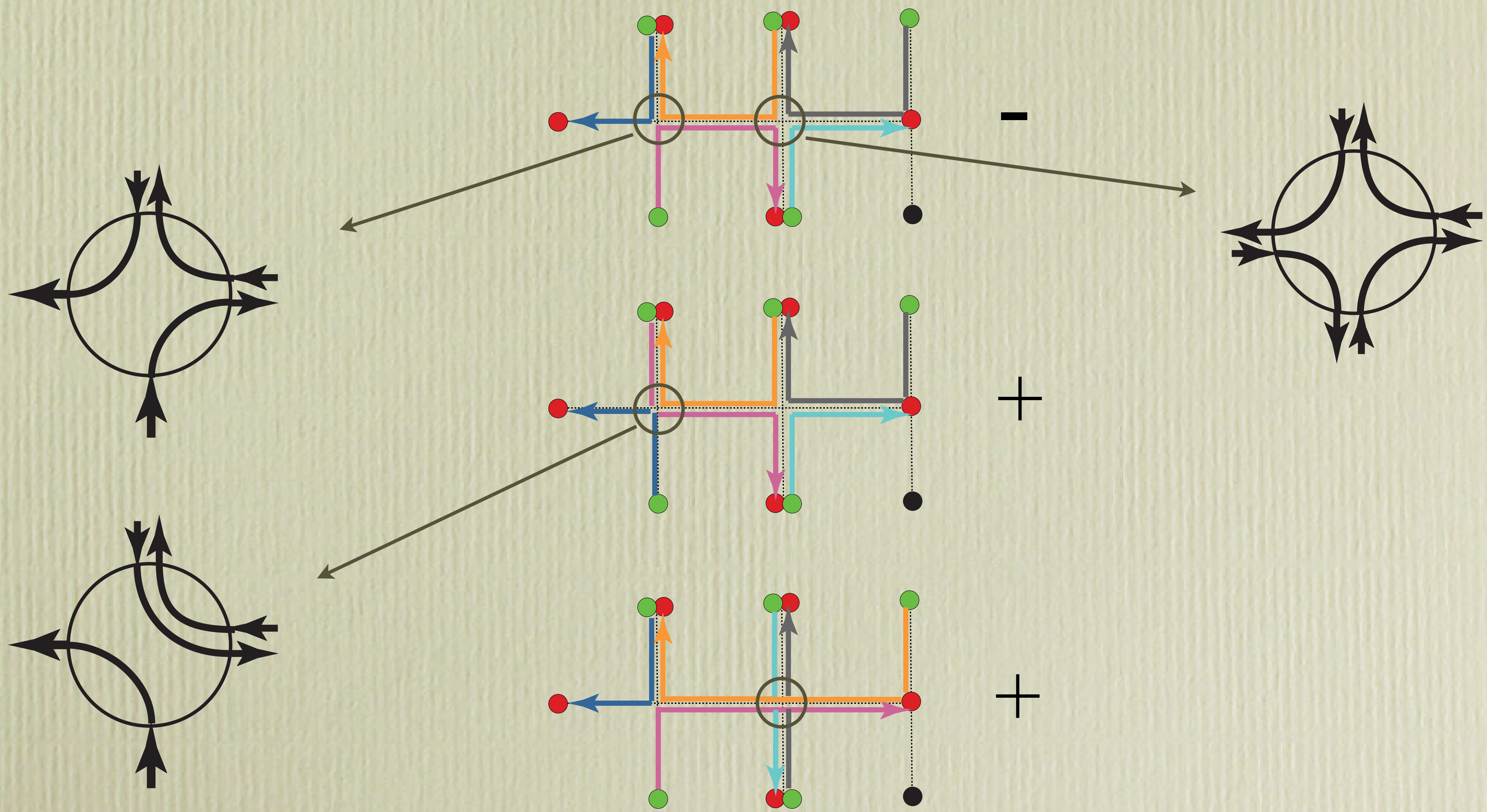

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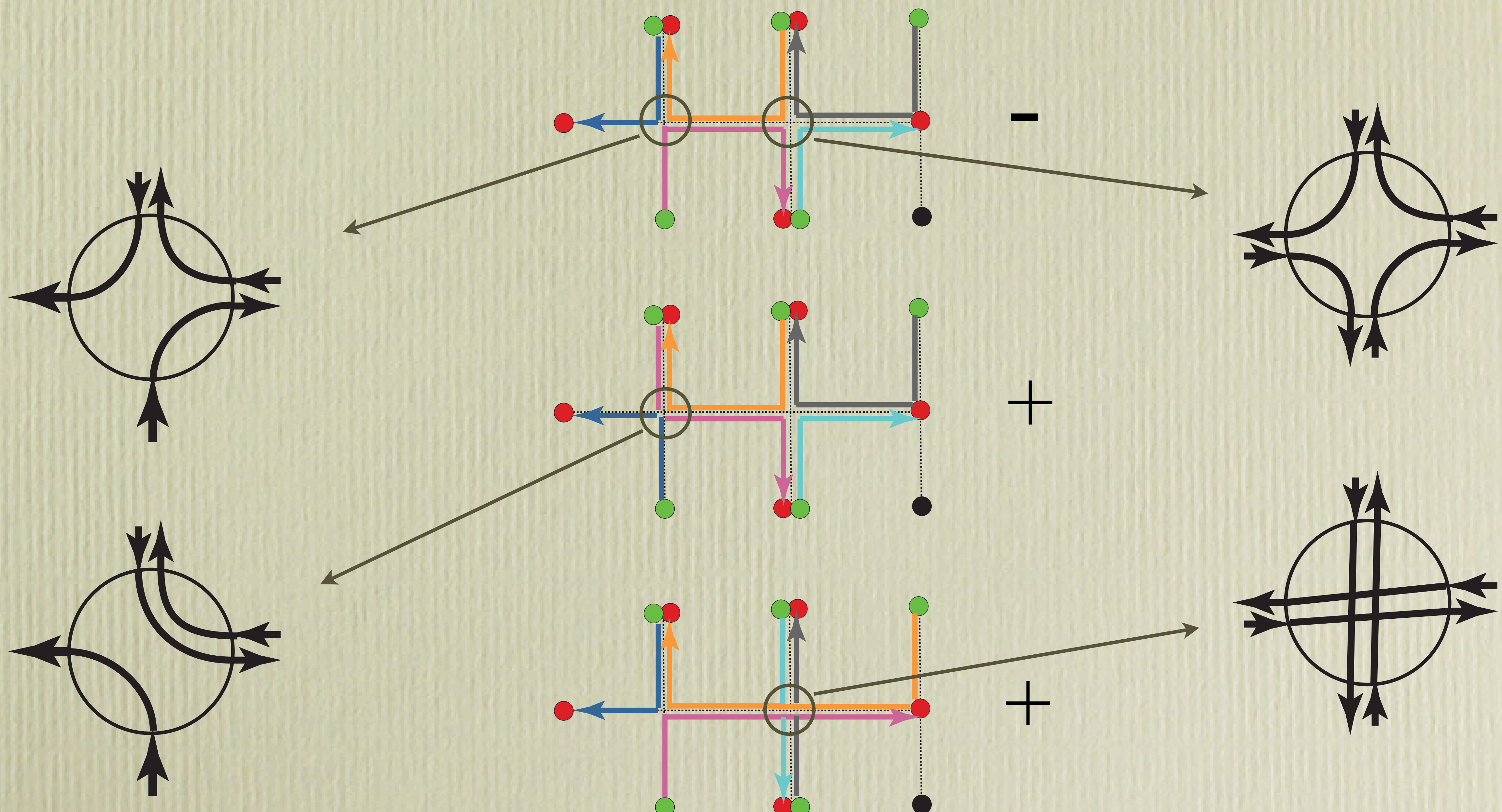

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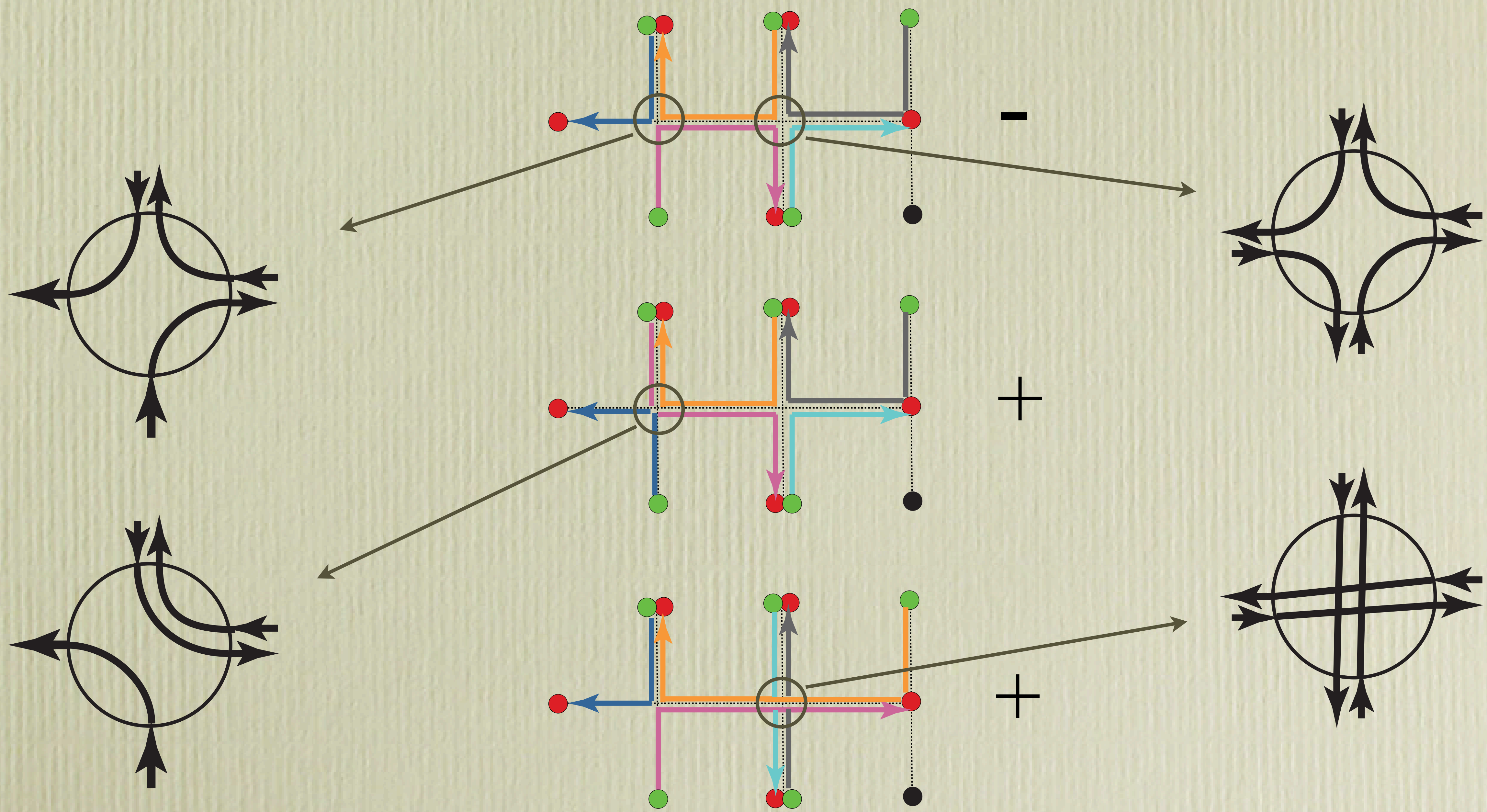

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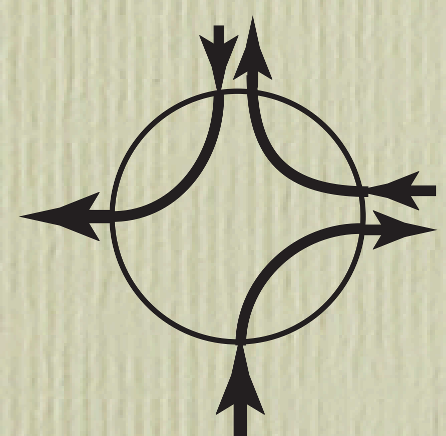

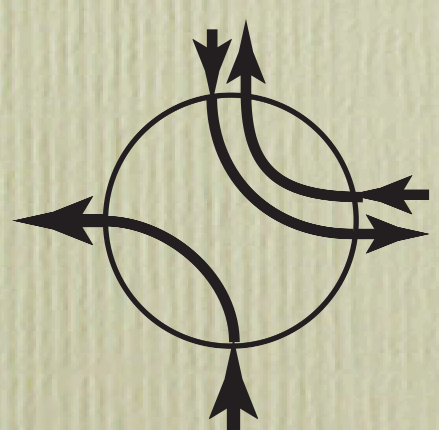

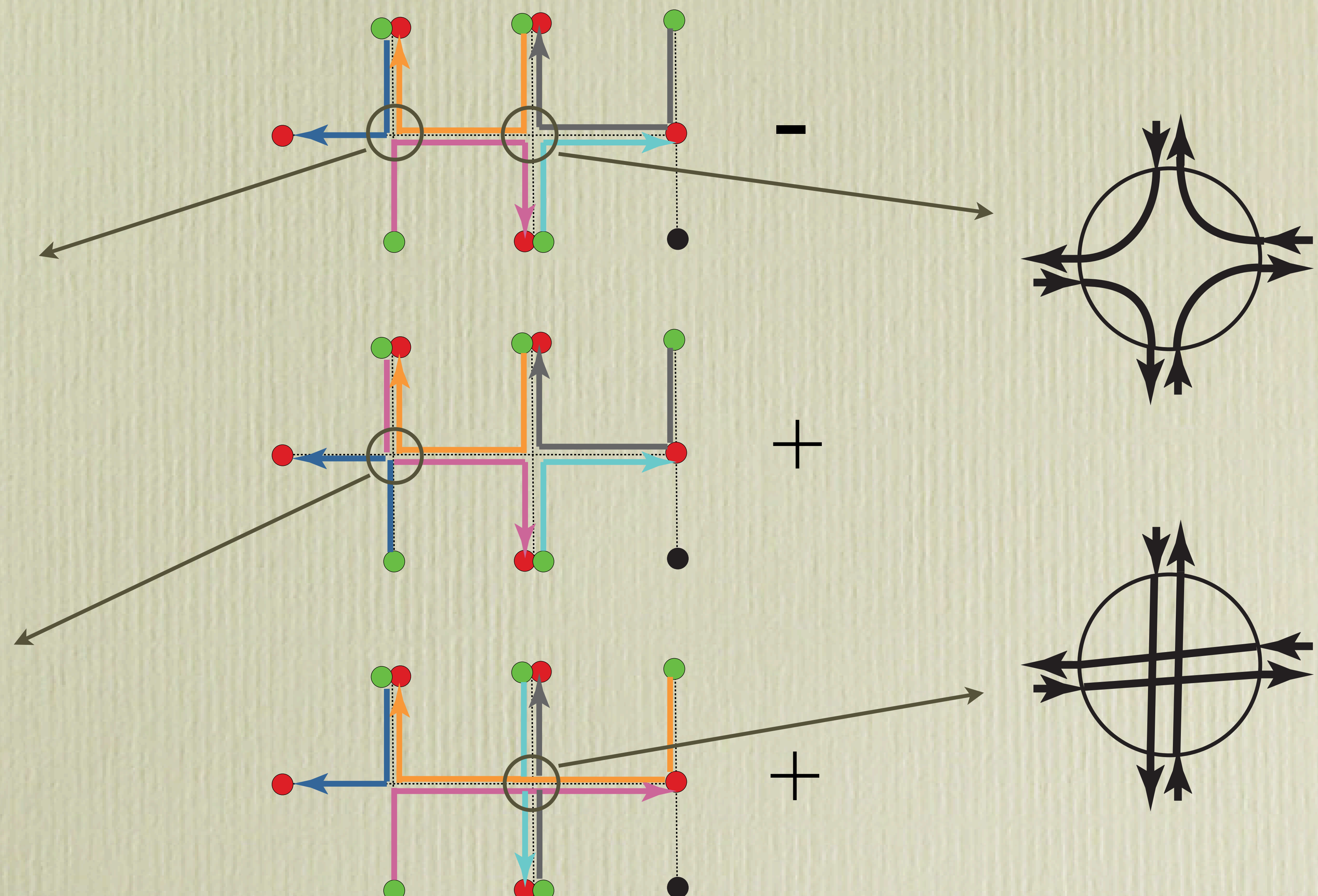

On the minors ... : Enumeration

- Configurations with fixed weight differ by the way the arrows are connected at each vertex
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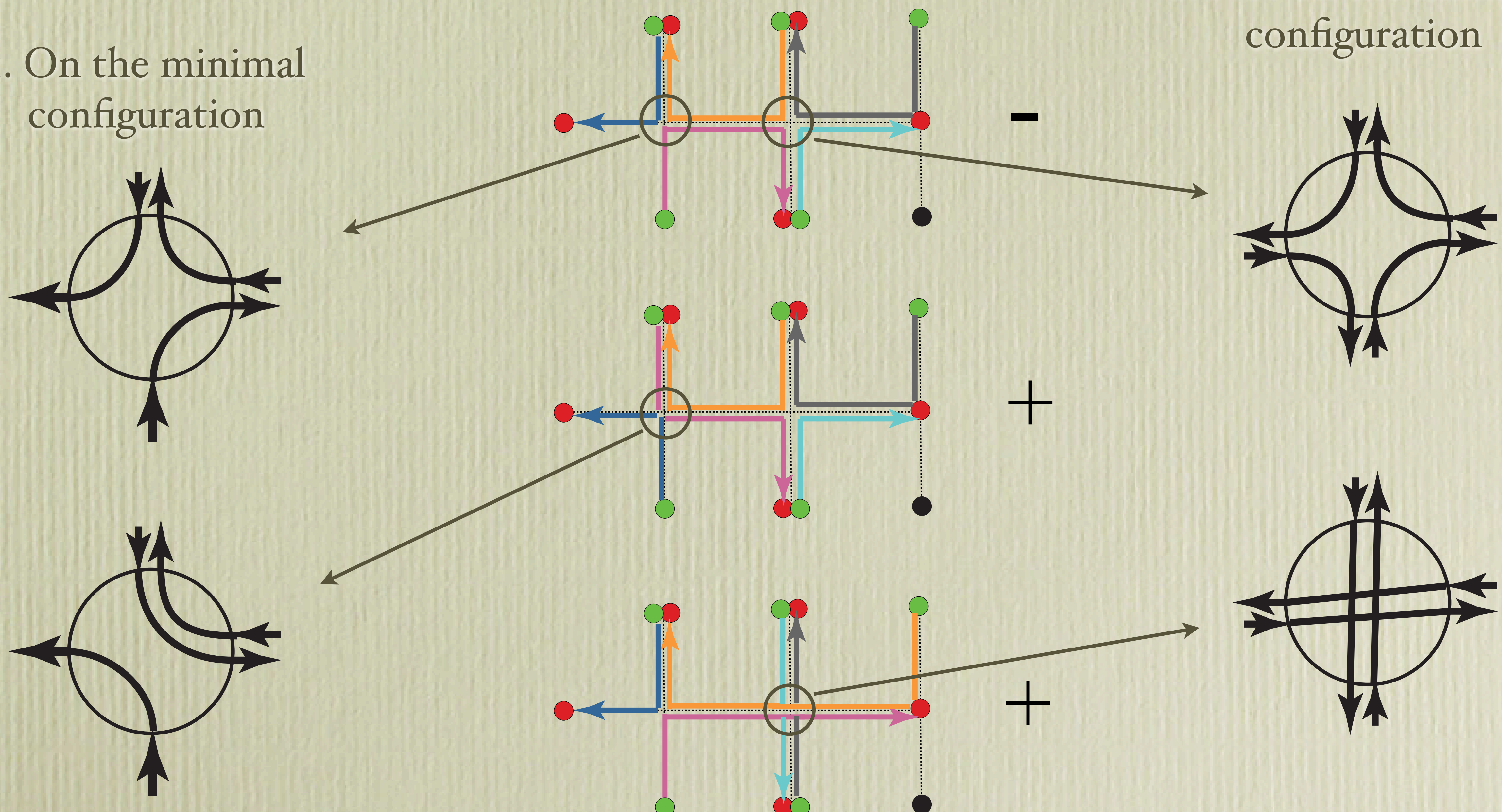
1. On the minimal configuration

2. Not on the minimal configuration



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On the minors ... : Enumeration

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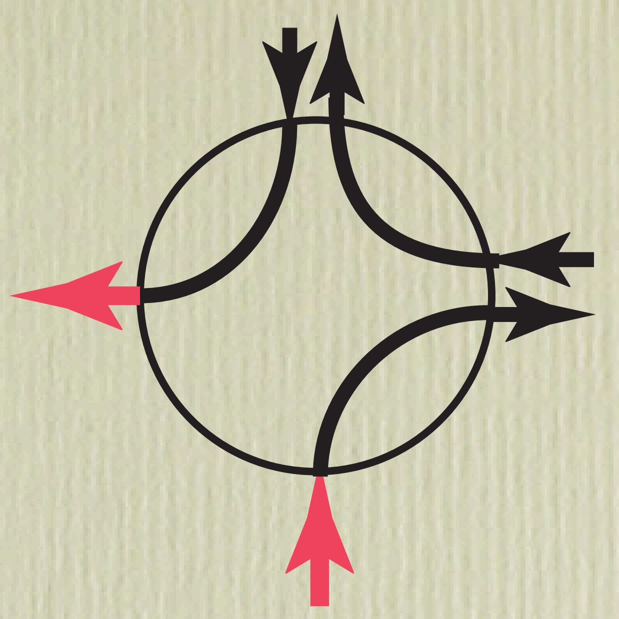

