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I.14. Pathological Solutions

Haim Brezis

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A well-known result from 1957, due to De Giorgi and Nash (see [3]), asserts that if $u \in H^1(\Omega)$ satisfies

$$\sum_{ij} \int_\Omega a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j} = 0 \quad \forall \varphi \in C_c^\infty(\Omega), \tag{1}$$

where the coefficients $a_{ij} \in L^\infty(\Omega)$ are uniformly elliptic, then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha > 0$, with a corresponding estimate. In particular

$$\|u\|_{L^\infty(\omega)} \le C\|u\|_{H^1(\Omega)}, \quad \forall \omega \quad \text{with } \overline{\omega} \subset \Omega. \tag{2}$$

Since (1) makes sense for functions $u \in W^{1,1}(\Omega)$, one may wonder whether the De Giorgi–Nash estimate (or the weaker form (2)) is still valid when H^1 is replaced by $W^{1,1}$. In his 1964 paper (paper [i] below) J. Serrin produced a striking example showing that such a stronger version of the De Giorgi–Nash estimate fails: Given any $p < 2$, J. Serrin constructed a function $u \in W^{1,p}(\Omega)$ in a ball $\Omega \subset \mathbb{R}^N, N \ge 2$, satisfying (1) for some elliptic coefficients $a_{ij} \in L^\infty$, and such that $u \notin L^\infty_{\mathrm{loc}}(\Omega)$. In the same paper he proposed the following:

Conjecture (J. Serrin, 1964). *Assume that the coefficients a_{ij} are Hölder and uniformly elliptic. Assume that $u \in W^{1,1}(\Omega)$ satisfies (1). Then u is "classical", i.e., $u \in H^1_{\mathrm{loc}}(\Omega)$* (and therefore u is Hölder by De Giorgi–Nash).

A partial answer to Serrin's conjecture was given in 1971 by two former PhD students of J. Serrin, R. Hager and J. Ross:

Theorem 1 **([4]).** *Assume that the coefficients a_{ij} are Hölder and uniformly elliptic. Assume that u belongs to $W^{1,p}(\Omega)$ for some $p > 1$, and satisfies (1). Then $u \in H^1_{\mathrm{loc}}(\Omega)$.*

H. Brezis gave a complete solution of Serrin's conjecture and also weakened the first assumption in the theorem of Hager–Ross. The results were announced in [1] and the detailed proofs are presented in [2] as an Appendix in a paper by A. Ancona (who, in turn, used this result while answering a question raised by H. Brezis and A. Ponce).

Theorem 2 **([1], [2]).** *Assume that the coefficients a_{ij} are continuous in $\overline{\Omega}$ and uniformly elliptic. Assume that u belongs to $W^{1,p}(\Omega)$ for some $p > 1$, and satisfies (1). Then u belongs to $W^{1,q}_{\mathrm{loc}}(\Omega)$ for every $q < \infty$. Moreover*

$$\|u\|_{W^{1,q}(\omega)} \le C\|u\|_{W^{1,p}(\Omega)} \quad \forall \omega \quad \textit{with } \overline{\omega} \subset \Omega, \tag{3}$$

where C depends only on ω, Ω, the ellipticity constant, p, q, and the modulus of continuity of the a_{ij}'s.

Theorem 3 **(**[1], [2]**).** *Assume that the coefficients a_{ij} are Dini-continuous, i.e.,*

$$|a_{ij}(x) - a_{ij}(y)| \leq \gamma(|x-y|) \quad \forall x, y \in \Omega, \qquad \text{with } \int_0^1 \frac{\gamma(t)}{t} dt < \infty$$

(e.g., a_{ij} are Hölder). Assume that u belongs to $W^{1,1}(\Omega)$ and satisfies (1). Then $u \in C^1(\Omega)$ and moreover

$$\|Du\|_{L^\infty(\omega)} \leq C\|u\|_{W^{1,1}(\Omega)} \quad \forall \omega \quad \text{with } \overline{\omega} \subset \Omega, \tag{4}$$

where C depends only on ω, Ω, and on the Dini-modulus of continuity of the a_{ij}'s.

Here are several comments showing that the results of Theorems 2 and 3 are sharp and more subtle than they seem! Under the assumptions of Theorem 3 the following estimate holds, for every $p < \infty$,

$$\|Du\|_{L^p(\omega)} \leq C\|u\|_{W^{1,1}(\Omega)}, \quad \forall \omega \quad \text{with } \overline{\omega} \subset \Omega \tag{5}$$

where C depends only on ω, Ω, p and the *modulus of continuity* of the a_{ij}'s (but *not* on the Dini-modulus of continuity!). Estimate (5) suggested that the answer to Serrin's conjecture might still be positive when the a_{ij}'s are merely continuous. H. Brezis raised that question in [2]. It turns out that the answer is negative! T. Jin, V. Maz'ya and J. Van Schaftingen [5] have constructed an ingenious example of a function $u \in W^{1,1}(\Omega)$ satisfying (1) for some elliptic coefficients $a_{ij} \in C^0$, such that Du does *not* belong even to the class $L \log L$! They also answered another question raised by H. Brezis: they found a function $\tilde{u}$ which belongs to $W^{1,p}(\Omega) \quad \forall p < \infty$, which satisfies (1) for some elliptic coefficients $a_{ij} \in C^0$, and such that $D\tilde{u} \notin \mathrm{BMO}$. Putting all those facts together one has now a clear picture of the "smoothing effects" for elliptic equations in divergence form with continous coefficients.

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PATHOLOGICAL SOLUTIONS OF ELLIPTIC DIFFERENTIAL EQUATIONS

by JAMES SERRIN

In this note we consider the simplest kind of divergence structure elliptic equation, namely

$$\Sigma \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) = 0 \tag{1}$$

where the coefficients a_{ij} are bounded measurable functions of $x = (x_1, \dots, x_n)$ satisfying the ellipticity condition

$$\lambda\xi^2 \leq a_{ij}\,\xi_i\,\xi_j \leq \Lambda\xi^2, \quad \lambda, \Lambda > 0). \tag{2}$$

Because of the general assumptions made on the coefficients it is natural that the equation be interpreted in a weak sense. In particular, let $u = u(x)$ be a function having strong derivatives $\partial u/\partial x_i$ which are locally summable over a domain D. Then u may be called a *weak solution* of (1) over D if

$$\int a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \Phi}{\partial x_i}\,dx = 0 \tag{3}$$

for any continuously differentiable function $\Phi = \Phi(x)$ with compact support in D.

