which impelled her to go beyond the confines of the specialist's work. In many respects her inner vision complements that of C. F. Gauss. But, where it takes arduous studies and a life time of commitment to discover C. F. Gauss's program behind the cool marble of his finished works, Emmy Noether communicated her ideas in her active years freely and convincingly to many people and through them to subsequent generations of scholars. It is safe to predict that many more (and perhaps even more inspired) tributes to her life and work are going to appear in the future.

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An introduction to variational inequalities and their applications, by David Kinderlehrer and Guido Stampacchia, Academic Press, New York, 1980, xiv + 313 pp., $35.00.

The theory of variational inequalities (= V.I.) was born in Italy in the early sixties. The "founding fathers" were G. Stampacchia and G. Fichera. Stampacchia was motivated by potential theory, while Fichera was motivated by mechanics (problems in elasticity with unilateral constraints; see III.2). Less than twenty years later the theory of V.I. has become a rich source of inspiration *both* in pure and applied mathematics. On the one hand, V.I. have stimulated new and deep results dealing with nonlinear partial differential equations. On the other hand, V.I. have been used in a *large variety* of questions in mechanics, physics, optimization and control, linear programming, engineering, etc...; today V.I. are considered as an indispensable tool in various sectors of applied mathematics.

I. What is a V.I.? V.I. appears in a natural way in the *calculus of variations* when a function is minimized over a convex set of constraints. In this case the classical Euler equation must be replaced by a set of inequalities. Let us consider first a very simple example.

I.1. *V.I. in finite-dimensional spaces.* Let F denote a C^1 real valued function on $\mathbf{R}^n$ and let $K \subset \mathbf{R}^n$ be a closed convex set. If there is some $u \in K$ such that

$$F(u) = \min_{v \in K} F(v) \tag{1}$$

then u satisfies

$$\begin{cases} u \in K, \\ (F'(u), v - u) \geq 0 \quad \text{for all } v \in K. \end{cases} \tag{2}$$

In general a solution of (2) is not a solution of (1), unless F is convex.

EXAMPLE. $F(v) = |v - a|^2$ $(a \in \mathbf{R}^n)$, then (2) reduces to the well-known characterization of the projection of a on K.

A natural generalization of Problem (2) is the following. Suppose A is a continuous map from $\mathbf{R}^n$ to $\mathbf{R}^n$; find

$$(3) \qquad u \in K \text{ such that } (Au, v-u) \geqslant 0 \quad \text{for all } v \in K.$$

Problem (3) *is called a V.I.*

REMARK. When K is a convex *cone* with vertex at 0, then (3) is equivalent to

$$(3') \qquad \begin{cases} u \in K, \\ (Au, v) \geqslant 0 \quad \text{for all } v \in K \text{ and } (Au, u) = 0. \end{cases}$$

For example if $K = \mathbf{R}^n_+$ (the positive cone in $\mathbf{R}^n$) then (3′) becomes

$$(3'') \qquad u \geqslant 0, \qquad Au \geqslant 0 \qquad (Au, u) = 0.$$

When A is an affine map, then (3″) is the well-known *complementarity problem* of linear programming (see e.g. the lectures of Cottle, Giannessi and Lemke in [**1**]).

Here is a simple (but useful) existence result.

THEOREM 1 (HARTMAN-STAMPACCHIA, 1966). *Assume K is a compact convex set in $\mathbf{R}^n$. Then* (3) *admits a solution.*

Theorem 1 can easily be reduced (and in fact is equivalent) to Brouwer's fixed point theorem.

I.2. *V.I. in infinite-dimensional spaces.* Consider now the analogue of Problem (3) in an infinite-dimensional Hilbert space H. Historically, the first existence result for V.I. was the following:

THEOREM 2 (STAMPACCHIA, 1964). *Assume A has the form $Av = Bv - f$ where $f \in H$ is fixed and B is a bounded linear operator in H satisfying the "ellipticity" condition*

$$(4) \qquad (Bv, v) \geqslant \alpha |v|^2 \quad \textit{for all } v \in H, \alpha > 0.$$

Then there exists a unique solution u of (3). *Moreover if B is selfadjoint then u achieves* $\operatorname{Min}_{v \in K}\{\frac{1}{2}(Bv, v) - (f, v)\}$.