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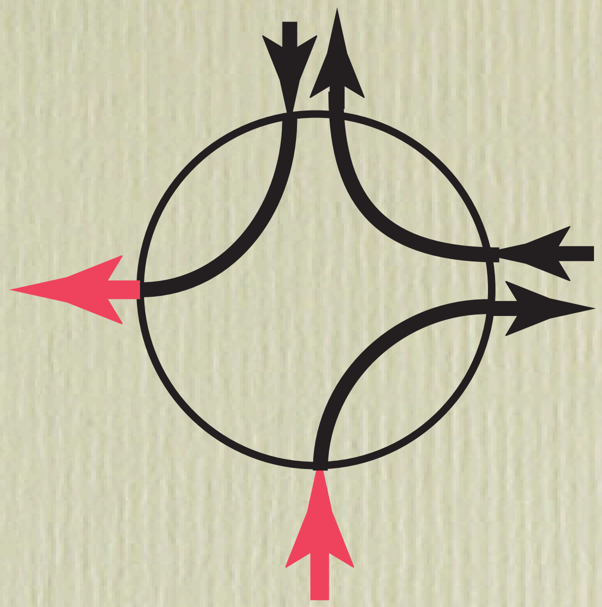
    - Choose a pair of opposites



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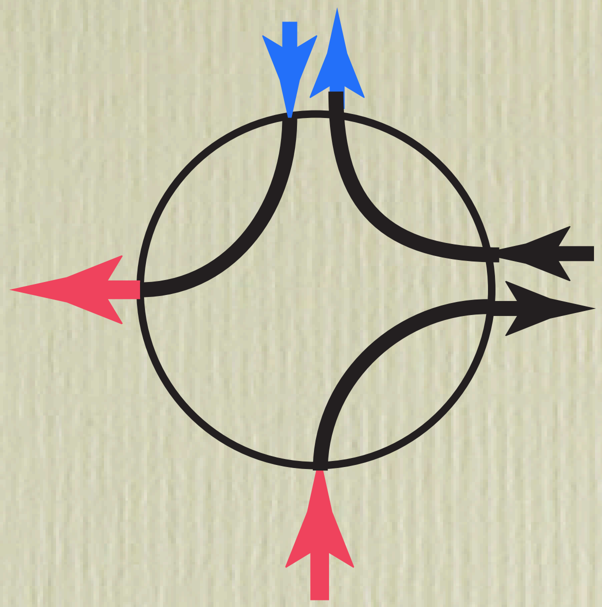
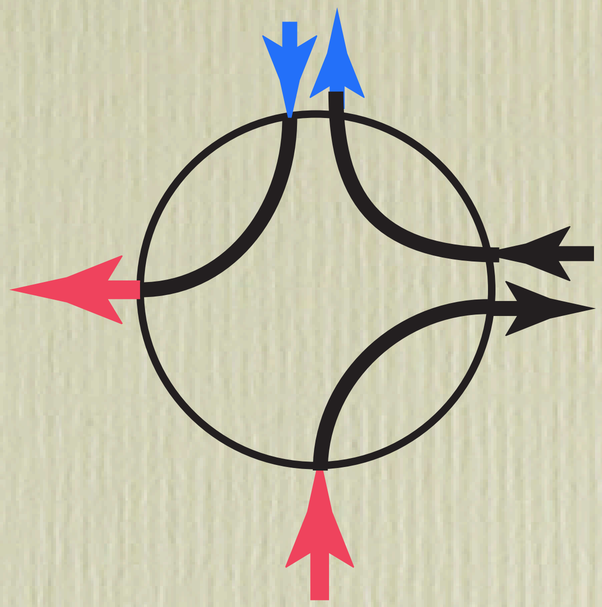


- 1. Vertex on the minimal configuration:
  - Choose a pair of opposites
  - Exchange the connections of the chosen incoming path and of the incoming single arrow ...



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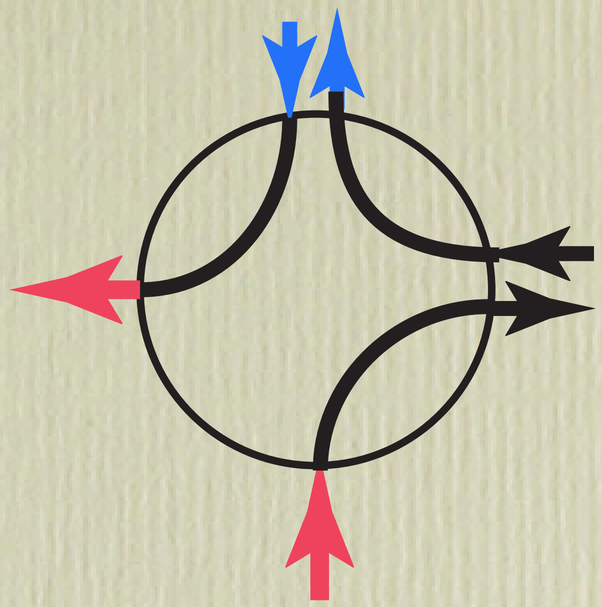

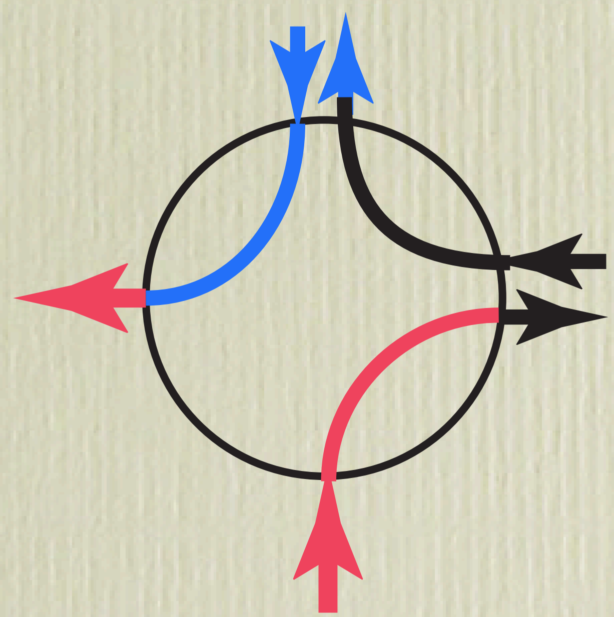

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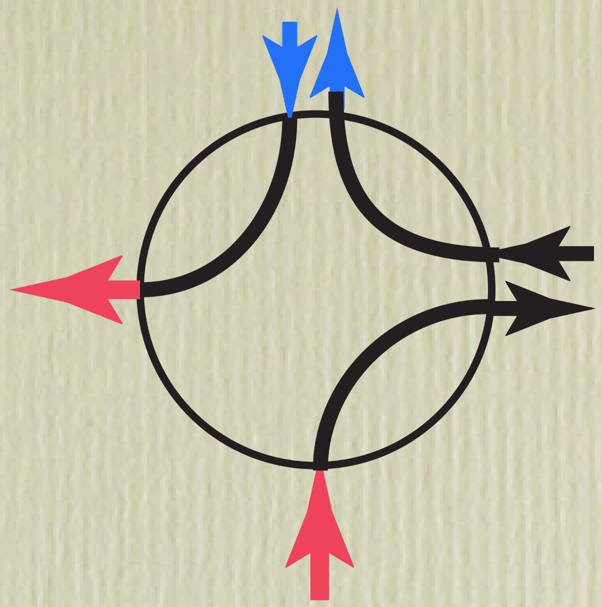

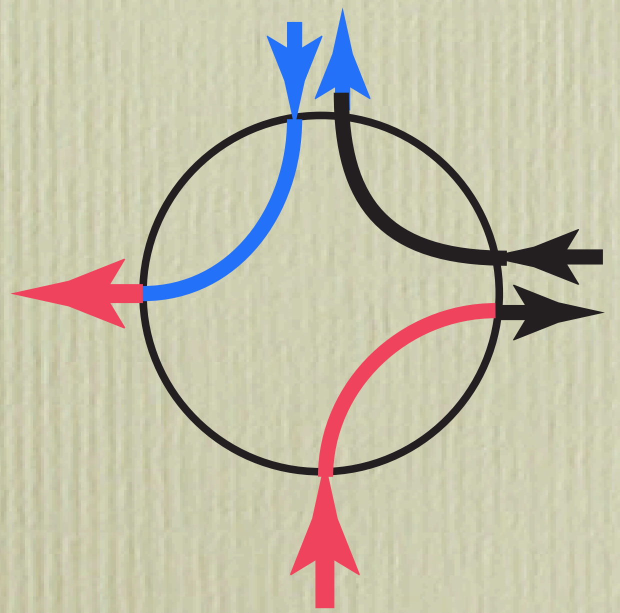

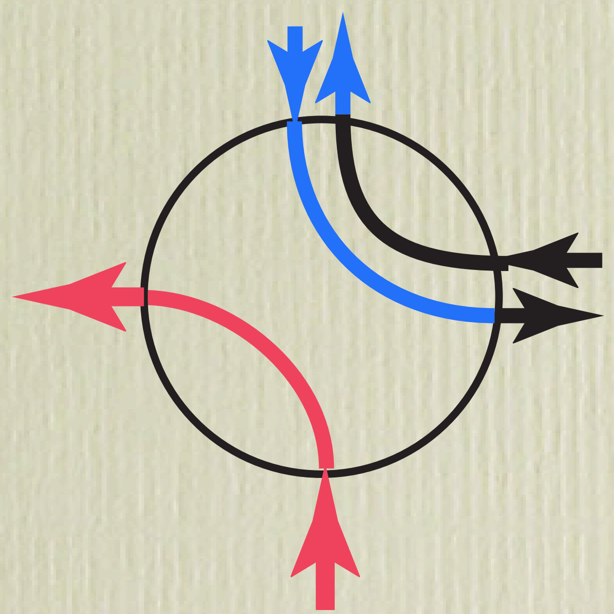

On the minors ... : Enumeration

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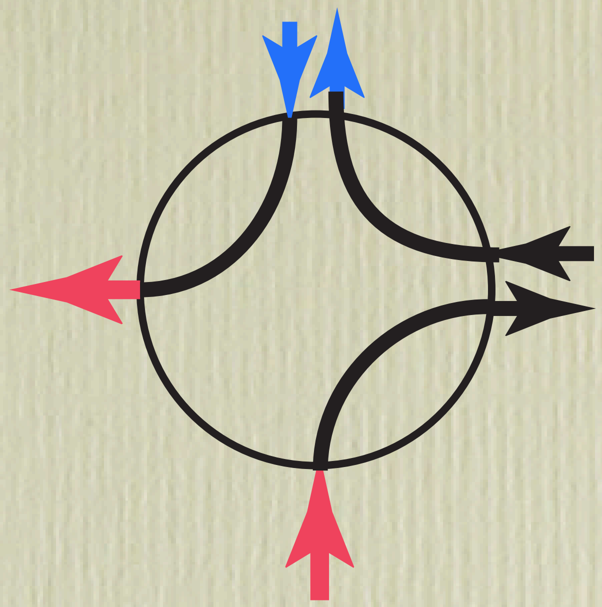
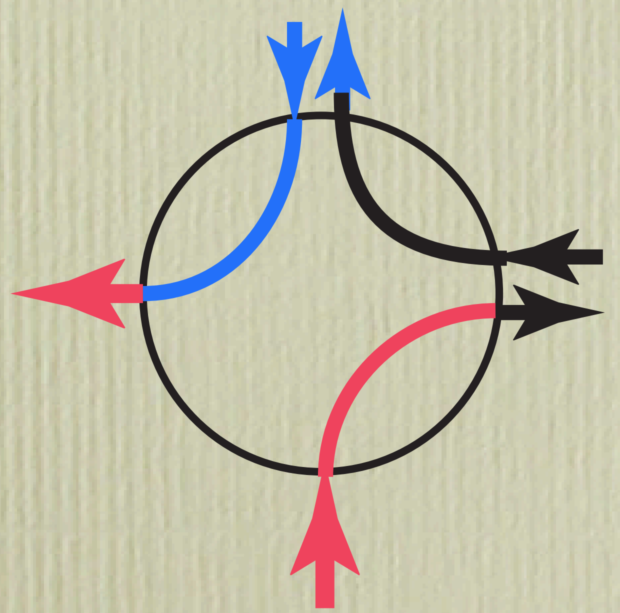
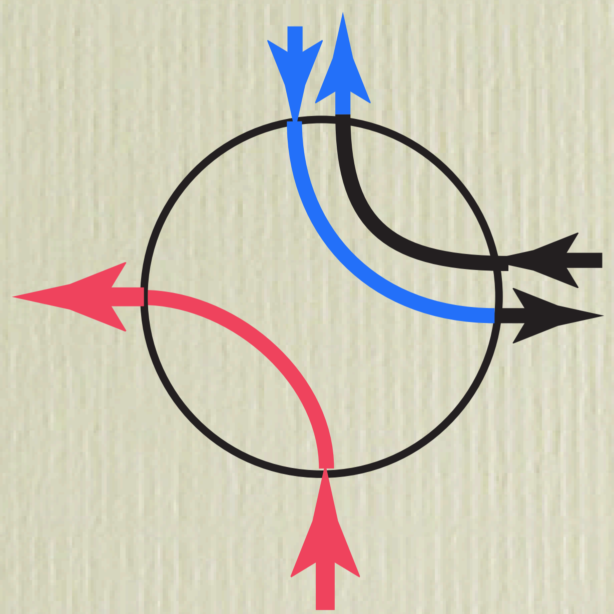
- ... causing a change of sign: cancellation







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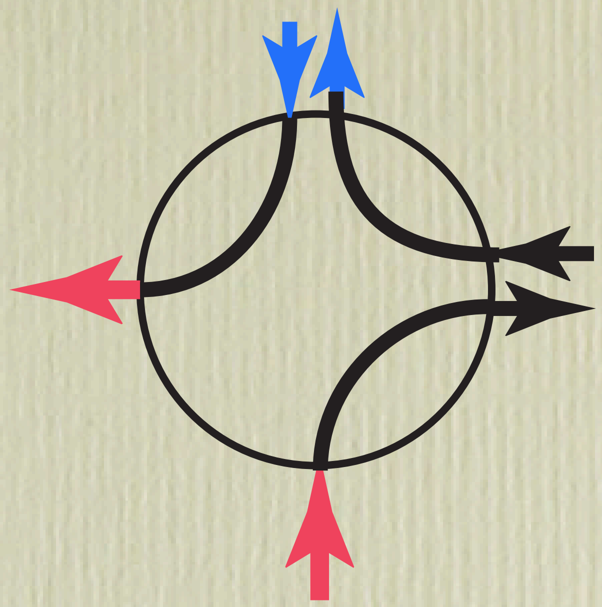
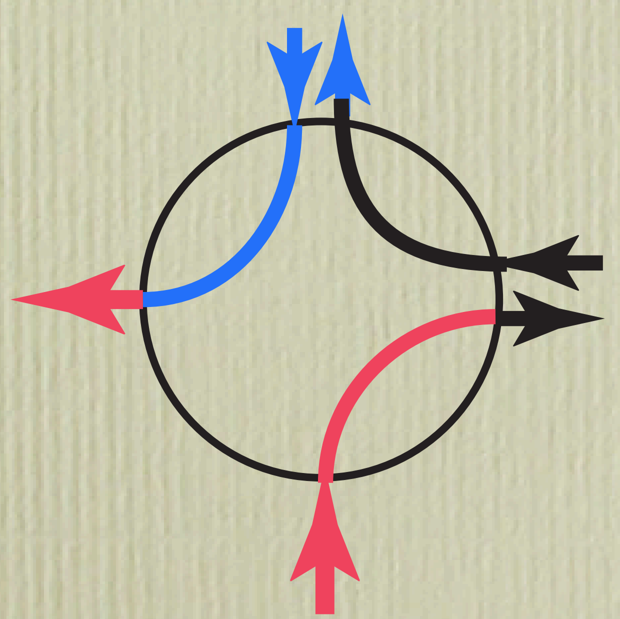
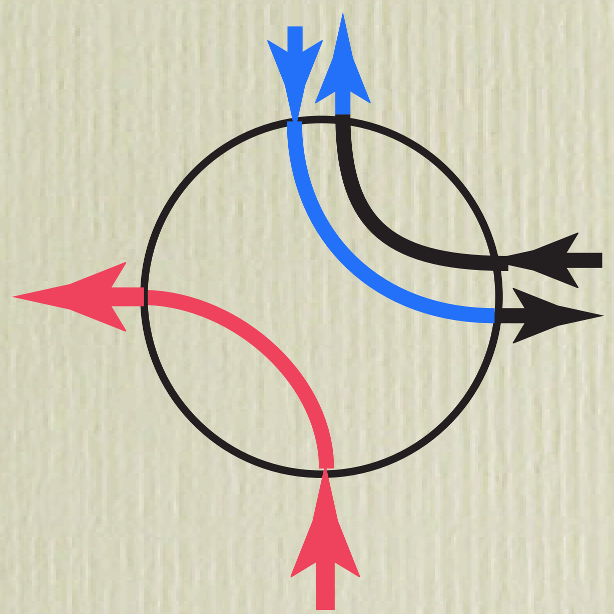
    - Exchange the connections of the chosen incoming path and of the incoming single arrow ...
- ... causing a change of sign: cancellation



−





+

On the minors ... : Enumeration

- A connection survives if the single incoming arrow is connected to the chosen pair

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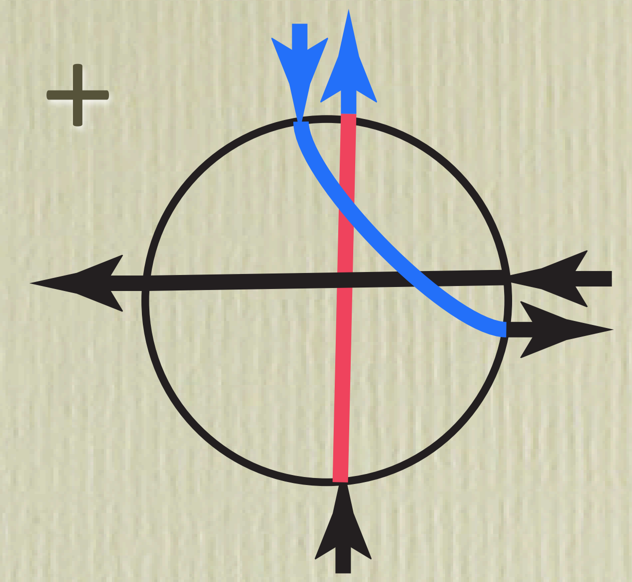
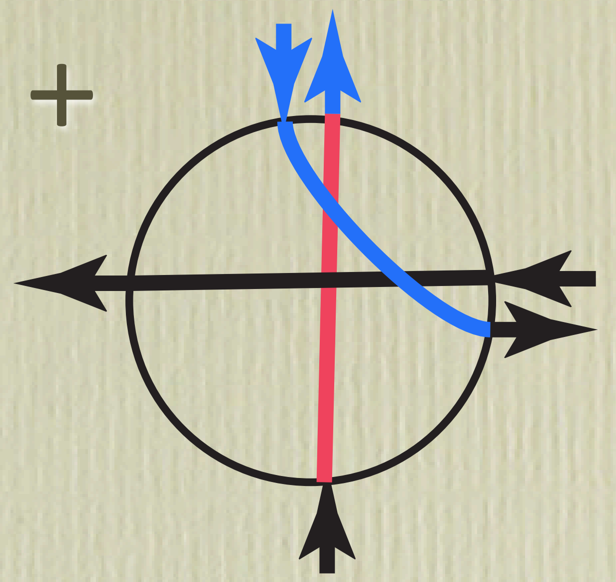


- A connection survives if the single incoming arrow is connected to the chosen pair
  - The exchange does not produce an allowed path



