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NONLINEAR SCHRÖDINGER EVOLUTION EQUATIONS

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LET Ω BE a domain in R^2 with compact smooth boundary Γ (Ω could be for example a bounded domain or an exterior domain). Consider the equation

$$\left.\begin{aligned} & i\frac{\partial u}{\partial t} - \Delta u + k|u|^2 u = 0 && \text{in} \quad \Omega \times [0, \infty) \\ & u(x, t) = 0 && \text{in} \quad \Gamma \times [0, \infty) \\ & u(x, 0) = u_0(x), && \end{aligned}\right\} \tag{1}$$

where $u(x, t)$ is a complex valued function and $k \in \mathbf{R}$ is a constant. Problem (1) which occurs in nonlinear optics when $\Omega = \mathbf{R}^2$ has been extensively studied in this case (see [1–3, 5, 8]), but we are not aware of any known result when $\Omega \neq R^2$.

Our main result is the following:

THEOREM 1. Let $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. Assume that *one* of the following conditions holds
(a) either $k \geqslant 0$,
(b) or $k < 0$ and $|k|\int|u_0(x)|^2\,dx < 4$.
Then there exists a unique solution of (1) such that

$u \in C([0, \infty);\quad H^2(\Omega)) \cap C^1([0, \infty);\quad L^2(\Omega))$.

The proof of Theorem 1 relies on several lemmas. The first lemma is of interest for its own sake; it is a new interpolation-embedding inequality.

In what follows we denote by C various constants depending only on Ω.

LEMMA 2. We have

$$\|u\|_{L^\infty} \leqslant C(1 + \sqrt{\log(1 + \|u\|_{H^2})}) \tag{2}$$

for every $u \in H^2(\Omega)$ with $\|u\|_{H^1} \leqslant 1$.

Proof. It is well known that an H^2 function on Ω can be extended by an H^2 function on R^2.

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More precisely one can construct an extension operator P such that:

P is a bounded operator from $H^1(\Omega)$ into $H^1(R^2)$

P is a bounded operator from $H^2(\Omega)$ into $H^2(R^2)$

$Pu_{|\Omega} = u$ for every $u \in H^1(\Omega)$.

Let $u \in H^2(\Omega)$ with $\|u\|_{H^1} \leqslant 1$. Let $v = Pu$ and denote by $\hat{v}$ the Fourier transform of v. We clearly have

$$\|(1 + |\xi|)\hat{v}\|_{L^2(R^2)} \leqslant C \tag{3}$$

$$\|(1 + |\xi|^2)\hat{v}\|_{L^2(R^2)} \leqslant C\|u\|_{H^2(\Omega)} \tag{4}$$

$$\|u\|_{L^\infty(\Omega)} \leqslant \|v\|_{L^\infty(R^2)} \leqslant C\|\hat{v}\|_{L^1(R^2)}. \tag{5}$$

For $R > 0$ we write

$$\begin{aligned}
\|\hat{v}\|_{L^1} &= \int_{|\xi|<R} |\hat{v}(\xi)|\,\mathrm{d}\xi + \int_{|\xi|\geqslant R} |\hat{v}(\xi)|\,\mathrm{d}\xi \\
&= \int_{|\xi|<R} (1 + |\xi|)\,|\hat{v}(\xi)|\,\frac{1}{1+|\xi|}\,\mathrm{d}\xi + \int_{|\xi|\geqslant R} (1 + |\xi|^2)\,|\hat{v}(\xi)|\,\frac{1}{1+|\xi|^2}\,\mathrm{d}\xi \\
&\leqslant C\left[\int\!\!\int_{|\xi|<R} \frac{1}{(1+|\xi|)^2}\,\mathrm{d}\xi\right]^{1/2} + C\|u\|_{H^2}\left[\int\!\!\int_{|\xi|\geqslant R} \frac{1}{(1+|\xi|^2)^2}\,\mathrm{d}\xi\right]^{1/2}
\end{aligned}$$

by Cauchy–Schwarz, (3) and (4). A straightforward computation leads to

$$\|\hat{v}\|_{L^1} \leqslant C[\log(1 + R)]^{1/2} + C\|u\|_{H^2}(1 + R)^{-1}$$

by every $R \geqslant 0$. We obtain (2) by choosing $R = \|u\|_{H^2}$.

