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ON A FREE BOUNDARY PROBLEM ARISING IN PLASMA PHYSICS

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INTRODUCTION

THE PROBLEM we study in this paper has its origin in plasma physics. It stems from a model describing the equilibrium of a plasma confined in a toroidal cavity (a "Tokamak machine"). For a detailed presentation of this model the reader is referred to [1, 2] and also to the appendix in [3].

Let Ω denote the meridian cross section of the cavity and $0z$ the axis of toroidal symmetry, $(\bar{\Omega} \cap 0z = \varnothing)$. The plasma occupies an unknown region $\Omega_p \subset \Omega$; let $\Gamma_p = \partial\Omega_p$ and $\Gamma = \partial\Omega$. The region $\Omega_v = \Omega - \bar{\Omega}_p$ is assumed to be vacuous (in particular, there are no external currents in Ω_v, cf. [4]). From the Maxwell equations and the magneto-hydrodynamic theory of macroscopic equilibrium in the plasma, one derives the following relations for u, the flux function of the meridian magnetic field:

$$\mathscr{L}u = g(r, u), \qquad \text{in } \Omega_p \tag{0.1}$$

$$\mathscr{L}u = 0, \qquad \text{in } \Omega_v, \tag{0.2}$$

$$u = 0, \qquad \text{on } \Gamma_p, \tag{0.3}$$

$$u \neq 0, \qquad \text{in } \Omega_p, \tag{0.4}$$

$$u \text{ is a constant on } \Gamma \text{ (whose value is unknown)}, \tag{0.5}$$

$$-\int_\Gamma \frac{1}{r}\frac{\partial u}{\partial n}\,\mathrm{d}\Gamma = I, \tag{0.6}$$

where

$$\mathscr{L}\bullet = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\bullet}{\partial r}\right) - \frac{1}{r}\frac{\partial^2\bullet}{\partial z^2},$$

in cylindrical coordinates, and $\partial/\partial n$ is the outward normal derivative.

Thus, problem (0.1)–(0.6) is a free boundary problem: Given the domain Ω, the function g† and the constant $I > 0$‡, one seeks to determine the region Ω_p and the function u.

†Actually, the precise form of the function g is not known, and this is one of the difficult questions from the physical viewpoint.

‡The condition (0.6) is a normalization. One requires that the total longitudinal current inside the plasma achieve a given value.

Assuming $g(r, u) > 0$ for $u > 0$, and defining $g(r, u) = 0$ for $u \leqslant 0$, it is easily seen, using the maximum principle and (0.6) that the relations (0.1)–(0.4) can be expressed in the single equation

$$\mathscr{L}u = g(r, u) \qquad \text{in } \Omega. \tag{0.7}$$

The problem (0.1)–(0.6) is thus equivalent to the nonlinear problem (0.5)–(0.7).

Free boundary problems having some similarities with the one we study here arise in different contexts, for example in astrophysics [5], or in the theory of steady vortex rings in an ideal fluid [6, 7]. (The analogy with problems in hydrodynamics is explained in [8]).

The problem and the precise hypotheses are stated in the first section of this paper. In Section 2, we establish the existence of a solution by introducing a variational formulation of the problem. Another approach to the question of existence, based on a topological degree argument, is presented in Section 4. This method requires some a priori estimates which are proved in Section 3. If g is Lipschitz with small Lipschitz constant, we show in Section 5 the uniqueness of the solution. Lastly, in Section 6, we present various results and open questions concerning the "model case", where $g(r, u) \equiv \lambda u^+$.

Problems of the type (0.5)–(0.7) have been considered by several authors. Temam [3] first proved the existence of at least one pair (λ, u) solution to a nonlinear eigenvalue problem

$$\begin{cases} (0.7)^{\text{bis}} \ \mathscr{L}u = \lambda g(r, u) & \text{in } \Omega \\ (0.5), (0.6) \end{cases}$$

subject to the constraint $\int_\Omega G(x, u(x))\,dx = C$, for any given $C > 0$, where $G(x, z) = \int_0^z g(x, s)\,ds$, under some restrictive hypotheses on g. In [9], we established the existence of a solution for this problem for a prescribed $\lambda > 0$, (thus dropping the constraint $\int_\Omega G(x, u(x))\,dx = C$). Subsequently, several papers have been devoted to the study of different aspects of this type of problem, in the framework of the "model case":

Existence of solutions: R. Temam [10], J. P. Puel [11],

Uniqueness: R. Temam [10], A. Damlamian, J. P. Puel [11]

Non uniqueness: D. G. Schaeffer [12, 13].

Geometric properties of the free boundary: D. Kinderlehrer [14], D. Kinderlehrer, J. Spruck [15] and D. Kinderlehrer, L. Nirenberg and J. Spruck [16].

Equivalence of the two variational formulations: A. Damlamian [17].

Behaviour of u as $\lambda \to +\infty$: T. Gallouet [18].

The main part of the results we present in this paper were announced in [9].

1. THE PROBLEM AND THE HYPOTHESES

Let Ω be a bounded domain of $\mathbf{R}^N$ with a smooth boundary $\Gamma = \partial\Omega$. Let $\mathscr{L}$ be the operator defined by:

$$\mathscr{L}u = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right),$$

with $a_{ij} \in C^1(\overline{\Omega})$; $a_{ij} = a_{ji}$. $\mathscr{L}$ is assumed to be uniformly elliptic in $\overline{\Omega}$, i.e. there exists a constant $\eta > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\,\xi_i\xi_j \geqslant \eta|\xi|^2, \qquad \forall x \in \overline{\Omega}, \qquad \forall \xi \in \mathbf{R}^N.$$

We denote by $\partial/\partial\nu$ the outward conormal derivative on Γ associated with the operator $\mathscr{L}$:

$$\frac{\partial\cdot}{\partial\nu} = \sum_{i,j=1}^{N} a_{ij}\cos(n, x_i)\frac{\partial\cdot}{\partial x_j},$$

(n being the outward normal to Γ).

I is a given positive constant. We consider the following problem:

PROBLEM 1. Find a function $u(x) \in H^2(\Omega)$ satisfying

$$\mathscr{L}u = g(x, u), \qquad \text{for } x \in \Omega, \tag{1.1}$$

$$u \text{ is a constant on } \Gamma \text{ (whose value is unknown)}, \tag{1.2}$$

$$-\int_\Gamma \frac{\partial u}{\partial\nu}\,\mathrm{d}\Gamma = I. \tag{1.3}$$

The function g is assumed to satisfy the following conditions:

$g: \overline{\Omega} \times \mathbf{R} \to [0, +\infty)$ is a continuous function such that

$$g(x, z) = 0, \qquad \forall x \in \overline{\Omega}, \quad \forall z \leqslant 0. \tag{1.4}$$

$$g(x, z) \leqslant g(x, z'), \qquad \forall x \in \overline{\Omega}, \quad \forall z, z' \in \mathbf{R} \quad \text{such that } z \leqslant z' \tag{1.5}$$

$$\lim_{z \to +\infty} : \frac{g(x, z)}{z^p} = 0, \qquad \text{uniformly in } x \in \overline{\Omega}, \tag{1.6}$$

for $p = N/N - 2$ if $N > 2$, or for at least one $p > 1$ if $N \leqslant 2$.

$$\lim_{z \to +\infty} \int_\Omega g(x, z)\,\mathrm{d}x > I. \tag{1.7}$$

It should be observed that condition (1.7) is almost necessary in order to solve problem 1, in the sense that the existence of a solution to Problem 1, together with conditions (1.4), (1.5) imply

$$\lim_{z \to +\infty} \int_\Omega g(x, z)\,\mathrm{d}x \geqslant I.$$

Indeed, if u is a solution to Problem 1, one has

$$\lim_{z \to +\infty} \int_\Omega g(x,)z\,\mathrm{d}x \geqslant \int_\Omega g(x, u(x))\,\mathrm{d}x = \int_\Omega \mathscr{L}u\,\mathrm{d}x = -\int_\Gamma \frac{\partial u}{\partial\nu}\,\mathrm{d}\Gamma = I. \tag{1.8}$$

Remark 1. Suppose g satisfies conditions (1.4) and (1.6); then, for any $u \in H^1(\Omega)$, $g(\cdot, u) \in L^2(\Omega)$.

Indeed, from (1.4) and (1.6) it follows that there exists a constant C such that

$$0 \leqslant g(x, z) \leqslant C + |z|^p, \qquad \forall x \in \overline{\Omega}, \quad \forall z \in \mathbf{R}. \tag{1.9}$$

For example, in case $N > 2$, $H^1(\Omega)$ is imbedded in $L^{2^*}(\Omega)$, where $1/2^* = 1/2 - 1/N = 1/2p$, and from (1.9) it follows that $g(\cdot, u) \in L^2(\Omega)$ as soon as $u \in L^{2^*}(\Omega)$.

Further regularity of a solution u to Problem 1 can of course be achieved by additional regularity hypotheses on the coefficients of $\mathscr{L}$ and on g. For instance, if one assumes $a_{ij} \in C^2(\overline{\Omega})$, and

$g(x, z)$ Lipschitzian in z uniformly with respect to $x \in \bar{\Omega}$, a standard argument shows that if $u \in H^2(\Omega)$ then in fact $u \in C^{2,\alpha}(\bar{\Omega})$, for any $\alpha \in (0, 1)$.

