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A Note on Limiting Cases of Sobolev Embeddings and Convolution Inequalities

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Section 1 - Introduction:

A classical theorem of Sobolev, see e.g. [N] or [S], asserts that if f and all k th order derivatives of f are in $L^p(R^d)$ and

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{d}$$

then f is in L^q provided $kp < d$ and so $q < \infty$. Also if $kp > d$, f is in L^∞. In case $kp = d$ and $p > 1$, it is well known that f need not be in L^∞.

Recently two variants of the limiting case $kp = d$ have been discovered, and it has turned out that these results have had various applications. See e.g. [BG], [H], [M], [ST], [T], and references cited there.

These theorems are expressed in terms of the spaces W^k_p. If k is an integer, W^k_p is the space of functions f in L^p such that all derivatives of f up to order k are in L^p, and the norm of a function is expressed by the formula

$$||f||_{k,p} = \{ \int (|f|^p + \sum_{|\alpha| \leq k} |D^\alpha f|^p) dx\}^{1/p} .$$

Trudinger [T] obtained the following theorem:

Theorem A: If $f \in W_p^k$ and $kp = d$, then $\exp(c_{d,k}|u|^{d/(d-1)})$ is locally integrable for some small constant $c_{d,k}$, and

$$\sup_{||u||_{k,p}\leq 1} \int_{|x|\leq 1} \exp\{c_{d,k}|u|^{d/(d-1)}\}dx < \infty .$$

Trudinger claims that the power $d/(d-1)$ is the best possible power. If $k = 1$, this is in fact correct. However, if k is strictly greater than one, Strichartz [ST] has pointed out that the power $d/(d-1)$ may be replaced by the larger power $p/(p-1)$. See also Hedberg [H]. It has also been pointed out to us by E. M. Stein that the correct power is obtained in [Z] in the framework of one dimensional fractional integration.

A second type of limiting case of Sobolev's inequalities was discovered by Brezis and Gallouet [B.G.]. They found Theorem B.

Theorem B: Suppose $f \in W_2^2$, $||f||_{1,2} \leq 1$ and $d = 2$. Then

$$||f||_\infty \leq C\{1 + \log^{1/2}(1 + ||f||_{2,2})\}$$

for some absolute constant C.

Theorems A and B raise two questions. Theorem B raises the question as to what is the general form of the theorem. Our first theorem will assert that in any number, d, of dimensions

$$||f||_\infty \leq C(1 + \log^{1/p'}(1+ \|f\|_{\ell,q}))$$

provided f is in W_q^ℓ for some ℓ and q with $\ell q > d$, $||f||_{k,p} \leq 1$ with $kp = d$, and $1 < p < \infty$. Theorem 1 will actually hold even if k and ℓ are non-integral.

Theorem A raises the question as to why Trudinger did not obtain the optimal power that Strichartz later obtained. Now Trudinger reduced the case

$k > 1$ to the case $k = 1$ by using a result of Sobolev. Namely if f is in W_p^k $k \geq 2$, $pk = d$, then f is in W_d^1. (Strichartz on the other hand uses a direct argument.) However, if f is in W_p^k, then f is actually in a space better than W_d^1, namely, the first derivatives are in the Lorentz space $L(d,p)$, at least if $p > 1$, see e.g. [SW]. (We shall give the precise definition later.) Thus one might ask whether functions, f, having first derivatives in $L(d,p)$ satisfy the condition that $\exp\{c|f|^{p/(p-1)}\}$ is locally integrable. Essentially, we are asking whether $\exp c|f * k|^{p/(p-1)}$ is integrable if f is in $L(d,p)$ where k is in $L(d/(d-1),\infty)$, and k is suitably small at infinity.

Thus we might ask more generally whether $f \in L(q_1,p_1)$ and $g \in L(q_2,p_2)$ with $\frac{1}{q_1} + \frac{1}{q_2} = 1$ and g suitably restricted at infinity imply $\exp\{c|f * g|^{r'}\}$ with $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$. This will be the content of Theorem 3. Theorem 3 itself is a limiting case of an inequality of O'Neil and Stein [O]. Their result concerns the case $\frac{1}{q_1} + \frac{1}{q_2} > 1$.