In developing the theory of equations (1) or (3) it is generally assumed that the derivatives $\partial u/\partial x_i$ are locally of class L_2 in D, and this additional requirement is accordingly built into the definition of weak solutions. This being done, one can then prove that solutions are locally bounded, have a local maximum principle, and that the Dirichlet problem for

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smooth data on smooth boundaries has one and only one solution. If, however, one does not introduce this additional requirement, the definition of weak solution still remains meaningful as we have given it. In fact, the above definition is in a sense the most « natural » one, in that it requires of the function $u(x)$ just exactly enough to make relation (3) well defined.

Nevertheless, the generality allowed in this definition is too great to allow a comprehensive theory to be developed. We shall show here that there exist weak solutions in the sense defined above that are neither locally bounded nor have a local maximum principle; and that, moreover, the Dirichlet problem need not have unique solutions in the class considered. In addition, the violation of local boundedness will be shown to hold even in the class of solutions whose derivatives are locally in $L_{1+\mu}$, where μ is any fixed number less than one. Thus the usual requirement that the derivatives be in L_2 forms an important and essential part of the theory.

Consider now equation (1) with the particular coefficients

$$a_{ij} = \delta_{ij} + (a - 1)\frac{x_i x_j}{r^2}, \qquad (r = |x|),$$

where a is a constant greater than one. It is easily checked that these coefficients are bounded and that (2) holds with $\lambda = 1$, $\Lambda = a$. Moreover one finds that the function

$$u = x_1 r^{1-n-\varepsilon}$$

is a classical solution in $|x| > 0$ provided a and ε are related by

$$a = \frac{n-1}{\varepsilon(\varepsilon + n - 2)}, \qquad 0 < \varepsilon < 1.$$

We shall now show that u is in fact a weak solution throughout E^n. First of all, since $u = 0(r^{2-n-\varepsilon})$, $u_x = 0(r^{1-n-\varepsilon})$, it is easily verified that u is strongly differentiable with $u_x \in L_\beta$ for any $\beta < n/(n + \varepsilon - 1)$. Moreover, if Φ is any continuously differentiable function with compact support in E^n, then obviously

$$\int a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \Phi}{\partial x_i}\,dx = \lim_{\varrho \to 0}\int_{r>\varrho} a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \Phi}{\partial x_i}\,dx$$
$$= -\lim_{\varrho \to 0}\oint_{r=\varrho} \Phi\, a_{ij}\frac{\partial u}{\partial x_j}\frac{x_i}{r}\,ds.$$

But by direct calculation

$$- \oint_{r=\varrho} \Phi \, a_{ij} \frac{\partial u}{\partial x_j} \frac{x_i}{r} \, ds = \frac{\omega_n (n-1)}{\varepsilon} \varrho^{-1-\varepsilon} \oint_{r=\varrho} \Phi \, x_1 \, d\omega .$$

Since Φ is continuously differentiable the last integral is easily seen to be $0(\varrho^2)$ as $\varrho \longrightarrow 0$, whence combining these results it follows that

$$\int a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \Phi}{\partial x_i} \, dx = 0.$$

This proves that u is a weak solution in E^n.

Clearly the function u is neither locally bounded nor has a local maximum principle. It can be used, moreover, to show that the Dirichlet problem need not have unique solutions. In fact, let v be the unique weak solution of (1) with L_2 derivatives, taking the same values on $r = 1$ as the function u. Then $u - v$ has zero data on $r = 1$, but is not identically zero. This shows clearly that the Dirichlet problem can have two solutions corresponding to the same data, provided we give up the requirement that these solutions have L_2 derivatives.

In case $n = 2$, we have $u_x \in L_\beta$ for any $\beta < 2/(1+\varepsilon)$, whence by choosing ε sufficiently near zero it is clear that any relaxation of the requirement that derivatives be in L_2 will lead to difficulties. When $n > 2$ the function

$$u(x) = x_1 (x_1^2 + x_2^2)^{-(1+\varepsilon)/2}$$

may be thought of as a solution (by descent) of another uniformly elliptic equations whose coefficients a_{11}, a_{12}, a_{22} are those associated with the case $n = 2$ above. Since this solution is not locally bounded, the proof of our assertions is complete.

Remarks. Although the above example shows that in general it is reasonable to require solutions to have strong derivatives which are locally in L_2, nevertheless, for any given equation (1) it can be shown that regularity properties remain valid in a somewhat more extended class of functions, depending in particular on the moduli of ellipticity [1]. From this point of view, the above example shows that there are definite limits beyond which one cannot go without losing the outlines of the theory.

In conclusion, we should like to conjecture that if the coefficients a_{ij} are Hölder continuous, then any weak solution as defined here must be in fact a classical solution.

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VII.1 Surfaces of Constant Mean Curvature

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In the early thirties J. Douglas and T. Rado independently solved the *Plateau problem* concerning the existence of a minimal surface which spans a given Jordan curve Γ in $\mathbb{R}^3$. The same question for surfaces of constant mean curvature $H > 0$, in brief H-surfaces, was tackled twenty years later. A well-known result of S. Hildebrandt [4] (which improves earlier contributions by E. Heinz and H. Werner) asserts that if $\Gamma \subset B_R = B_R(0)$ and $0 < H \leq 1/R$, there exists an H-surface S contained in B_R which spans Γ.

In the paper [i] below, J. Serrin established three remarkable additions:

A) Any H-surface S which spans Γ and is contained in $B_{1/H}$ must be contained in B_R.

B) Any H-surface S without self-intersections which spans Γ must be contained in $B_{R+2/H}$.

C) Under some additional geometric conditions on Γ, there are exactly two H-surfaces without self-intersections, contained in B_R, which span Γ.

If Γ is a circle of radius R centered at 0, and $0 < H < 1/R$, there are, of course, two *small* spherical caps, $\underline{S}_1$ and $\underline{S}_2$, of curvature H which span Γ. Moreover there are two *large* spherical caps $\overline{S}_1$ and $\overline{S}_2$ of curvature H which span Γ. The surfaces $\underline{S}_1$ and $\underline{S}_2$ are precisely the two H-surfaces which appear in assertion C), while $\overline{S}_1$ and $\overline{S}_2$ are *not* contained in B_R.