2. EXISTENCE OF SOLUTIONS: A VARIATIONAL METHOD

THEOREM 1. Suppose g satisfies conditions (1.4)–(1.7), then, there exists at least one solution to Problem 1.

Let $G: \bar{\Omega} \times \mathbf{R} \to [0, +\infty)$ be the function defined by

$$G(x, z) = \int_0^z g(x, s)\, ds.$$

$G(x, \cdot)$ is a proper convex function on $\mathbf{R}$. Let $G^*(x, \cdot)$ be its convex conjugate function, that is:

$$G^*(x, \zeta) = \sup_{z \in \mathbf{R}} \{\zeta z - G(x, z)\}, \quad x \in \bar{\Omega}.$$

LEMMA 2.1. For any $\alpha > 0$, there exists a constant C_α such that $G^*(x, \zeta) \geqslant \alpha|\zeta|^r - C_\alpha$, $\forall x \in \bar{\Omega}$, $\forall \zeta \in \mathbf{R}$, where $r = 1 + 1/p$ and p is the exponent defined in (1.6).

Proof. By (1.4) and (1.6), for any $\varepsilon > 0$, there exists a constant C_ε such that

$$g(x, z) \leqslant C_\varepsilon + \varepsilon|z|^p, \qquad \forall x \in \bar{\Omega}, \quad \forall z \in \mathbf{R}.$$

Therefore, for any $\varepsilon > 0$, there exists another constant C_ε such that

$$G(x, z) \leqslant C_\varepsilon + \varepsilon|z|^{1+p}, \qquad \forall x \in \bar{\Omega}, \quad \forall z \in \mathbf{R}.$$

The function $h_\varepsilon(z) = C_\varepsilon + \varepsilon|z|^{1+p}$ is convex, and since $G(x, z) \leqslant h_\varepsilon(z)$, it follows that $G^*(x, \zeta) \geqslant h_\varepsilon^*(\zeta)$, $\forall x \in \bar{\Omega}$, $\forall \zeta \in \mathbf{R}$. It is easily seen that h_ε^* has the form

$$h_\varepsilon^*(\zeta) = -C_\varepsilon + \frac{C}{\varepsilon^{r-1}}|\zeta|^r,$$

where C depends only on p. Since $\varepsilon > 0$ was arbitrary, we obtain the lemma. The notation $r = 1 + 1/p$ will be maintained in the following.

Let $S: L^r(\Omega) \to W^{2,r}(\Omega)$ be the inverse operator of $\mathscr{L}$ with homogeneous Dirichlet condition, that is, for $\rho \in L^r(\Omega)$

$$S\rho = \xi \Leftrightarrow \begin{cases} \mathscr{L}\xi = \rho & \text{in } \Omega, \\ \xi = 0 & \text{on } \Gamma. \end{cases}$$

By Sobolev's theorem, $W^{2,r}(\Omega)$ is imbedded in $L^{r'}(\Omega)$, $(r' = p + 1)$, so that $\int_\Omega S\rho \,.\, \rho\, dx$ is defined for any $\rho \in L^r(\Omega)$. Let

$$K = \left\{\rho \in L^r(\Omega); \quad \rho \geqslant 0 \text{ a.e. in } \Omega, \quad \int_\Omega \rho(x)\, dx = I\right\}.$$

K is a closed convex subset of $L^r(\Omega)$. For $\rho \in K$, define

$$J(\rho) = \int_\Omega G^*(x, \rho(x))\, dx - 1/2\left(\int_\Omega S\rho(x) \,.\, \rho(x)\, dx\right).$$

Consider the following variational problem:

$$\text{minimize } \{J(\rho);\ \rho \in K\}. \tag{2.1}$$

The existence result of Theorem 1 is obtained by solving problem (2.1) and then deriving from a solution of (2.1) a solution to Problem 1.

THEOREM 2. There exists a solution $\rho_0 \in K$ to problem (2.1). Furthermore, there exists a constant $\theta_0 \in \mathbf{R}$ such that

$$u_0 = S\rho_0 + \theta_0 \text{ is a solution to Problem 1.}$$

Remark 2. J is not everywhere infinite on K. Indeed, let $\hat{\rho} = g(x, \hat{z})$, where $\hat{z} \in \mathbf{R}$ is a constant such that $\int_\Omega g(x, \hat{z})\,\mathrm{d}x = I$. The existence of such a constant is implied by (1.4) and (1.7), Thus, $\hat{\rho} \in K$. Since $\hat{\rho} = G_z'(x, \hat{z})$, clearly, $G^*(x, \hat{\rho}) = \hat{z}\hat{\rho} - G(x, \hat{z})$. Therefore, $J(\hat{\rho}) < +\infty$.

The first step of the proof is to check that J is bounded from below on K.

LEMMA 2.2. Let q be defined by $1/q = 1 - 1/2p$, $(1 < q < 2)$. There exists a constant C such that for any $\rho \in L^q(\Omega)$, one has

$$0 \leqslant \int_\Omega S\rho \,.\, \rho\,\mathrm{d}x \leqslant C\|\rho\|_{L^q}^2.$$

In the following, we will use the same generic notation C for all various positive constants that will be needed.

Proof of Lemma 2.2. Let $u = S\rho$; we have

$$\eta\|\nabla u\|_{L^2}^2 \leqslant \int_\Omega S\rho\,.\,\rho\,\mathrm{d}x = \int_\Omega u\rho\,\mathrm{d}x \leqslant \|u\|_{L^{2*}}\|\rho\|_{L^q}, \qquad \text{where } \eta > 0, \tag{2.2}$$

and $1/2^* + 1/q = 1$.

By Sobolev's inequality, there exists a constant $C > 0$ such that $\|u\|_{L^{2*}} \leqslant C\|\nabla u\|_{L^2}$. Using this inequality in (2.2) yields $\|u\|_{L^{2*}} \leqslant C\|\rho\|_{L^q}$.

Therefore, from (2.2) it follows

$$0 \leqslant \int_\Omega S\rho\,.\,\rho\,\mathrm{d}x \leqslant C\|\rho\|_{L^q}^2.$$

We also observe from (2.2) that $\int_\Omega S\rho\,.\,\rho\,\mathrm{d}x = 0$ if and only if $\rho = 0$.

Proof of Theorem 2. We first check that J is bounded from below on K. Since $1 < q < r$, we have by Hölder's inequality

$$\|\rho\|_{L^q} \leqslant \|\rho\|_{L^r}^\theta \|\rho\|_{L^1}^{1-\theta}, \quad \text{with } 1/q = \theta/r + (1-\theta)/1 \text{ that is } \theta = r/2.$$

Thus, for $\rho \in K$ we have

$$\|\rho\|_{L^q} \leqslant C\|\rho\|_{L^r}^{r/2}. \tag{2.3}$$

Combining Lemmas (2.1) and (2.2), and inequality (2.3), we see that

$$J(\rho) \geqslant \alpha \|\rho\|_{L^r}^r - C_\alpha - C\|\rho\|_{L^r}^r,$$

whenceupon, by choosing α large enough, we have

$$J(\rho) \geqslant \|\rho\|_{L^r}^r - C \geqslant -C > -\infty. \tag{2.4}$$

The second step of the proof is to show that J achieves its minimum on K. Let (ρ_n) be a minimizing sequence:

$$\rho_n \in K \quad \text{and} \quad \lim_{n\to\infty} J(\rho_n) = \operatorname{Inf}\{J(\rho): \rho \in K\}.$$

By (2.4), $\|\rho_n\|_{L^r}$ is bounded; we may therefore assume that ρ_n converges weakly in $L^r(\Omega)$ to some $\rho_0 \in K$. Let $u_n = S\rho_n$ and $u = S\rho_0$. Clearly, u_n converges weakly to u in $W^{2,r}(\Omega)$. Since $1/2 > 1/r - 1/N$, by Sobolev's theorem, $W^{2,r}(\Omega)$ is imbedded in $H^1(\Omega)$ with compact injection. Hence u_n converges strongly in $H^1(\Omega)$ to u, and consequently,

$$\lim_{n\to+\infty} \int_\Omega S\rho_n \cdot \rho_n \, dx = \lim_{n\to+\infty} \sum_{i,j=1}^N \int_\Omega a_{ij} \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial x_j} dx = \int_\Omega S\rho_0 \cdot \rho_0 \, dx. \tag{2.5}$$

The functional $\rho \mapsto \int_\Omega G^*(x, \rho)\, dx$ being convex, is lower semi-continuous for the weak topology on $L^r(\Omega)$. Thus,

$$\lim_{n\to+\infty} \inf \int_\Omega G^*(x, \rho_n)\, dx \geqslant \int_\Omega G^*(x, \rho_0)\, dx. \tag{2.6}$$

(2.5) and (2.6) show that $\lim_{n\to+\infty} \inf J(\rho_n) \geqslant J(\rho_0)$, whence, $J(\rho_0) = \min\{J(\rho): \rho \in K\}$, that is, ρ_0 is a solution to problem (2.1).

We now prove that the existence of ρ_0 leads to a solution of Problem 1. The function τ, $\mathbf{R} \to [0, +\infty)$ defined by

$$\tau(\theta) = \int_\Omega g(x, S\rho_0 + \theta)\, dx,$$

is continuous, monotone increasing, and $\lim_{\theta\to-\infty} \tau(\theta) = 0$, $\lim_{\theta\to+\infty} \tau(\theta) > I$, by (1.4) and (1.7). There exists $\theta_0 \in \mathbf{R}$ such that $\tau(\theta_0) = I$.