Section 2: General Form of Theorem B

In Theorem 1 below we consider also derivatives of fractional order, that is spaces W_p^k, where k is any positive number. We shall prove the following theorem:

Theorem 1: Let $f \in W_q^{\ell}(R^d)$ with $\ell q > d$, $1 \leq q \leq \infty$, and let $kp = d$, $1 < p < \infty$. If $||f||_{k,p} \leq 1$, then

$$||f||_{\infty} \leq C\,\{1 + \log^{1/p'}(1 + ||f||_{\ell,q})\}\,.$$

Remark 1: The case $p = 1$, $k = d$ is known. This is in a fact a part of [G]. We thank Bob Turner for pointing out to us that, trivially if $\frac{\partial^d f}{\partial x_d \ldots \partial x_1}$ is integrable, f is bounded.

Remark 2: Let $\psi(r)$ be in $C_0^{\infty}[0,\infty)$ with the further condition that $\psi(r)$ be 1 near $r = 0$.

Set

$$u_\delta(x) = \frac{\log\left(\frac{1}{|x|^2 + \delta^2}\right)}{(\log \frac{1}{\delta})^{1/p}} \psi(|x|).$$

Then by considering the functions $u_\delta(x)$ one can show the power $1/p'$ in Theorem 1 is sharp and in fact no estimate of the form.

$$||u||_\infty \leq \varepsilon \log^{1/p'}(||u||_{\ell,q} + 1) + C_\varepsilon$$

is possible.

Proof of Theorem 1: We write

$$f(x) = \int e^{i\xi\cdot x}\hat{f}(\xi)\phi(\xi/R)d\xi$$

$$+ \int e^{i\xi\cdot x}\hat{f}(\xi)\psi(\xi/R)$$

$$= f_1(x) + f_2(x) ,$$

where $\phi \in C_0^\infty$ $\phi = 1$ near the origin, $\phi + \psi = 1$, and $R > 2$ is a positive constant to be determined later. We shall prove two facts

i) $$||f_1||_{L^\infty} \leq C(\log R)^{1/p'}$$

and

ii) $$||f_2||_{L^\infty} \leq \frac{C}{R^\eta}||f||_{\ell,q} \quad \text{for some} \quad \eta > 0 .$$

We then finish the proof by taking $R = \max(2, ||f||_{\ell,q}^{1/\eta})$.

We first prove i)

$$f_1(x) = \int e^{i\xi\cdot x}(1 + |\xi|^2)^{k/2}\,\hat{f}(\xi)\,\frac{\phi(\xi/R)}{(1+|\xi|^2)^{k/2}}d\xi$$

$$= g(x) * K_R(x)$$

where $g(x) \in L^p$, and $||g||_{L^p} = ||f||_{k,p} \leq 1$, and

$$\hat{K}_R(\xi) = \frac{\phi(\xi/R)}{(1 + |\xi|^2)^{k/2}} .$$

Thus it suffices to show $||K_R(x)||_{L^{p'}} \leq C(\log R)^{1/p'}$, $R > 2$.

$$K_R(x) = L_R(x) * G_k(x)$$

where

$$L_R(x) = R^d L(Rx)$$

and

$$\hat{L}(\xi) = \phi(\xi) .$$

Thus L is in $L^1(R^d) \cap C^\infty(R^d)$ and $G_k(x) \leq \frac{C_1}{|x|^{d-k}} e^{-C_2|x|}$ (see [S], p. 132).

Let $\chi(t)$ be the characteristic function of the unit interval, and set

$$G_k^1(x) = G_k(x)\chi(|x|R) ,$$

and

$$G_k^2(x) = G_k(x)(1 - \chi(|x|R)) .$$

So $K_R(x) = L_R(x) * G_k^{(1)}(x) + L_R(x) * G_k^2(x)$. One checks that

$$||L_R(x)||_{L^1} = ||L||_{L^1} \leq C$$

$$||L_R(x)||_{L^{p'}} \leq CR^{d/p},$$

$$||G_k^1(x)||_{L^1} \leq CR^{-k} = CR^{-d/p}$$

and

$$||G_k^2(x)||_{L^{p'}} \leq C(\log R)^{1/p'} .$$

Conclusion i) now follows from Young's inequality.