Rellich's conjecture concerning the existence of a *large* H-surface for any Jordan curve $\Gamma \subset B_R$ and any $H \in (0, 1/R)$ was settled in 1982 by H. Brezis and J.M. Coron [2], (see also related works by K. Steffen and M. Struwe). A year later I visited the U. of Minnesota and had a very stimulating conversation with J. Serrin about H-surfaces during a cruise over the Mississippi river. In the course of our discussion he raised the following:

Question (J. Serrin, 1983). *Let (Γ_n) be a sequence of Jordan curves such that $\Gamma_n \subset B_{R_n}$ with $R_n \to 0$. Assume that, for each n, S_n is a large H-surface with H=1, which spans Γ_n. Does S_n converge (modulo a subsequence) to a sphere of radius 1 containing 0?*

In a joint work with J.M. Coron [3] we gave two partial answers:

Theorem 1 ([3]). *Assume that the area of the surface S_n remains bounded as $n \to \infty$. Then a subsequence of (S_n) converges to a "bouquet" of spheres of radius 1 containg 0,*

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i.e., a finite connected union of spheres of radius 1 *such that, at least, one of them contains* 0.

In general we do not have more precise information about the limiting configuration. Indeed, it would be interesting to determine whether an arbitray bouquet may be achieved as a limit of some sequence (S_n) corresponding to appropriate Γ_n's. However we do have a positive answer to Serrin's question when the S_n's are chosen in a *special way.*

Theorem 2 ([3]). *Assume that S_n is a large* 1*-surface obtained through the construction of Brezis-Coron* [2]. *Then a subsequence S_{n_k} converges to a sphere of radius* 1 *containing* 0.

A key ingredient in the proofs is the following sharp description of the blow-up mechanism for a sequence $u^n \in H_0^1(\Omega; \mathbb{R}^3)$, ($\Omega$ = unit disc in $\mathbb{R}^2$), of solutions of

$$\begin{aligned} &\Delta u^n = 2u_x^n \wedge u_y^n + f^n \text{ in } \Omega, \text{ where} \\ &\qquad \|f^n\|_{H^{-1}} \to 0 \text{ and } \|u^n\|_{H^1} \text{ remains bounded.} \end{aligned} \tag{1}$$

Theorem 3 ([3]). *There exist an integer k, functions $\omega_1, \omega_2, \ldots, \omega_k : \mathbb{R}^2 \to \mathbb{R}^3$, sequences $a_1^n, a_2^n, \ldots, a_k^n$ in Ω, sequences of positive numbers $\varepsilon_1^n, \varepsilon_2^n, \ldots, \varepsilon_k^n$ with $\lim_{n\to\infty} \varepsilon_i^n = 0\,(\forall i)$, such that, (modulo a subsequence),*

$$\lim_{n\to\infty} \left\| u^n(\cdot) - \sum_{i=1}^{k} \omega_i \left(\frac{\cdot - a_i^n}{\varepsilon_i^n} \right) \right\|_{H^1} = 0, \tag{2}$$

$$\lim_{n\to\infty} \frac{1}{\varepsilon_i^n} \operatorname{dist}(a_i^n, \partial\Omega) = \infty \quad \forall i, \tag{3}$$

and

$$\lim_{n\to\infty} \left\{ \frac{\varepsilon_i^n}{\varepsilon_j^n} + \frac{\varepsilon_j^n}{\varepsilon_i^n} + \frac{|a_i^n - a_j^n|}{\varepsilon_j^n + \varepsilon_j^n} \right\} = \infty \quad \forall i \neq j.$$

The functions ω_i are of the form $\omega(z) = \Pi(R(z))$+Const, where R is a rational function and $\Pi : \mathbb{C} \to S^2$ is a stereographic projection. Properties (2) and (3) imply that (u^n) behaves like a superposition of well-differentiated peaking waves. As pointed out in [1] the functions $\omega_i\left(\frac{\cdot - a_i^n}{\varepsilon_i^n}\right)$ are almost orthogonal (in H^1). In particular $\int_\Omega |\nabla u^n|^2$ tends to $\sum_i \int_{\mathbb{R}^2} |\nabla \omega_i|^2$ which is an integer multiple of 8π. A similar conclusion (without (3)) was obtained independently by M. Struwe [5] for an elliptic equation with critical exponent. Such a technique, with roots in the 1981 paper by J. Sacks and K. Uhlenbeck, is a refinement of the concentration-compactness principle of P.L. Lions. Subsequently, it has been widely used under the name "profile decomposition" in a large variety of problems; in particular, Yamabe-type equations (e.g., by A. Bahri, J.M. Coron, O. Druet, E. Hebey), nonlinear wave and nonlinear dispersive equations, such as nonlinear Schrödinger equations (e.g., by H. Bahouri, P. Gérard, C. Kenig, F. Merle, R. Killip, M. Visan), the Navier–Stokes equation (e.g., by I. Gallagher, G. Koch).

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On Surfaces of Constant Mean Curvature which Span a Given Space Curve

JAMES SERRIN

Let Γ be a Jordan curve in E^3, which is contained in the closed unit ball B about the origin. Recently it has been shown by Heinz [6] and Werner [18], and by Hildebrandt [7], that there exists a surface of constant mean curvature which spans the curve Γ and lies inside B, provided the mean curvature in question does not exceed 1. The surface is of course defined parametrically as in the case of Plateau's problem for minimal surfaces, and accordingly may have self-intersections, multiple coverings, and branch points.

The purpose of this note is to describe a sufficient condition for this surface to be unique, and to obtain an *a priori* limitation on the form of the region in which it is contained. As a corollary, the results include a well-known criterion due to Radó for a solution of Plateau's problem to be unique and regular.

Besides the results just noted, Theorems 1 and 1′ contain a general limitation on the diameter of the region in which a surface of constant mean curvature must lie, assuming that it spans a curve Γ contained in B. Finally, Theorem 4 gives an interesting maximum principle for non-parametric surfaces of constant mean curvature.