Let $\zeta = g(x, S\rho_0 + \theta_0)$. $\zeta \in L^2(\Omega) \subset L^r(\Omega)$ and thus, $\zeta \in K$. We claim that $\zeta = \rho_0$. Indeed, let $\rho = \frac{1}{2}(\rho_0 + \zeta) = \rho_0 + \frac{1}{2}(\zeta - \rho_0) \in K$. Using the convexity of $G^*(x, \cdot)$ and the fact that $S: L^r(\Omega) \to L^{r'}(\Omega)$ is self-adjoint, we derive from $J(\rho_0) \leqslant J(\rho)$:

$$\int_\Omega G^*(x, \zeta)\, dx - \int_\Omega G^*(x, \rho_0)\, dx - \tfrac{1}{4}\int_\Omega S(\zeta - \rho_0).(\zeta - \rho_0)\, dx \geqslant \int_\Omega S\rho_0.(\zeta - \rho_0)\, dx. \tag{2.7}$$

Since $\partial G^*(x, \cdot)$ and $g(x, \cdot)$ are inverse graphs, $S\rho_0 + \theta_0 \in \partial G^*(x, \zeta)$. By the convexity of $G^*(x, \cdot)$, we then have

$$\int_\Omega G^*(x, \zeta)\, dx - \int_\Omega G^*(x, \rho_0)\, dx \leqslant \int_\Omega (S\rho_0 + \theta_0)(\zeta - \rho_0)\, dx. \tag{2.8}$$

Observing that $\int_\Omega (\zeta - \rho_0)\,dx = 0$, we derive from (2.7) and (2.8) that

$$\int_\Omega S(\zeta - \rho_0).(\zeta - \rho_0)\,dx \leqslant 0,$$

whence $\zeta = \rho_0$. (This fact could also have been obtained as a consequence of a more general result of Benilan and Brezis [19]). Thus,

$$\rho_0 = g(x, S\rho_0 + \theta_0). \tag{2.9}$$

Let $u_0 = S\rho_0 + \theta_0$. From (2.9) it follows that $\rho_0 \in L^2(\Omega)$; whence $u_0 \in H^2(\Omega)$. We have

$$\mathscr{L}u_0 = \rho_0 = g(x, u_0), \qquad \text{in } \Omega$$

$$u_0 = \theta_0, \qquad \text{on } \Gamma,$$

$$-\int_\Gamma \frac{\partial u_0}{\partial \nu}\,d\Gamma = \int_\Omega \mathscr{L}u_0\,dx = \int_\Omega \rho_0\,dx = I.$$

Hence, u_0 is a solution to Problem 1.

Remark. Similar methods to the one presented in this section have been used in [5] for a model of rotating stars, and in [19, 20] for the Thomas–Fermi equation.

Dual formation of the variational problem. Let E denote the space of functions of $H^1(\Omega)$ whose trace on Γ is a constant. Let

$$V = \left\{u \in E; \int_\Omega g(x, u)\,dx = I\right\}.$$

V is endowed with the $H^1(\Omega)$ topology

It is easily checked that under conditions (1.4) and (1.7), V is a non-empty subset of $H^1(\Omega)$. Let $a(.,.)$ be the symmetric bilinear form associated with $\mathscr{L}$:

$$a(u, v) = \sum_{i,j=1}^{N} \int_\Omega a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\,dx.$$

For $u \in V$, define

$$\Phi(u) = \tfrac{1}{2}a(u, u) - \int_\Omega G(x, u)\,dx + I\,u(\Gamma).$$

We consider the following variational problem:

$$\text{minimize}\,\{\Phi(u);\, u \in V\}. \tag{2.10}$$

This minimization problem has been introduced by Temam [10]. By solving (2.10), he obtained another proof of the existence result of Theorem 1, in the framework of the model case, that is, when $g(x, u) = \lambda u^+$. In fact, as pointed out by Damlamian [17], the two variational problems (2.1) and (2.10) are equivalent, due to the following proposition which is related to a general non convex duality principle of Toland [21, 22].

PROPOSITION. (i) $\min\{\Phi(u); u \in V\} = \min\{J(\rho); \rho \in K\}$.

(ii) If $\Phi(u_0) = \min\{\Phi(u); u \in V\}$, then $\rho_0 = g(x, u_0)$ satisfies $J(\rho_0) = \min\{J(\rho); \rho \in K\}$.

(iii) If $J(\rho_0) = \min\{J(\rho); \rho \in K\}$, then there exists $\theta_0 \in \mathbf{R}$ such that $u_0 = S\rho_0 + \theta_0 \in V$ and $\Phi(u_0) = \min\{\Phi(u); u \in V\}$.

Remark. The constant θ_0 given in (iii) is unique if one assumes $g(x, z)$ to be strictly increasing in $z > 0$.

Proof. (a) Let $\rho \in K$; there exists $u \in V$ such that $\Phi(u) \leqslant J(\rho)$. By (1.4), (1.7), there exists $\theta \in \mathbf{R}$ such that

$$\int_\Omega g(x, S\rho + \theta)\,\mathrm{d}x = I. \qquad \text{Let } u = S\rho + \theta; \quad u \in V.$$

By Young's inequality, $G(x, u) + G^*(x, \rho) \geqslant u\rho$, hence,

$$\Phi(u) \leqslant 1/2(a(u, u)) + \int_\Omega G^*(x, \rho)\,\mathrm{d}x - \int_\Omega u\rho\,\mathrm{d}x + Iu(\Gamma),$$

or

$$\Phi(u) \leqslant 1/2\left(\int_\Omega S\rho . \rho\,\mathrm{d}x\right) + \int_\Omega G^*(x,\rho)\,\mathrm{d}x - \int_\Omega (S\rho + \theta)\,\rho\,\mathrm{d}x + I\theta = J(\rho).$$

(b) Let $u \in V$: there exists $\rho \in K$ such that $J(\rho) \leqslant \Phi(u)$.
Let $\rho = g(\cdot, u)$; then $\rho \in L^r(\Omega)$ and $\rho \in K$. By Young's equality, we have

$$G(x, u) + G^*(x, \rho) = u\rho.$$

Thus,

$$\Phi(u) - J(\rho) = 1/2\left(\int_\Omega S\rho . \rho\,\mathrm{d}x\right) - \int_\Omega u\rho\,\mathrm{d}x + Iu(\Gamma) + 1/2(a(u, u)).$$

Let $v = u - u(\Gamma)$ and $w = S\rho$. We have

$$-\int_\Omega u\rho\,\mathrm{d}x + Iu(\Gamma) = -\int_\Omega v\rho\,\mathrm{d}x = -a(v, w).$$

Hence,

$$\Phi(u) - J(\rho) = 1/2(a(v - w, v - w)) \geqslant 0. \tag{2.11}$$

The proposition obviously results from (a) and (b).

Construction of a solution by an iteration scheme. The technique used in the preceding proof leads to an iteration scheme for solving Problem 1, as well as to another proof (using Schauder's fixed point theorem) for the existence of solution. We assume here that $g(x, z)$ is strictly increasing in $z > 0$, and that $\partial\Omega$ is smooth.

$$\text{Define } \mathscr{R}u = g(.,u); \qquad \mathscr{R} \text{ maps } V \text{ into } K.$$

Since the injection $H^1(\Omega) \hookrightarrow L^{pr}(\Omega)$ is compact, and $\mathscr{R}$ maps continuously $L^{pr}(\Omega)$ into $L^r(\Omega)$, it follows that $\mathscr{R}$: $V \to K$ is a compact operator (i.e. $\mathscr{R}$ is continuous and maps bounded sets of V

into relatively compact sets of K). Define $\mathscr{S}: K \to V$ by $\mathscr{S}\rho = S\rho + \theta$, where $\theta = \theta(\rho)$ is uniquely determined from

$$\int_\Omega g(x, S\rho + \theta)\,\mathrm{d}x = I.$$

The function $\rho \in K \mapsto \theta(\rho) \in \mathbf{R}$ is continuous. Indeed, by Fatou's lemma it is easily seen that if $\rho_n \to \rho$ in $L^r(\Omega)$, then $\theta(\rho_n)$ is bounded; furthermore, $S\rho_n \to S\rho$ in $L^p(\Omega)$ since $W^{2,r}(\Omega)$ is embedded in $L^p(\Omega)$. Thus, a subsequence of $S\rho_n + \theta(\rho_n)$ can converge only to $S\rho + \theta(\rho)$, which implies $\theta(\rho_n) \to \theta(\rho)$. Hence, $\mathscr{S}$ is continuous from K into $W^{2,r}(\Omega)$ with compact injection; $\mathscr{S}$ is a compact operator from K into V.

