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On a Characterization of Flow-Invariant Sets*

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Let E be a finite-dimensional Euclidean space. Let Ω be an open set in E and let $F \subset \Omega$ be relatively closed in Ω. For $x \in E$ we use the notation $d(x, F) = \min_{y \in F} |x - y|$. Let A be a locally Lipschitz mapping of Ω into E.

If $x \in \Omega$, we denote by $S(t)x$ the solution of the differential equation

$$\dot{x}(t) = A\,x(t)\,, \qquad t \in [0, T_x)\,, \quad x(0) = x\,.$$

THEOREM 1. *The following conditions are equivalent:*

$$\lim_{h \downarrow 0} \frac{d(x + hA\,x, F)}{h} = 0 \qquad \text{for every} \qquad x \in F\,, \tag{1}$$

$$S(t)x \in F \qquad \text{for every} \qquad x \in F \qquad \text{and} \qquad t \in [0, T_x]\,. \tag{2}$$

Proof: It is clear that (2) implies (1), since

$$d(x + hA\,x, F) \leqq \|S(h)x - x - hA\,x\|$$

and $\|(S(h)x - x)/h - A\,x\|$ tends to zero as $h \to 0$.

Conversely, it is sufficient to prove that $S(t)x \in F$ for $x \in F$ and small t. We may assume that $\|A y_1 - A y_2\| \leqq L\|y_1 - y_2\|$ for all y_1, y_2 in an open ball $B(x, r)$. Consequently,

$$\|S(t)y_1 - S(t)y_2\| \leqq e^{Lt}\,\|y_1 - y_2\|$$

if $y_1, y_2 \in B(x, r)$ and t is small enough.

We choose $T < T_x$ so that $\|S(t)x - x\| < \frac{1}{2}r$ for $0 \leqq t < T$. We set $\phi(t) = d(S(t)x, F)$. Let $y_t \in F$ be such that $\|y_t - S(t)x\| = \phi(t)$. Therefore, $\|y_t - r\| < r$ for $0 \leqq t < T$. In the following, x and $0 \leqq t < T$ are fixed.

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For $h > 0$ sufficiently small, $S(h)y_t \in B(x, r)$ and therefore

$$\begin{aligned}\phi(t + h) &\leqq \|S(t + h)x - S(h)y_t\| + \|S(h)y_t - y_t - hAy_t\| + d(y_t + hAy_t, F)\\ &\leqq e^{Lh}\|y_t - S(t)x\| + \|S(h)y_t - y_t - hAy_t\| + d(y_t + hAy_t, F).\end{aligned}$$

Thus,

$$\frac{\phi(t + h) - \phi(t)}{h} \leqq \left(\frac{e^{Lh} - 1}{h}\right)\phi(t) + \left\|\frac{S(h)y_t - y_t}{h} - Ay_t\right\| + \frac{d(y_t + hAy_t, F)}{h}.$$

We deduce from (1) that

$$\limsup_{h \downarrow 0} \frac{\phi(t + h) - \phi(t)}{h} \leqq L\phi(t) \qquad \text{for every} \qquad 0 \leqq t < T. \tag{3}$$

It follows from (3), by a standard argument, that $e^{-Lt}\phi(t)$ is nonincreasing on $[0, T)$. Since $\phi(0) = 0$, we conclude that $\phi(t) = 0$ for all $t \in [0, T)$.

Remarks. 1. Theorem 1 is well known in the case when F is a closed set bounded by a "smooth surface".

2. Theorem 1 can be easily extended to general Banach spaces if we replace (1) by

$$\begin{aligned}&\text{for every } x \in F, \text{ there is a neighborhood } U \text{ of } x \text{ such that}\\ &\lim_{h \downarrow 0} d(y + hAy, F)/h = 0 \text{ uniformly in } y \in U.\end{aligned} \tag{1'}$$

The following application is useful in [1] and was suggested to me by L. Nirenberg.

