COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. XXX, 1–11 (1977)

Some First-Order Nonlinear Equations on a Torus*

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Dedicated to Carl Ludwig Siegel
on the occasion of his 80th birthday

1. The Main Result

We consider here real functions of $x=(x_1, \cdots, x_n)$ in $\mathbb{R}^n$ which are periodic of period 2π in each variable, i.e., functions defined on the torus Ω, and on Ω we consider the constant coefficient operator

$$Au=\sum a^j \frac{\partial u}{\partial x_j}, \qquad a^j \in R.$$

Let $g:\Omega\times\mathbb{R}\to\mathbb{R}$ be a C^∞ function periodic in x such that

(1) $$g_u(x,u)>0 \quad \text{for all} \quad x, u.$$

Our purpose is to find a real C^∞ periodic function u on the torus satisfying the first-order differential equation

(2) $$Au+g(x,u)=0.$$

We shall give necessary and sufficient conditions for a solution to exist. In the study of such problems one usually encounters difficulty with small divisors—in trying to invert A using Fourier series. It is because of (1) that this difficulty can be avoided. Let $N(A)=\{u\in L^2 \mid Au=0\}$ (to be understood in the distribution sense). Let P denote the L^2 projection on $N(A)$. P has the important property that $Pf\geqq 0$ when $f\geqq 0$ (this follows from the fact that $P=\lim_{\lambda\to+\infty}(I+\lambda A)^{-1}$). Since $P(1)=1$, P is a contraction in L^∞. Our main result is the following.

THEOREM 1. *Equation* (2) *has a* (*unique*) C^∞ *solution if and only if*

(3) *there exist constants* $\delta>0$, $\underline{M}$, $\bar{M}$ *such that*

$$Pg(x,\bar{M})\geqq\delta, \qquad Pg(x,\underline{M})\leqq-\delta,$$

at every point of Ω.

* The second author was partially supported by the Army Research Office Grant No. DAHCO4-75-G-0149, and by the John Simon Guggenheim Memorial Foundation. Reproduction in whole or in part is permitted for any purpose of the U.S. Government.

Observe first that since $(Au, u)=0$ for every u, it follows easily that $R(A)$ and $N(A)$ are orthogonal.

Proof: *Necessity.* If u is a solution of (2) we have

$$Pg(x, u)=0\,.$$

Let $\bar{M}>\sup u$, $\underline{M}<\inf u$; then for some $\delta>0$

$$g(x, \underline{M})+\delta \leqq g(x, u) \leqq g(x, \bar{M})-\delta\,.$$

Therefore,

$$Pg(x, \underline{M})+\delta \leqq Pg(x, u) \leqq Pg(x, \bar{M})-\delta$$

by the properties of P mentioned earlier.

Uniqueness follows from the maximum principle.

Sufficiency. To prove existence we approximate (2) by

$$(2_\varepsilon) \qquad -\varepsilon\Delta u_\varepsilon+\varepsilon u_\varepsilon+Au_\varepsilon+g(x, u_\varepsilon)=0\,, \qquad \varepsilon>0\,,$$

and we show that (2_ε) has a solution. First we derive an *a priori* bound for the solution of (2_ε) with the aid of the maximum principle, used in a slightly unusual manner. The following lemma plays an essential role.

LEMMA 1. *The set of functions of the form*

$$Av+\zeta\,, \qquad v\in C^\infty\,, \qquad \zeta\in C^\infty\cap N(A)\,,$$

is dense in the space C of continuous functions on Ω.

Proof: If $f\in C$, we may approximate f arbitrarily closely in the maximum norm by a real finite trigonometric sum

$$\tilde{f}=\sum_{|k|\leqq N} c_k e^{ik\cdot x}\,.$$

Here $k=(k_1, \cdots, k_n)$ represents a multi-index of integers, $k\cdot x=\sum k_j x_j$, and $|k|=\sum |k_j|$. Then we have

$$\tilde{f}=Av+\zeta\,,$$

with

$$v=\frac{1}{i}\sum_{k\in J}\frac{c_k}{a\cdot k}e^{ik\cdot x}\,, \qquad \zeta=\sum_{k\in J'} c_k e^{ik\cdot x}\,,$$

where

$$J=\{k \mid |k|\leqq N,\ a\cdot k\neq 0\}\,,$$
$$J'=\{k \mid |k|\leqq N,\ a\cdot k=0\}\,.$$

Note that since $\tilde{f}$ is real-valued, so are v and ζ.