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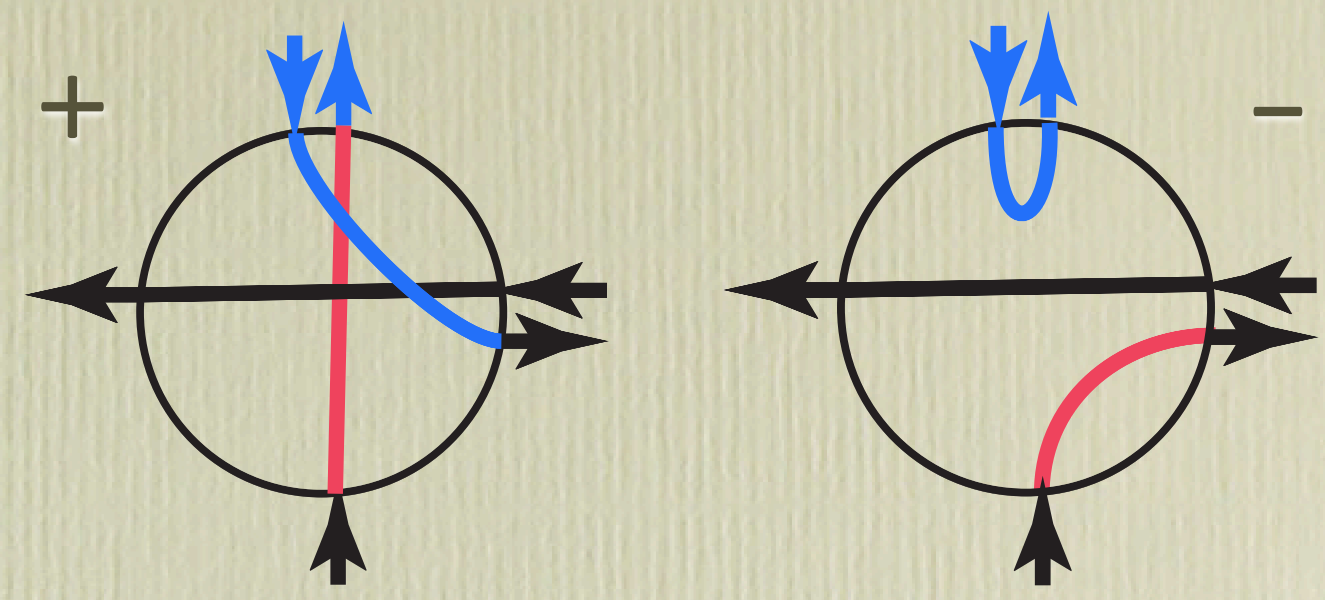
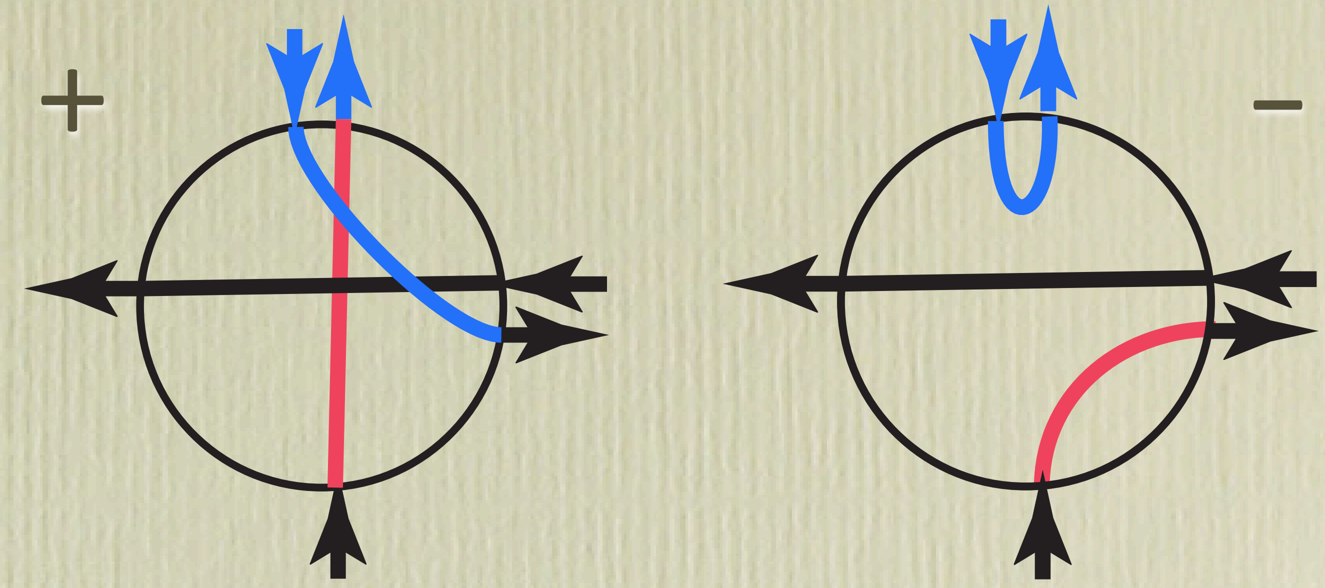


- A connection survives if the single incoming arrow is connected to the chosen pair
    - The exchange does not produce an allowed path
- Do it anyway! But create a new source-target vertex for the illegal path



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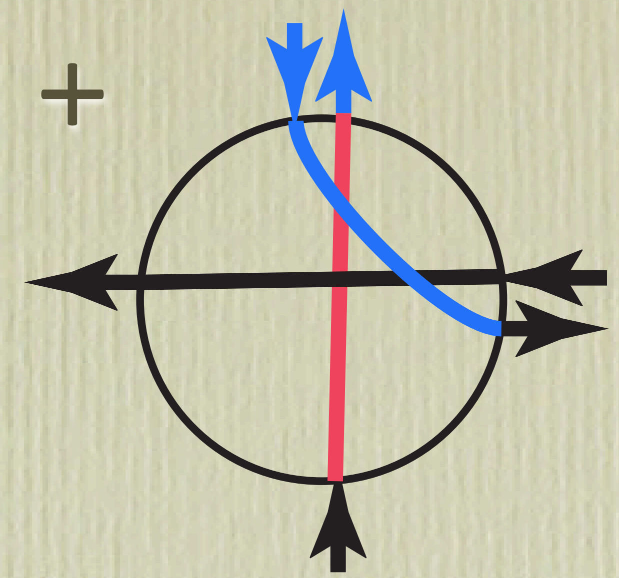

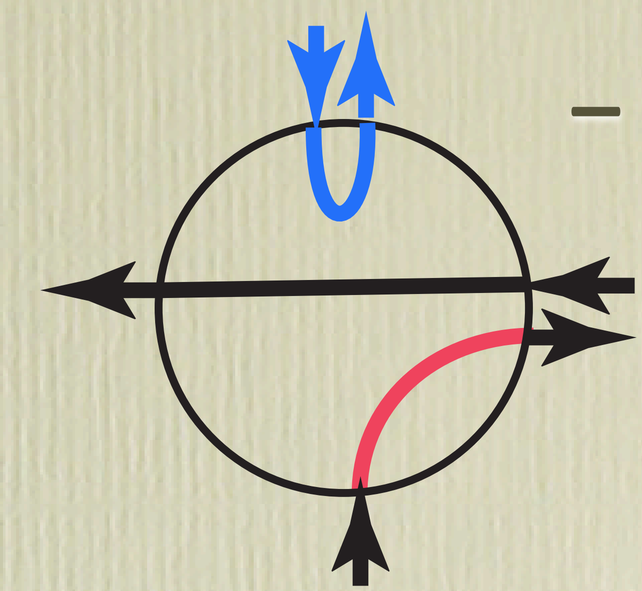

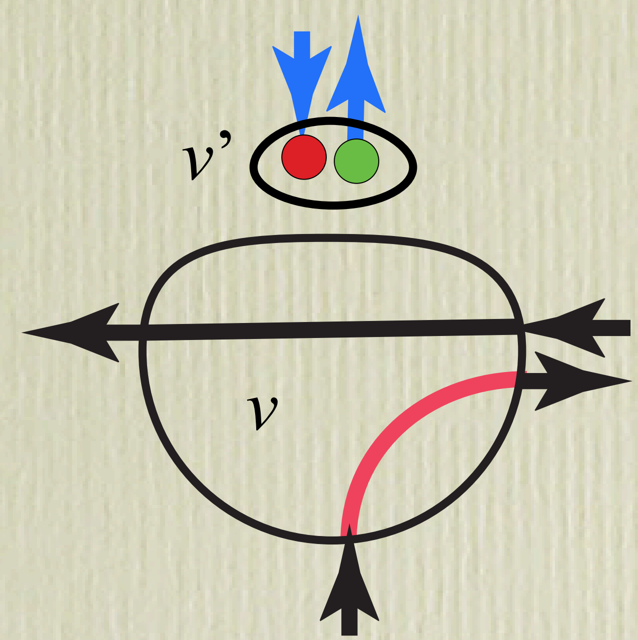

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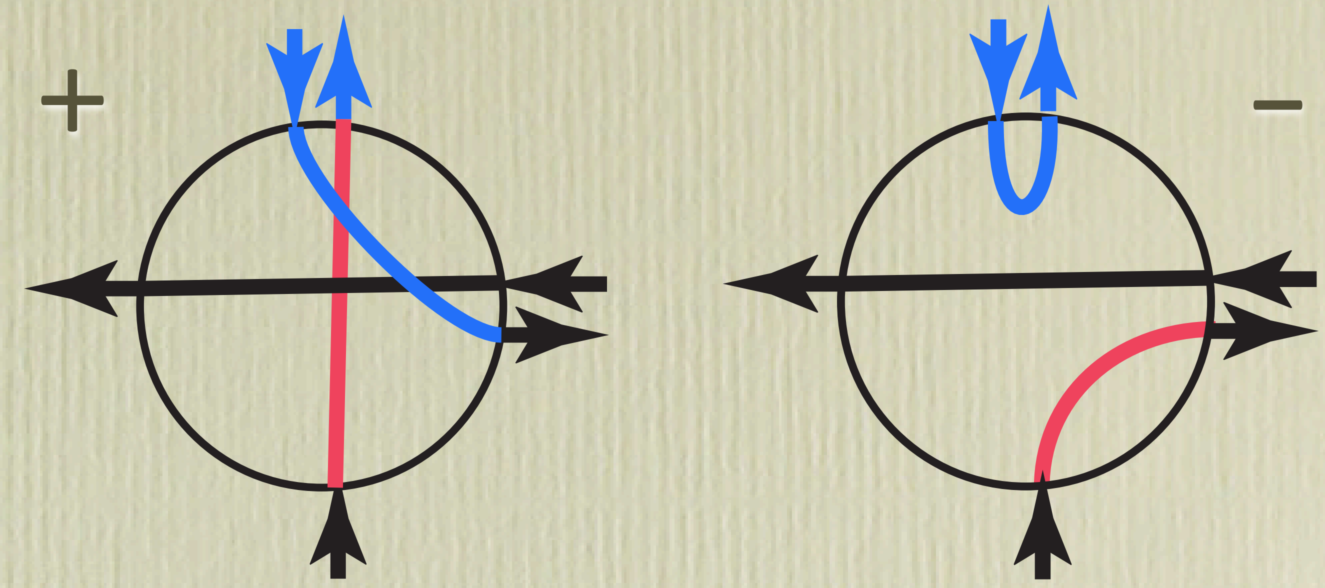
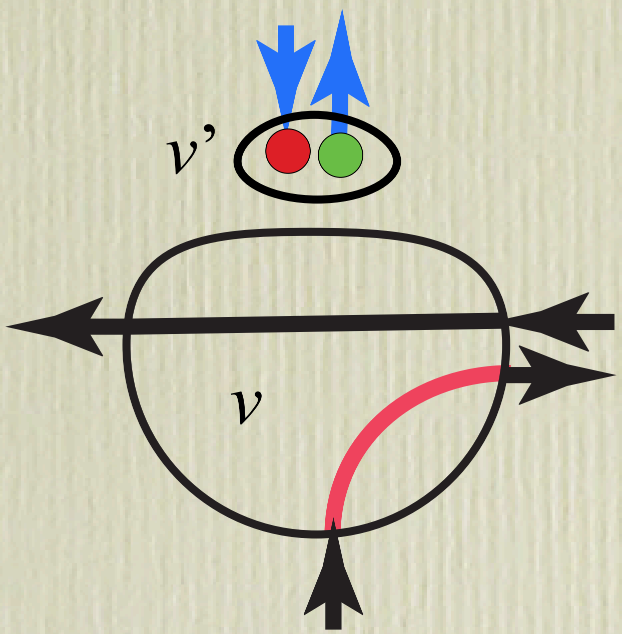


- Do it anyway! But create a new source-target vertex for the illegal path
    - This change the associated permutation by a 3-cycle. No sign change



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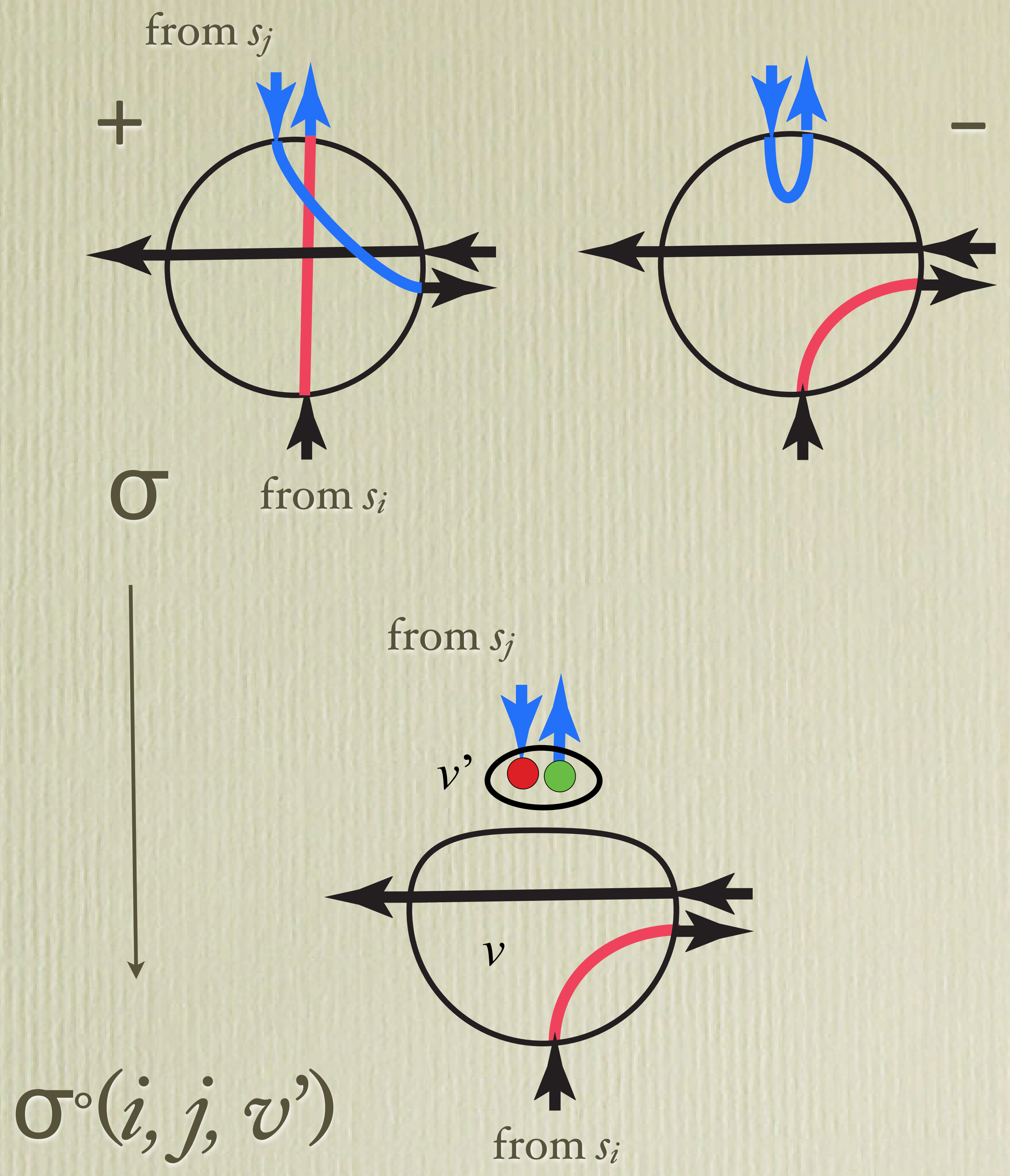
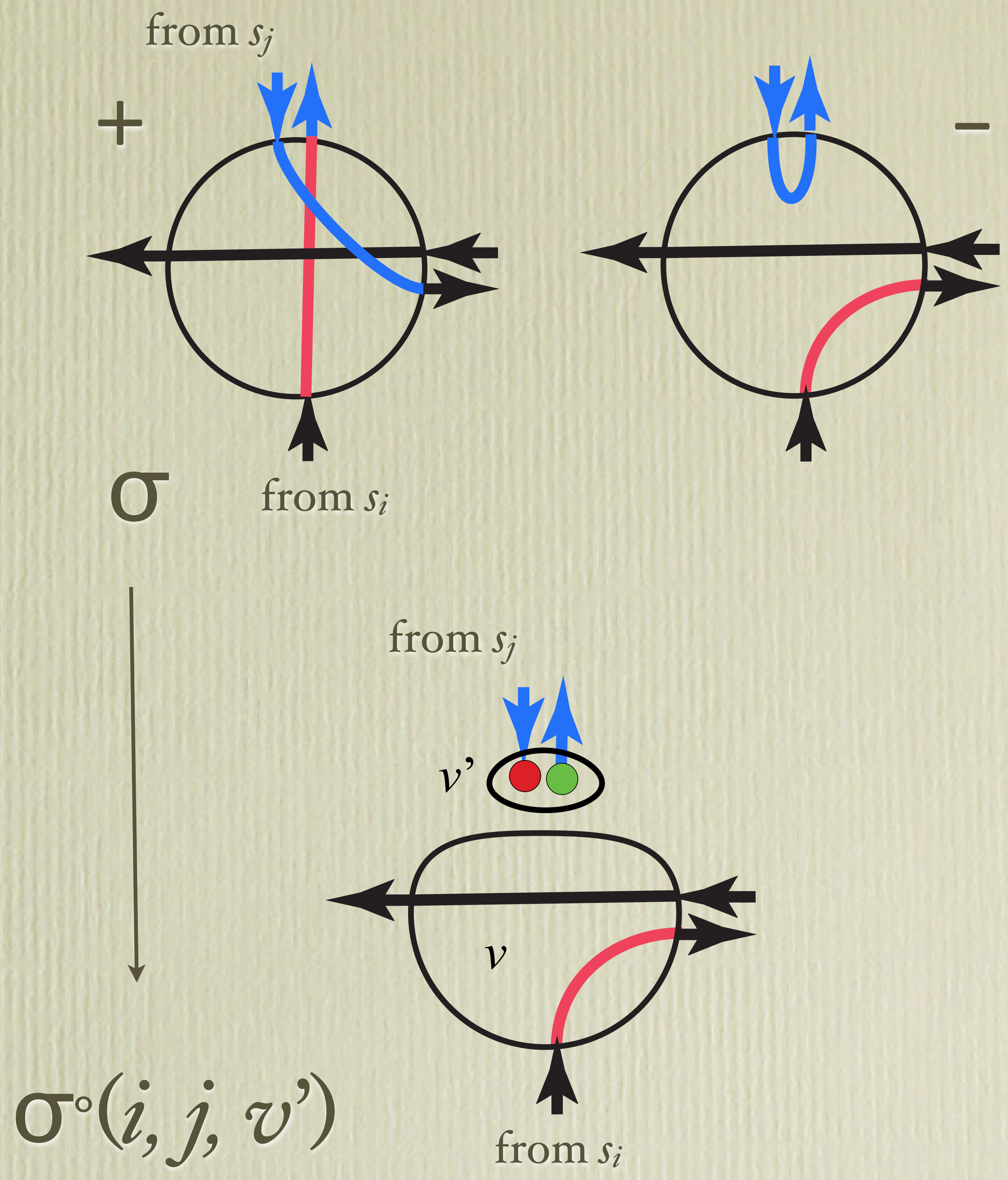


- A connection survives if the single incoming arrow is connected to the chosen pair
  - The exchange does not produce an allowed path
- Do it anyway! But create a new source-target vertex for the illegal path
  - This change the associated permutation by a 3-cycle. No sign change
- This is the bijection



On the minors ... : Enumeration

- Repeat with the others pairs of opposites adjacent to the vertex. Only one connection survives.