The methods by which we arrive at our conclusions belong to the realm of differential geometry. Nevertheless the ultimate basis of the results comes from an analysis of the partial differential equations obeyed by the surfaces in question. To illustrate, we require at one stage the construction of a non-parametric surface of constant mean curvature spanning a given space curve. This surface, once its existence is established, can be treated as a geometric structure or entity; but the basic existence theorem is a matter depending heavily on the theory of *a priori* estimates of non-linear partial differential equations and the delicate fixed point analysis of Leray and Schauder (see [16]). Again, we make use of the fact that two surfaces of constant mean curvature which touch at a point, but do not cross one another, must in fact coincide. This is a geometric statement, but its proof relies on the analytical maximum principle of E. Hopf (see footnote 4). The result just noted is a fundamental one in surface theory, and might in fact appropriately be called the "touching principle".

Another construction, used here as a geometrical tool apparently for the first time, is due to Bonnet, and states that parallel to any regular surface of constant mean curvature there is a second surface also having constant mean curvature. In point of fact, to apply Bonnet's construction requires an analysis of the behavior of the second surface in the neighborhood of points which are images of umbilical points of the original surface, an analysis which can be carried through only after a careful discussion of the local behavior of surfaces of constant mean curvature near an umbilic.

We note that a similar interplay between analytic and geometrical concepts also occurs in other work in the theory of surfaces of constant mean curvature, notably that of H. Hopf and A. D. Alexandrov.

1. A Priori Limitations

We shall retain the notations of the introduction throughout the paper. Thus Γ will denote a Jordan curve (not necessarily rectifiable) contained in the closed unit ball B about the origin. Let D denote the parameter domain $u^2+v^2<1$ and let $\bar{D}$ be its closure. A surface S will be called a solution of Plateau's problem for constant mean curvature H provided that S can be represented by a vector function $\vec{x}(u,v)\in C(\bar{D})\cap C^2(D)$ such that both

$$\Delta \vec{x}=2H\,\vec{x}_u\times\vec{x}_v$$

and

$$|\vec{x}_u|=|\vec{x}_v|, \qquad x_u\cdot x_v=0$$

in D, while the boundary of D is topologically mapped onto Γ. We may obviously assume that $H>0$ without loss generality.

Before turning to the main *a priori* limitations, we note a few simple results concerning the local behavior of solutions of Plateau's problem. A *branch point* $\vec{x}_0$ of S is a point where the surface has the local representation

$$\vec{x}=\vec{x}_0+\sum_{i+j\geqq\mu}\vec{a}_{ij}(u-u_0)^i(v-v_0)^j,$$

where $\mu\geqq 2$ is the order of the branch point. We can assume without loss of generality that $\vec{x}_0=0$ and $u_0=v_0=0$. Then exactly as in the case of Plateau's problem for minimal surfaces, it follows that

$$\begin{aligned} x+iy&=A(u+iv)^\mu+O(|u+iv|^{\mu+1})\\ z&=\qquad\qquad\quad\; O(|u+iv|^{\mu+1})\end{aligned}$$

for appropriately oriented coordinates $\vec{x}=(x,y,z)$, where A is a suitable positive constant[1]. It is an immediate consequence that the surface S has a unique tangent plane at each point not on Γ (points of S which arise from different values (u,v) are considered to be different). Moreover, it is clear that branch points of S must necessarily be isolated[2].

1 The result for minimal surfaces is due to Chen [4]; cf. also [12], p. 235. I first learned of its generalization to surfaces of constant mean curvature in a communication of Hildebrandt.

2 Hildebrandt, ibid.

Somewhat less immediate, but still a consequence of the development above, the surface S must have self-intersections in the neighborhood of any branch point [3]. Indeed, it is easily verified that any suitably small neighborhood of $(0, 0)$ in the (u, v) plane is mapped by

$$x+iy=A(u+iv)^{\mu}+O(|u+iv|)^{\mu}$$

onto a neighborhood of the origin of the Riemann surface for $(u+iv)^{\mu}$. Consequently we can find a curve γ surrounding the origin in the (u, v) plane, whose image in the (x, y) plane is the circle $x^2+y^2=\delta^2$ covered μ times. Consider the values of (x, y, z) as (u, v) traverses γ once. If S has no self-intersections, then each time the point (x, y) circles the origin once the value of z must increase (or decrease). But this is impossible because after μ circuits z returns to its initial value. Consequently S must have self-intersections in the neighborhood of a branch point, as asserted.

We may now turn to our main results. Recall here that S denotes a solution of the Plateau problem spanning a Jordan curve Γ contained in the unit ball B.

Theorem 1. *Suppose that $H \leqq 1$ and that S is contained in the closed ball of radius $1/H$ about the origin. Then S is contained in B.*

Before proving this result, we observe that the conclusion becomes false without the hypothesis that S is contained in the closed ball of radius $1/H$ about the origin, as is apparent by considering a sphere of radius $1/H$ which intersects B. Whether the radius $1/H$ in the theorem is best possible is not known, but the same example shows that the conclusion is definitely untrue if $1/H$ is replaced by any number greater than $(2/H)-1$. (We note that if the hypothesis $H \leqq 1$ is dropped the theorem remains true, but becomes trivial.)

Proof of Theorem 1. Let us suppose for contradiction that S contains points outside B. Let K denote the smallest closed ball with center at the origin which contains S, and let Σ be the spherical boundary of K. Evidently Σ has mean curvature $\geqq H$ and is tangent to S at some point $P \notin \Gamma$. We now consider two cases.

1. P is a regular point of S. Then, in a rectangular coordinate system with the z-axis oriented along the exterior normal to Σ at P, the surfaces Σ and S can be represented in the neighborhood of P by

$$z=U(x, y) \quad \text{and} \quad z=u(x, y)$$

respectively, where the functions $U(x, y)$ and $u(x, y)$ satisfy the non-parametric equations

$$(1+U_y^2)\,U_{xx}-2\,U_x\,U_y\,U_{xy}+(1+U_x^2)\,U_{yy}=2\Lambda(1+U_x^2+U_y^2)^{\frac{3}{2}}$$
$$(1+u_y^2)\,u_{xx}-2\,u_x\,u_y\,u_{xy}+(1+u_x^2)\,u_{yy}=2\lambda(1+u_x^2+u_y^2)^{\frac{3}{2}}$$

3 The corresponding result for the case of minimal surfaces is due to Chen [4]; see also Nitsche [13]. The present proof is similar to that for minimal surfaces.