LEMMA 3. We have

$$\||u|^2 u\|_{H^2} \leqslant C\|u\|_{L^\infty}^2 \|u\|_{H^2} \qquad \text{for every} \quad u \in H^2(\Omega). \tag{6}$$

Proof of Lemma 3. Let D denote any first order differential operator. For $u \in H^2$ we have

$$|D^2(|u|^2 u)| \leqslant C(|u|^2|D^2 u| + |u||Du|^2),$$

and so

$$\||u|^2 u\|_{H^2} \leqslant C\|u\|_{L^\infty}^2\|u\|_{H^2} + C\|u\|_{L^\infty}\|u\|_{W^{1,4}}^2. \tag{7}$$

On the other hand an inequality of Gagliardo–Nirenberg (see [6]) implies that

$$\|u\|_{W^{1,4}} \leqslant C\|u\|_{L^\infty}^{1/2}\|u\|_{H^2}^{1/2}. \tag{8}$$

Combining (7) and (8) we obtain (6).

Finally we recall the following well known result essentially due to Segal [7]:

LEMMA 4. Assume H is a Hilbert space and $A\colon D(A) \subset H \to H$ is an m-accretive linear operator. Assume F is a mapping from $D(A)$ into itself which is Lipschitz on every bounded set of $D(A)$.

Then for every $u_0 \in D(A)$, there exists a unique solution u of the equation

$$\left.\begin{aligned} &\frac{\mathrm{d}u}{\mathrm{d}t} + Au = Fu \\ &u(0) = u_0 \end{aligned}\right\}$$

defined for $t \in [0, T_{\max})$ such that

$$u \in C^1([0, T_{\max}); H) \cap C([0, T_{\max}); D(A))$$

with the additional property that

$$\left.\begin{aligned} &\text{either} \quad T_{\max} = \infty \\ &\text{or} \quad T_{\max} < \infty \quad \text{and} \quad \lim_{t \uparrow T_{\max}} \|u(t)\| + \|Au(t)\| = \infty. \end{aligned}\right\}$$

Proof of Theorem 1. We apply Lemma 4 in $H = L^2(\Omega)$ to $Au = i\Delta u$, $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, $Fu = ik|u|^2u$. We shall show that $T_{\max} = \infty$ by proving that $\|u(t)\|_{H^2}$ remains bounded on every finite time interval.

First we multiply (1) by $\bar{u}$ and consider the imaginary part. This leads to

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}. \tag{9}$$

Next we multiply (1) by $\partial\bar{u}/\partial t$ and consider the real part. This leads to

$$\tfrac{1}{2}\int |\nabla u(x, t)|^2 \,\mathrm{d}x + \frac{k}{4}\int |u(x, t)|^4 \,\mathrm{d}x \equiv E_0 \tag{10}$$

where

$$E_0 = \tfrac{1}{2}\int_\Omega |\nabla u_0(x)|^2 \,\mathrm{d}x + \frac{k}{4}\int_\Omega |u_0(x)|^4 \,\mathrm{d}x.$$

We claim that $\|u(t)\|_{H^1}$ remains bounded for $t > 0$. Indeed, this is clear when $k \geqslant 0$. While if $k < 0$ we have

$$\int |\nabla u(x, t)|^2 \leqslant \frac{|k|}{2}\int |u(x, t)|^4 \,\mathrm{d}x + 2E_0. \tag{11}$$

On the other hand an inequality of Gagliardo and Nirenberg ([6]) shows that*

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* In order to obtain the constant $\frac{1}{2}$ one proceeds as follows. For $\varphi \in C_0^\infty(\mathbf{R}^2)$ we have

$$|\varphi(x_1, x_2)| \leqslant \tfrac{1}{2}\int_{-\infty}^{+\infty} |\varphi_{x_1}(t, x_2)| \mathrm{d}t, \quad |\varphi(x_1, x_2)| \leqslant \tfrac{1}{2}\int_{-\infty}^{+\infty} |\varphi_{x_2}(x_1, s)| \mathrm{d}s.$$