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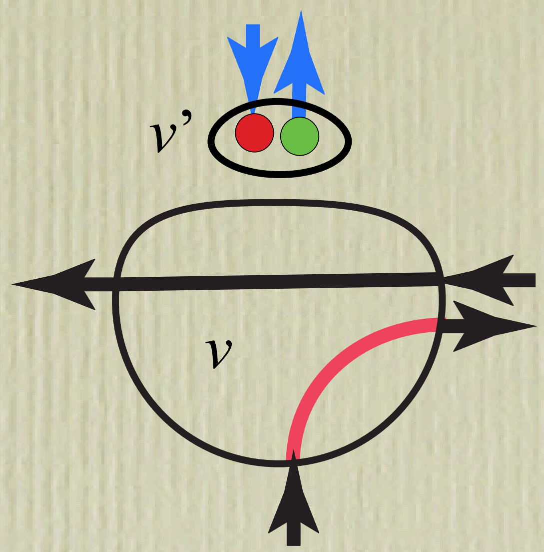

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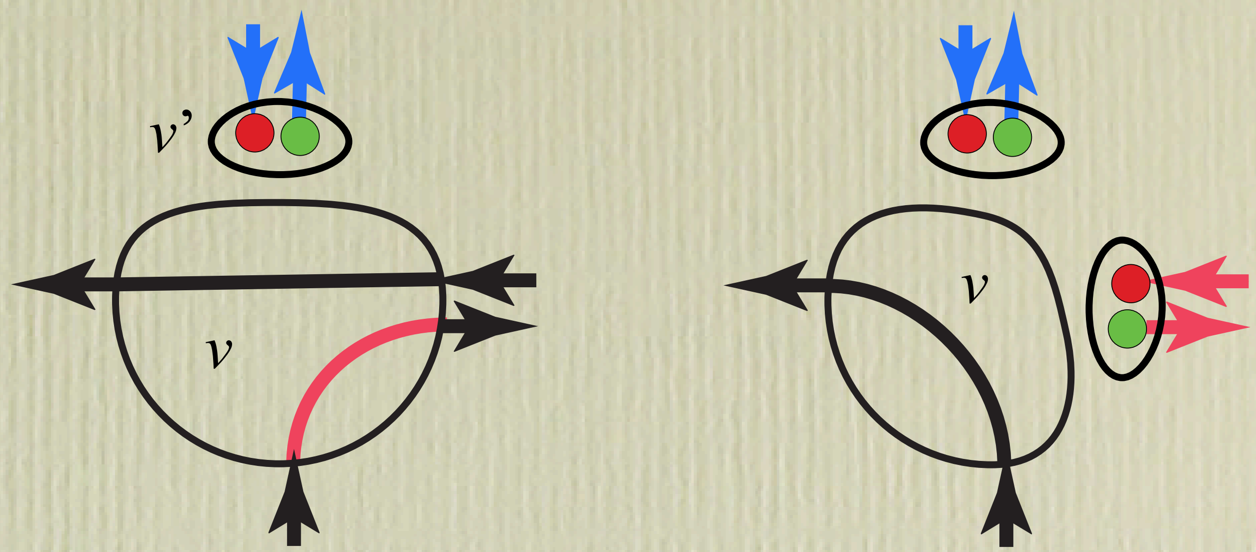
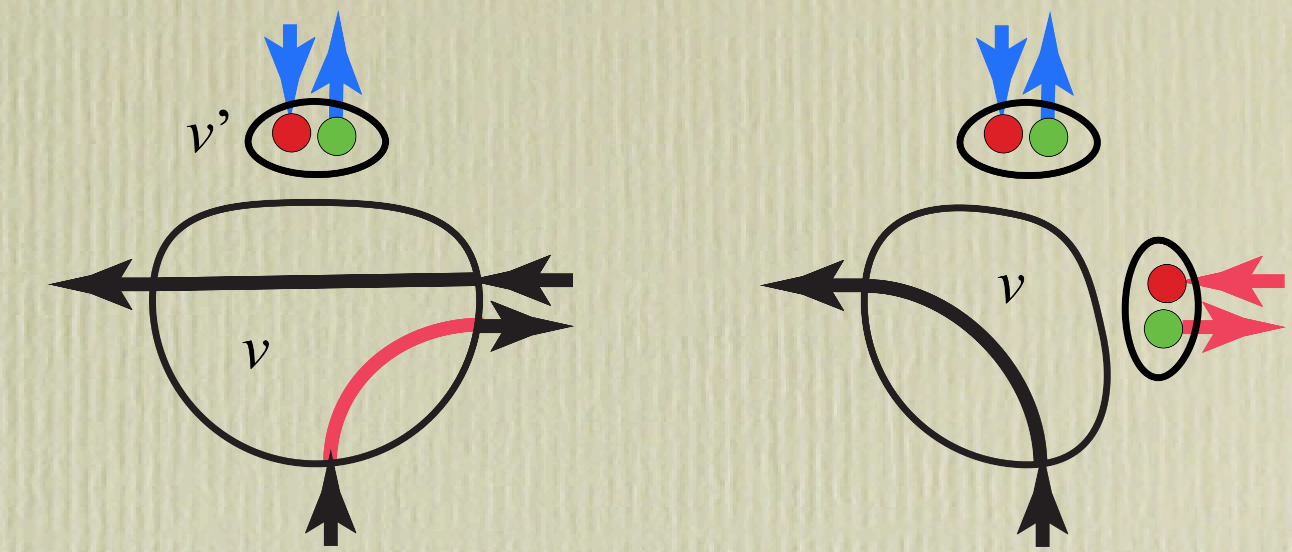


On the minors ... : Enumeration



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- Repeat with every vertex on the minimal configuration

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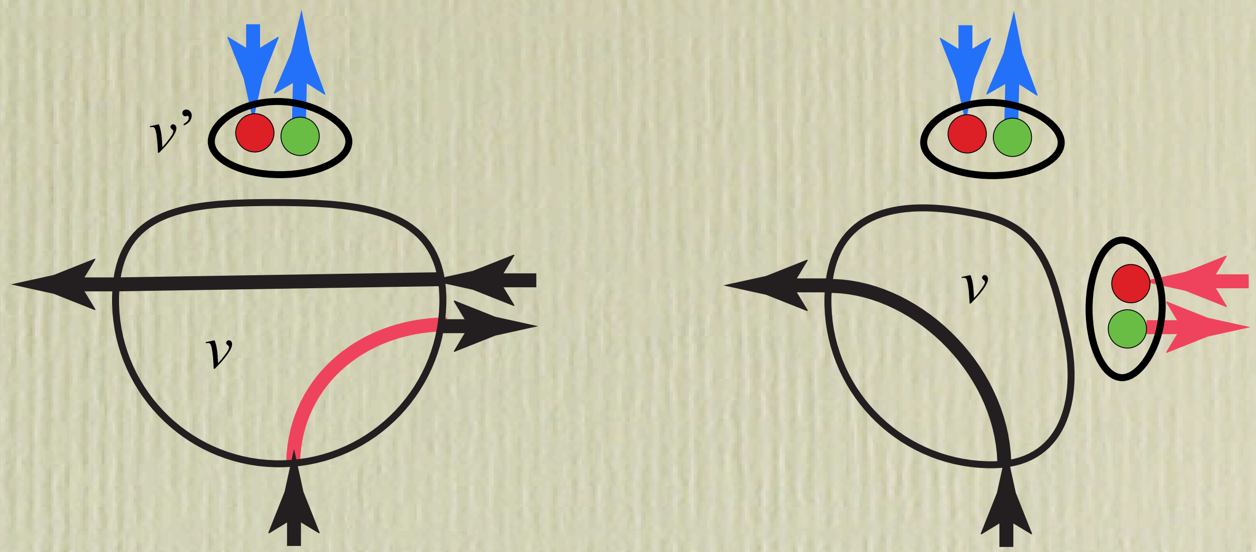

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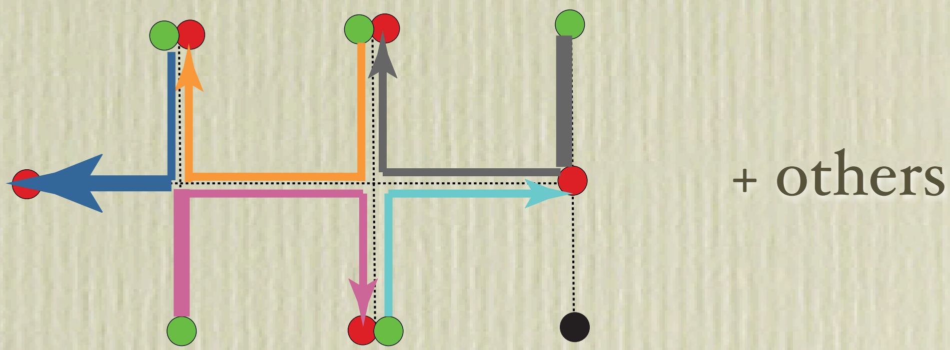

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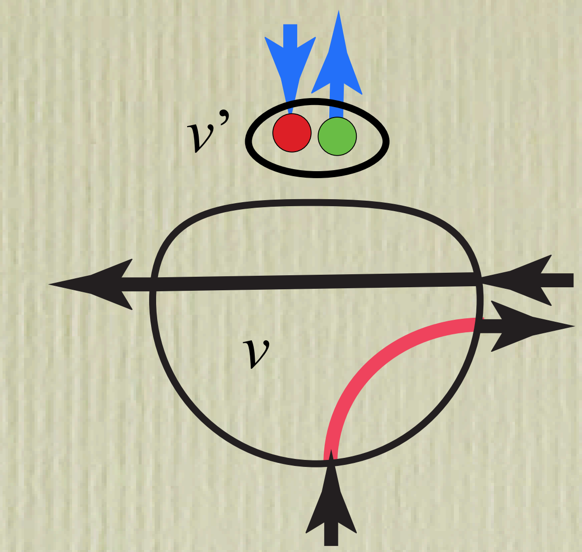

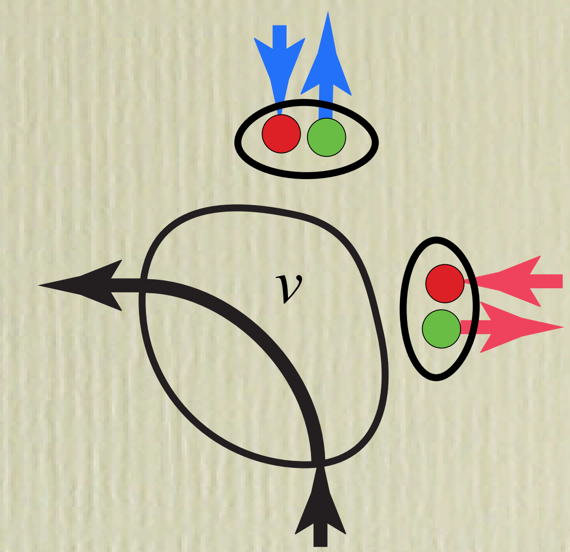

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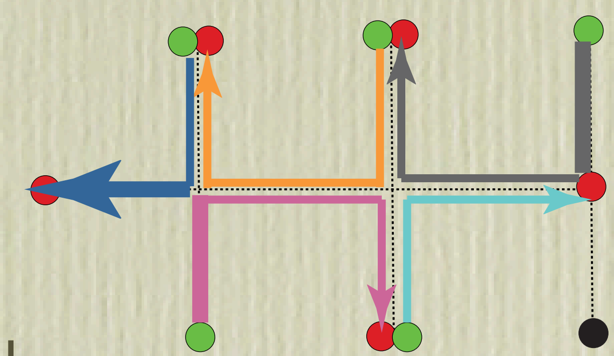
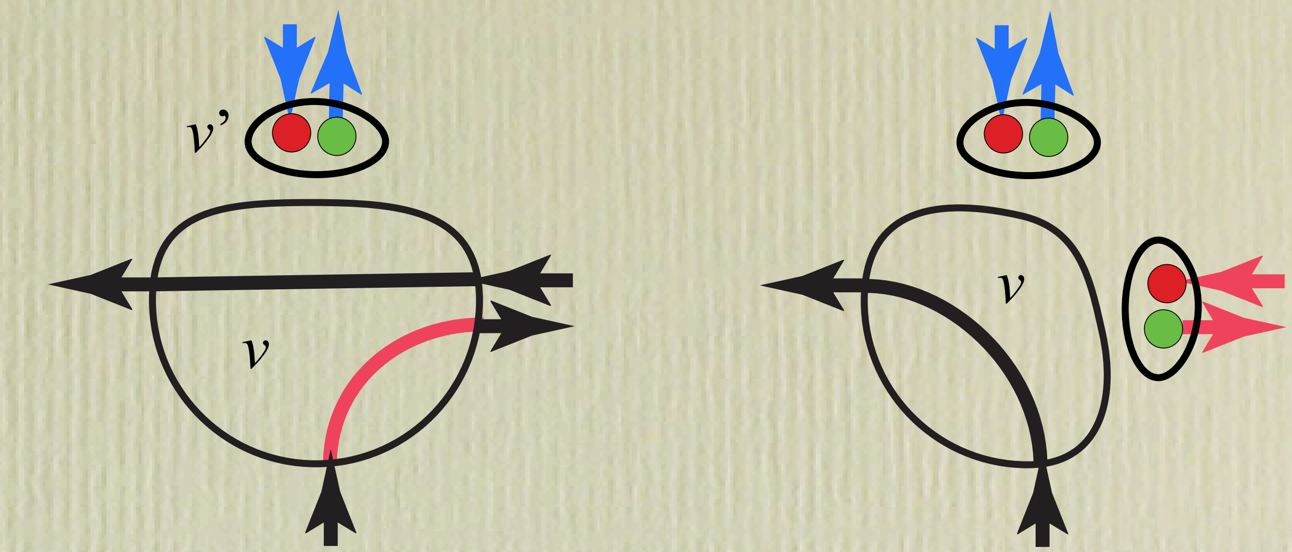


+ others

Cancellations
Bijections



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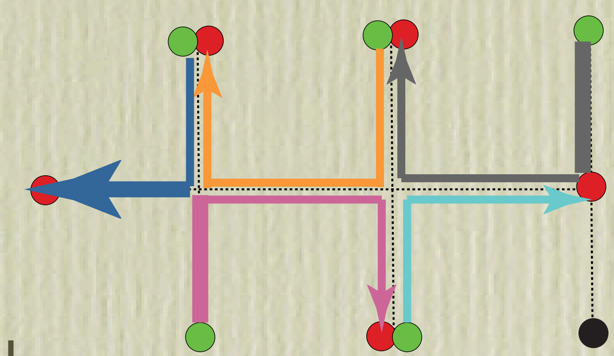

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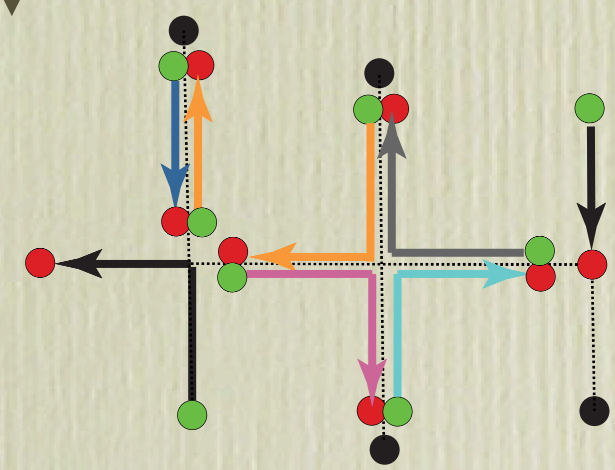

+ others

On the minors ... : Enumeration

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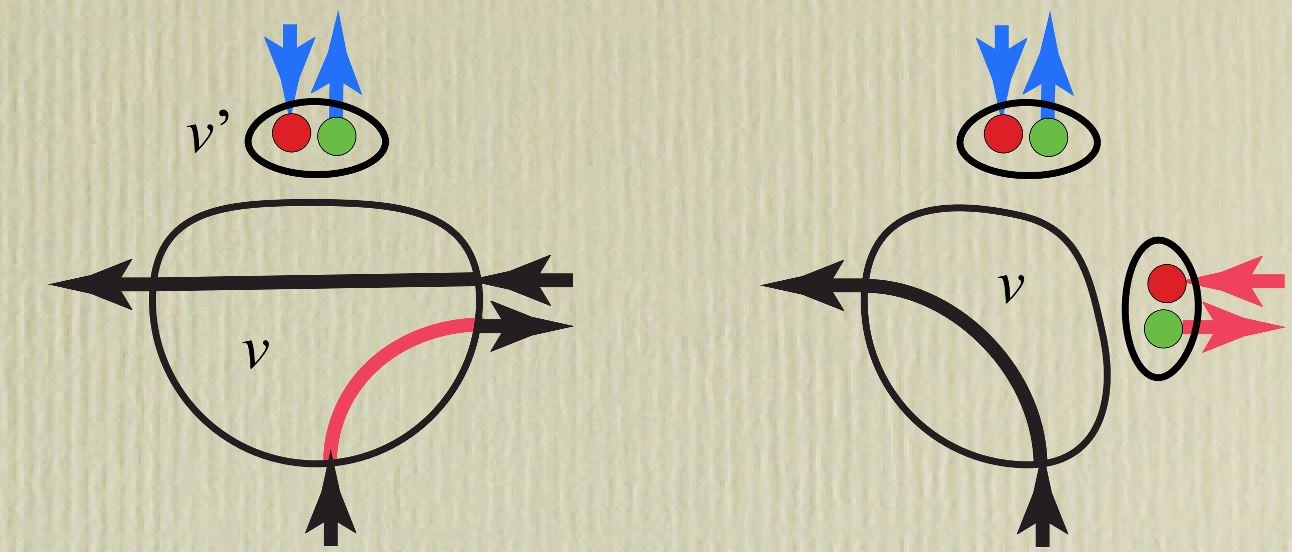

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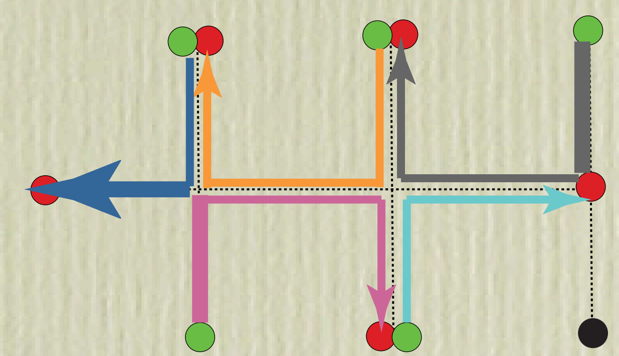
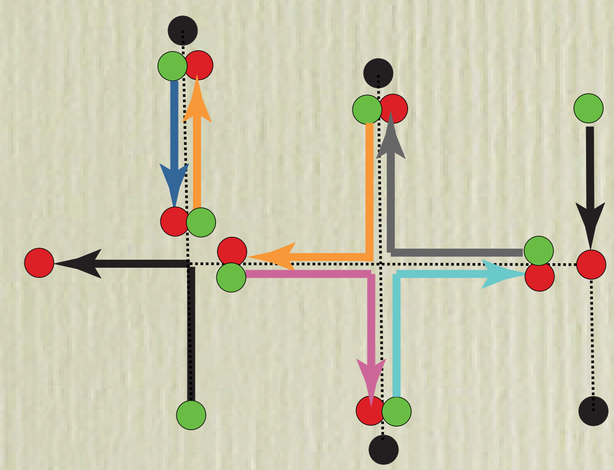


+ others

- We can even extract the minimal configuration



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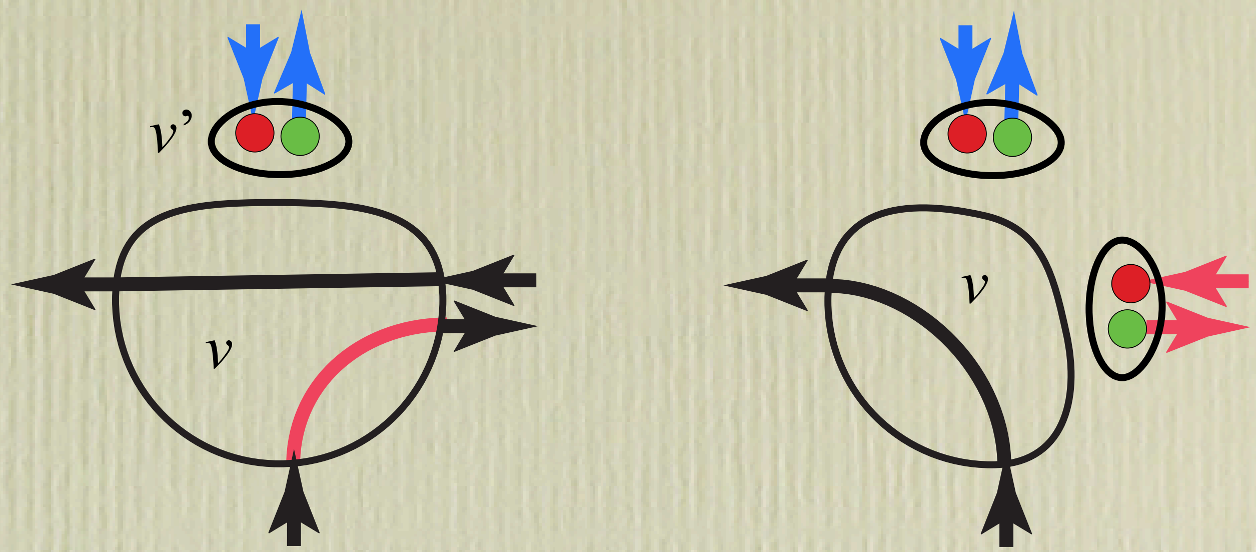
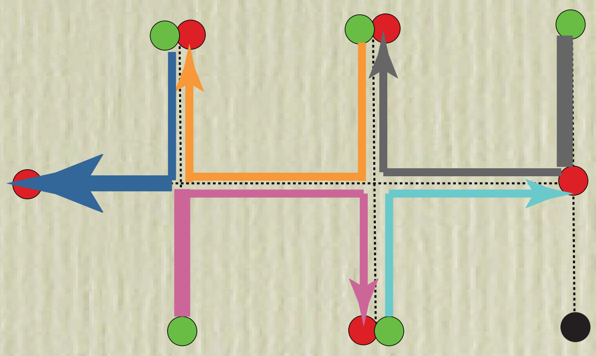


- Repeat with every vertex on the minimal configuration



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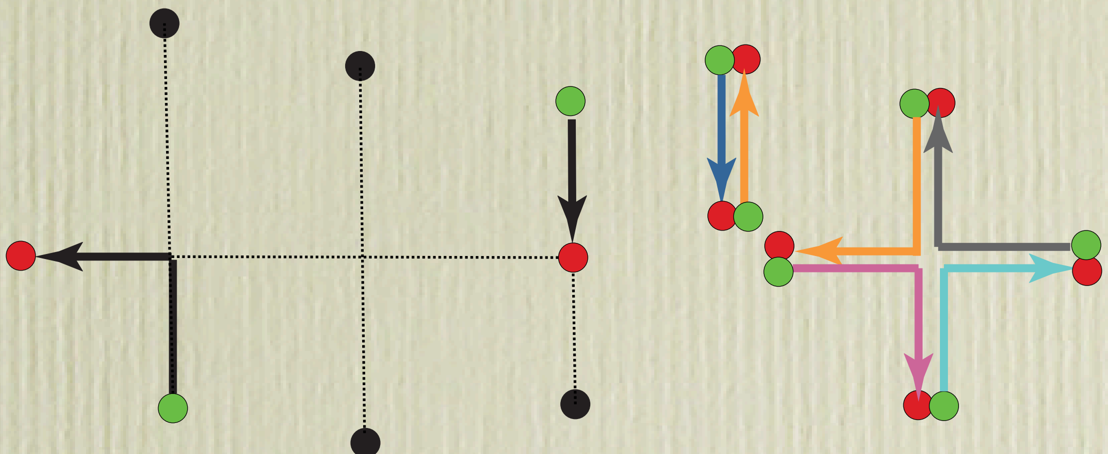