Let Ω be an open set in $\mathbb{R}^N \times \mathbb{R}$; let $b(y, t) : \Omega \to \mathbb{R}$ be a C^2 function and let $Y(y, t) : \Omega \to \mathbb{R}^N$ be locally Lipschitz. We assume that the following conditions hold:

$$\begin{gathered}\text{If } b(y, s) < 0 \text{ for some } (y, s) \in \Omega, \text{ then } b(y, t) \leqq 0\\ \text{for } (y, t) \in \Omega \text{ with } t \geqq s.\end{gathered} \tag{4}$$

$$Y(y, t) \cdot b_y(y, t) \leqq 0 \text{ at any point } (y, t) \in \Omega, \text{ where } b(y, t) = 0. \tag{5}$$

$$\begin{gathered}Y(y, t) = 0 \text{ at any point } (y, t) \in \Omega, \text{ where } b(y, t) = 0,\\ b_y(y, t) = 0 \text{ and } b_t(y, t) = 0.\end{gathered} \tag{6}$$

THEOREM 2. *Let* $(y_0, t_0) \in \Omega$ *and let* $y(t)$ *be the solution of*

$$\dot{y}(t) = Y(y(t)), \qquad t \in [t_0, t_1), \quad y(t_0) = y_0.$$

If $b(y_0, t_0) < 0$, *then* $b(y(t), t) \leqq 0$ *for* $t \in [t_0, t_1)$.

Proof: We define F_0 by $(x, s) \in F_0$ if and only if $(x, s) \in \Omega$ and there is some $(x, t) \in \Omega$ with $t \leqq s$ such that $b(x, t) < 0$. Let F be the closure of F_0 relative to Ω. Clearly, F has the property that if $(y_0, t_0) \in F$, then $(y_0, t) \in F$ for $t_0 \leqq t < t_0 + h$, h small enough. By (4) we have $b(y, t) \leqq 0$ for all $(y, t) \in F$. We are going to prove that (1) holds with $A(y, t) = (T(t, y), 1)$. Let $(y_0, t_0) \in F$; we consider three cases.

a. $b(y_0, t_0) < 0$, so that (y_0, t_0) lies in the interior of F and (1) clearly holds at (y_0, t_0).

b. $b(y_0, t_0) = 0$, $b_y(y_0, t_0) = 0$, $b_t(y_0, t_0) = 0$. By (6), $Y(y_0, t_0) = 0$ so that $A(y_0, t_0) = (0, 1)$ and $(y_0, t_0) + hA(y_0, t_0) = (y_0, t_0 + h)$ is in F for h small enough. Hence (1) holds at (y_0, t_0).

c. $b(y_0, t_0) = 0$, $|\nabla b(y_0, t_0)| \neq 0$, where $\nabla b(y_0, t_0) = (b_y(y_0, t_0), b_t(y_0, t_0))$. Thus there is a constant $c \geqq 0$ such that $b(y, t) < 0$ on the set

$$\{(y, t) \in \Omega : (y - y_0) \cdot b_y(y_0, t_0) + (t - t_0) b_t(y_0, t_0) + c(|t - t_0|^2 + |y - y_0|^2) < 0\}.$$

Therefore,

$$d((y_0, t_0) + hA(y_0, t_0), F) \leqq \max\left\{0, \frac{hY(y_0, t_0) \cdot b_y(y_0, t_0) + hb_t(y_0, t_0) + ch^2(1 + |Y(y_0, t_0)|^2)}{|\nabla b(y_0, t_0)|}\right\}.$$

Since $b(y_0, t_0) = 0$ and $Y(y_0, t_0)$, $b_y(y_0, t_0) \leqq 0$ by (5), condition (4) implies that $b_t(y_0, t_0) \leqq 0$, Consequently,

$$\frac{d((y_0, t_0) + hA(y_0, t_0), F)}{h} \leqq \frac{ch(1 + |Y(y_0, t_0)|^2)}{|\nabla b(y_0, t_0)|}$$

and (1) holds at (y_0, t_0).

Bibliography

[1] Nirenberg, L. and Treves, J. F., *Local solvability of linear partial differential equations II; sufficient conditions*, Comm. Pure Appl. Math., Vol. 23, 1970, to appear.

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