Thus

$$\int_{\mathbf{R}^2} |\varphi|^2 \,\mathrm{d}x \leqslant \tfrac{1}{4}\int_{\mathbf{R}^2} |\varphi_{x_1}| \mathrm{d}x \int_{\mathbf{R}^2} |\varphi_{x_2}| \mathrm{d}x.$$

Choosing $\varphi = |u|^2$ leads to

$$\int |u|^4 \,\mathrm{d}x \leqslant \int |u|^2 \,\mathrm{d}x \Big(\int |u_{x_1}|^2 \,\mathrm{d}x\Big)^{1/2} \Big(\int |u_{x_2}|^2 \,\mathrm{d}x\Big)^{1/2} \leqslant \tfrac{1}{2}\int |u|^2 \,\mathrm{d}x \int |\nabla u|^2 \mathrm{d}x.$$

$$\int |u|^4 \, \mathrm{d}x \leqslant \tfrac{1}{2} \int |u|^2 \, \mathrm{d}x \int |\nabla u|^2 \, \mathrm{d}x \tag{12}$$
$$= \tfrac{1}{2} \int |u_0|^2 \, \mathrm{d}x \int |\nabla u|^2 \, \mathrm{d}x.$$

Combining (11), (12) and assumption (b) in Theorem 1 we see that

$$\|u(t)\|_{H^1} \leqslant C \tag{13}$$

where C is independent of t.

We now denote by $S(t)$ the L^2 isometry group generated by $-A$. From (1) we have

$$u(t) = S(t)u_0 + ik \int_0^t S(t-s) \, |u(s)|^2 u(s) \, \mathrm{d}s$$

and so

$$Au(t) = S(t)Au_0 + ik \int_0^t S(t-s) A \, [|u(s)|^2 \, u(s)] \, \mathrm{d}s.$$

Thus

$$\|Au(t)\|_{L^2} \leqslant \|Au_0\|_{L^2} + |k| \int_0^t \|A[|u(s)|^2 u(s)]\|_{L^2} \, \mathrm{d}s. \tag{14}$$

Lemma 3 implies that

$$\|A[|u(s)|^2 \, u(s)]\|_{L^2} \leqslant C \|u(s)\|_{L^\infty}^2 \, \|u(s)\|_{H^2}.$$

From Lemma 2 and estimate (13) we deduce that

$$\|u(s)\|_{L^\infty} \leqslant C(1 + \sqrt{\log(1 + \|u(s)\|_{H^2})}).$$

Hence (14) leads to

$$\|u(t)\|_{H^2} \leqslant C + C \int_0^t \|u(s)\|_{H^2}[1 + \log(1 + \|u(s)\|_{H^2})] \, \mathrm{d}s. \tag{15}$$

We denote by $G(t)$ the RHS in (15); thus

$$G'(t) = C\|u(t)\|_{H^2}[1 + \log(1 + \|u(t)\|_{H^2})] \leqslant CG(t)[1 + \log(1 + G(t))].$$

Consequently

$$\frac{\mathrm{d}}{\mathrm{d}t} \log[1 + \log(1 + G(t))] \leqslant C$$

and we find an estimate for $\|u(t)\|_{H^2}$ of the form

$$\|u(t)\|_{H^2} \leqslant \mathrm{e}^{\alpha \mathrm{e}^{\beta t}}$$

for some constants α and β. Therefore $\|u(t)\|_{H^2}$ remains bounded on every finite time interval and so we must have $T_{\max} = \infty$.

Remarks. (1) The proof of Theorem 1 leads to an estimate of the form $\|u(t)\|_{L^\infty} \leqslant \alpha\, e^{\beta t}$. We do not know whether $\|u(t)\|_{L^\infty}$ remains actually bounded as $t \to \infty$.

(2) When $k < 0$ and $|k|\int|u_0|^2 > 4$, it is known (see [4] and [2]) if $\Omega = R^2$ that the solution of (1) corresponding to some intial conditions may blow up in finite time. A similar phenomenon presumably occurs when $\Omega \neq R^2$.

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