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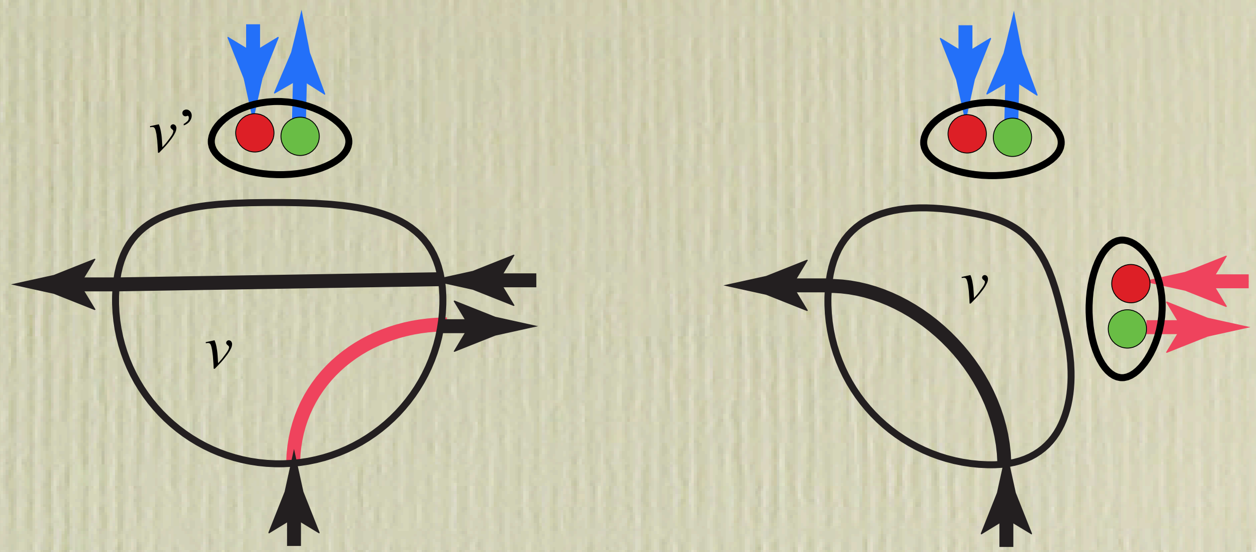

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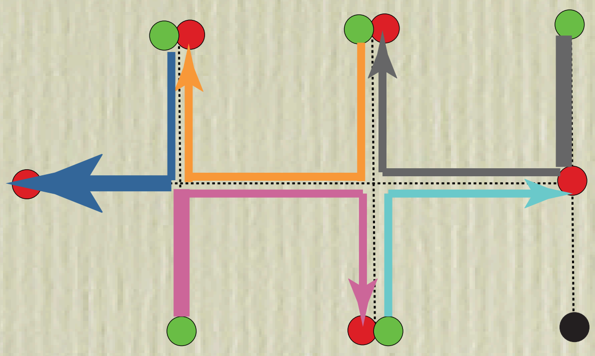

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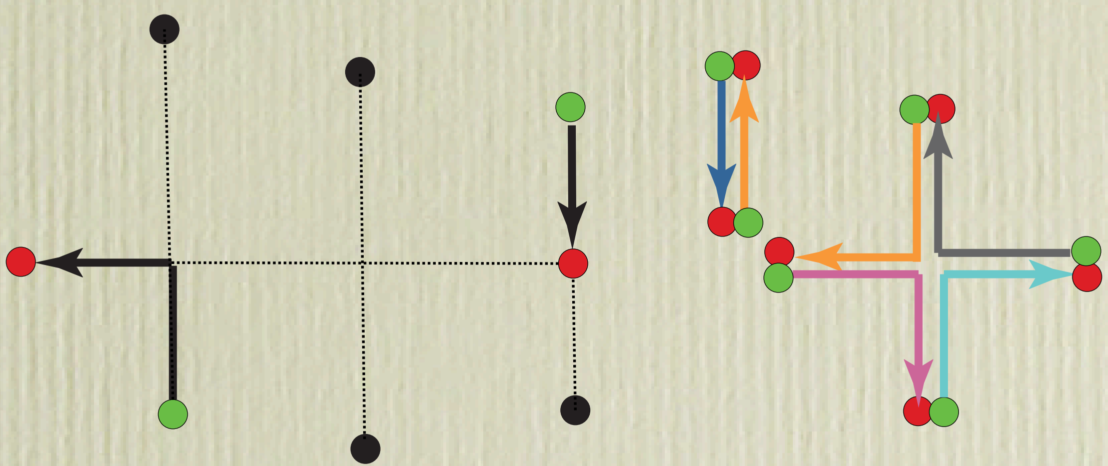

$$\det(P[S,T]) = (-1)^{\Omega_0}\, \mathrm{wt}(\Omega_0) \times ??$$

On the minors ... : Enumeration

- 2. Vertex not on the minimal configuration. Choose a leaf to become the root

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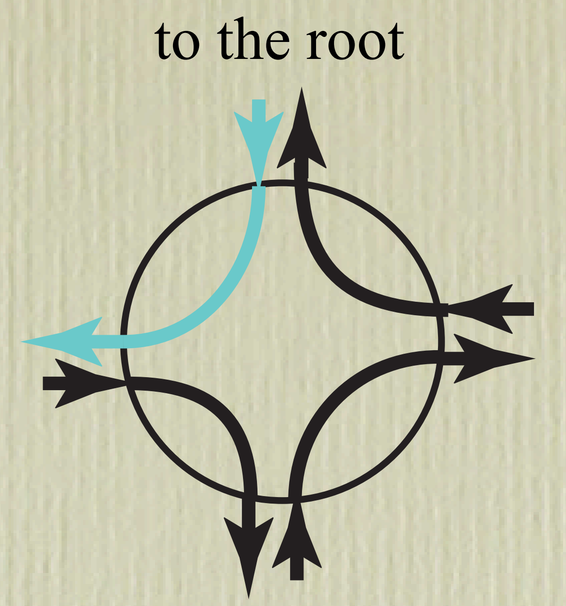

- 2. Vertex not on the minimal configuration. Choose a leaf to become the root
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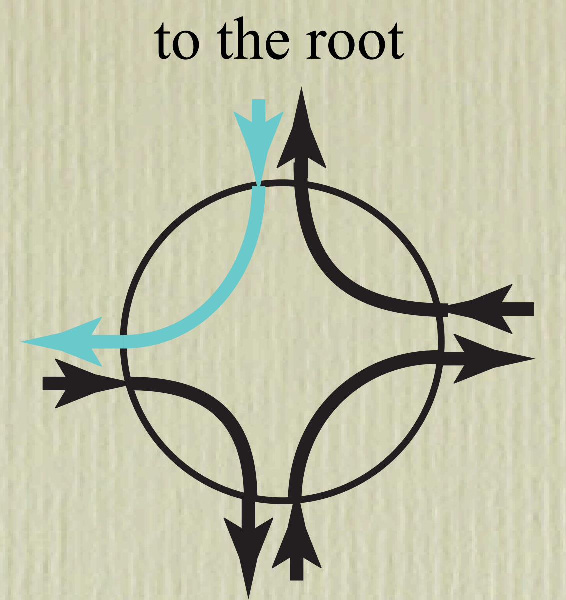
  - This defines two opposite arrows



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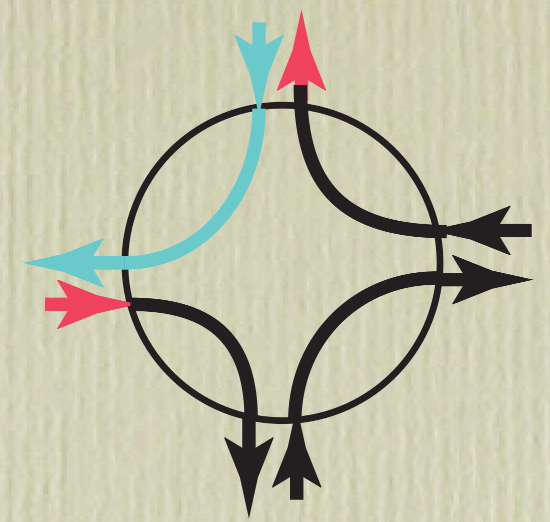

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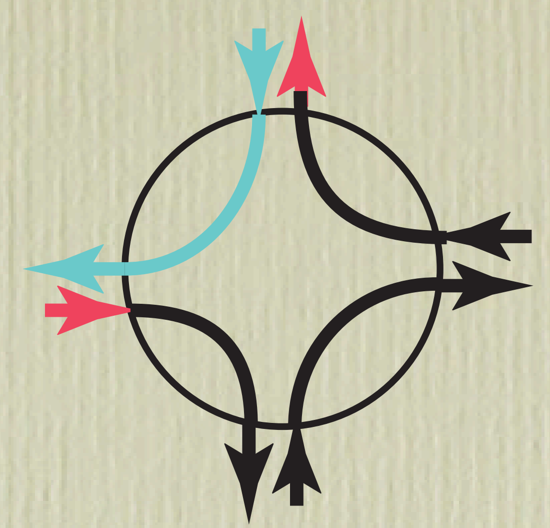
  - Cancellations/bijection



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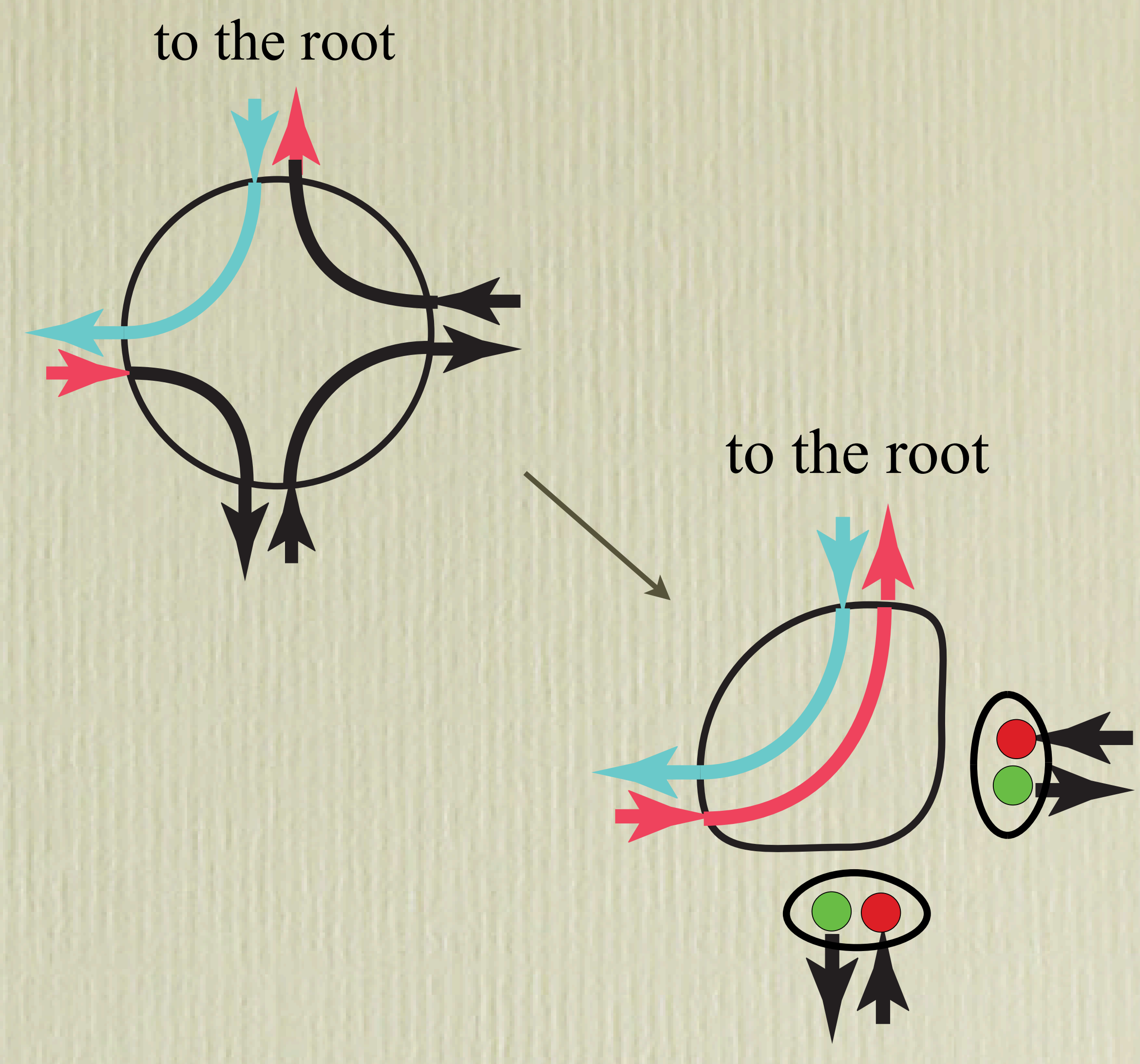

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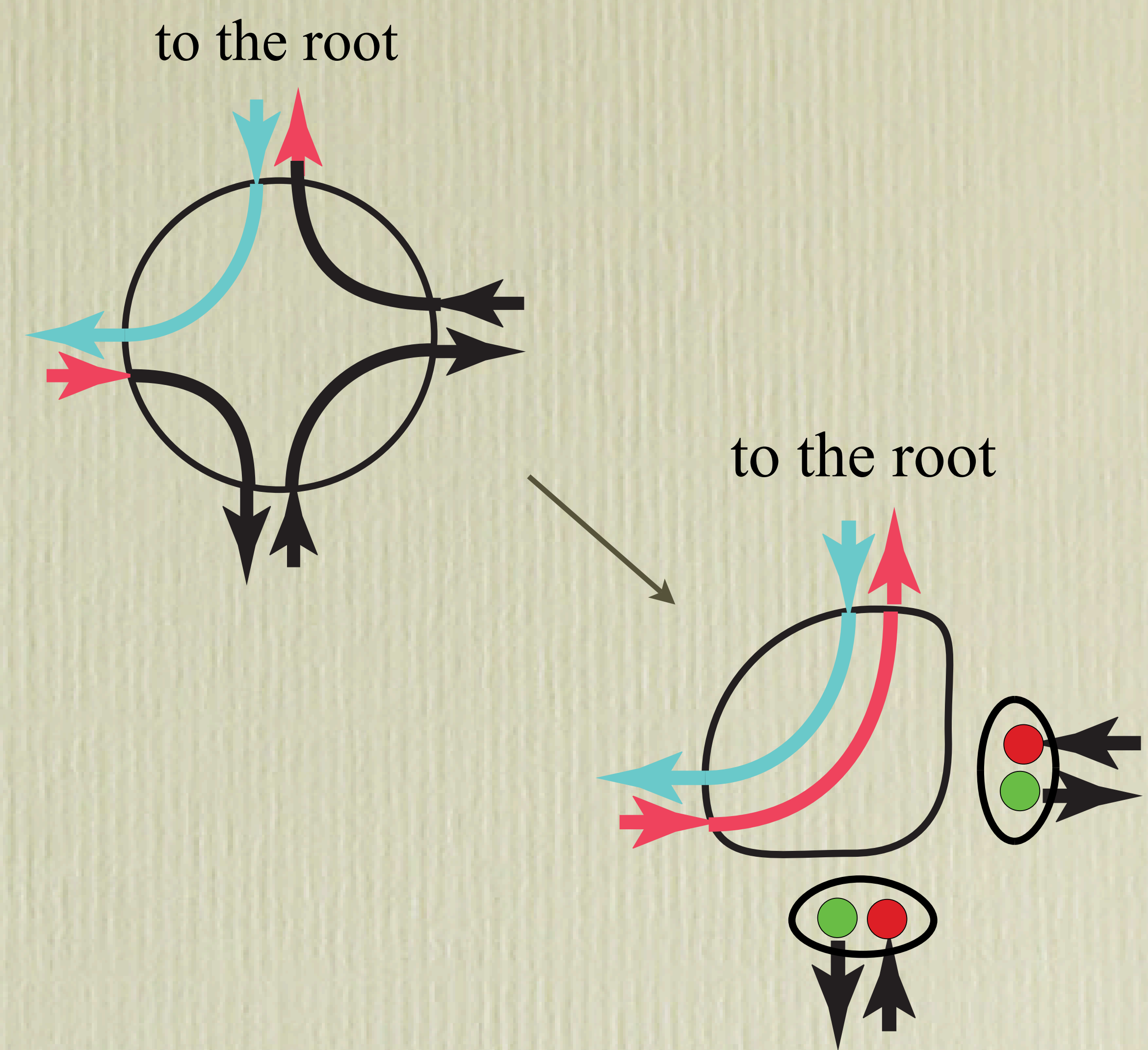
- 2. Vertex not on the minimal configuration. Choose a leaf to become the root
  - Take the incoming path from the edge that leads to the root
  - This defines two opposite arrows
  - Cancellations/bijection
  - One last cut



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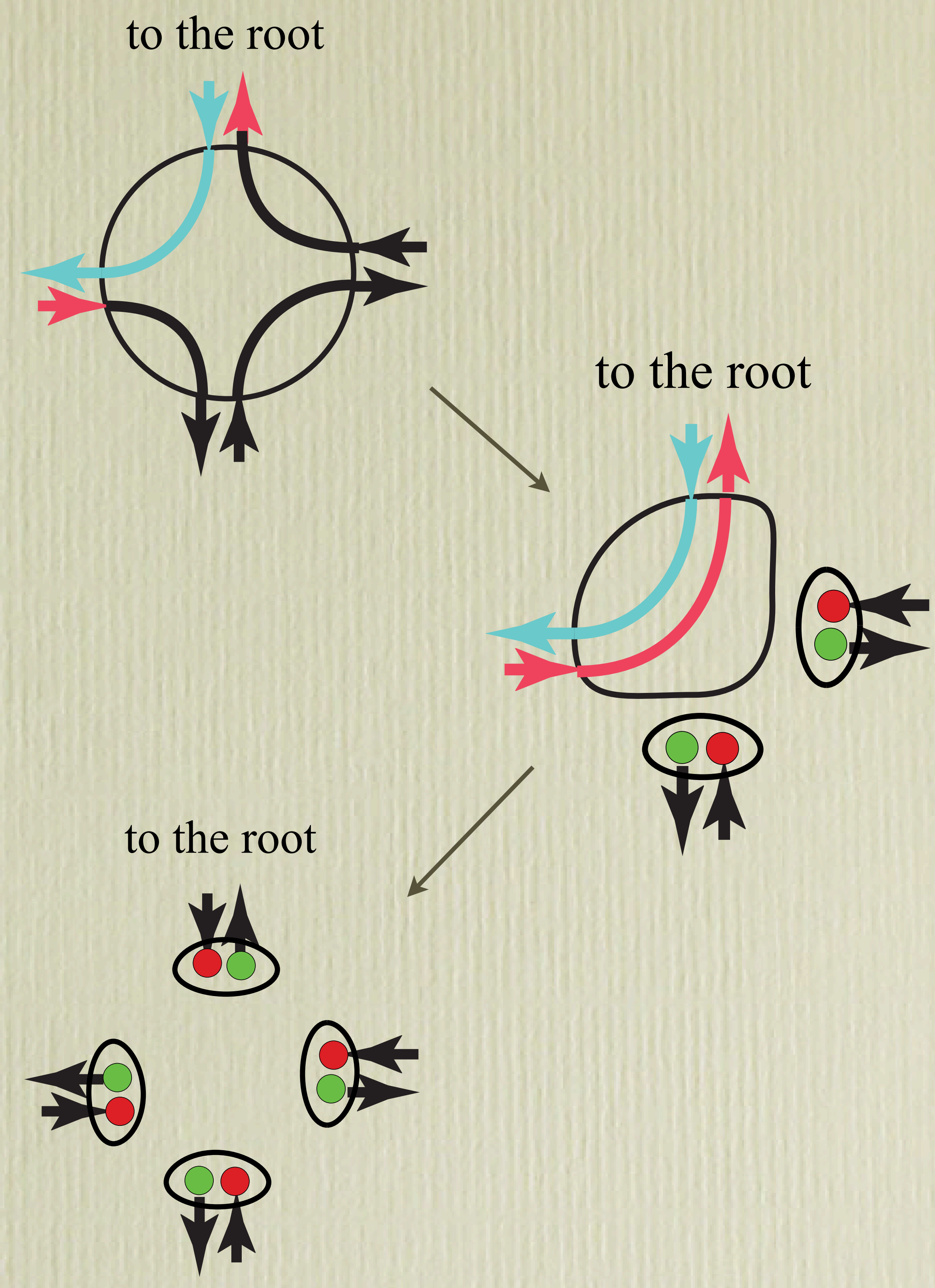