We turn now to ii):

$$f_2(x) = \int e^{i\xi \cdot x}\,\hat{f}(\xi)(1 + |\xi|^2)^{\ell/2} \frac{\psi(\frac{\xi}{R})}{(1+|\xi|^2)^{\ell/2}}\, d\xi.$$

Define $g(x)$ so that

$$\hat{g}(\xi) = \hat{f}(\xi)(1 + |\xi|^2)^{\ell/2} .$$

By hypothesis g is in L^q. Thus, we wish to show for some $\eta > 0$,

$$||K_R(x)||_{L^{q'}} \leq \frac{C}{R^\eta}$$

where

$$\hat{K}_R(\xi) = \frac{\psi(\frac{\xi}{R})}{(1 + |\xi|^2)^{\ell/2}}$$

If $q = 1$, $\ell > d$; so clearly

$$||K_R(x)||_{L^\infty} \leq \frac{C}{R^{\ell - d}} .$$

Suppose now $q > 1$, so $q' < \infty$; we may assume $\ell < d$. Then note that

$$\hat{K}_R(\xi) = \frac{1}{(1 + |\xi|^2)^{\ell/2}} - \frac{\phi(\frac{\xi}{R})}{(1 + |\xi|^2)^{\ell/2}} .$$

So

$$K_R(x) = h(x) - h * L_R(x) ,$$

where

$$\hat{h}(\xi) = \frac{1}{(1 + |\xi|^2)^{\ell/2}} ,$$

$$L_R(x) = R^d L(Rx) ,$$

and

$$\hat{L}(\xi) = \phi(\xi) .$$

Thus $K_R(x) = \int [h(x) - h(x - y)] L_R(y) dy$

As in [S], p. 132, we see $|h(x)| \leq \frac{C_1}{|x|^{d - \ell}} e^{-C_2|x|}$; a similar argument shows

$$|\nabla h| \leq \frac{C_1}{|x|^{d + 1 - \ell}} e^{-C_2|x|}$$

Since $\ell q > d$, we see h is in $L^{q'}$ and

$$\int |h(x - y) - h(x)|^{q'} dx \leq C|y|^\delta$$

for some $c_\delta > 0$. It then follows by Minkowski's inequality that $||K_R(x)||_{q'} \leq C(1/R)^\eta$, for some $\eta > 0$.

Section 3: Convolution inequalities in Lorentz spaces for the limiting cases.

We first recall the definition of the Lorentz spaces $L(p,q)$. Given a measurable function on $\mathbb{R}^d$ we set

$$\lambda(s) = \text{meas } \{x \in \mathbb{R}^d ; |f(x)| > s\} \qquad \text{for} \quad s > 0 ,$$

$$f^*(t) = \inf \{s > 0; \lambda(s) \leq t\} \qquad \text{for} \quad t > 0$$

(f^* is the decreasing rearrangement of f),

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)ds,$$

so that $f^*(t) \leq f^{**}(t)$ and f^{**} is also non-increasing. For $1 < p < \infty$ and $1 \leq q \leq \infty$, the Lorentz space $L(p,q)$ is defined as

$$L(p,q) = \{f ; t^{1/p} f^*(t) \in L^q(0,\infty; \frac{dt}{t})\}$$

or equivalently,

$$L(p,q) = \{f ; t^{1/p} f^{**}(t) \in L^q(0, \infty; \frac{dt}{t}\} ;$$

$L(p,q)$ is provided with the norm

$$(1) \qquad ||f||_{L(p,q)} = (\frac{q}{p})^{1/q} ||t^{1/p} f^{**}(t)||_{L^q(0,\infty;\frac{dt}{t})}$$

(for an expository presentation of $L(p,q)$ spaces, see e.g. [S.W.] or [H]). In particular we recall that

$$L(p,p) = L^p \quad \text{and} \quad L(p,\infty) = M^p$$

(M^p denotes the weak L^p space or Marcinkiewicz space). It is clear from (1) that

$$(2) \qquad f^*(t) \le f^{**}(t) \le \frac{1}{t^{1/p}} \|f\|_{L(p,q)} \qquad \forall t > 0.$$

A result of O'Neil and Stein (see [0]) asserts that if $f \in L(p_1,q_1)$ and $g \in L(p_2,q_2)$ with $\frac{1}{p_1} + \frac{1}{p_2} > 1$, then $u = f * g \in L(p_3,q_3)$ with $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2} - 1$ and $q_3 \ge 1$ is any number such that $\frac{1}{q_3} \le \frac{1}{q_1} + \frac{1}{q_2}$.

Our main result deals with the limiting case where $\frac{1}{p_1} + \frac{1}{p_2} = 1$. We shall actually obtain a result slightly sharper than Theorem 3 promised in the introduction, and then deduce Theorem 3 from it.