In the preceding paragraph we have seen that

$$\Phi(\mathscr{S}\rho) \leqslant J(\rho), \qquad \forall \rho \in K, \tag{2.12}$$

$$J(\mathscr{R}u) \leqslant \Phi(u), \qquad \forall u \in V. \tag{2.13}$$

In fact, equality holds in (2.12) (resp. (2.13)) if and only if $\mathscr{S}\rho$ (resp. u) is a solution to Problem 1. Indeed, from part (a) above, $\Phi(\mathscr{S}\rho) = J(\rho)$ holds if and only if $G(x, \mathscr{S}\rho) + G^*(x, \rho) = \mathscr{S}\rho\rho$. This is equivalent to $\rho = G'_u(x, \mathscr{S}\rho) = g(x, \mathscr{S}\rho)$, that is, $\mathscr{S}\rho$ is a solution (cf. the end of the proof of Theorem 2). From (2.11) we have

$$\Phi(u) - J(\mathscr{R}u) = 1/2(a(u - \mathscr{S}\mathscr{R}u, \quad u - \mathscr{S}\mathscr{R}u)). \tag{2.14}$$

Hence, $\Phi(u) = J(\mathscr{R}u)$ if and only if $u = \mathscr{S}\mathscr{R}u$ ($u - \mathscr{S}\mathscr{R}u$ is a constant which has to be zero), that is, u is a solution to Problem 1. Indeed, Problem 1 is clearly equivalent to finding a fixed point of the mapping $\hat{\mathscr{C}} = \mathscr{S}\mathscr{R}: V \to V$. The fixed points of $\hat{\mathscr{C}}$ are the images under $\mathscr{S}$ of the fixed points of $\mathscr{C} = \mathscr{R}\mathscr{S}$. $\mathscr{C}$ is a compact operator from the closed convex set K into itself. The existence of a solution to Problem 1 is thus obtained by Schauder's fixed point theorem. Indeed, it is a direct consequence of the next lemma that there exists a constant $C > 0$ such that $\mathscr{C}$ maps the intersection of K with the ball of radius C in $L^r(\Omega)$ into itself.

LEMMA 2.3. For any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$\|\mathscr{C}\rho\|_{L^r(\Omega)} \leqslant \varepsilon \|\rho\|_{L^r(\Omega)} + C_\varepsilon, \quad \forall \rho \in K.$$

Proof. Let $\zeta = \mathscr{C}\rho$, i.e. $\zeta = g(., u)$, with $u = S\rho + \theta$ and $\zeta \in K$. From (1.4) and (1.6), we know that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$0 \leqslant g(x, u) \leqslant C_\varepsilon + \varepsilon(u^+)^p, \qquad \forall x \in \bar{\Omega}, \quad u \in \mathbf{R}.$$

Hence,

$$\|g(\cdot, u)\|_{L^{p+1}(\Omega)} \leqslant C_\varepsilon + \varepsilon \|u^+\|^p_{L^{p(p+1)}(\Omega)}. \tag{2.15}$$

Observe that $p(p+1) = r^{**}$ (where $1/r^{**} = 1/r - 2/N$, in the case $N > 2$). Therefore, by Lemma 3.1 (see Section 3 below) and by the L^p estimate [23] for the operator $\mathscr{L}$, we have

$$\|u^+\|_{L^{p(p+1)}(\Omega)} \leqslant C \|\nabla u\|_{L^{r^*}(\Omega)} \leqslant C \|\rho\|_{L^r(\Omega)}. \tag{2.16}$$

The assumption on the regularity of $\partial\Omega$ was used here for applying the L^p estimate).

For (2.15), (2.16), we derive

$$\|\zeta\|_{L^{p+1}(\Omega)} \leqslant C_\varepsilon + \varepsilon \|\rho\|^p_{L^r(\Omega)}. \tag{2.17}$$

Since $1 < r < 2 < p + 1$, we have, by Hölder's inequality,

$$\|\zeta\|_{L^r(\Omega)} \leqslant I^{1-\theta}\|\zeta\|^{\theta}_{L^{p+1}(\Omega)},$$

with $1/r = 1 - \theta + \theta/(p+1)$, that is $\theta = 1/p$. Thus, (2.17) yields

$$\|\zeta\|_{L^r(\Omega)} \leqslant C_\varepsilon + \varepsilon\|\rho\|_{L^r(\Omega)}.$$

The preceding construction leads to an algorithm for finding a solution to Problem 1. Let u_0 be any function in V. Define

$$\begin{aligned} &\rho_1 = \mathscr{R}u_0, \quad u_1 = \mathscr{S}\rho_1, \ldots \\ &\rho_n = \mathscr{R}u_{n-1}, u_n = \mathscr{S}\rho_n, \ldots \text{ etc} \ldots \end{aligned} \tag{2.18}$$

The sequence u_n satisfies the induction equation

$$\begin{aligned} &\mathscr{L}u_n = g(x, u_{n-1}) \quad \text{in } \Omega \\ &u_{n|\Gamma} \text{ is a constant} \\ &\int_\Omega g(x, u_n)\,\mathrm{d}x = I. \end{aligned} \tag{2.19}$$

PROPOSITION. There exists a subsequence of (u_n) that converges strongly in $H^1(\Omega)$ to a solution of Problem 1. Furthermore, any convergent subsequence of (u_n) converges to a solution of Problem 1.

Proof. By (2.12), (2.13), we have

$$\Phi(u_0) \geqslant J(\rho_1) \geqslant \Phi(u_1) \geqslant \ldots \geqslant \Phi(u_{n-1}) \geqslant J(\rho_n) \geqslant \Phi(u_n) \geqslant J(\rho_{n+1}) \geqslant \ldots \tag{2.20}$$

From (2.4), we see that ρ_n is bounded in $L^r(\Omega)$.

Since $\mathscr{S}$ and $\mathscr{R}$ are compact operators, the sequences (u_n) and (ρ_n) are relatively compact in $H^1(\Omega)$ and $L^r(\Omega)$ respectively. Hence, there exists a subsequence ρ_{n_k} which converges strongly to ρ in $L^r(\Omega)$. Then, $u_{n_k} = \mathscr{S}\rho_{n_k}$ converges strongly to $u = \mathscr{S}\rho$ in $H^1(\Omega)$. Since Φ is continuous on $H^1(\Omega)$ and J is weakly lower semi-continuous on $L^r(\Omega)$, we have from (2.20):

$$\Phi(u) = \lim_{n \to +\infty} \Phi(u_n) = \lim_{n \to +\infty} J(\rho_n) \geqslant J(\rho).$$

But, by (2.12) we have

$$\Phi(u) = \Phi(\mathscr{S}\rho) \leqslant J(\rho).$$

Hence, $\Phi(\mathscr{S}\rho) = J(\rho)$, and $u = \mathscr{S}\rho$ is a solution to Problem 1.

Remark 1. Using (2.14), we see that

$$\sum_{n=0}^{+\infty} \|\nabla(u_n - u_{n+1})\|^2_{L^2(\Omega)} < +\infty.$$

Remark 2. It would be of interest, especially from the numerical viewpoint, to know under what conditions is the whole sequence (u_n) convergent to the same solution u.

3. SOME ESTIMATES

For the topological method we present in the next section, we require several inequalities.

In the next three lemmas, it is not necessary to assume that the domain Ω has a smooth boundary. We recall that

$$E = \{u \in H^1(\Omega);\ u_{|\Gamma} \text{ is a constant}\},$$

$$u^+ = \max(u, 0), \quad \text{and for } 1 < s < N, \quad 1/s^* = 1/s - 1/N, \quad \text{and} \quad s^* > 1 \quad \text{if} \quad s \geqslant N.$$

LEMMA 3.1. Suppose g satisfies conditions (1.4), (1.5) and (1.7). Then there exists a constant C such that for any $u \in E$ with $\int_\Omega g(x, u)\,\mathrm{d}x \leqslant I$, one has $\| u^+ \|_{L^{s*}} \leqslant C(\| \nabla u \|_{L^s} + 1)$.

Proof. We argue by contradiction and suppose the existence of a sequence $(u_n) \subset E$ such that

$$\int_\Omega g(x, u_n)\,\mathrm{d}x \leqslant I \qquad \text{and} \qquad \| u_n^+ \|_{L^{s*}} > n(\| \nabla u_n \|_{L^s} + 1).$$

Thus, in particular, $\| u_n^+ \|_{L^{s*}}$ converges to $+\infty$.

Let

$$v_n = \frac{u_n^+}{\| u_n^+ \|_{L^{s*}}},$$

so that $\| v_n \|_{L^{s*}} = 1$, and

$$\| \nabla v_n \|_{L^s} \leqslant \frac{\| \nabla u_n \|_{L^s}}{\| u_n^+ \|_{L^{s*}}} < 1/n$$

(cf. [24]). By the Sobolev inequality we have

$$\| v_n - v_n(\Gamma) \|_{L^{s*}} \leqslant C \| \nabla v_n \|_{L^s} < C/n.$$

Hence, $\| v_n(\Gamma) \|_{L^{s*}} \leqslant 1 + C/n$. Therefore, a subsequence $v_{nj}(\Gamma)$ converges to $l \in \mathbf{R}$ and $l \neq 0$ since $\| l \|_{L^{s*}} = 1$. Thus, $l > 0$ and since v_{n_j} converges to l almost everywhere, $u_{n_j}^+$ and consequently u_{n_j} converge to $+\infty$ a.e. in Ω. This is a contradiction, since by (1.7) and using Fatou's Lemma, it would lead to

$$I \geqslant \liminf_{nj \to +\infty} \int_\Omega g(x, u_{n_j})\,\mathrm{d}x > I.$$

LEMMA 3.2. Suppose g satisfies conditions (1.4) and (1.6). Then, for any $\varepsilon > 0$, there exists a constant C_ε such that for any $u \in E$, one has

$$\| g(., u) \|_{L^1}^{1/p} . u(\Gamma) \geqslant -\varepsilon \| \nabla u \|_{L^2}^2 - C_\varepsilon.$$

(C_ε depends only on ε, g and the measure of Ω).