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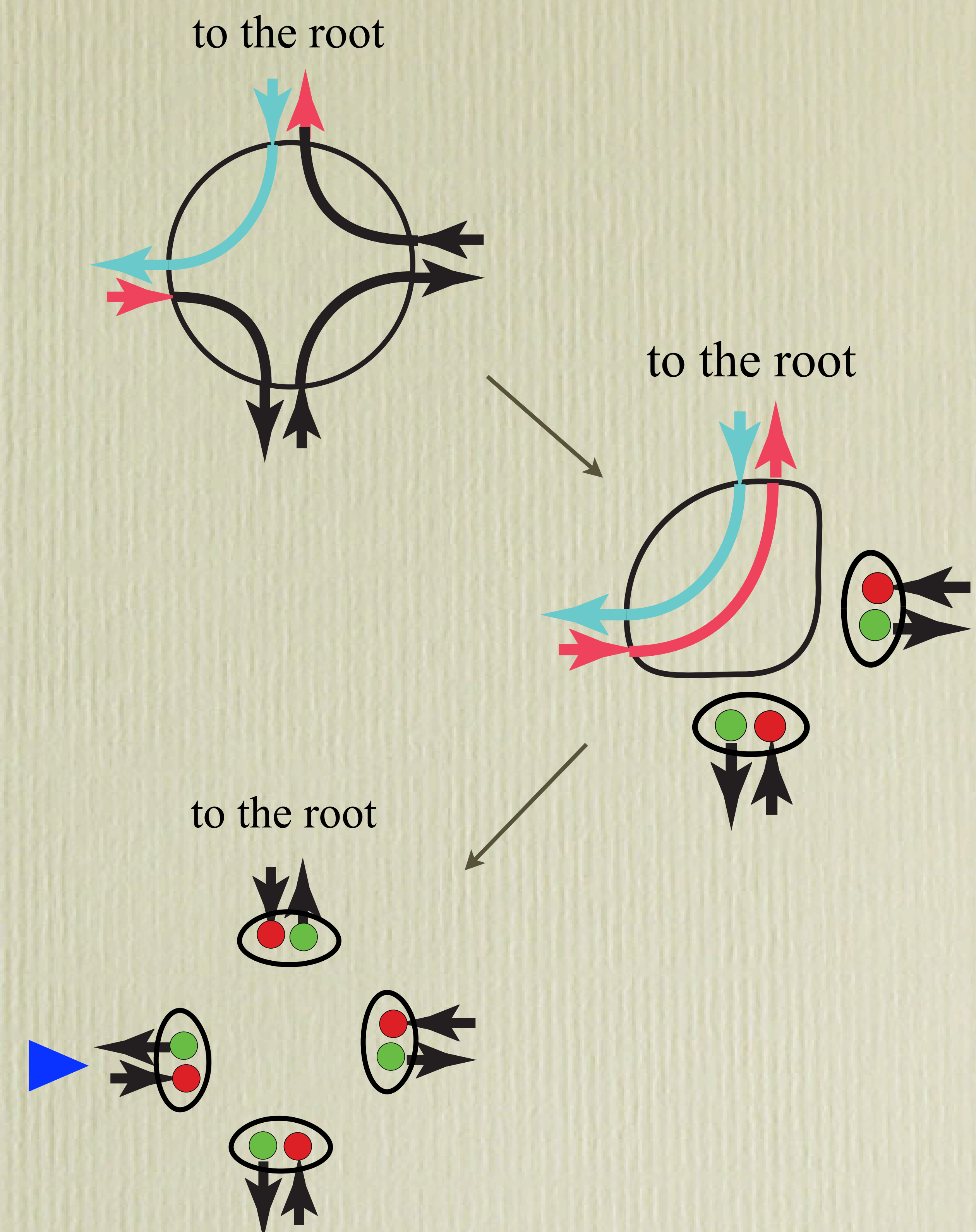
  - Take the incoming path from the edge that leads to the root
  - This defines two opposite arrows
  - Cancellations/bijection
  - One last cut
    - Record the edge that was connected to edge leading to the root



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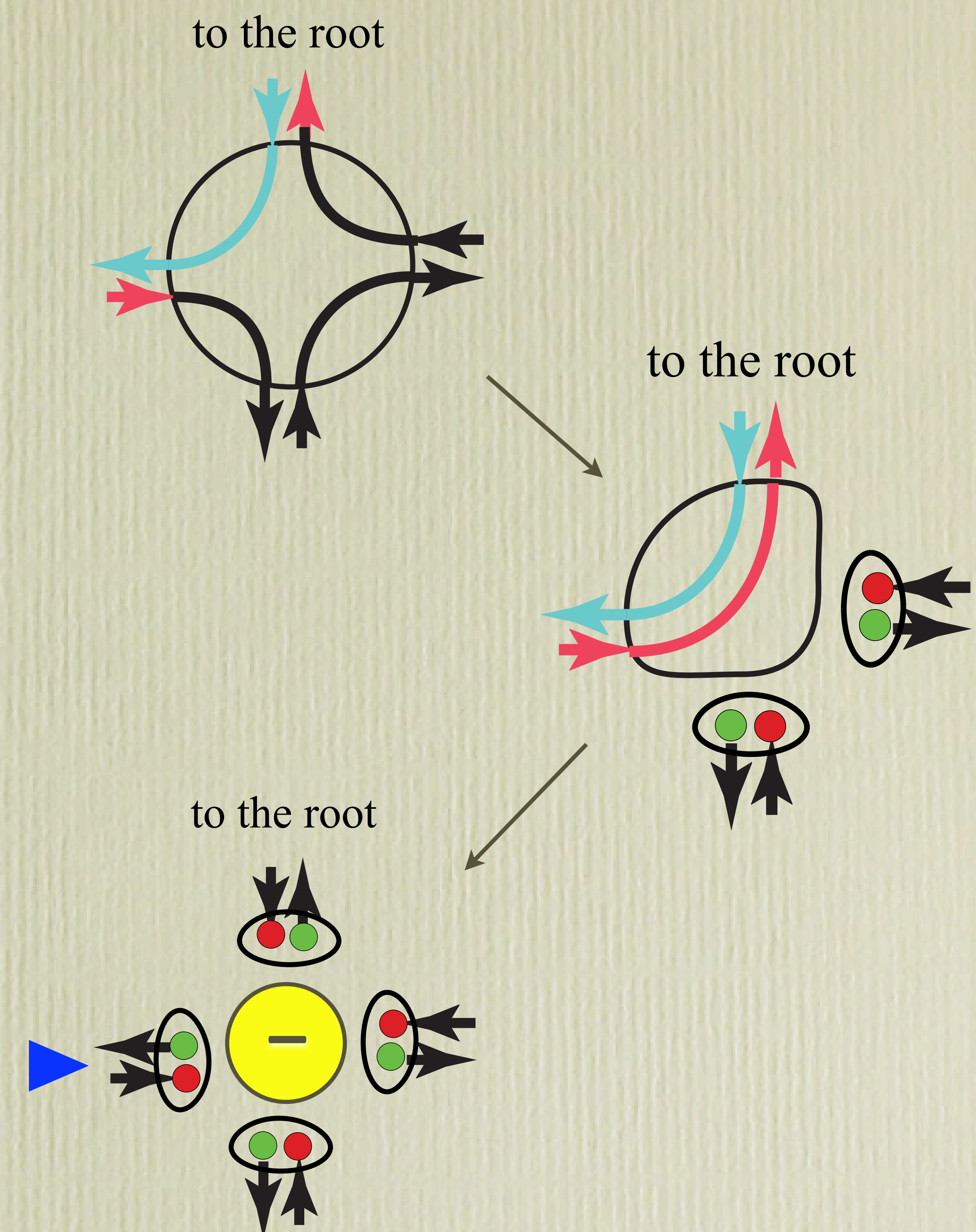
  - One last cut
    - Record the edge that was connected to edge leading to the root
    - The last cut changes the sign



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    - This defines two opposite arrows
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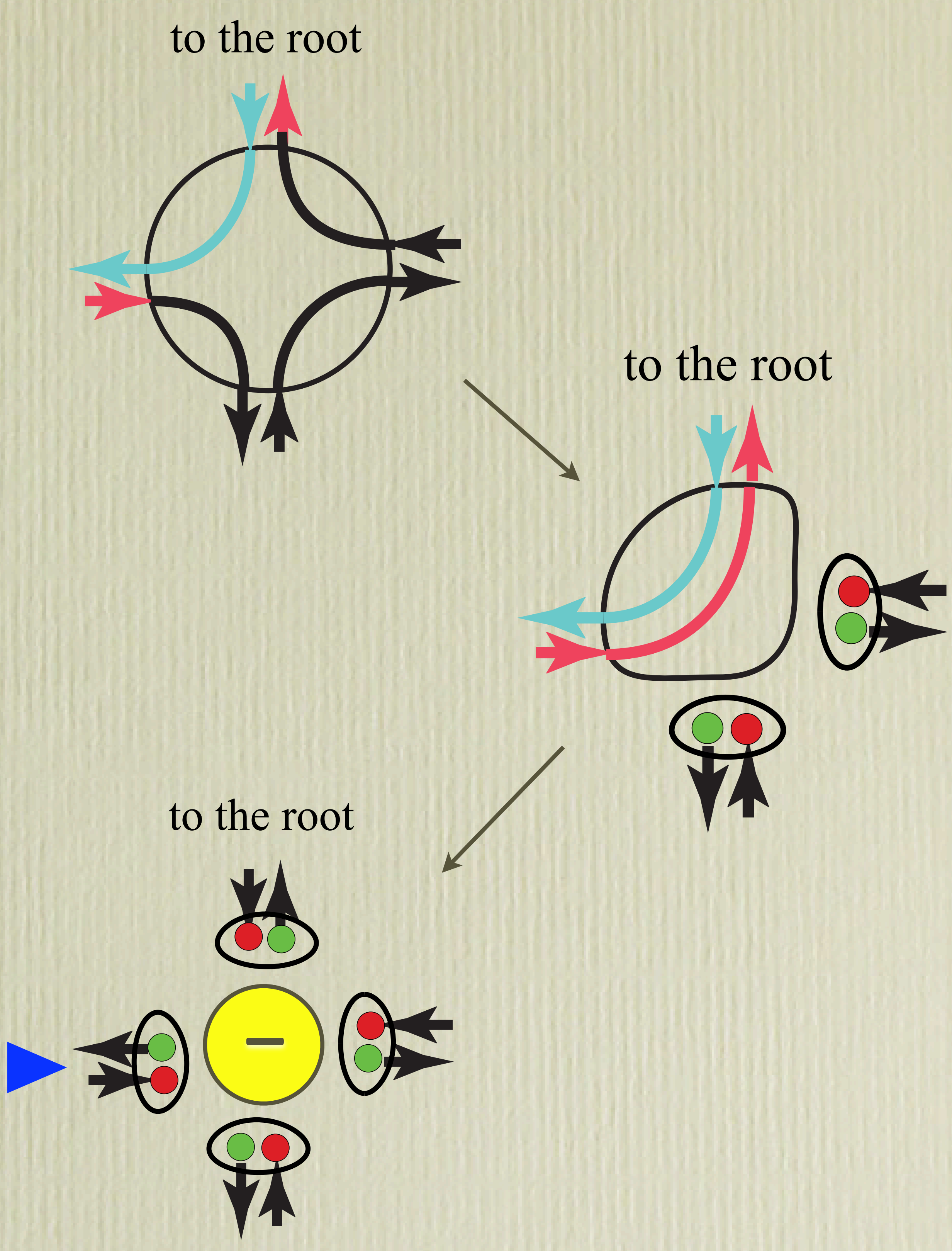
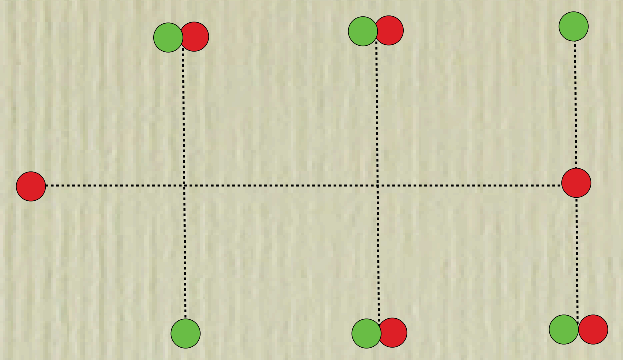
    - One last cut
        - Record the edge that was connected to edge leading to the root
        - The last cut changes the sign
        - All pairs of opposite are now separated

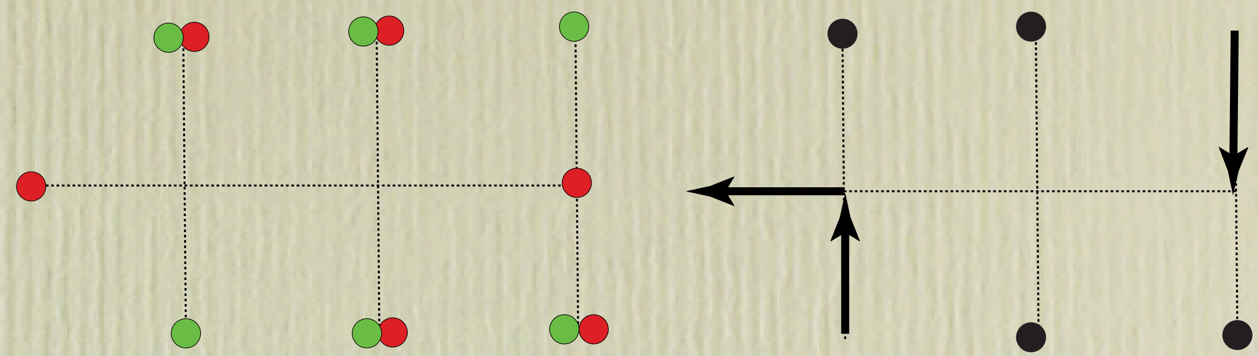


- Thus given S,T



On the minors ... : Enumeration

- Thus given S,T
- with an unique minimal configuration



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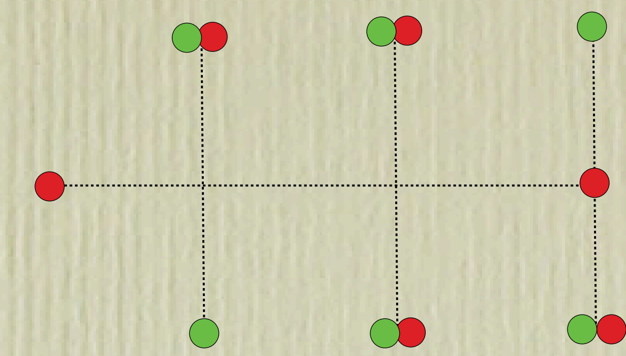
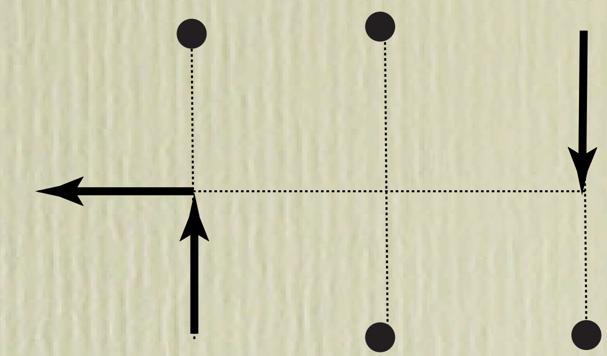
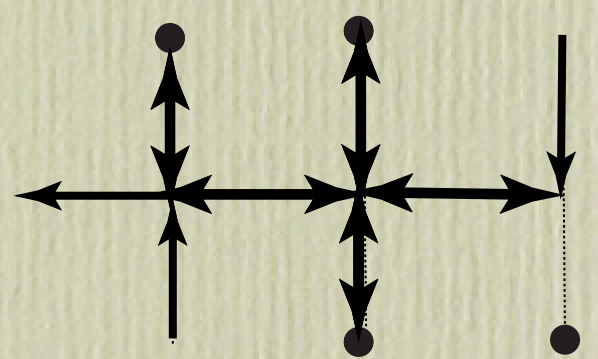




- How many surviving configurations have weight:



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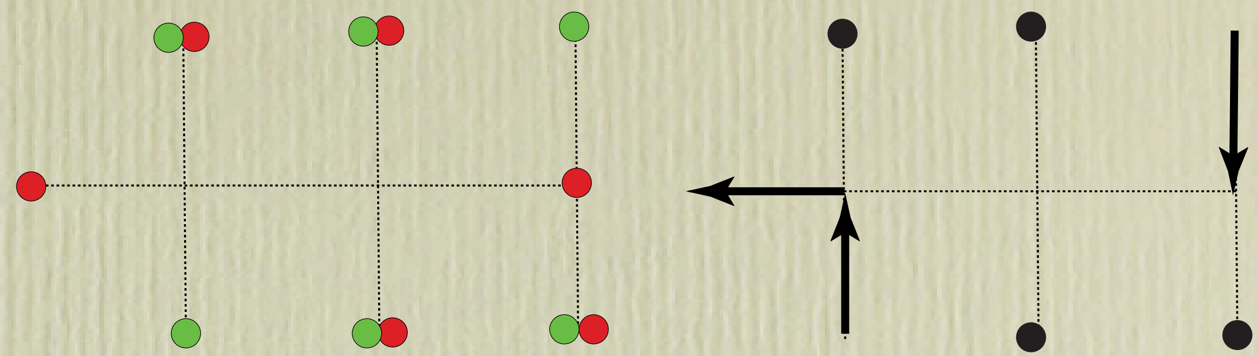

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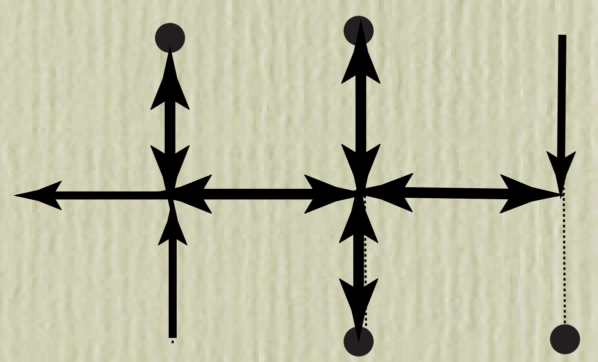
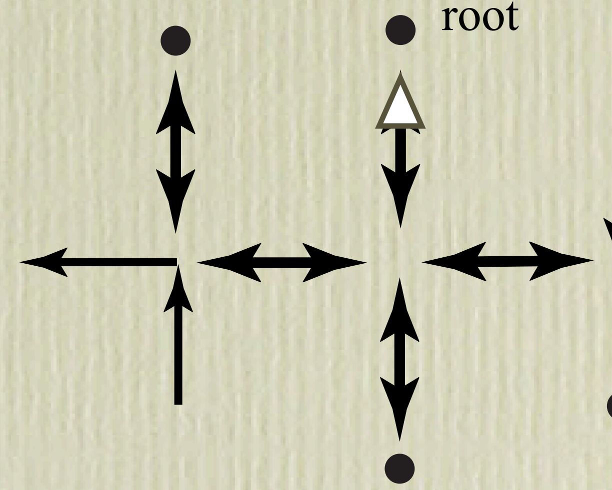


separate



On the minors ... : Enumeration

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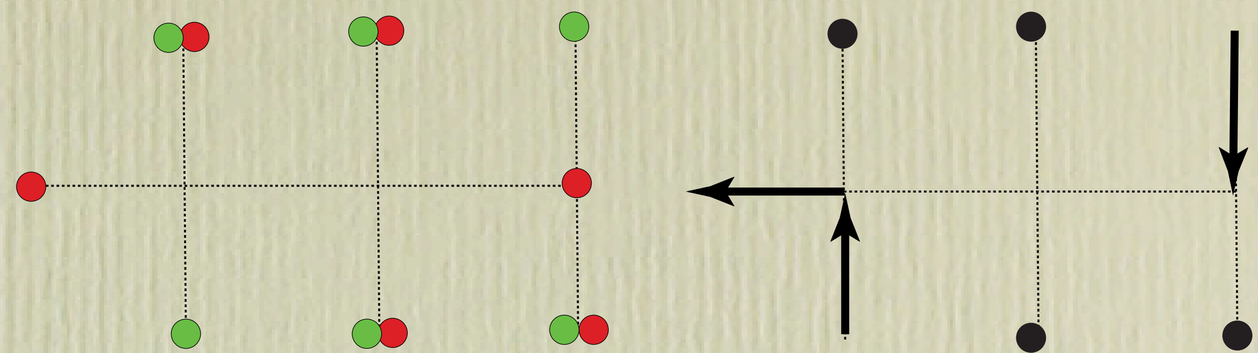

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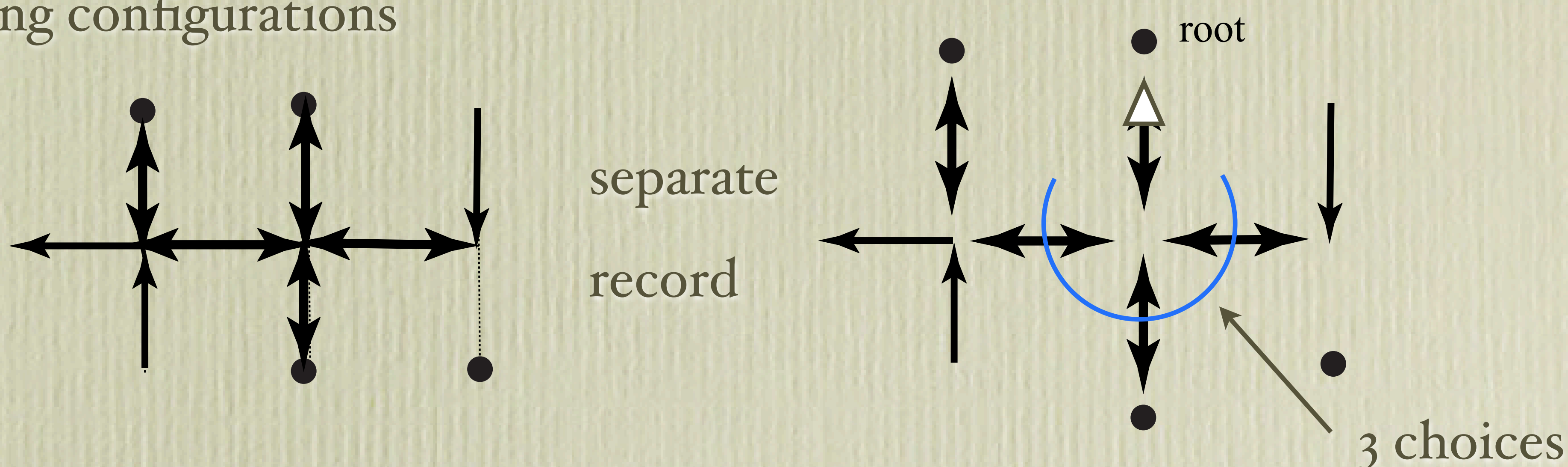
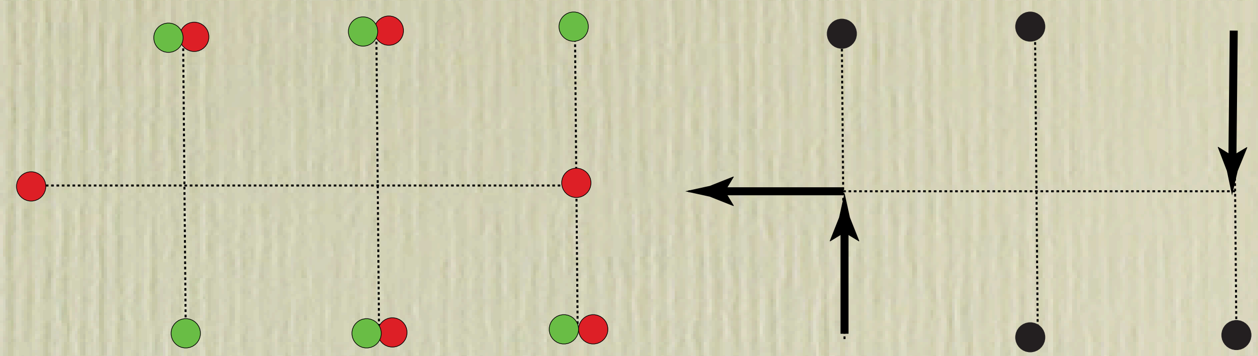


On the minors ... : Enumeration



- Thus given S,T
- with an unique minimal configuration
- How many surviving configurations have weight:
- What is the sign?