<u>Theorem 2</u>. Let $1 < p < \infty$, $1 \le q_1 \le \infty$, $1 \le q_2 \le \infty$ be such that $\frac{1}{q_1} + \frac{1}{q_2} < 1$ and set $\frac{1}{r} = \frac{1}{q_1} + \frac{1}{q_2}$. Assume $f \in L(p,q_1)$ and $g \in L(p',q_2) \cap L^1$ so that $u = f * g$ is defined[(1)]. Then

$$\frac{u^*(t)}{1 + |\log t|} \in L^r(0,1;\frac{dt}{t})$$

and $$\left\|\frac{u^*(t)}{1 + |\log t|}\right\|_{L^r(0,1;\frac{dt}{t})} \le C\|f\|_{L(p,q_1)} (\|g\|_{L(p',q_2)} + \|g\|_{L^1})$$

where C depends only on p, q_1 and q_2.

<u>Remark</u> If $f \in L(p,q_1)$ and $g \in L(p',q_2)$ with $\frac{1}{q_1} + \frac{1}{q_2} \ge 1$, then $u \in L^\infty$.

(1) Note that $f * g$ need not be defined if we assume only $f \in L(p,\infty)$ and $g \in L(p',\infty)$.

Proof of Theorem 2 For simplicity we set $||f|| = ||f||_{L(p,q_1)}$ and $||g|| = ||g||_{L(p',q_2)} + ||g||_{L^1}$. We shall distinguish two cases:

a) $r < \infty$

b) $r = \infty$ (i.e. $q_1 = q_2 = \infty$).

a) The case $r < \infty$

We shall make use of the following interesting inequality due to O'Neil (see [0], Theorem 1.7):

$$u^{**}(t) \leq t\, f^{**}(t)g^{**}(t) + \int_t^\infty f^*(s)g^*(s)ds, \qquad \forall t > 0. \tag{3}$$

Note that the integral on the right hand side of (3) is finite since we have

$$f^*(s) \leq f^{**}(s) \leq \frac{1}{s^{1/p}} ||f||_{L(p,q_1)}, \tag{4}$$

$$g^*(s) \leq g^{**}(s) \leq \frac{1}{s^{1/p'}} ||g||_{L(p',q_2)}, \tag{5}$$

$$g^*(s) \leq \frac{1}{s} ||g||_{L^1}. \tag{6}$$

It follows easily from (3), (4), (5) and (6) that for $t < 1$

$$u^{**}(t) \leq p||f||\,||g|| + \int_t^1 f^*(s)g^*(s)ds. \tag{7}$$

In order to finish we shall need the following logarithmic variant of Hardy's inequality

Lemma 1 Assume $t\varphi(t) \in L^r(0,1;\frac{dt}{t})$. Then

$$||(1 + |\log t|)^{-1} \int_t^1 \varphi(s)ds||_{L^r(0,1;\frac{dt}{t})} \leq \frac{r}{r-1} ||t\varphi(t)||_{L^r(0,1;\frac{dt}{t})}.$$

Proof of Lemma 1 We can always assume that $\varphi \geq 0$ and φ is bounded. Set $I = \int_0^1 (1 + |\log t|)^{-r} \left(\int_t^1 \varphi(s)ds\right)^r \frac{dt}{t}$; so that integrating by parts we find

$$I = \frac{1}{(r-1)} \int_0^1 \Big(\int_t^1 \varphi(s)\,ds\Big)^r \, d(1-\log t)^{-r+1}$$

$$= \frac{r}{r-1} \int_0^1 \Big(\int_t^1 \varphi(s)\,ds\Big)^{r-1} (1-\log t)^{-r+1} \varphi(t)\,dt$$

We deduce from Holder's inequality that

$$I \leq \frac{r}{r-1} \Big\| (1+|\log t|)^{-1} \int_t^1 \varphi(s)\,ds \Big\|^{r-1}_{L^r(0,1;\frac{dt}{t})} \| t\varphi(t) \|_{L^r(0,1,\frac{dt}{t})} ,$$

and the conclusion follows.

<u>Proof of Theorem 2 concluded</u> By Lemma 1 and (7) we have

$$\| (1+|\log t|)^{-1} u^{**}(t) \|_{L^r(0,1;\frac{dt}{t})}$$

$$\leq C\|f\|\,\|g\| + C\| t f^*(t) g^*(t) \|_{L^r(0,1;\frac{dt}{t})}$$

$$= C\|f\|\,\|g\| + C\| t^{1/p} f^*(t) t^{1/p'} g^*(t) \|_{L^r(0,1;\frac{dt}{t})}$$

$$\leq C\|f\|\,\|g\| + C\|f\|_{L(p,q_1)} \|g\|_{L(p',q_2)}$$

b) <u>The case $r = \infty$</u> (i.e. $q_1 = q_2 = \infty$).