Very soon it was realized by Lions, Stampacchia, F. Browder and others that the assumptions of Theorem 2 (especially, the linearity and ellipticity of A) could be considerably weakened so as to include more general differential operators (for example parabolic operators).

II. Two model problems. We shall now examine the interpretation of (3) in two simple cases.

II.1. *The obstacle problem.* Let $\Omega \subset \mathbf{R}^N$ be a smooth bounded domain and let $H_0^1(\Omega)$ denote the usual Sobolev space with its scalar product $(u, v) = \int_\Omega \nabla u \nabla v \, dx$. Let $K = \{v \in H_0^1(\Omega);\ v \geqslant \psi \text{ a.e. on } \Omega\}$ where $\psi(x)$ is a given function. Theorem 2 (applied with $f = 0$, $B = \mathrm{Id}$) shows that there is a unique function u such that

$$(5) \qquad u \in K \quad \text{and} \quad \int_\Omega \nabla u(\nabla v - \nabla u)\,dx \geqslant 0 \quad \text{for all } v \in K.$$

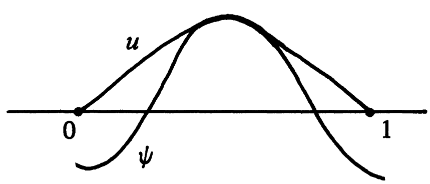
Moreover u achieves $\text{Min}_{v\in K} \int_\Omega |\nabla v|^2 \, dx$. *Formally*—that is, assuming u regular—(5) is equivalent to

$$(5') \qquad u \geqslant \psi, \; -\Delta u \geqslant 0, \; \Delta u(u-\psi) = 0 \quad \text{on } \Omega; \qquad u = 0 \quad \text{on } \partial\Omega.$$

We may divide Ω into 2 regions: $I = \{x \in \Omega;\ u(x) = \psi(x)\}$ = the coincidence set, and $\Omega - I = \{x \in \Omega;\ u(x) > \psi(x)\}$. The common boundary (in Ω), $\partial I = \partial(\Omega - I)$ is called the *free boundary* or *interface*. We have

$$(6) \qquad \begin{cases} \Delta u = 0 & \text{on } \Omega - I, \\ u = \psi \text{ and } \nabla u = \nabla\psi & \text{on } \partial(\Omega - I). \end{cases}$$

Physically, u represents the equilibrium position of a membrane constrained to lie *above the obstacle* ψ. For example when $\Omega = (0, 1)$ the solution u is materialized by stretching an elastic rubber over ψ:



Once the (weak) solution—$u \in H^1$—of (5) has been obtained, many natural questions may be asked:

(a) How *regular* is the function u? The answer is $u \in C^{1,1}(\overline{\Omega})$ provided ψ is smooth enough. (This problem has been investigated by H. Lewy, G. Stampacchia, Frehse, Kinderlehrer, the reviewer, Jensen.)

(b) What are the *geometrical* properties of the regions I and $\Omega - I$? Is the free boundary a regular surface? If not, analyze its singularities. These difficult questions have been considered by H. Lewy, Stampacchia, Kinderlehrer, Nirenberg, Spruck, Caffarelli, Riviere, D. Schaeffer, Mallet-Paret, Friedman, and others.

(c) Is the free boundary stable under perturbations of the data? Estimate the location of the free boundary. Find qualitative properties of the free boundary (monotonicity, etc...); see the works of Baiocchi, Alt, Dervieux and others.

II.2. *V.I. associated with evolution operators*. For simplicity let us examine now the case of a V.I. associated with an ordinary differential operator. Let N: $\mathbf{R}^n \to \mathbf{R}^n$ be a Lipschitz mapping and let $K \subset \mathbf{R}^n$ be a closed convex set. Given $u_0 \in K$, the problem

$$(7) \qquad \begin{cases} \dfrac{du}{dt}(t) = Nu(t), & t \geqslant 0, \\ u(0) = u_0 \end{cases}$$

has a unique solution; but in general the trajectory leaves K. If we insist that $u(t)$ remains in K[1] it is of course necessary to modify (7). Instead of (7) we consider now the V.I. (3) where $Au = du/dt - N(u)$; more precisely we look for a function $u(t)$ such that

$$\text{(8)} \quad \begin{cases} u(t) \in K \quad \text{for all } t \geq 0,\ u(0) = u_0, \text{ and} \\ \left(\dfrac{du}{dt}(t) - Nu(t), v - u(t) \right) \geq 0 \quad \text{for all } t \geq 0 \text{ and all } v \in K. \end{cases}$$