On the minors ... : Enumeration

- Thus given S,T
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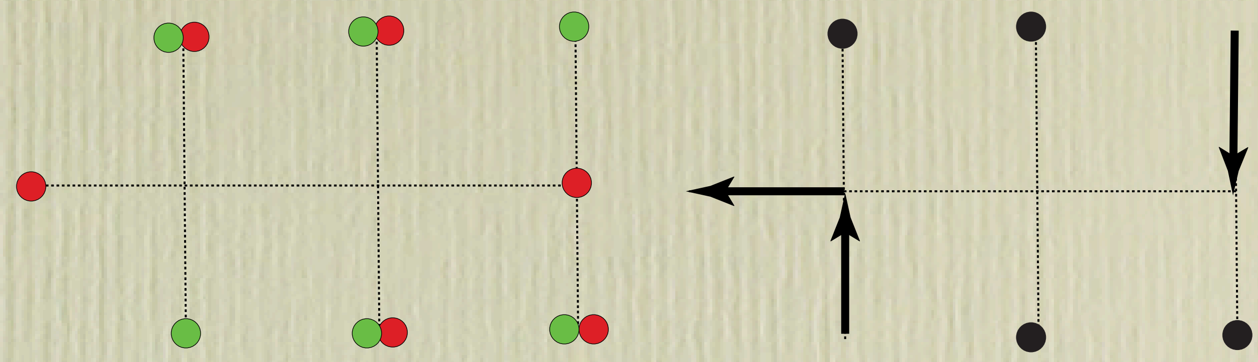

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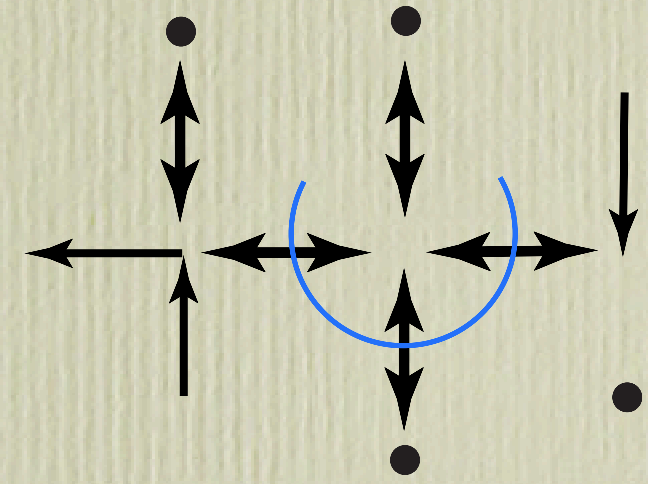

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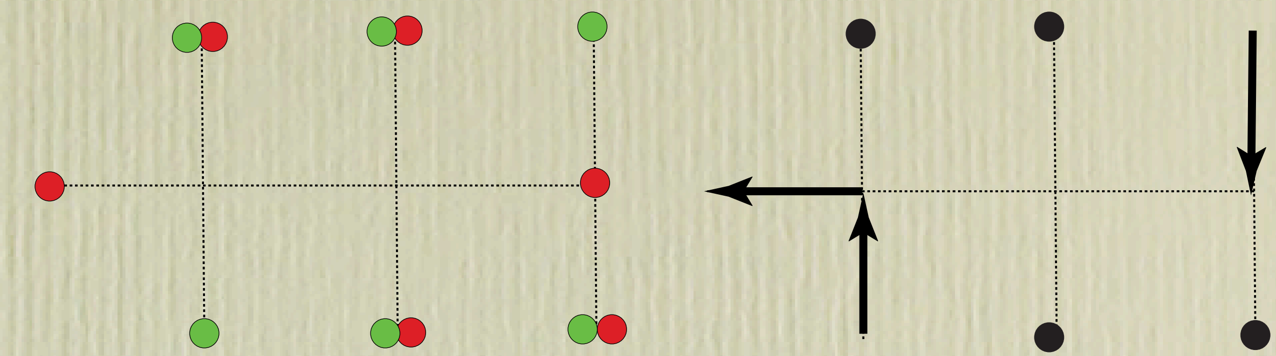

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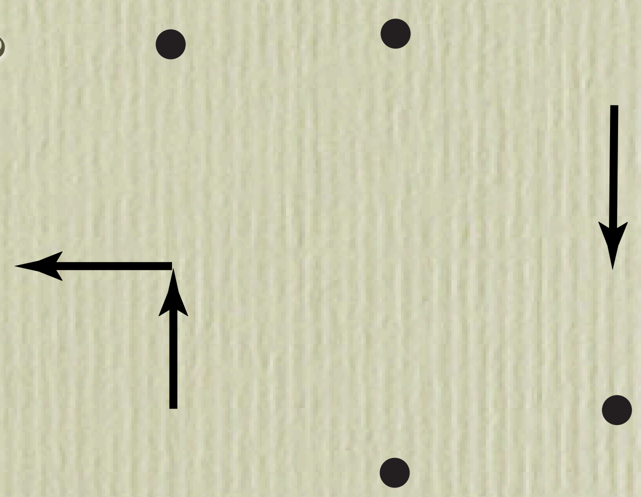
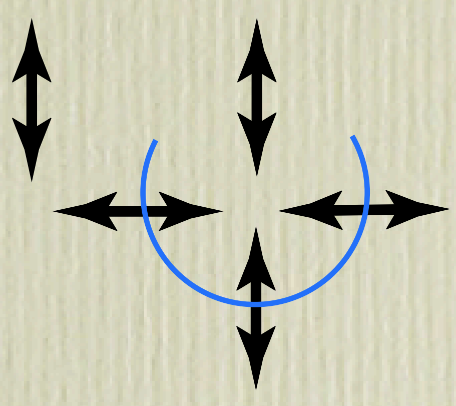
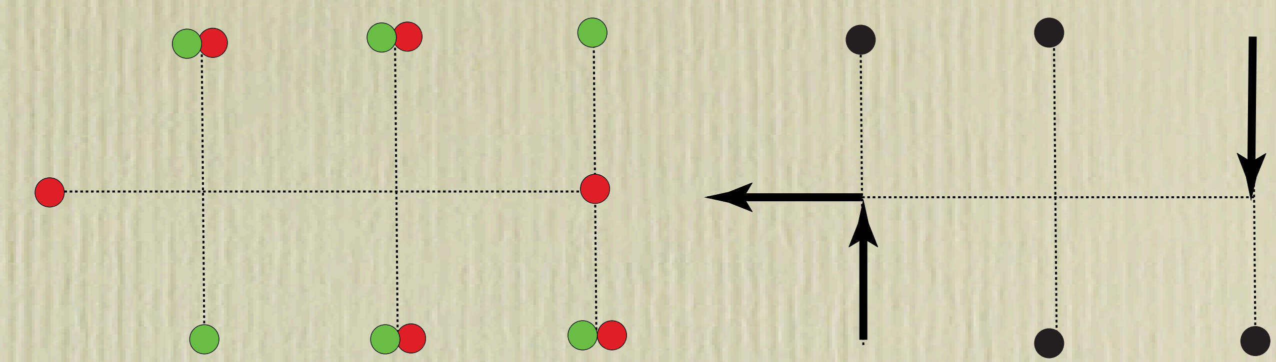


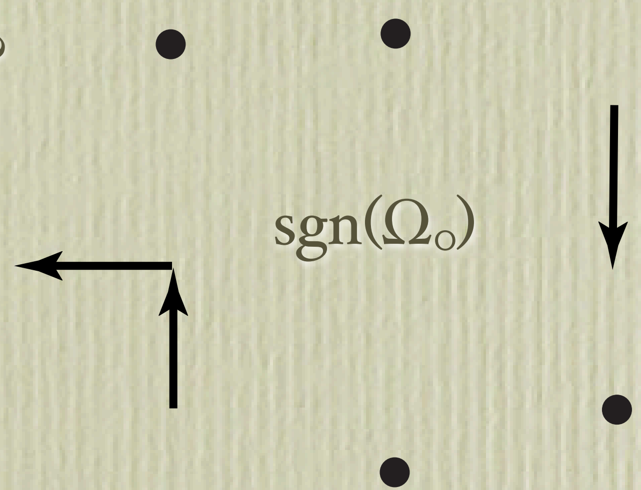


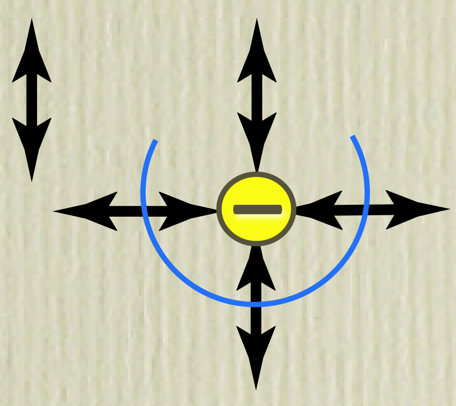
On the minors ... : Enumeration

- Thus given S, T
- with an unique minimal configuration



- How many surviving configurations have weight:
- What is the sign?



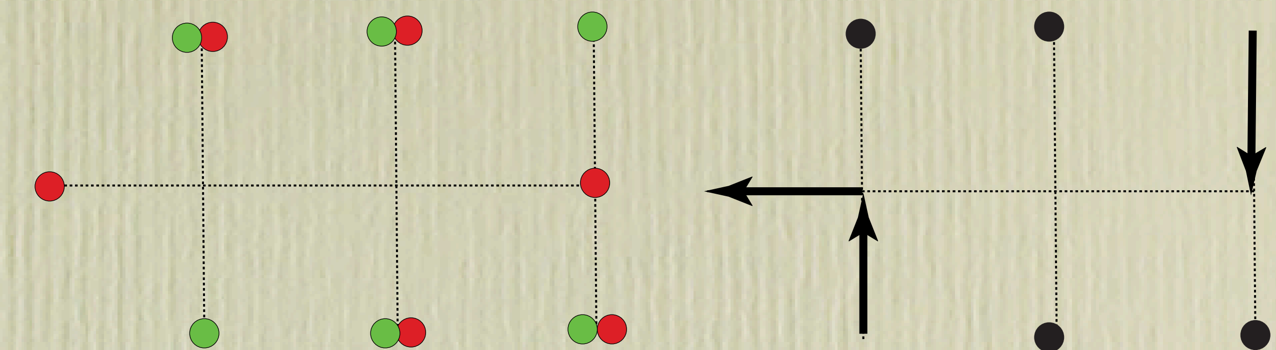


$\#$transposition = $\#F$ = 5

$\#$sign change = 1

On the minors ... : Enumeration

- Thus given S, T
- with an unique minimal configuration



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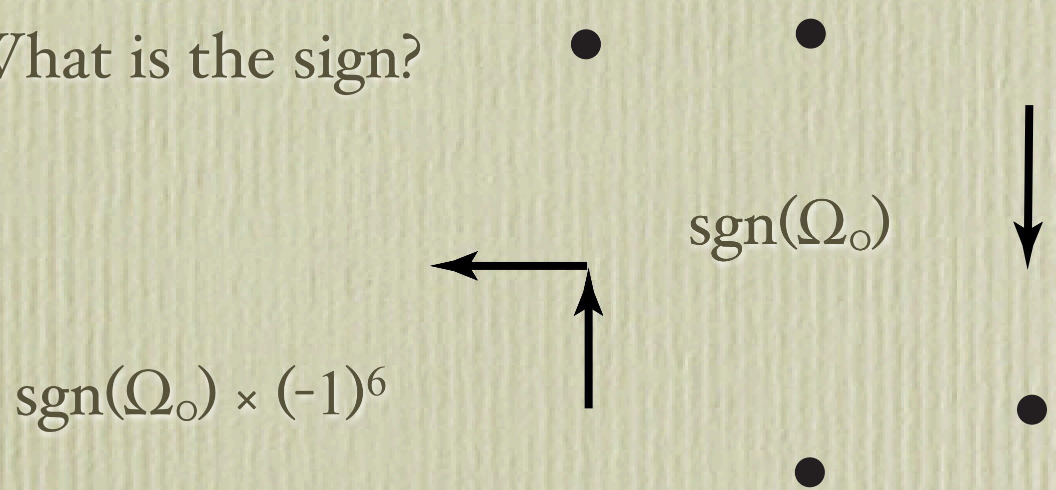

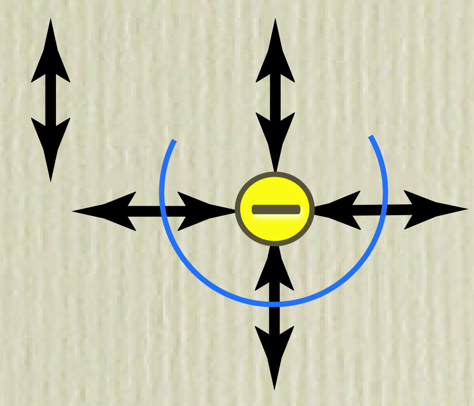

#transposition = $\#F = 5$

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- **Main theorem**: Let $d_F(v)$ be the degree of v in F. Then:

$$\det(P[S,T]) = (-1)^{\Omega_0}\,\mathrm{wt}(\Omega_0) \times \sum_{F \subseteq E^+ - E(\Omega_0)} (-1)^{|F|}\mathrm{wt}(F)\mathrm{wt}(\bar{F}) \prod_{v \in V - V(\Omega_0)} (1 - d_F(v))$$

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sign-weight due to the minimal configuration

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choose the pairs
of opposites

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due to the transpositions

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don't forget the opposite arrows

- **Main theorem**: Let $d_F(v)$ be the degree of v in F. Then:

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for all vertex not on the minimal configuration

record the choices and change sign

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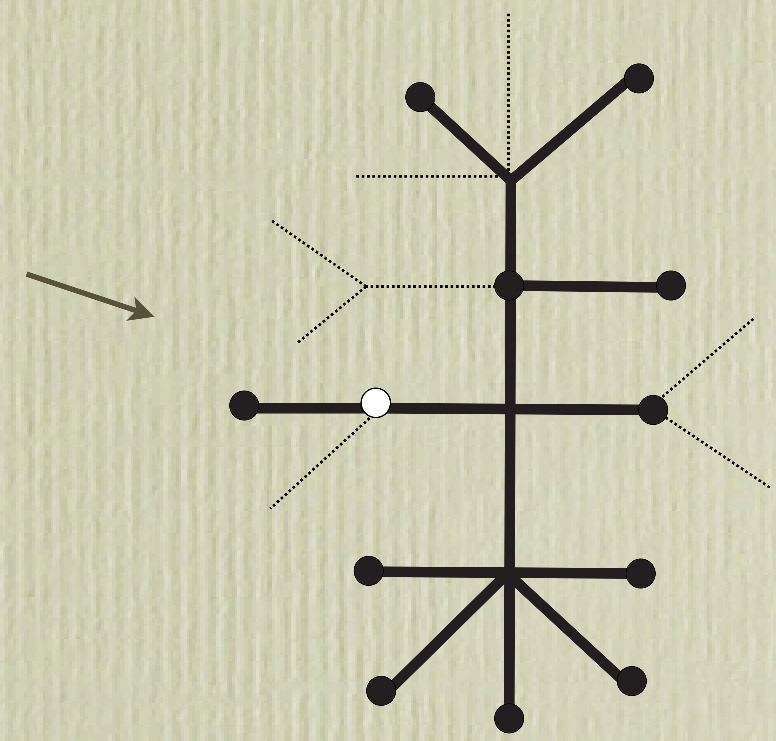
- For instance: let $S = T$.

  What is the coefficient of

  (within some forest)

  in $\det(P[S,T])$?



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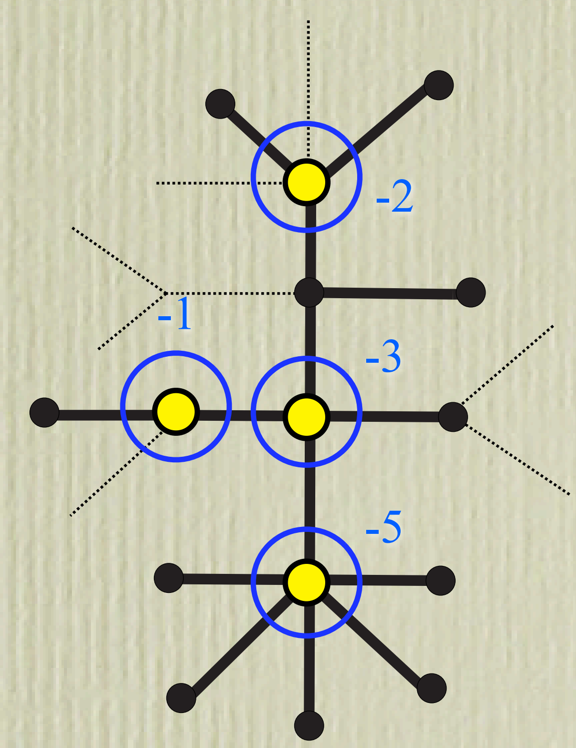

$|F| = 14$

#sign changes = 4

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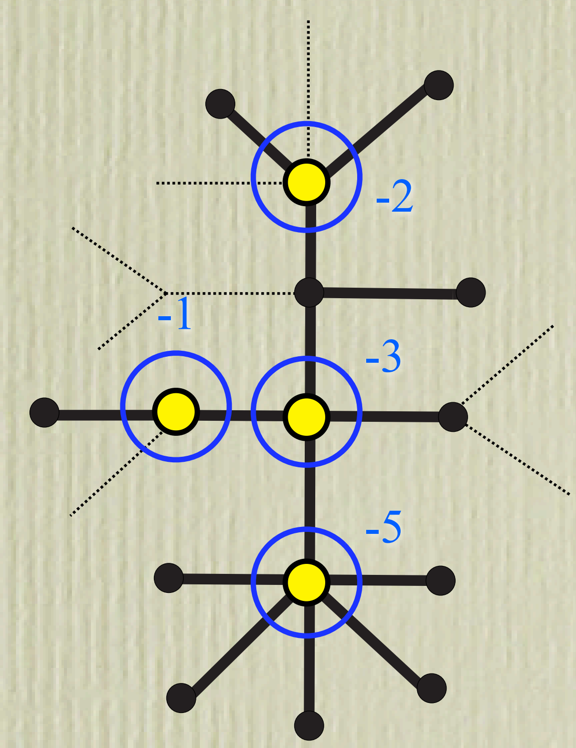

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$(-1)^{14}\,(-2)(-1)(-3)(-5) = 30$

On the minors ... : Consequences

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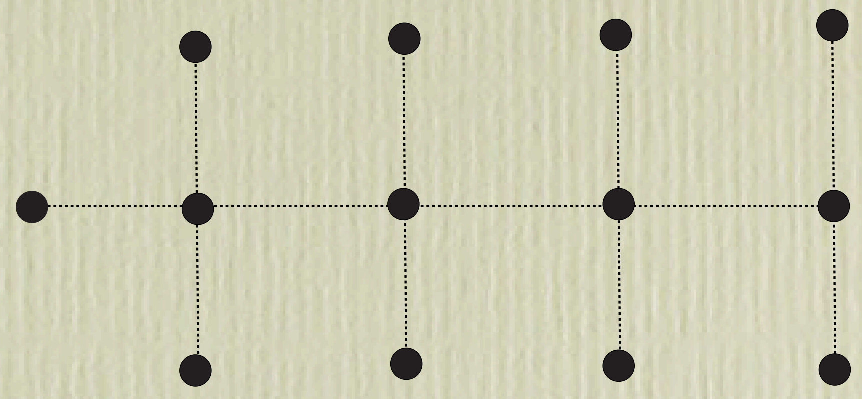