By (7) we have, for $t < 1$,

$$u^{**}(t) \leq p\|f\|\,\|g\| + |\log t|\,\|f\|_{L(p,\infty)} \|g\|_{L(p',\infty)}$$

and therefore

$$\| (1+|\log t|)^{-1} u^{**}(t) \|_{L^\infty(0,1)} \leq C\|f\|\,\|g\|.$$

An easy consequence of Theorem 2 is the following:

Theorem 3: Under the assumptions of Theorem 2 we have:

a) if $r < \infty$, then $e^{\lambda|u|^{r'}} \in L^1_{loc}$ for every $\lambda > 0$,

b) if $r \leq \infty$, then there exist $\theta > 0$ which depends only on p, q_1, q_2 and C which depends only on $m(Q)$, such that

$$\int_Q e^{\theta|u|^{r'}} dx \leq C \qquad \forall f,g \quad \text{with} \quad ||f|| \leq 1 \quad \text{and} \quad ||g|| \leq 1 .$$

Proof of Theorem 3: Since $u^*(t)$ is non increasing we have by Theorem 2

$$|u^*(t)|^r \int_0^t (1 - \log s)^{-r} \frac{ds}{s} \leq C||f||^r \; ||g||^r \qquad \forall t < 1 .$$

Therefore

$$(8) \qquad |u^*(t)|^{r'} \leq C(1 + |\log t|) \; ||f||^{r'} \; ||g||^{r'} \qquad \forall t < 1 .$$

On the other hand

$$\int_Q e^{\theta|u|^{r'}} dx = \int_0^{m(Q)} e^{\theta|u^*|^{r'}} dt$$

and b) follows easily from (8) by choosing $\theta C < 1$. Next we prove a). Set $\varepsilon(t) = \int_0^t (1 + |\log s|)^{-r} |u^*(s)|^r \frac{ds}{s}$, so that $\varepsilon(t) \to 0$ as $t \to 0$. As above we have

$$|u^*(t)|^{r'} \leq C(1 + |\log t|)\varepsilon(t)^{\frac{1}{r-1}} .$$

Given $\lambda > 0$ we write

$$\int_Q e^{\lambda|u|^{r'}} dx = \int_0^{m(Q)} e^{\lambda|u^*|^{r'}} dt = \int_0^{t_0} + \int_{t_0}^{m(Q)}$$

and we choose $t_0 < m(Q)$ so that $\lambda C \; \varepsilon(t_0)^{\frac{1}{r-1}} < 1$.

Section 4: Some corollaries

Our results here will be of two types, namely:

4.1. Results "dual" to Theorem 2

4.2. Embedding of Sobolev spaces into spaces of functions which are "almost" lipschitz.

4.1. Results "dual" to Theorem 2

Corollary 4 Let $1 < p < \infty$, $1 \leq q \leq \infty$, $1 \leq r \leq \infty$. Assume $f \in L(p,q) \cap L^1$ and g is such that $|g|(1 + \log^+ g)^{1/r} \in L^1$. Assume $\frac{1}{q} + \frac{1}{r} \leq 1$ and set $\frac{1}{s} = \frac{1}{q} + \frac{1}{r}$. Then $u = f * g \in L(p,s)$.

Example Assume $d \geq 3$, $u \in L^1_{loc}(\mathbb{R}^d)$, $\Delta u = g$ and $|g|(1 + \log^+ g)^{\frac{d-2}{d}} \in L^1_{loc}(\mathbb{R}^d)$. Then $u \in L^{d/d-2}_{loc}(\mathbb{R}^d)$. [Indeed, one uses Corollary 4 with $p = r = s = \frac{d}{d-2}$, $q = \infty$, and $\hat{f}(\xi) = \frac{1}{1+|\xi|^2}$].

Proof of Corollary 4 We use a duality argument. Assume for example $||f||_{L(p,q)} + ||f||_{L^1} \leq 1$. Let $\varphi \in L(p',s')$ - the dual space of $L(p,s)$ - with $||\varphi||_{L(p',s')} \leq 1$. We have

$$\int u\varphi \, dx = \int gv \, dx \leq \int_0^\infty g^* \, v^* \, dt$$

where $v = \check{f} * \varphi$ $(\check{f}(x) = f(-x))$. From (8) we deduce that