Problem (8) admits a unique solution; moreover the interpretation is the following:

$$\begin{cases} \dfrac{du}{dt}(t) = Nu(t) & \text{if } u(t) \in \operatorname{Int} K, \\ \dfrac{du}{dt}(t) = \operatorname{Proj}_{\Pi u(t)} Nu(t) & \text{if } u(t) \in \partial K, \end{cases}$$

where $\Pi u(t) = \overline{U_{\lambda \geq 0} \lambda(K - u(t))}$ denotes the "tangent cone" to K at $u(t)$. In other words we follow (7) until the trajectory hits ∂K; when $u(t)$ reaches ∂K we modify the vector field N in the "minimal" way so as to keep the trajectory within K. The same "philosophy" applies to partial differential operators such as the heat operator $\partial/\partial t - \Delta$. (This problem was introduced by Lions and Stampacchia.) After proving the existence and uniqueness of a weak solution we are led to the same questions as in II.1 (regularity of the solution, regularity of the free boundary, etc...).

III. Applications and variants. We indicate briefly some of the main applications of V.I.

III.1. *Free boundary value problems.* As a "by-product" of having solved (5) we have determined a domain $G(= \Omega - I)$ and a function u satisfying

$$\text{(9)} \quad \begin{cases} \Delta u = 0 \quad \text{on } G, \\ u = \psi \text{ and } \dfrac{\partial u}{\partial n} = \dfrac{\partial \psi}{\partial n} \quad \text{on } \partial G. \end{cases}$$

This is a typical example of a *free boundary value problem*: the domain G is *not prescribed* but instead, the boundary conditions are "*over determined*". Many free boundary value problems arise in physics and mechanics. *Some of them* may be solved with the help of V.I.,[2] either *directly* or after an *appropriate transformation* (for example, Baiocchi's transform). V.I. have been especially useful on the following questions:

(a) Problems in *hydraulics* (filtration through a porous medium). The reduction to a V.I. has been discovered by Baiocchi in 1971 and since then extensively studied in the Pavia group (Magenes, Maione, Comincioli, Pozzi,

[1] For example, if $u(t)$ represents a system of prices and we want to avoid negative prices!

[2] Three conferences have recently been devoted to free boundary problems (see [**2**, **3**, **4**]). It is clear that V.I. have produced a tremendous impact in this field.

and Torelli); see also the works by Stampacchia, Benci, Alt, Caffarelli, Friedman, Jensen, Cryer, and others.

(b) The one phase *Stefan problem* occurring in the melting of ice (also in alloy solidification, frost propagation, etc...). The reduction to a V.I. has been discovered by Duvaut and Fremond in 1973; see also further investigations by Friedman, Kinderlehrer, Nirenberg, Rodrigues and others.

(c) *Subsonic flows* past given profiles. The reduction to a V.I. has been done by Stampacchia and the reviewer in 1973; see also subsequent works by Duvaut, Bourgat, Ciavaldini, Tournemine, Hummel, and others.

III.2. *Further applications*. Some other topics where V.I. play a role include:

(a) *Stochastic control*. The connection between V.I. and the question of *optimal stopping time* has been first pointed out by Bensoussan and Lions in 1973 see [**9** and **10**]; see also the works of Friedman, van Moerbeke, and Bismut.

(b) *Elasticity with unilateral constraints*. The *Signorini problem* in mechanics (that is, the equilibrium configuration of an elastic body under assigned forces in contact with a rigid support) leads to an elliptic V.I. associated with the convex set of the type

$$K = \{v \in H^1(\Omega)^n, v.\nu \geq 0 \text{ on } \partial\Omega\}$$

where ν denotes the normal direction. This was first observed by Fichera in 1963; subsequent works were done by Duvaut, Lions, Caffarelli, Kinderlehrer.

(c) *Elasto-plastic torsion*. The stress function of a twisted elasto-plastic cylindrical bar satisfies a V.I. associated with the convex set

$$K = \{v \in H_0^1(\Omega); |\nabla v| \leq 1 \text{ a.e. on } \Omega\}.$$

This has been noticed in 1967 by Lanchon and Duvaut; see related works by Ting, Stampacchia and the reviewer, Caffarelli, Riviere, Friedman, Gerhardt, and others.