By saying that S has *self-intersections* we mean here that there are at least two distinct points (u_1, v_1), (u_2, v_2) for which $\vec{x}(u_1, v_1)=\vec{x}(u_2, v_2)$.

with $\Lambda \leqq -H$ and $\lambda = H$ or $-H$. Since S is internally tangent to Σ at P, we have

$$u \leqq U \text{ near } P, \qquad u = U \text{ at } P.$$

This immediately implies $\Lambda \geqq \lambda$, whence the cases $\Lambda < -H$ and $\lambda = H$ are ruled out. It follows finally from E. Hopf's maximum principle that $u = U$ near P.[4] Clearly then S coincides with Σ and consequently cannot span Γ.

We note that the same argument applies even if two or more branches of S should intersect at P, provided P is regular for at least one of the branches.

2. P is a branch point of S. Since branch points are isolated, it is evident from the previous case that Σ cannot touch S anywhere in the neighborhood of P except for the point P itself. Consider then a small spherical cap of Σ about P. If this cap is tilted slightly its boundary will still not touch S. Then letting the cap retreat from S (in the direction normal to S at P) it is clear that the cap will have a last position where it touches S, say at P'. Evidently P' will be a regular point of S, or at worst a point where regular branches intersect. But then we obtain a contradiction exactly as in case 1, and this completes the proof. (The same proof in fact shows that S can intersect the boundary of B only at points of Γ, interior points of S being strictly contained in the interior of B.)

Theorem 1 yields an interesting qualitative addition to the result of Heinz, Werner, and Hildebrandt noted at the beginning of the paper. In particular, while they prove that there exists a solution of Plateau's problem which is contained in B, Theorem 1 shows that there can be no solution which extends outside B and yet remains in the closed ball of radius $1/H$ about the origin. We remark that the same result can be obtained by considering the differential equation

$$\Delta |\vec{x}|^2 = 2(|\vec{x}_u|^2 + |\vec{x}_v|^2) + 4H\vec{x} \cdot (\vec{x}_u \times \vec{x}_v)$$

4 Cf. [8], p. 152. To give the argument in detail, let the nonlinear operator $\mathscr{L}v$ be defined by

$$\mathscr{L}v = (1+v_y^2)\, v_{xx} - 2 v_x v_y v_{xy} + (1+v_x^2)\, v_{yy} + 2H(1+v_x^2+v_y^2)^{\frac{3}{2}}.$$

Then we have $\mathscr{L}u = \mathscr{L}U = 0$. The difference $w = U - u$ therefore satisfies a certain linear elliptic differential equation

$$a\, w_{xx} + 2b\, w_{xy} + c\, w_{yy} + d\, w_x + e\, w_y = 0,$$

and consequently cannot have an interior minimum unless it is constant. [The argument can also be handled in an alternate way to avoid use of the maximum principle. In particular, if $u = u(x, y)$ and $v = v(x, y)$ are solutions of the equation of constant mean curvature, such that $u \not\equiv v$ and

$$u = v \quad \text{and} \quad \nabla u = \nabla v = 0 \qquad \text{at } 0,$$

then there is an integer $\nu \geqq 2$ such that

$$u - v = H(x, y) + O(r^{\nu+1})$$

where $H(x, y)$ is a harmonic polynomial of degree ν (see [17], Lemma 1). In consequence, it is again evident that a one-sided tangency of two solutions is possible only if the solutions are identical.

The main advantage of the argument as given is that it applies to quite general partial differential equations and does not make use of the analyticity of solutions.]

satisfied by the function $|\vec{x}|^2$, but we have given the present proof because of its separate geometric interest and relation to later results in the paper.

When the spanning surface S has no self-intersections an overall *a priori* limitation on its spatial location can be given.

Theorem 1′. *Suppose that S is a solution of Plateau's problem for constant mean curvature $H>0$. Then if S has no self-intersections it must be contained in the open ball of radius $1+2H^{-1}$ about the origin. The number $1+2H^{-1}$ is best possible.*

Proof. Recall that S spans a curve Γ located in the unit ball B. We show first that no radius smaller than $1+2H^{-1}$ suffices for the conclusion. For let $\tilde{S}$ be a sphere of radius $1/H$ which intersects the boundary of B along a circle Γ. The portion S of $\tilde{S}$ which lies outside B and spans Γ then satisfies the hypotheses of the theorem. Clearly the distance from the outermost point of S to the origin can be made arbitrarity near $1+2H^{-1}$ by a suitable choice of $\tilde{S}$, proving that the radius $1+2H^{-1}$ cannot be improved.

Now consider a solution S of the Plateau problem which extends outside B, and has no self-intersections. It is clear from the remarks preceding Theorem 1 that S can have no branch points and hence is regular. Denote by ρ the maximum distance from the origin to points of S. We must show that $\rho<1+2H^{-1}$.

Let P be a point of S whose distance from the origin is ρ, and consider the tangent plane T_0 to S at P. We now suppose this plane to be continuously moved normal to itself to positions nearer the origin. At each stage of the motion the resulting plane T will cut off from S a (closed) cap $\Sigma(T)$. We assert that this cap has a single-valued projection onto T as long as the distance from T to the origin is not less than $(1+\rho)/2$; and even more, that no point of the cap has a tangent plane orthogonal to T.

This result is clear for all positions of T sufficiently near T_0. Moreover (by continuity) the set of positions of T where it holds is certainly open and connected. It remains to show that the set is also closed, provided the distance from T to the origin is at least $(1+\rho)/2$.

Thus let T_1 denote the limiting position of a set of planes T for which the result holds, the distance from T_1 to the origin being at least $(1+\rho)/2$. It must be shown that $\Sigma(T_1)$ has a single-valued projection onto T, and that no tangent plane of this cap is orthogonal to T_1.