Proof. Let $\gamma = u(\Gamma)$; we need only to consider the case $\gamma < 0$. By Sobolev's imbedding theorem, we know that

$$\| u - \gamma \|_{L^{2p}} \leqslant C \| \nabla u \|_{L^2},$$

(recall that $2p = 2^*$, when $N > 2$). Thus

$$\int_{\{x\in\Omega:\, u(x)\geq 0\}} (u^+ + |\gamma|^{2p})\,dx \leqslant C\|\nabla u\|_{L^2}^{2p}. \tag{3.1}$$

By (1.4), (1.6), there exists, for any $\varepsilon > 0$, a constant C_ε such that

$$g(x, z) \leqslant C_\varepsilon + \varepsilon(z^+)^p, \qquad \forall x \in \overline{\Omega}, \quad \forall z \in R. \tag{3.2}$$

Using the same notation C_ε for various constants depending on ε, we have

$$|\gamma|^p g(x, z) \leqslant \varepsilon((z^+)^{2p} + |\gamma|^{2p}) + C_\varepsilon|\gamma|^p,$$

hence

$$|\gamma|^p g(x, z) \leqslant \varepsilon(z^+ + |\gamma|)^{2p} + C_\varepsilon. \tag{3.3}$$

Thus, from (3.1) and (3.3) we derive

$$|\gamma|^p.\|g(.,u)\|_{L^1} \leqslant \varepsilon\|\nabla u\|_{L^2}^{2p} + C_\varepsilon,$$

which yields the inequality of the Lemma.

LEMMA 3.3. Suppose g satisfies conditions (1.4) and (1.6), then, for any $\varepsilon > 0$, there exists a constant C_ε such that for any $u \in H^1(\Omega)$, one has

$$\|g(.,u)\|_{L^{(2p)'}} \leqslant (\varepsilon\|u^+\|_{L^{2p}} + C_\varepsilon).\|g(.,u)\|_{L^1}^{1-1/p},$$

(where $1/(2p)' + 1/2p = 1$).

Proof. Since $1 < (2p)' < 2$. Hölder's inequality gives

$$\|g(.,u)\|_{L^{(2p)'}} \leqslant \|g(.,u)\|_{L^2}^{\theta}.\|g(.,u)\|_{L^1}^{1-\theta},$$

with $\theta/2 + (1-\theta)/1 = 1/(2p)'$, that is $\theta = 1/p$.
From (3.2) we derive

$$\|g(.,u)\|_{L^2} \leqslant \varepsilon\|u^+\|_{L^{2p}}^{p} + C_\varepsilon.$$

Therefore

$$\|g(.,u)\|_{L^2}^{\theta} \leqslant \varepsilon\|u^+\|_{L^{2p}} + C_\varepsilon,$$

which leads to the inequality of the lemma.

Remark 1. Let $\Lambda \subset (0, +\infty)$ be a compact interval, $\Lambda = [\underline{\lambda}, \bar{\lambda}]$. Suppose $\lim_{z\to+\infty} \underline{\lambda}\int_\Omega g(x,z)\,dx > I$. Let $g_t(x,z) = (1-t)\,g(x,z) + tz^+$, $t \in [0,1]$. It is easily seen that the results of lemmas 3.1, 3.2 and 3.3 remain valid if g is replaced by the function λg_t, *and this, in a uniform sense with respect to* $t \in [0,1]$ and $\lambda \in \Lambda$. That is, the various constants, the existence of which Lemmas 3.1–3.3 assert, can be chosen independently of $t \in [0,1]$ and of $\lambda \in \Lambda$. It just suffices to observe that λg_t satisfies (1.4)–(1.7). Furthermore, (1.6) and (1.7) hold uniformly with respect to $t \in [0,1]$ and $\lambda \in \Lambda$: For any $\varepsilon > 0$, there exists a constant C_ε such that $\lambda g_t(x,z) \leqslant \varepsilon(z^+)^p + C_\varepsilon$, $\forall x \in \overline{\Omega}$, $\forall x \in R$ and $\forall t \in [0,1]$, $\forall \lambda \in \Lambda$. There exists $I' > I$ and $A \in R$ such that $z \geqslant A$ implies $\int_\Omega \lambda g_t(x,z)\,dt \geqslant I'$, *for all* $t \in [0,1]$ and all $\lambda \in \Lambda$.

Combining the preceding lemmas, it is easy to show that all solutions of Problem 1 are *a priori* bounded:

LEMMA 3.4. Under conditions (1.4)–(1.7) for g, there exists a constant C such that for any solution u to Problem 1, one has $\|u\|_{H^1} \leqslant C$. Hence, the solutions of Problem 1 are *a priori* bounded in $W^{2,p}(\Omega)$ as well, for all $p > 1$.

Proof. By (1.8), one has $\|g(.,u)\|_{L^1} = I$. Denote $\gamma = u(\Gamma)$. From (1.1)–(1.3), one derives

$$\eta\|\nabla u\|_{L^2}^2 \leqslant \int_\Omega \mathscr{L}u.(u-\gamma)\,\mathrm{d}x = \int_\Omega g(x,u)\,u^+\,\mathrm{d}x + \gamma\int_\Gamma \frac{\partial u}{\partial v}\,\mathrm{d}\Gamma,$$

that is

$$\eta\|\nabla u\|_{L^2}^2 + \gamma I \leqslant \int_\Omega g(x,u)\,u^+\mathrm{d}x. \tag{3.4}$$

By Lemma 3.2 we have

$$\gamma \geqslant -\varepsilon\|\nabla u\|_{L^2}^2 - C_\varepsilon.$$

By Lemmas 3.1 and 3.3 we have,

$$\|g(.,u)\|_{L^{(2p)'}} \leqslant \varepsilon\|\nabla u\|_{L^2} + C_\varepsilon$$

and

$$\|u^+\|_{L^{2p}} \leqslant C(\|\nabla u\|_{L^2} + 1).$$

Hence, by Hölder's inequality

$$\int_\Omega g(x,u).u^+\,\mathrm{d}x \leqslant \varepsilon\|\nabla u\|_{L^2}^2 + C_\varepsilon. \tag{3.5}$$

From (3.4), we then have $(\eta - 2\varepsilon)\|\nabla u\|_{L^2}^2 \leqslant C_\varepsilon$. Choosing ε sufficiently small yields: $\|\nabla u\|_{L^2} \leqslant C$. Hence, by Lemma 3.2, $\gamma \geqslant -C$ and by (3.5), $\int_\Omega g(x,u).u^+\,\mathrm{d}x \leqslant C$. Thus, by (3.4) $\gamma \leqslant C$ which yields $|\gamma| \leqslant C$. Therefore, $\|u\|_{H^1} \leqslant C$. A standard bootstrap argument then shows that $\|u\|_{W^{2,p}}$ is bounded for all $p > 1$.

Remark 2. Let $\Lambda = [\underline{\lambda}, \bar{\lambda}] \subset (0, +\infty)$; assume that $\lim_{z\to+\infty} \int_\Omega \underline{\lambda} g(x,z)\,\mathrm{d}x > I$. Let $g_t(x,z) - (1-t)\,g(x,z) + tz^+$, $t \in [0,1]$. By using Remark 1 above and the preceding proof, it is easily checked that the same *a priori* estimate holds if g is replaced by the function λg_t, uniformly in $\lambda \in \Lambda$ and $t \in [0,1]$, (that is, the constant C can be chosen independently of $\lambda \in \Lambda$ and $t \in [0,1]$). This version of Lemma 3.4 will be used in the next section.

4. EXISTENCE OF SOLUTIONS: A TOPOLOGICAL METHOD

We now consider a non linear eigenvalue problem obtained from Problem 1 by allowing the right hand side in (1.1) to depend on a parameter $\lambda > 0$.

PROBLEM 2. Find $u \in H^2(\Omega)$ satisfying

$$\mathscr{L}u = \lambda g(x, u), \qquad \text{in } \Omega, \tag{4.1}$$

$$u_{|\Gamma} \text{ is a constant} \tag{4.2}$$

$$-\int_\Gamma \frac{\partial u}{\partial \nu}\, \mathrm{d}\Gamma = I. \tag{4.3}$$

We suppose g satisfies conditions (1.4)–(1.6). Condition (1.7), however, will be replaced in this section by the following assumption:

$$g \not\equiv 0. \tag{4.4}$$

We define λ^* by

$$\frac{I}{\lambda^*} = \lim_{z \to +\infty} \int_\Omega g(x, z)\, \mathrm{d}x, \quad \text{so that} \quad 0 \leqslant \lambda^* < +\infty.$$

Theorem 1 shows the existence of at least one solution to Problem 2 for any $\lambda > \lambda^*$. In this section, we establish the following result, which is more precise.

THEOREM 3. (i) Suppose g satisfies conditions (1.4)–(1.6) and (4.4). Then, for any compact interval $\Lambda \subset (\lambda^*, +\infty)$, there exists a connected component $\mathscr{C}_\Lambda$ of solutions (λ, u) to Problem 2 in $(\lambda^*, \infty) \times E$ such that the projection of $\mathscr{C}_\Lambda$ on $(\lambda^*, +\infty)$ covers Λ.