$$|v^*(t)|^r \leq C(1 + |\log t|) \qquad \text{for} \quad 0 < t < 1 .$$

On the other hand $v \in L(p',s')$ (since $\varphi \in L(p',s')$ and $f \in L^1$); thus $v^*(t) \leq \frac{1}{t^{1/p'}}$ and in particular $\int_1^\infty g^* \, v^* \, dt \leq C||g||_{L^1}$. Finally we estimate $\int_0^1 g^* \, v^* \, dt$ using Young's inequality:

$$ab \leq e^{\theta b^r} + Ca(1 + \log^+ a)^{1/r} \qquad \forall a, b \geq 0$$

where C depends only on θ and r. Thus

$$g^* v^* \leq e^{\theta|v^*|^r} + C g^*(1 + \log^+ g^*)^{1/r}$$

and the conclusion follows since

$$\int_0^1 g^*(1 + \log^+ g^*)^{1/r} \leq \int |g|(1 + \log^+|g|)^{1/r} .$$

4.2. Embeddings of Sobolev spaces into spaces of functions which are "almost" Lipschitz

Our main result is the following

Corollary 5 Assume $u \in W_p^{k+1}(\mathbb{R}^d)$ with $kp = d$, $1 < p < \infty$. Then for every $x,y \in \mathbb{R}^d$

$$|u(x) - u(y)| \leq C||u||_{W_p^{k+1}} |x - y|[1 + |\log|x - y||]^{1/p'}$$

where C depends only on k and p.

The proof of Corollary 5 follows Morrey's technique. We first prove the following

Lemma 2 Assume $f \in W_p^k(\mathbb{R}^d)$ and $kp = d$, with $1 < p < \infty$. Then

$$\int_Q |f(x)|dx \leq C||f||_{W_p^k} m(Q)[1 + |\log m(Q)|]^{1/p'} ,$$

for every Q with finite measure; C depends only on k and p.

Proof of Lemma 2 We have

$$\int_Q |f(x)|dx = \int_0^{m(Q)} f^*(t)dt .$$

Assume first $m(Q) < 1$; we deduce from (8) that

$$f^*(t) \le C\|f\|_{W_p^k}[1 + |\log t|]^{1/p'} \qquad \text{for} \quad 0 < t < 1.$$

The conclusion follows easily since

$$\int_0^{m(Q)} [1 + |\log t|]^{1/p'}\, dt = \int_0^1 [1 + |\log s\, m(Q)|]^{1/p'}\, m(Q)\,ds$$

$$\le Cm(Q)[1 + |\log m(Q)|]^{1/p'} .$$

When $m(Q) > 1$ we write

$$\int_0^{m(Q)} f^*(t)\,dt = \int_0^1 f^*(t)\,dt + \int_1^{m(Q)} f^*(t)\,dt$$

$$\le C\|f\|_{W_p^k} + m(Q)f^*(1) \le C\, m(Q)\|f\|_{W_p^k} .$$

<u>Proof of Corollary 5</u> Let Q be a cube in $\mathbb{R}^d$ with side $\rho = |x - y|$ containing x and y. Let $z \in Q$; we have

$$u(z) - u(x) = \int_0^1 \nabla u(tz + (1 - t)x)(z - x)\,dt$$

and so

$$|u(z) - u(x)| \le \sqrt{d}\,\rho \int_0^1 |\nabla u(tz + (1 - t)x)|\,dt \quad . \tag{9}$$

Integrating (9) with respect to z over Q we find

$$|\bar{u} - u(x)| \le \frac{\sqrt{d}}{\rho^{d-1}} \int_Q dz \int_0^1 |\nabla u(tz + (1 - t)x)|\,dt$$

where $\bar{u} = \frac{1}{m(Q)} \int_Q u(z)\,dz$. Thus, if we set $\zeta = tz + (1 - t)x$ we obtain

$$|\bar{u} - u(x)| \le \frac{\sqrt{d}}{\rho^{d-1}} \int_0^1 \frac{dt}{t^d} \int_{tQ+(1-t)x} |\nabla u(\zeta)|\,d\zeta .$$

From Lemma 2 we deduce that

$$\int_{tQ+(1-t)x} |\nabla u(\zeta)|\,d\zeta \leq C||u||_{W_p^{k+1}} t^d\rho^d [1 + |\log t^d\rho^d|]^{1/p'} .$$

Therefore

$$|\bar{u} - u(x)| \leq C||u||_{W_p^{k+1}}\,\rho \int_0^1 [1 + |\log t\rho|]^{1/p}\,dt$$

$$\leq C||u||_{W_p^{k+1}}\rho[1 + |\log \rho|]^{1/p'} .$$

The same estimate holds if we replace x by y and the conclusion follows.

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