For any cap $\Sigma(T)$ let $\Sigma'(T)$ be the reflection of this cap in T. We claim to begin with that $\Sigma'(T_1)$ lies in the region bounded by S and the surface of the ball B. Indeed if this were not true, it is evident that by continuously withdrawing the plane T from a position coincident with T_1 we would arrive at a position T_2 intermediate between T_1 and T_0 for which the cap $\Sigma'(T_2)$ is internally tangent to S (here we use the fact that for all planes T between T_1 and T_0 the cap $\Sigma'(T)$ lies outside B and has a single-valued projection on T). The maximum principle of Hopf then shows that the part of S on the same side of T_2 as $\Sigma'(T_2)$ must coincide with $\Sigma'(T_2)$, a patent absurdity. We have thus shown that $\Sigma'(T_1)$ is interior to S, as claimed.

Now suppose that there were a point P on $\Sigma(T_1)$ with a tangent plane V orthogonal to T_1. Clearly P must lie on T_1, by the assumption that T_1 is the limit of planes for which our result holds. Let us introduce rectangular coordinates with origin at P, with the z-axis orthogonal to V, and with the x-axis along the intersection of V and T_1. Then locally S and $\Sigma'(T_1)$ have representations $z=u(x, y)$ and $z=U(x, y)$ for $y \leqq 0$ (say), and either $u \leqq U$ or $u \geqq U$ since $\Sigma'(T_1)$ is interior to S. Clearly $\partial u/\partial y=\partial U/\partial y=0$ at $(0, 0)$ since both $\Sigma'(T_1)$ and S have V as tangent plane at P. By Hopf's boundary point lemma [9], it follows that the part of S below T_1 must coincide with $\Sigma'(T_1)$, which is certainly impossible. Thus no tangent plane of $\Sigma(T_1)$ is orthogonal to T_1.

We must still show that $\Sigma(T_1)$ has a single-valued projection onto T_1. From what has gone before, it is clear that $\Sigma(T_1)$ consists of one or more components each of which *separately* has a single valued projection. The only overlapping of these projections which could occur would be due to new components appearing at the instant T reaches coincidence with T_1. But these latter components must also have projections which are distinct from other components, in view of the fact that the reflected cap $\Sigma'(T_1)$ is interior to S. The original assertion is thereby proved; that is, for any plane T whose distance from the origin is at least $(1+\rho)/2$ the corresponding cap $\Sigma(T)$ has a single-valued projection onto T, and no point of $\Sigma(T)$ has a tangent plane orthogonal to T.

Consider the cap $\tilde{\Sigma}$ cut off from S when the distance from T to the origin is exactly $(1+\rho)/2$. By what we have just shown, $\tilde{\Sigma}$ can be considered a non-parametric solution of the equation of constant mean curvature over a portion $\tilde{T}$ of T, which takes on the value zero on the boundary of $\tilde{T}$. According to the maximum principle proved in the final section of this paper, the distance from points of $\tilde{\Sigma}$ to the set $\tilde{T}$ cannot exceed $1/H$. Consequently, by considering the point P on $\tilde{\Sigma}$ we obtain the relation

$$\rho-\frac{\rho+1}{2} \leqq \frac{1}{H},$$

that is $\rho \leqq 1+2H^{-1}$. If equality holds, then by Theorem 4 the cap $\tilde{\Sigma}$ must be a hemisphere, whence S is a portion of a sphere. But then S does not after all extend the full distance $1+2H^{-1}$ from the origin. This completes the proof of the theorem [5].

We note that the assumption that S has no self-intersections is similar in both concept and use with the corresponding assumption in Alexandrov's proof that a closed surface of constant mean curvature having no self-intersections is a sphere.

We suppose from here on that the curve Γ has a single-valued parallel projection Γ^* onto some plane L in E^3, and that each point P of Γ^* (viewed as a plane curve in L) admits a supporting circle of radius $\frac{1}{2}$ which surrounds Γ^*.

5 *Note added in proof.* A generalization of Theorem 1′ to apply to elliptic Weingarten surfaces has been obtained by A. Aeppli and the author.

(If Γ^* is of class C^2 this condition is equivalent to the requirement that the curvature of Γ^* is everywhere greater than or equal to $\frac{1}{2}$.) A surface $\bar{S}$ which spans Γ and has a single-valued projection on L will be called a non-parametric spanning surface. The following result then holds.

Lemma. *For any constant Λ satisfying $0<\Lambda\leqq 1$ there exist exactly two regular non-parametric spanning surfaces of Γ having (unsigned) constant mean curvature Λ.*

To see this, we first introduce a fixed rectangular coordinate system with the z-axis normal to L. Then any regular non-parametric spanning surface may be represented in the form $z=U(x,y)$ where $U(x,y)$ must satisfy either

$$(1+U_y^2)\,U_{xx}-2\,U_x\,U_y\,U_{xy}+(1+U_x^2)\,U_{yy}=2\Lambda(1+U_x^2+U_y^2)^{\frac{3}{2}}$$

or

$$(1+U_y^2)\,U_{xx}-2\,U_x\,U_y\,U_{xy}+(1+U_x^2)\,U_{yy}=-2\Lambda(1+U_x^2+U_y^2)^{\frac{3}{2}},$$

and where $U(x,y)$ takes on appropriately defined values on the curve Γ^*. The assertion is therefore equivalent to the existence of a unique solution of the Dirichlet problem for each equation. In view of the results of [16], both problems are solvable provided that the curve Γ is of class C^2. An additional limiting argument based on an interior estimate for the gradient of solutions then shows that the problems are solvable even for continuous Γ, cf. [17]. Uniqueness follows at once from the maximum principle for the difference of solutions.

Let the two surfaces described above be denoted by $\bar{S}_+$ and $\bar{S}_-$ respectively, and let $R(\Lambda)$ be the closed region bounded by these surfaces [note that $\bar{S}_+$ lies below $\bar{S}_-$ if the upward direction is associated with the positive z-axis].

Theorem 2. *Suppose $0<H\leqq 1$. Let Γ have the properties noted above and let S be a solution of Plateau's problem for constant mean curvature H, contained in the closed ball of radius $1/H$ about the origin. Then S is contained in $R(H)$.*

Proof. We show first that S must be contained in the right cylinder with cross section $\Gamma^*+\text{Interior}\,(\Gamma^*)$, and can intersect the boundary of this cylinder only along the curve Γ. Suppose in fact that this were not the case. Then, recalling that S is contained in B according to Theorem 1, it is evident that there would exist a right circular cylinder of radius $\frac{1}{2}$ which is tangent to S at some point $P\notin\Gamma$. (To see this, begin with the cylinder in a position tangent to B and translate it in an appropriate direction parallel to L until it first touches S.) Since the cylinder has mean curvature 1, our assertion now follows as in the concluding argument of the proof of Theorem 1.