(ii) If, in addition, $\lambda^* = 0$ and $g(x, z)$ is Lipschitz continuous in $z \in R$, uniformly with respect to $x \in \overline{\Omega}$, then there exists a connected component $\mathscr{C}$ of solutions (λ, u) to Problem 2 in $\mathbf{R}^+ \times E$ whose projection on $\mathbf{R}^+$ covers all of $\mathbf{R}^+$.

Proof. The main idea in the proof of this theorem is to show that Problem 2 and a problem where $g(x, u) = u^+$ are homotopic. Then, we use a topological degree argument to establish an existence result for the latter problem.

A. Formulation of Problem 2 as an abstract functional equation

By the Lax–Milgram theorem, for any $f \in L^2(\Omega)$, there exists a unique $u \in E$, which we denote $u = Lf$, such that

$$a(u, \varphi) + \int_\Omega u\varphi\, \mathrm{d}x = \int_\Omega f\varphi\, \mathrm{d}x, \qquad \forall \varphi \in E.$$

It is easily checked that u is the unique solution to the problem

$$\begin{aligned} &\mathscr{L}u + u = f, \quad \text{in } \Omega, \\ &u \in E \\ &\int_\Gamma \frac{\partial u}{\partial \nu}\, \mathrm{d}\Gamma = 0. \end{aligned} \tag{4.5}$$

Clearly, $L = L^2(\Omega) \to L^2(\Omega)$ is a compact linear self-adjoint operator, and there exists a constant C such that

$$\| Lf \|_{H^2} \leqslant C \| f \|_{L^2}, \qquad \forall f \in L^2(\Omega). \tag{4.6}$$

Let $\mu_1 < \mu_2$ denote the first two characteristic values of L; we denote by λ_1 the first eigenvalue of the homogeneous Dirichlet problem in Ω:

$$\begin{aligned} &\mathscr{L}v = \lambda v, \quad \text{in } \Omega, \\ &v_{|\Gamma} = 0. \end{aligned} \tag{4.7}$$

The following information about μ_1 and μ_2 will be useful:

LEMMA 4.1. (i) $\mu_1 = 1$; μ_1 is simple and the associated eigenfunctions are the constants.
(ii) $\mu_2 > \lambda_1 + 1$.

Proof. Let $v = \mu L v$ with $\mu \leqslant \lambda_1 + 1$. This means that

$$\begin{aligned} &\mathscr{L}v = (\mu - 1)\, v, \quad \text{in } \Omega, \\ &v \in E; \quad \text{we denote } \gamma = v(\Gamma), \\ &\int_\Gamma \frac{\partial v}{\partial \nu} \mathrm{d}\Gamma = 0. \end{aligned} \tag{4.8}$$

Obviously, if $v \neq 0$, then $\mu \geqslant 1$. It is also easy to show that $\ker(I-L) = \ker(I-L)^2$ is the subspace of constant functions on Ω, so that $\mu_1 = 1$ is simple. Suppose now $1 < \mu \leqslant \lambda_1 + 1$. From (4.8) and the variational characterization of λ_1 [25], we have

$$\int_\Omega \mathscr{L}v(v - \gamma)\,\mathrm{d}x = (\mu - 1)\int_\Omega v(v - \gamma)\,\mathrm{d}x \geqslant \lambda_1 \int (v - \gamma)^2\,\mathrm{d}x.$$

Observing that $\int_\Omega v\,\mathrm{d}x = 0$, we derive

$$(\mu - 1)\int_\Omega v^2\,\mathrm{d}x \geqslant \lambda_1 \int_\Omega v^2\,\mathrm{d}x + \lambda_1 \gamma^2 |\Omega|,$$

where $|\Omega|$ is the measure of Ω. This inequality implies $\gamma = 0$. If $v \neq 0$, we also have $\mu = \lambda_1 + 1$. But then v is an eigenfunction of (4.7) associated to λ_1, and since $\int_\Omega v\,\mathrm{d}x = 0$, necessarily $v = 0$, a contradiction.

From condition (1.6), it follows that the operator $u \to g(.,u)$ acting from $H^1(\Omega)$ into $L^2(\Omega)$ is continuous, bounded, and even compact (*cf.* Section 2, Dual variational formulation). Therefore, $Ru = L(g(.,u))$ is well defined for any $u \in H^1(\Omega)$ and $R: E \to E$ is a compact (non linear) operator.

Let w_0 denote the unique solution of the problem

$$\begin{aligned} &\mathscr{L}w_0 + w_0 = 0, \quad \text{in } \Omega, \\ &w_0 \in E, \\ &-\int_\Gamma \frac{\partial w_0}{\partial \nu} \mathrm{d}\Gamma = I. \end{aligned} \tag{4.9}$$

(w_0 is given by $w_0 = tw$ with a suitable choice of $t \in R$, where w is the solution of the Dirichlet problem: $\mathscr{L}w + w = 0$ in Ω, and $w_{|\Gamma} = 1$).

Clearly, Problem 2 is equivalent to the functional equation

$$u \in E, \qquad u = \lambda Ru + Lu + w_0. \tag{4.10}$$

B. Computation of the topological degree

Let $\Phi_0(\lambda, u) = \lambda Ru + Lu + w_0$. We denote by B_C the ball in E, centered at the origin of radius C. Let $\Lambda \subset (\lambda^*, +\infty)$; by the *a priori* estimate of Lemma 3.4 (*cf.* also Remark 2), we know that there exists a constant $C > 0$ (depending on Λ) such that $u = \Phi_0(\lambda, u)$ and $\lambda \in \Lambda$ imply $\| u \|_{H^1} < C$. Hence the topological degree† $d(I - \Phi_0(\lambda, .), B_C, 0)$ is well defined and independent of $\lambda \in \Lambda$.

LEMMA 4.2. $d(I - \Phi_0(\lambda, .), B_C, 0) = -1, \quad \forall \lambda \in \Lambda.$

Proof. For $t \in [0, 1]$, we define

$$\Phi_t(\lambda, u) = \lambda[(1 - t)\, Ru + tL(u^+)] + Lu + w_0,$$

and consider the equation

$$u = \Phi_t(\lambda, u), \tag{4.11}$$

which is equivalent to

$$\begin{aligned} &\mathscr{L}u = \lambda g_t(x, u), \qquad \text{in } \Omega, \\ &u \in E, \\ &-\int_\Gamma \frac{\partial u}{\partial \nu}\, \mathrm{d}\Gamma = I, \end{aligned}$$

with $g_t(x, u) = (1 - t)\, g(x, u) + tu^+$. When $t = 0$, (4.11) reduces to (4.10), that is Problem 2, while for $t = 1$, (4.11) is equivalent to

$$\begin{aligned} &\mathscr{L}u = \lambda u^+, \qquad \text{in } \Omega, \\ &u \in E, \\ &-\int \frac{\partial u}{\partial \nu}\, \mathrm{d}\Gamma = I. \end{aligned} \tag{4.12}$$

By Remark 2 after Lemma 3.4, the constant C can be chosen so that $u = \Phi_t(\lambda, u)$ and $\lambda \in \Lambda$, $t \in [0, 1]$, always imply $\|u\|_{H^1} < C$. Hence, the topological degree $d(I - \Phi_t(\lambda, .), B_C, 0)$ is well defined for all $\lambda \in \Lambda$ and $t \in [0, 1]$. Clearly, this degree is independent of $\lambda \in \Lambda$ and $t \in [0, 1]$. Thus,

$$d(I - \Phi_0(\lambda, .), B_C, 0) = d(I - \Phi_1(\lambda, .), B_C, 0). \tag{4.13}$$

Again by Lemma 3.4, Remark 2, there exists a constant C_1 such that $u = \Phi_1(\sigma, u)$, and σ in between λ and λ_1, imply $\|u\|_{H^1} < C_1$. Hence, using the excision and homotopy invariance properties of the degree, we see that

$$d(I - \Phi^1(\lambda, .), B_C), 0 = d(I - \Phi_1(\lambda_1, .), B_{C_1}, 0). \tag{4.14}$$

†*cf.* [26]. The definitions and properties of the topological degree can also be found e.g. in [27].

For $t \in [0, 1]$, let $\psi_t: E \to E$ be the compact operator defined by $\psi_t(u) = (\lambda_1 + 1) Lu + t\lambda_1 L(u^-) + w_0$, (where $u^- = \max(-u, o)$); thus $\psi_1 = \Phi(\lambda_1, .)$. In the next lemma, we show that ψ_t is an admissible homotopy for computing the right hand side degree in (4.14).

LEMMA 4.3. Let v_1 be the unique eigenfunction of (4.7), associated with λ_1, such that $-\int_\Gamma(\partial v_1/\partial v)\,\mathrm{d}\Gamma = I$. For any $t \in [0, 1]$, v_1 is the unique solution of the equation $u - \psi_t(u) = 0$.