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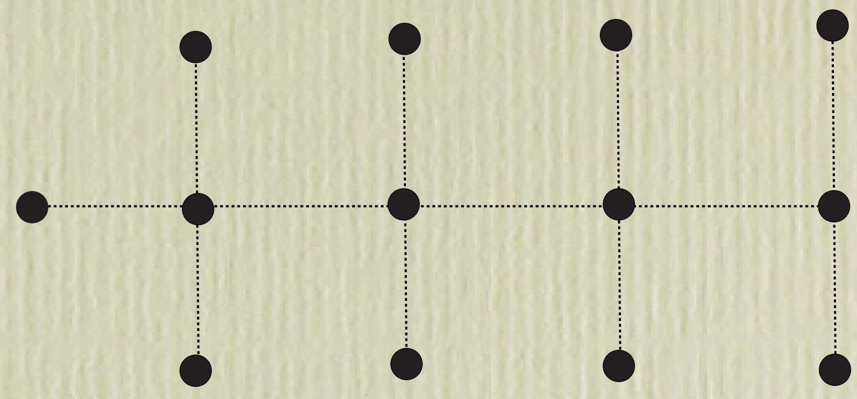

$$\begin{aligned}\det(P) &= \sum_{F \subseteq E^+} (-1)^{|F|}\mathrm{wt}(F)\mathrm{wt}(\bar{F}) \\ &= \prod_{e \in E^+} (1 - e\bar{e})\end{aligned}$$

(Yan-Yeh 06)

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- Cofactors: Let $S = V - \{j\}$, $T = V - \{i\}$.

$$(-1)^{i+j}\det(P[S,T]) = \begin{cases} 0 & \text{if } i \neq j \text{ and } (i,j) \text{ is not form an edge,} \\ -\frac{e}{1-e\bar{e}}\,|P| & \text{if } (i,j) \text{ is the arrow } e, \\ \left(1 + \sum_{e\in t^{-1}(i)} \frac{e\bar{e}}{1-e\bar{e}}\right)|P| & \text{if } i = j. \end{cases}$$

(all $e = q$: Bapat, Lal, Sukanta Pati 06)

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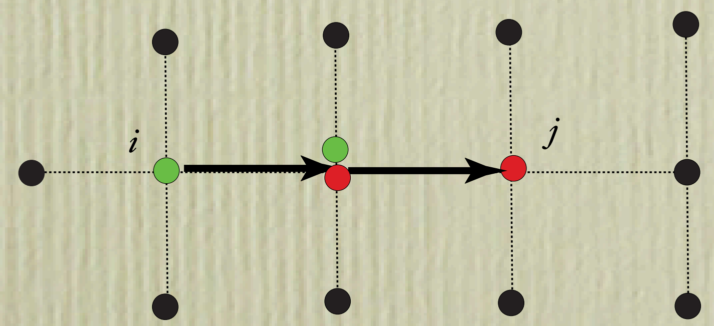

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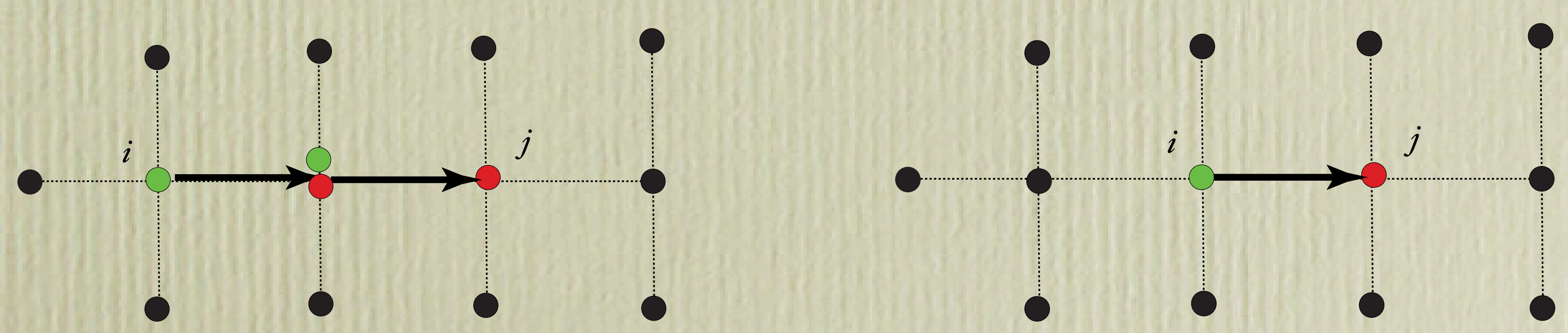

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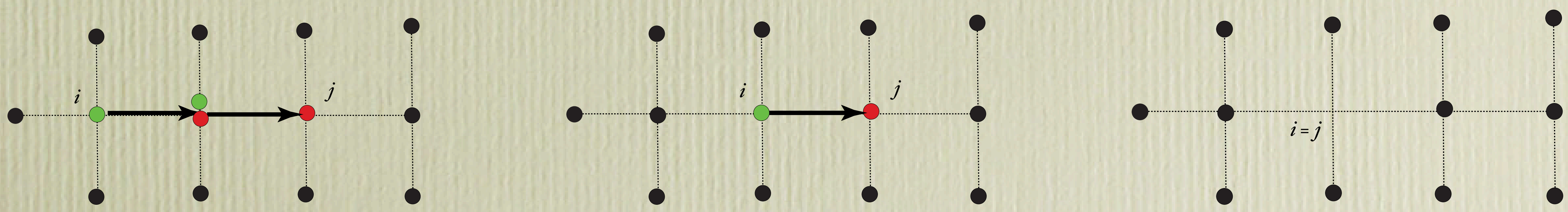

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- Let J be the all 1's matrix and c the number of trees in the forest. Then

$$\begin{aligned}\det(P + xJ) &= |P| + x\,(\text{sum of the cofactors of } P) \\ &= (1 + cx)\,|P| + x\left(\sum_{e \in E} \frac{(1-e)(1-\bar{e})}{1 - e\bar{e}}\right)|P|\end{aligned}$$

Bapat, Kirkland, Neumann (05): D instead of P.

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- When $S = T$ ...

$$\det(P[U,U]) = \sum_{F \subseteq E^+} (-1)^{|F|}\mathrm{wt}(F)\mathrm{wt}(\bar{F}) \prod_{v \in V - U} (1 - d_F(v))$$

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- For $e \in E^+ \cup E^-$, let $\delta_e(\text{monomial}) = \begin{cases} 0 & \text{if } e \notin \text{monomial}, \\ \text{monomial} & \text{if } e \in \text{monomial}. \end{cases} \quad = e\,\dfrac{\partial}{\partial e}$

On the minors ... : Consequences

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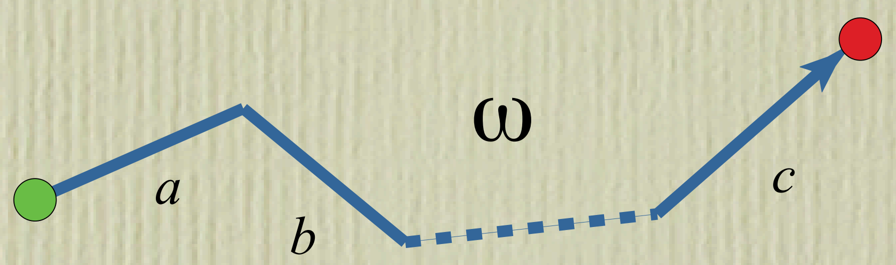
On the minors ... : Additive weight

- From multiplicative to additive weight:



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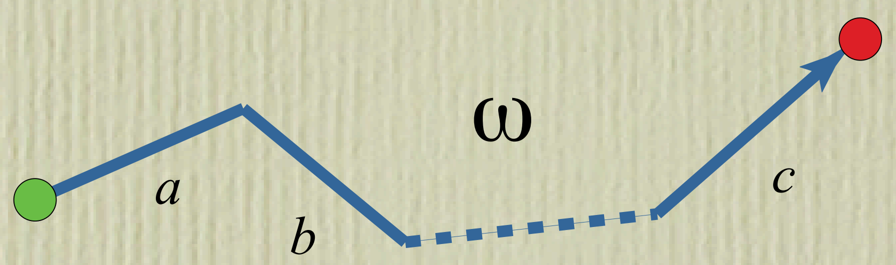

multiplicative	additive
$\mathrm{wt}(\omega) = ab \cdots c$	$\mathrm{wt}(\omega) = a + b + \cdots + c$
matrix: P	matrix: D

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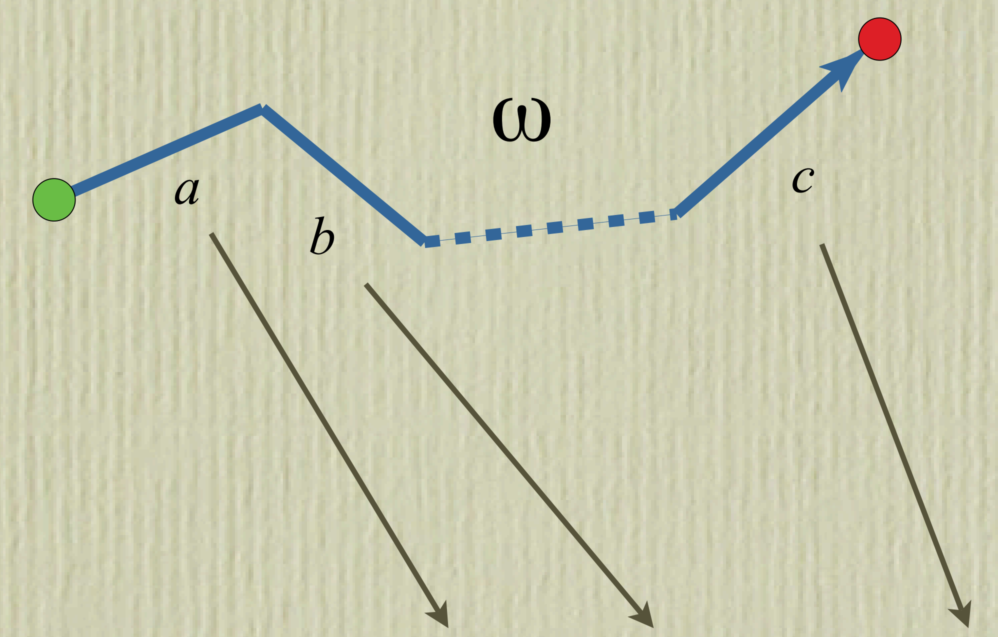

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$$\mathrm{wt}(\omega) = (1+at)(1+bt)\cdots(1+ct) = 1 + (a+b+\cdots+c)t + O(t^2)$$

matrix: P^+

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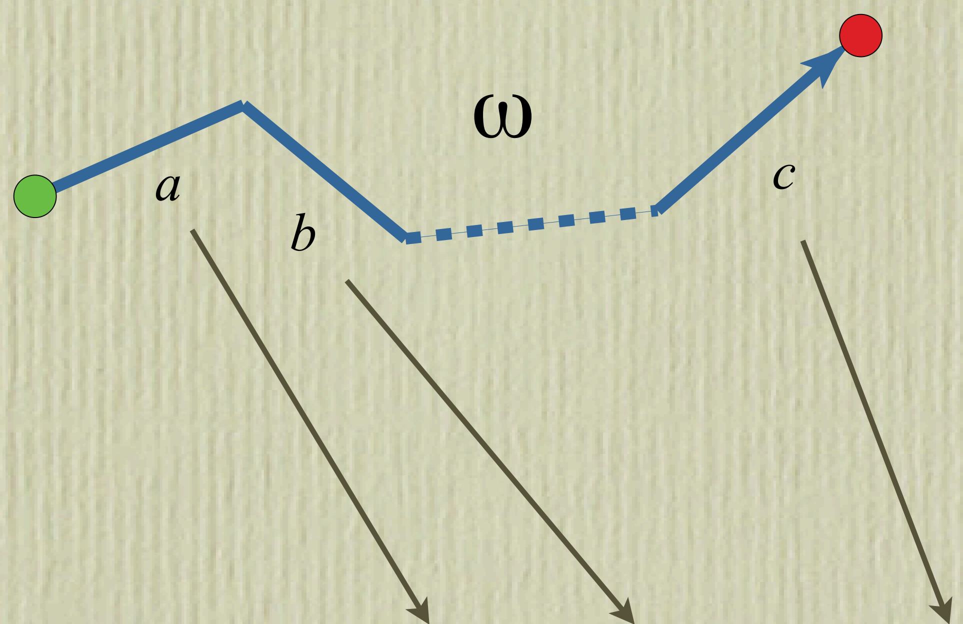

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matrix: P^+

$$P^+ = J + Dt + O(t^2)$$

$$[t^n]\ \det(P^+ - J) = \det(D)$$

On the minors ... : Additive weight

On the minors ... : Additive weight

- $[t^n]\ \det(P^+ + (xt-1)J) = \det(D + xJ) = (-1)^{n-1}\left(x + \sum_{e \in E} \frac{e\bar{e}}{e+\bar{e}}\right) \prod_{e \in E}(e + \bar{e})$

- $[t^n]\ \det(P^+ + (xt-1)J) = \det(D + xJ) = (-1)^{n-1}\left(x + \sum_{e\in E}\frac{e\bar{e}}{e+\bar{e}}\right)\prod_{e\in E}(e+\bar{e})$

$e = \bar{e}$: Bapat, Kirkland, Neumann (05)

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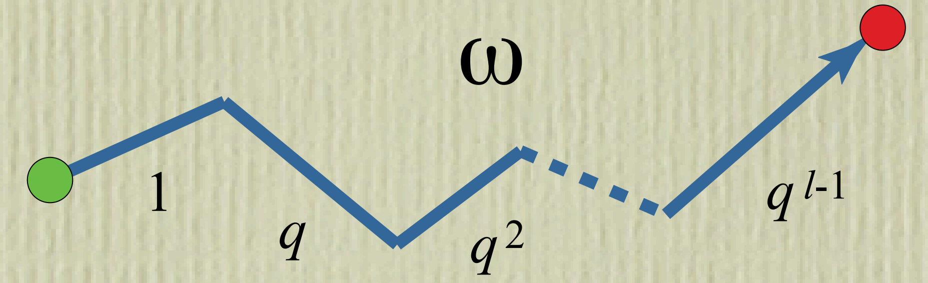
  - $x = 0,\ \ e=\bar{e}=1$

    $$\det(D) = (-1)^{n-1}(n-1)\,2^{n-2}$$

    Graham, Pollak (71)

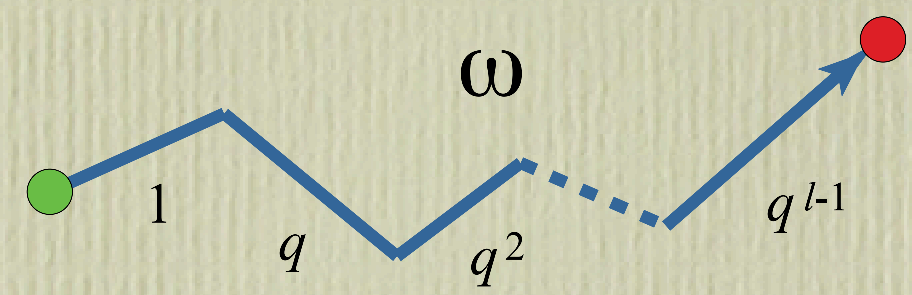
- *q*-analogue of the distance



$$\begin{aligned} \mathrm{wt}(\omega) &= 1 + q + q^2 + \cdots + q^{l-1} \\ &= \frac{q^l - 1}{q - 1} \\ &= [l] \end{aligned}$$

Matrix: D_q

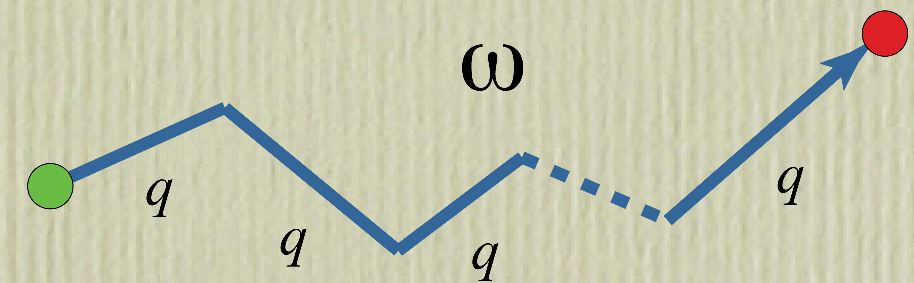
- q-analogue of the distance



$$\mathrm{wt}(\omega) = 1 + q + q^2 + \cdots + q^{l-1} = \frac{q^l - 1}{q - 1} = [l]$$

Matrix: D_q

- multiplicative weight q on each arrow

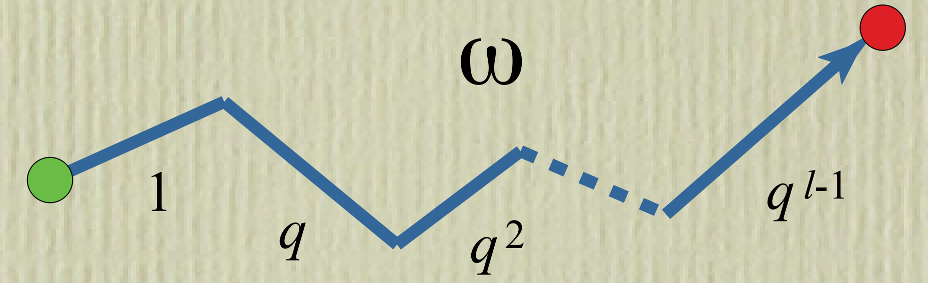


$$\mathrm{wt}(\omega) = q^l$$

Matrix: P

On the minors ... : q-analogues

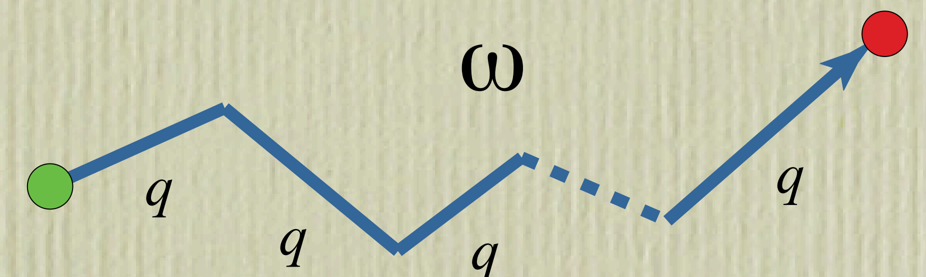
- q-analogue of the distance



$$\begin{aligned}\mathrm{wt}(\omega) &= 1 + q + q^2 + \cdots + q^{l-1} \\ &= \frac{q^l - 1}{q - 1} \\ &= [l]\end{aligned}$$

Matrix: D_q

- multiplicative weight q on each arrow

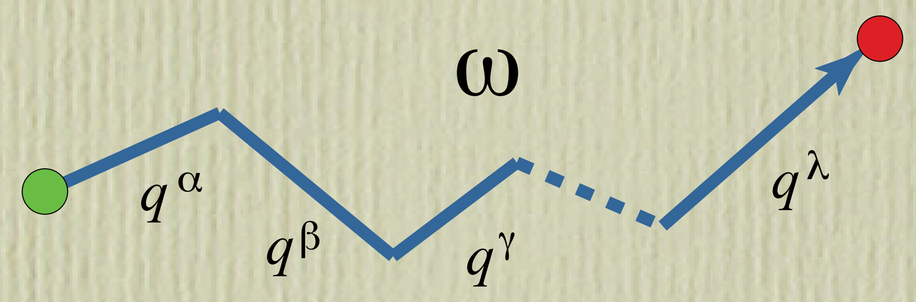


$$\mathrm{wt}(\omega) = q^l$$

Matrix: P

- $$\begin{aligned}\det\left(\frac{P - J}{q - 1}\right) &= \det(D_q) \\ &= (-1)^{n-1}(n-1)(1+q)^{n-2}\end{aligned}$$

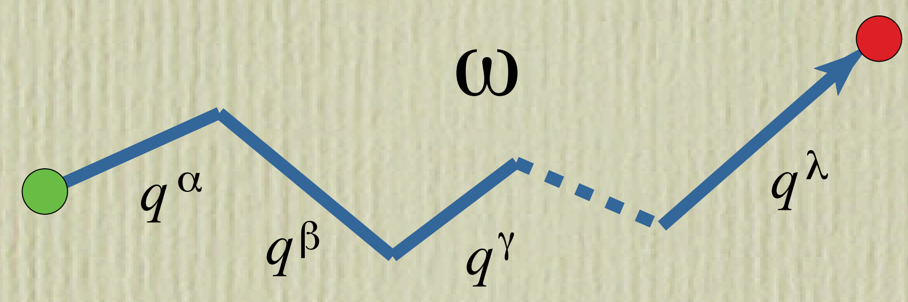
- Generalization (Yan-Yeh, 06): arrows $a, b, \ldots, \bar{a}, \bar{b}, \ldots$ have (multiplicative) weight $q^{\alpha}, q^{\beta}, \ldots, q^{\bar{\alpha}}, q^{\bar{\beta}}, \ldots$



$$\mathrm{wt}(\omega) = q^{\alpha+\beta+\cdots+\lambda}$$

Matrix: P

- Generalization (Yan-Yeh, 06): arrows $a, b, \ldots, \bar{a}, \bar{b}, \ldots$ have (multiplicative) weight $q^{\alpha}, q^{\beta}, \ldots, q^{\bar{\alpha}}, q^{\bar{\beta}}, \ldots$



$$\mathrm{wt}(\omega) \quad = \quad q^{\alpha+\beta+\cdots+\lambda}$$

Matrix: P

$$\det\left(\frac{P-J}{q-1}\right) = (-1)^{n-1} \prod_{\epsilon} [\epsilon + \bar{\epsilon}] \sum_{\epsilon} \frac{[\epsilon]\,[\bar{\epsilon}]}{[\epsilon + \bar{\epsilon}]}$$

Yan-Yeh (06)