Next suppose that S is not contained in $R(H)$. Let $\bar{S}_+$ and $\bar{S}_-$ be the lower and upper bounding surfaces of this region, as before. Then we can translate $\bar{S}_+$ downwards (or $\bar{S}_-$ upwards) so that it reaches a final position tangent to S at some point $P\notin\Gamma$. This being done, we again reach a contradiction as in the proof of Theorem 1, and the demonstration of Theorem 2 is completed.

The following result is due to Radó [14], though the proof we give is quite different and independent of the monodromy theorem which was required by Radó.

Theorem 2′. *Let S be a minimal surface spanning a Jordan curve C in E^3, where C has a single-valued convex projection C^* onto a plane L in E^3. Then S is a regular non-parametric spanning surface, and is unique.*

Proof. By replacing the right circular cylinders used in the preceding proof by planes normal to L, it is easy to see that S must be contained in the right cylinder with cross section $C^*+\text{Interior}\,(C^*)$, and can intersect the boundary of this cylinder only along C.

We now observe that there is exactly one regular non-parametric minimal surface $\bar{S}_0$ spanning Γ.[6] By the argument of Theorem 2, it follows that S lies both above and below $\bar{S}_0$, and hence coincides with $\bar{S}_0$. This completes the proof.

Remark. Theorems 1 and 2 have been stated in terms of a Jordan curve Γ contained in the unit ball. This has been done in order to facilitate comparison with the existence theorem of Heinz, Werner, and Hildebrandt, but it is quite apparent that the results would apply equally after an arbitrary change of scale, that is, with Γ contained in a ball of radius $1/\rho$ and with $H \leqq \rho$ where $0<\rho<\infty$. In this way, it becomes clear that Theorem 2′ is precisely the limiting case $H \to 0$ of Theorem 2.

2. A Uniqueness Theorem

The method of proof introduced in Section 1 allows us to obtain a uniqueness theorem for spanning surfaces of constant mean curvature, which is a partial analogue of Radó's theorem for minimal surfaces (Theorem 2′ of Section 1).

We suppose as before that Γ is contained in the unit ball B, has a single valued parallel projection Γ^* onto a plane L in E^3, and that each point of Γ^* admits a supporting circle of radius $\frac{1}{2}$ which surrounds Γ^*. Then we have the following theorem.

Theorem 3. *Suppose that $0<H\leqq 1$. Then there are exactly two solutions of the Plateau problem for constant mean curvature H which are contained in B and have no self-intersections. Moreover, each of these surfaces is regular and has a single-valued parallel projection on L.*

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6 This is of course well-known and was proved first by Radó, *making use of the result of Theorem 2′*. Since we are here proving Theorem 2′ on the basis of the *existence* of $\bar{S}_0$ it is necessary to cite an independent proof of the latter result in order to avoid circularity. Should C be smooth and C^* have positive curvature this goes back to Bernstein [2]; see also [11] and [15] for simpler proofs of the same result. For the general case the first independent proof was given by Finn, [5], p. 415. Another demonstration was given by Jenkins and Serrin, [10], pp. 323–324, and an outline proof appears in the article of Nitsche, [12], p. 201. From a more general point of view the result can be considered as the special case $\Lambda=0$ of the existence theorem which was used in the proof of the lemma preceding Theorem 2.

Proof. Let S be a solution of the Plateau problem which is contained in B and has no self-intersections. As in the proof of Theorem 1′ it follows that S can have no branch points and hence is regular. We shall show that S must coincide with one of the surfaces $\bar{S}_+$ or $\bar{S}_-$ defined previously, which will complete the proof.

Thus suppose for contradiction that $S \neq \bar{S}_+, \bar{S}_-$. Then since S is contained in $R(H)$ according to Theorem 2, it is apparent that S lies strictly between $\bar{S}_+$ and $\bar{S}_-$. This being the case, we can translate the (upper) boundary surface $\bar{S}_-$ of $R(H)$ downwards until it reaches a final position tangent to S at some point $P \notin \Gamma$. Then by the argument of Theorem 1 the signed mean curvature of S at P with respect to the upward directed normal must be $+H$ (if it were $-H$ then S would coincide with the translation of $\bar{S}_-$ and would not span Γ).

In the same way, we can translate $\bar{S}_+$ upwards until it reaches a final position tangent to S at some point $Q \notin \Gamma$. The mean curvature of S at Q would then be $-H$.

Now the surface S must be oriented, and, moreover, the upward normals to S at both P and Q must point to the same side of S. (This can be seen most easily by considering the two relatively open subsets of $R(H)$ which are separated from each other by S.) Accordingly the signed mean curvatures at P and Q must be equal, and this contradicts the first part of the proof. Consequently S must coincide with $\bar{S}_+$ or $\bar{S}_-$ and the proof is completed.

3. A Maximum Principle

In this section we prove a maximum principle for non-parametric solutions of the equation of constant mean curvature. Besides being required for the proof of the earlier Theorem 1′, this result has a basic interest in its own right.

Theorem 4. *Let Λ be a positive constant, and suppose $u = u(x, y)$ is a solution of the equation*

$$(1+u_y^2)\,u_{xx} - 2u_x\,u_y\,u_{xy} + (1+u_x^2)\,u_{yy} = 2\Lambda(1+u_x^2+u_y^2)^{\frac{3}{2}}$$

in a bounded domain Ω. If u is continuous up to the boundary of Ω, then we have

$$m - 1/\Lambda \leqq u < M \qquad \text{in } \Omega$$

where m and M are respectively the minimum and maximum boundary values of u. Equality is attained on the left only if u represents a hemisphere of radius $1/\Lambda$.

Proof. The right hand inequality is obvious, since a surface of positive mean curvature surely cannot posses an interior maximum point.

In order to prove the left hand inequality, we shall use of a theorem of Bonnet ([3], p. 119) which states that if S is a surface of constant mean curvature Λ, then the parallel surface to S at a distance $1/\Lambda$ in the direction of the preferred normal to S has mean curvature $-\Lambda$. In point of fact, this applies only at points of the parallel surface which are regular, since focal points of S must obviously be excluded. Such focal points can arise only as images of umbilics of S, for at a non-umbilical point the principal curvatures k_1 and k_2 (being unequal and summing to the value 2Λ) could not have the value Λ. More precisely, then, Bonnet's theorem asserts that the parallel surface (or set) is regular and has constant mean curvature $-\Lambda$ at all points which are not the image of an umbilical point of S.