III.3. *Generalizations*.

(a) Instead of a closed convex *set* we may consider a convex lower semicontinuous *function* j: $H \to (-\infty, +\infty]$ and ask the question: Find $u \in H$ such that

$$(Au, v - u) + j(v) - j(u) \geq 0 \quad \text{for all } v \in H.$$

Such a problem occurs for example in the theory of nonnewtonian fluids (Bingham fluids for example); see [**8**].

(b) *Quasi V.I.* (= *QVI*). Assume now that the convex set K depends on u; that is, for each $u \in H$, $K(u)$ is a given closed convex set and we ask the question: Find u such that $u \in K(u)$ and $(Au, v - u) \geq 0$ for all $v \in K(u)$. This problem was introduced by Bensoussan and Lions in 1973 in connection with *impulse control*. Subsequent works concerning the regularity have been done by Hanouzet, Joly, Mosco, Troianiello, Friedman, and Caffarelli.

IV. About the book of Kindelehrer and Stampacchia. The book starts with the theory of V.I. at its most basic level. The beginning may be read by someone with a limited background in functional analysis or partial differential equations (there is a detailed presentation of Sobolev spaces and even fixed point theorems are recalled). Slowly the book brings us to the frontiers of current research concerning the regularity of the free boundary. Finally the authors present a large variety of applications. I have noted some omissions:

(a) QVI are hardly mentioned. I would refer the reader, for example, to the presentations in [**5** or **6**] (and their bibliographies).

(b) There is no reference to the numerical aspects. This is unfortunate since one of the attractions of the theory of V.I. is that it provides highly efficient methods to solve for example free boundary problems. I would refer the reader to [**7**] and [**11**]. Besides these minor points, I strongly recommend the book to someone who wishes to teach a course on V.I. (Earlier books on V.I. were intended for specialists; for example, in [**8**] or in [**9**] the emphasis was put essentially on mechanics or control). The book is also very attractive for experts from other fields who wish to be introduced to the world of V.I.

Our pleasure to see this book published is only tarnished by the untimely death of Guido Stampacchia. We are grateful to David Kinderlehrer for having completed their collaboration.

REFERENCES

1. R. W. Cottle, F. Giannessi, and J. L. Lions (eds.), *Variational inequalities*, Wiley, New York, 1980.

2. D. G. Wilson, A. D. Solomon, and P. T. Boggs (eds.), *Moving boundary problems*, Academic Press, New York, 1978.

3. *Free boundary problems*, Proc. Sem. (Pavia), Ist. Naz. Alta Mat. Roma, 1980.

4. *Free boundary problems*, Proc. Sympos. (Montecatini) (A. Fasano and M. Primicerio (eds.)), Pitman, London, 1982.

5. J. L. Lions, *Quelques questions d'analyse de mécanique et de contrôle optimal*, Presses de l'Univ. de Montréal, 1976.

6. C. Baiocchi and A. Capelo, *Disequazioni variazionali e quasivariazionali. Appliazioni a problemi di frontiera libera*, Pitagora Editrice, Bologna, 1978; English transl. (to appear).

7. R. Glowinski, J. L. Lions and R. Trémolières, *Numerical analysis of variational inequalities*, North-Holland, Amsterdam, 1981.

8. G. Duvaut and J. L. Lions, *Les inéquations en mécanique et en physique*, Dunod, Paris, 1972; English transl., Springer-Verlag, Berlin and New York, 1976.

9. A. Bensoussan and J. L. Lions, *Applications des inéquations variationnelles en contrôle stochastique*, Dunod, Paris, 1978. English Transl. North-Holland, Amsterdam, 1982.

10. A. Bensoussan, *Stochastic control by functional analysis methods*, North-Holland, Amsterdam, 1982.

11. R. Glowinski, *Numerical methods for non-linear variational problems*, Tata Institute Lecture Notes, Springer-Verlag, Berlin and New York, 1980.

12. C. M. Elliott and J. R. Ockendon, *Weak and variational methods for moving boundary problems*, Pitman, London, 1982.

13. A. Friedman, *Variational principles and free boundary problems*, Wiley, New York (to appear).

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