Proof. The relation $u = \psi_t(u)$ is equivalent to:

$$\mathscr{L}u = \lambda_1 u + \lambda_1 t u^-, \quad \text{in } \Omega \tag{4.15}$$

$$u \in E; \quad \text{we denote } \gamma = u(\Gamma),$$

$$-\int_\Gamma \frac{\partial u}{\partial v}\,\mathrm{d}\Gamma = I.$$

By Green's formula, we have

$$\int_\Omega (v_1 \mathscr{L}u - u\mathscr{L}v_1)\,\mathrm{d}x = \lambda_1 t \int_\Omega u^- v_1\,\mathrm{d}x = -\gamma I.$$

Hence, $\gamma \leqslant 0$, so that $u^+ \in H_0^1(\Omega)$ (*cf.* [24]). Multiply the first equation in (4.15) by u^+ to obtain

$$a(u^+, u^+) = \lambda_1 \int_\Omega (u^+)^2\,\mathrm{d}x. \tag{4.16}$$

By the variational characterization of λ_1 as the first eigenvalue of (4.7) (cf. [25]), it follows from (4.16) that $u^+ = kv_1$, $k \geqslant 0$. Since from (4.15) we also have

$$I = \int_\Omega \mathscr{L}u\,\mathrm{d}x = \lambda_1 \int_\Omega u^+\,\mathrm{d}x - \lambda_1(1 - t)\int_\Omega u^-\,\mathrm{d}x,$$

necessarily $k \neq 0$, whence $u = kv_1$, since $v_1 > 0$ in Ω. Then, $I = k\lambda_1 \int_\Omega v_1\,\mathrm{d}x$ shows that $k = 1$, or $u = v_1$.

Proof of Lemma 4.2. By the preceding lemma, the Leray–Schauder index $i(I - \psi_t, v_1, 0)$ is well defined and independent of $t \in [0, 1]$. Thus,

$$d(I - \Phi_1(\lambda_1, .), B_{C_1}, 0) = i(I - \psi_0, v_1, 0). \tag{4.17}$$

Let $u = v + v_1$; one has $u - \psi_0(u) = v - (\lambda_1 + 1) Lv$. Thus, by translating the index computation to 0, we obtain

$$i(I - \psi_0, v_1, 0) = i(I - (\lambda_1 + 1) L, 0, 0) = (-1)^\beta, \tag{4.18}$$

where β denotes the sum of the multiplicities of the characteristic values of L in $(0, \lambda_1 + 1)$, (*cf.* [26]). Hence, by Lemma 4.1, $\beta = 1$, which yields Lemma 4.2.

C. *Proof of Theorem* 3

Since $d(I - \Phi_0(\lambda, .)\,B_C, 0) \neq 0$, $\lambda \in \Lambda$, by a theorem of Leray–Schauder [26], there exists a connected component $\mathscr{C}_\Lambda$ of solutions to Problem 2 in $(\lambda^*, +\infty) \times E$ whose projection on $(\lambda^*, +\infty)$ covers Λ.

The second part in Theorem 3 is a simple consequence of part (i). Indeed, suppose that

$$|g(x, z) - g(x, z')| \leqslant K|z - z'|, \qquad \forall x \in \bar{\Omega}, \qquad \forall z, z' \in \mathbf{R}.$$

Choose a λ_0 such that $0 < \lambda_0 < \lambda_2^*/K$, where $\mu_2 = \lambda_2^* + 1$ is the second characteristic value of L. Then, the function $\lambda_0 g$ satisfies the assumption of Theorem 4 in section 5 below. Therefore the solution to Problem 2 corresponding to $\lambda = \lambda_0$ is unique; we call it u_0. Let $\mathscr{C}$ be the connected component of the set of solutions to Problem 2 in $\mathbf{R}^+ \times E$ which contains (λ_0, u_0). Clearly, for any compact interval $\Lambda \subset (0, +\infty)$ such that $\lambda_0 \in \Lambda$, the component Λ must contain (λ_0, u_0) and therefore coincides with $\mathscr{C}$. Hence, the projection of $\mathscr{C}$ on $\mathbf{R}^+$ covers all of $\mathbf{R}^+$.

5. A UNIQUENESS RESULT

Let $\lambda_1^* < \lambda_2^*$ denote the two first eigenvalues of the problem

$$\begin{aligned} &\mathscr{L}v = \lambda v, \qquad \text{in } \Omega, \\ &v \in E, \\ &\int_\Gamma \frac{\partial v}{\partial \nu}\, d\Gamma = 0. \end{aligned} \tag{5.1}$$

Thus, by Lemma 4.1, $\lambda_1^* = 0 < \lambda_1 < \lambda_2^* = \mu_2 - 1$. λ_2^* is characterized by the following inequality (cf. e.g. [28]):

$$a(w, w) \geqslant \lambda_2^* \|w\|_{L^2}^2, \qquad \forall w \in E \quad \text{such that} \int_\Omega w\, dx = 0. \tag{5.2}$$

Equality holds in (5.2) if and only if w is an eigenfunction of (5.1) associated with λ_2^*.

The main result of this section is the following uniqueness result.

THEOREM 4. Suppose $g: \bar{\Omega} \times \mathbf{R} \to \mathbf{R}$ is continuous and satisfies the following two conditions:

(i) $\exists \bar{x} \in \Omega$ such that $g(\bar{x}, z) < g(\bar{x}, z')$, $\quad \forall z, z', 0 \leqslant z < z'$.

(ii) $|g(x, z) - g(x, z')| \leqslant K|z - z'|$, $\quad \forall x \in \bar{\Omega}, \forall z, z' \in \mathbf{R}$, with $0 < K < \lambda_2^*$. Then, there exists at most one solution to Problem 1.

Proof. Under the above hypotheses one has

$$|g(x, z) - g(x, z')|^2 \leqslant K(g(x, z) - g(x, z'))(z - z'), \qquad \forall x \in \bar{\Omega}, \qquad \forall z, z' \in \mathbf{R}.$$

Let u and $\hat{u}$ be two solutions of Problem 1. The preceding observation leads to

$$a(u - \hat{u}, u - \hat{u}) = \int_\Omega \mathscr{L}(u - \hat{u}).(u - \hat{u})\, dx \geqslant \frac{1}{K}\|g(., u) - g(., \hat{u})\|_{L^2}^2. \tag{5.3}$$

Let $u = t + w$, $\hat{u} = \hat{t} + \hat{w}$ be the decompositions of u and $\hat{u}$ along $E = \mathbf{R} \oplus E_1$, where $E_1 = \{w \in E; \int_\Omega w\, dx = 0\}$, that is $t, \hat{t} \in \mathbf{R}$; $w, \hat{w} \in E_1$. (t and $\hat{t}$ are the respective averages of u and $\hat{u}$ in Ω). From (5.2), one has

$$\lambda_2^* \|w - \hat{w}\|_{L^2}^2 \leqslant a(w - \hat{w}, w - \hat{w}) = a(u - \hat{u}, u - \hat{u}). \tag{5.4}$$

Since $\int_\Omega \mathscr{L}(u - \hat{u})\,\mathrm{d}x = 0$, we observe that

$$a(u - \hat{u}, u - \hat{u}) = \int_\Omega \mathscr{L}(u - \hat{u}).(u - \hat{u})\,\mathrm{d}x = \int_\Omega [g(x, u) - g(x, \hat{u})][w - \hat{w}]\,\mathrm{d}x. \tag{5.5}$$

Combining (5.4) and (5.5) yields

$$\| w - \hat{w} \|_{L^2} \leqslant \frac{1}{\lambda_2^*} \| g(.,u) - g(.,\hat{u}) \|_{L^2}.$$

Thus, from (5.5) we derive

$$a(u - \hat{u}, u - \hat{u}) \leqslant \frac{1}{\lambda_2^*} \| g(.,u) - g(.,\hat{u}) \|_{L^2}^2. \tag{5.6}$$

Comparing (5.3) and (5.6) yields $g(x, u) = g(x, \hat{u})$, which implies $u - \hat{u}$ is a constant, whence $u = \hat{u}$.

Remarks. (1) In the model case considered in the next section, where $g(x, u)$ has the form λu^+, Temam [10] and Puel and Damlamian [11] have established the uniqueness of the solution to Problem 1 for $0 < \lambda < \lambda_2$, λ_2 being the second eigenvalue of the homogeneous Dirichlet problem (4.7). It would be of interest to know whether the result of Theorem 4 can be extended by letting the Lipschitz constant K satisfy $0 < K < \lambda_2$. It should be noted that one always has $\lambda_1 < \lambda_2^* \leqslant \lambda_2$, (*cf.* [18]). Indeed, let v_1 and v_2 be two eigenfunctions of (4.7) respectively associated to λ_1 and λ_2. Let $w = tv_1 + t_2$, $t \in R$ being chosen such that $\int_\Omega w\,\mathrm{d}x = 0$. Applying the characterization (5.2), one has

$$a(w, w) = t^2 a(v_1, v_1) + a(v_2, v_2) \geqslant \lambda_2^*(t^2 \| v_1 \|_{L^2}^2 + \| v_2 \|_{L^2}^2).$$

Since $\lambda_2^* > \lambda_1$ and $a(v_i, v_i) = \lambda_i \| v_i \|_{L^2}^2$, $i = 1, 2$, one derives $\lambda_2 \| v_2 \|_{L^2}^2 \geqslant \lambda_2^* \| v_2 \|_{L^2}^2$, whence $\lambda_2^* \leqslant \lambda_2$.

(2) Those results about uniqueness are related to the more general works of Ambrosetti and Prodi [29], Berger and Podolak [30] and Fucik [31].

6. SOME ADDITIONAL REMARKS ON THE MODEL CASE

The following equations constitute a simple model for problems of type 1 or 2.

PROBLEM 3.

$$-\Delta u = \lambda u^+, \quad \text{in } \Omega,$$

$$u \in E,$$

$$-\int_\Gamma \frac{\partial u}{\partial n}\,\mathrm{d}\Gamma = I.$$

Here, $\partial./\partial n$ is the outward normal derivative on Γ.