Now let S denote specifically the surface $z=u(x,y), (x,y)\in\Omega$. If S is a hemisphere, then the left hand inequality is trivial, and we must therefore show, when S is *not* a hemisphere, that

$$u>m-\frac{1}{\Lambda} \quad \text{in } \Omega.$$

Let S^* be the parallel surface associated to S by Bonnet's theorem. The required inequality will follow at once if we can prove that S^* lies entirely above the plane $z=m$.

Assume first for contradiction that S^* contains some point below this plane. Since the image of points of S which are sufficiently near its boundary clearly lies above any plane $z=m-\varepsilon$, it follows that S^* contains some point P with minimum ordinate $m'<m$. We consider two cases.

1. P is a regular point of S^*. Then in the neighborhood of P the set S^* is a regular surface having constant mean curvature $-\Lambda$. Such a surface obviously cannot possess an interior minimum point, whence we obtain a contradiction and need consider only case

2. P is the image of an umbilical point Q of S. Introduce new rectangular coordinates (x',y',z') with origin at Q and with the z'-axis directed along the (upper) normal to S. The spherical bowl of radius $1/\Lambda$ which is tangent to S at Q is represented in this coordinate system by the relation

$$z'=v(x',y')=\frac{1}{\Lambda}\left(1-\sqrt{1-\Lambda^2 r'^2}\right), \qquad r'^2=x'^2+y'^2.$$

Similarly let S be locally represented in the new coordinates by $z'=u(x',y')$. Then since the normal curvatures of S at Q are all equal to Λ, the first differing terms in the Taylor expansions of u and v about $(0,0)$ will be of degree $\nu\geqq 3$. Using Lemma 1 of [17], we find therefore

$$u(x',y')=v(x',y')+H(x',y')+O(r'^{\nu+1})$$

where $H(x',y')$ is a harmonic polynomial of degree ν.

Now using (x',y') for local coordinates in the set S^* near P, the points (x^*,y^*,z^*) of this set are described parametrically by the relations

$$x^*=x'-\frac{u_{x'}}{\Lambda(1+u_{x'}^2+u_{y'}^2)^{\frac{1}{2}}}, \qquad y^*=y'-\frac{u_{y'}}{\Lambda(1+u_{x'}^2+u_{y'}^2)^{\frac{1}{2}}}$$

$$z^*=u(x',y')+\frac{1}{\Lambda(1+u_{x'}^2+u_{y'}^2)^{\frac{1}{2}}},$$

as one sees easily. By expansion around the function v we obtain without difficulty

$$(1+u_{x'}^2+u_{y'}^2)^{-\frac{1}{2}}=(1+v_{x'}^2+v_{y'}^2)^{-\frac{1}{2}}\{1-v_{x'}H_{x'}-v_{y'}H_{y'}+O(r'^{\nu+1})\}$$

(this calculation is made under the assumption $\nu \geqq 3$; if $\nu=2$ an extra term would appear). Hence by a straightforward evaluation we get

$$x^*-i y^*=c(x'+i y')^{\nu-1}+O(r'^{\nu})$$

$$z^*=\frac{1}{\Lambda}+\Lambda\left(1-\frac{1}{\nu}\right) \text{Real}\{c(x'+i y')^{\nu}\}+O(r'^{\nu+1}),$$

where c is a complex constant. In accordance with these formulae, it is apparent that the set S^* must have a unique tangent plane at P and must furthermore extend both above and below the horizontal plane $z=m'$ through P, which contradicts the fact that P is a point of S^* with minimum ordinate. Thus S^* cannot extend below the plane $z=m$.

Next suppose that S^* contained a point of the plane $z=m$. This point would have a minimum ordinate, and the same proof as before then leads to a contradiction.

Having shown that S^* lies entirely above $z=m$, we note that each point Q of S is associated with a point Q^* of S^* such that $|QQ^*|=1/\Lambda$. Hence S must lie everywhere above the plane $z=m-1/\Lambda$, completing the proof of the theorem.

It is remarkable that the bound given by the left hand inequality of Theorem 4 tends to minus infinity as the value of Λ tends to zero, even though in this limit the equation of constant mean curvature comes more and more nearly to resemble the minimal surface equation, for which the maximum principle holds in the form

$$m \leqq u \leqq M \qquad \text{in } \Omega.$$

This anomaly is partly explained by the fact that the domain must become increasingly large if the lower bound $m-1/\Lambda$ is to be attained. For a *fixed* domain, and for suitally small values of Λ, an alternate lower bound is available which (while depending on the size of the domain) does at least tend to the value m as Λ tends to zero. More precisely, we have the following result complementing the conclusion of Theorem 4.

Theorem 4′. *Suppose that the hypotheses of Theorem 4 are satisfied, and that in addition Ω is contained in a disk of radius $a \leqq 1/\Lambda$. Then we have*

$$m-\Lambda a^2 \leqq u < M \qquad \text{in } \Omega.$$

The equality can hold on the left hand side only if u represents a hemisphere.

Proof. With the help of Hopf's maximum principle it is evident that in Ω

$$M > u \geqq m-\frac{1}{\Lambda}\{\sqrt{1-\Lambda^2(x^2+y^2)}-\sqrt{1-\Lambda^2 a^2}\}$$

$$\geqq m-\frac{1}{\Lambda}\{1-\sqrt{1-\Lambda^2 a^2}\} \geqq m-\Lambda a^2$$

as required.

In this form of the maximum principle, it is clear that one recovers the conclusion $m \leqq u \leqq M$ in the limit as Λ tends to zero. We note finally that Theorems 4 and 4′ can be combined into a single statement, as follows.

Let the hypotheses of Theorem 4 be satisfied, and suppose that Ω is contained in a disk of radius a. Then we have

$$m - \mathrm{Min}(\Lambda^{-1}, \Lambda a^2) \leqq u < M \qquad \text{in } \Omega,$$

where the equality can hold on the left hand side only when u represents a hemisphere.

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