A. Condition for the existence of a free boundary

Let (λ, u) be a solutjon to Problem 3. Let $\Omega_p = \{x \in \Omega;\, u(x) > 0\}$ and $\Gamma_p = \partial\Omega_p$. Γ_p is called the free boundary. If $u(\Gamma) \geqslant 0$, then by the maximum principle, $u(x) > 0$ in Ω, and hence there is no free boundary. On the other hand, if $u(\Gamma) < 0$, then there exists a free boundary since $u^+ \not\equiv 0$

(indeed, $\int_\Omega u^+ \, dx > 0$). We denote by λ_1 the first eigenvalue of the homogeneous Dirichlet problem in Ω:

$$\begin{aligned} -\Delta v &= \lambda v, \quad \text{in } \Omega, \\ v_{|\Gamma} &= 0, \end{aligned} \tag{6.1}$$

and by v_1 the unique eigenfunction associated with λ_1, that satisfies $-\int_\Gamma (\partial v_1/\partial n) \, d\Gamma = I$. The existence of a free boundary is determined by the position of λ with respect to λ_1.

PROPOSITION 1. Let (λ, u) be a solution to Problem 3.

(i) If $0 < \lambda < \lambda_1$, then $u(x) \geqslant u(\Gamma) > 0$, $\quad \forall x \in \bar{\Omega}$.

(ii) If $\lambda = \lambda_1$, then $v = v_1$.

(iii) If $\lambda > \lambda_1$, then $u(\Gamma) < 0$: there is a free boundary.

Proof. We denote $\gamma = u(\Gamma)$.

(i) Suppose $0 < \lambda \leqslant \lambda_1$ and $\gamma < 0$. Then (*cf.* [24]) $u^+ \in H_0^1(\Omega)$. Using the variational characterization of λ_1, we have

$$\lambda_1 \int_\Omega (u^+)^2 \, dx \leqslant \int_\Omega |\nabla u^+|^2 \, dx = \lambda \int_\Omega (u^+)^2 \, dx \tag{6.2}$$

Since equality must hold in (6.2), we derive that $u^+ = Cv_1$, $C > 0$. Hence, $u = Cv_1$, which implies $\gamma = 0$, a contradiction. Thus, for $0 < \lambda < \lambda_1$, we have $\gamma \geqslant 0$ and $u = u^+$. If $0 < \lambda < \lambda_1$, since λ is not an eigenvalue of (6.1), $\gamma \neq 0$; therefore $\gamma > 0$.

(ii) Suppose $\lambda \geqslant \lambda_1$ and $\gamma > 0$. By Green's formula one has

$$-\gamma I = \int_\Omega (u\Delta v_1 - v_1 \Delta u) \, dx = \int_\Omega (\lambda u^+ v_1 - \lambda_1 u v_1) \, dx \geqslant 0,$$

thus $\gamma \leqslant 0$. Now, if $\lambda > \lambda_1$, we have $\gamma < 0$. Indeed $\gamma = 0$ would imply that u is a positive eigenfunction of (6.1) associated to $\lambda > \lambda_1$, which is impossible.

(iii) Let $\lambda = \lambda_1$. From what precedes, we know that $\gamma = 0$. Hence, u is an eigenfunction of (6.1). The fact that λ_1 is simple then shows that $u = v_1$.

Remark. The conditions for the existence of a free boundary in the more general Problems 1 or 2 are not yet clear.

B. The question of uniqueness

Two interesting examples of non uniqueness in Problem 3 have been given by Schaeffer [12, 13]. The problem of uniqueness remains open however when the domain Ω is assumed to have some geometrical properties. We mention in particular the question of knowing whether the solution is unique in the case of a convex domain Ω. The nature of the set of solutions needs to be clarified. Does the non uniqueness correspond to bifurcation points or to separate branches?. In the particular case when Ω is a ball in $\mathbf{R}^N$, Gallouët [18] has shown that the variational solution is unique and is radial. Furthermore, there is no other radial solution, and no bifurcation can occur from the branch of radial solutions†.

Another important aspect of this problem which remains open, concerns the study of the stability of the solutions.

† Actually, using a recent result of Gidas, Wei-Ming Ni and Nirenberg [32], we know that the solution to problem 3 is unique (and thus radial), when Ω is a ball centered at the origin.

APPENDIX (added in proofs)

CONNECTEDNESS OF Ω_p FOR VARIATIONAL SOLUTIONS

Under an additional convexity hypothesis on the nonlinear term $g(x,\cdot)$, we prove in this appendix that the "plasma region" $\Omega_p = \{x \in \Omega; u(x) > 0\}$ is connected when u is a solution of problem 1 obtained by solving the variational problem (2.1) or (2.10). Since these two problems are equivalent, we will work with the formulation (2.10). We assume the following:

The function $s \mapsto g(x, s)$ is of class C^2 and strictly convex in $s \in (0, +\infty)$, for any fixed $x \in \Omega$. (A.1)

Proposition. Suppose g satisfies conditions (1.4)–(1.7) and (A.1). Then, if u is a solution of (2.10), the region $\Omega_p = \{x \in \Omega; u(x) > 0\}$ is connected.

Remark. This proposition complements a preceding result of [15] establishing the connectedness of Ω_p in the particular case when $g(x, u) = \lambda u^+$. The same method can also be adapted to generalize the results in [6, 7] concerning connectedness in the context of the free boundary problem arising in vortex rings theory.

Proof of Proposition. We use an idea similar to that of [15]. Suppose A and B are two distinct components of Ω_p: $A, B \subset \Omega_p$, $A \neq \emptyset$, $B \neq \emptyset$, and $\bar{A} \cap \bar{B} = \emptyset$. Then, $u > 0$ in $A \cup B$ and $u = 0$ on $\partial A \cup \partial B$,

For $\alpha, \beta \geqslant 0$, define $w = w(\alpha, \beta)$ by setting

$$w = \begin{cases} \alpha u \text{ in } A \\ \beta u \text{ in } B \\ u \text{ in } \Omega - (A \cup B). \end{cases} \tag{A.2}$$

Thus, $w \in E$ (for $u = 0$ on ∂A and ∂B) and $w_{|\Gamma} = u_{|\Gamma}$. The condition $w \in K$ reads:

$$\int_A g(x, \alpha u)\, dx + \int_B g(x, \beta u)\, dx = \int_{A \cup B} g(x, u)\, dx \tag{A.3}$$

From (A.3), the conditions $w \in K$ is clearly seen to be equivalent to $\beta = \varphi(\alpha)$, where φ is a monotone, strictly decreasing function, from an interval $[0, \alpha^*]$ onto an interval $[0, \beta^*]$ such that $\varphi(0) = \beta^* > 1$ and $\varphi(\alpha^*) = 0$, $\alpha^* > 1$. Observe that $\varphi(1) = 1$. Furthermore, using hypothesis (A.1) and the implicit function theorem one sees that φ is of class C^2. Denote $w = w(\alpha)$ the function constructed in (A.2) when choosing $\beta = \varphi(\alpha)$. Thus, $w(\alpha) \in K, \forall \alpha \in [0, \alpha^*]$ and $1 < \alpha^*$; in particular, $w(1) = u$.

Define

$$D(\alpha) = J(w(\alpha)) - J(u),$$

and let

$$m_\omega(\varphi) = \sum_{i,j=1}^{N} \int_\omega a_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j} dx$$

for $\omega c \Omega$ and $\varphi \in H^1(\Omega)$. Then,

$$D(\alpha) = \tfrac{1}{2}[\alpha^2 - 1] m_A(u) + \tfrac{1}{2}[\varphi(\alpha)^2 - 1] m_B(u)$$
$$- \int_A \{G(x, \alpha u) - G(x, u)\} dx - \int_B \{G(x, \varphi(\alpha) u) - G(x, u)\}\, dx.$$

Hence,

$$D'(\alpha) = \alpha m_A(u) - \int_A g(x, \alpha u) u\, dx + \varphi'(\alpha) \{\varphi(\alpha) m_B(u) - \int_B g(x, \varphi(\alpha) u) u\, dx\}.$$

$$D''(1) = \int_A \{g(x, u) - g'(x, u) u\} u\, dx + \varphi'(1)^2 \int_B \{g(x, u) - g'(x, u) u\} u\, dx$$

Observe that

$$\left\{ \begin{aligned} m_A(u) &= \int_A g(x, u) u\, dx. \\ m_B(u) &= \int_B g(x, u) u\, dx. \end{aligned} \right\} \tag{A.4}$$

Indeed, since $u = 0$ on ∂A and ∂B, (A.4) obtains from multiplying the equation $\mathcal{L}_u = g(x, u)$ by u and integrating by parts on A and B respectively. Using (A.4) we derive $D(1) = D'(1) = 0$ and

$$D''(1) = \int_A \{g(x,u) - g'(x,u)u\}u \, dx + \varphi'(1)^2 \int_B \{g(x,u) - g'(x,u)u\}u \, dx$$

From hypothesis (A.1), it follows that $g(x, u) - g'(x, u)u < 0$ for $x \in A \cup B$.

Hence, $D(1) = D'(1) = 0$ and $D''(1) < 0$ which contradicts the fact that $D(\alpha) \geqslant 0$; $\forall\alpha$ (since u is a solution of (2.10)). This concludes the proof of the proposition.

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