A BOUND ON u_ε. Using the lemma we write

$$g(x,\bar{M})=-Av+\zeta+R$$

with $|R|<\frac{1}{4}\delta$ and $\zeta\in N(A)$. From (3) we see that $\zeta+PR\geqq\delta$ and hence $\zeta\geqq\frac{3}{4}\delta$. From (2_ε) we have

$$-\varepsilon\Delta u_\varepsilon+\varepsilon u_\varepsilon+Au_\varepsilon+g(x,u_\varepsilon)=Av-\zeta-R+g(x,\bar{M}),$$

that is

$$\begin{aligned}-\varepsilon\Delta(u_\varepsilon-v)+\varepsilon(u_\varepsilon-v)+A(u_\varepsilon-v)+g(x,u_\varepsilon)&\\ =g(x,\bar{M})-\zeta-R+\varepsilon\Delta v-\varepsilon v&<g(x,\bar{M}),\end{aligned}$$

provided ε is small enough.

We wish to estimate u_ε from above. At the point where $(u_\varepsilon-v)$ take its maximum (which we may suppose is positive) we have $g(x,u_\varepsilon)<g(x,\bar{M})$, so $u_\varepsilon\leqq\bar{M}$. Therefore, $u_\varepsilon-v\leqq\bar{M}+\max|v|$ everywhere and hence in any case $u_\varepsilon\leqq|\bar{M}|+2\max|v|$. Similarly we obtain an estimate from below and therefore $|u_\varepsilon|\leqq M$ independent of ε. (We can always assume that $M>\max(|\bar{M}|,|\underline{M}|)$.)

Existence. We prove now that (2_ε) has a solution for ε small by a standard truncation. Let $\tilde{g}(x,u)$ be a C^∞ bounded function with $g_u>0$ which agrees with g for $|u|\leqq M$. With the aid of the Schauder fixed point theorem and standard regularity results for elliptic equations it follows that there exists a C^∞ solution u_ε of

$$-\varepsilon\Delta u_\varepsilon+\varepsilon u_\varepsilon+Au_\varepsilon=-\tilde{g}(x,u_\varepsilon).$$

This is because the linear elliptic operator on the left-hand side is invertible. By the maximum principle the solution is unique. Since $\tilde{g}(x,\bar{M})=g(x,\bar{M})$ and $\tilde{g}(x,\underline{M})=g(x,\underline{M})$, assumption (3) holds. Therefore, $|u_\varepsilon|\leqq M$ and thus u_ε is also a solution of (2_ε). Using the maximum principle we now obtain bounds for all derivatives of u_ε independent of ε. Indeed, if we differentiate (2_ε) with respect to x_i we find

$$-\varepsilon\Delta\left(\frac{\partial}{\partial x_i}u_\varepsilon\right)+\varepsilon\frac{\partial}{\partial x_i}u_\varepsilon+A\left(\frac{\partial}{\partial x_i}u_\varepsilon\right)+g_u(x,u_\varepsilon)\frac{\partial}{\partial x_i}u_\varepsilon=-g_{x_i}(x,u_\varepsilon).$$

Since $g_u(x,u_\varepsilon)\geqq\alpha>0$ and $|g_{x_i}(x,u_\varepsilon)|$ is bounded, we obtain a bound for $\max(\partial/\partial x_i)u_\varepsilon$ and for $\min(\partial/\partial x_i)u$. If we keep differentiating (2_ε) we see that the C^k norm of u_ε is bounded independent of ε, for $k=1,2,\cdots$. We may therefore let $\varepsilon\to 0$ through a sequence, and obtain a limit solution $u\in C^\infty$ concluding the proof of Theorem 1.

We illustrate Theorem 1 by two examples.

EXAMPLE 1. Assume $a^1, a^2, \cdots, a^n$ are linearly independent over the rationals (i.e., $a \cdot k = 0$ with $k \in Z^n$ implies $k = 0$). Then (2) has a solution if and only if there exist $\underline{M}$ and $\bar{M}$ such that

$$\int_\Omega g(x, \bar{M})\, dx > 0 \qquad \text{and} \qquad \int_\Omega g(x, \underline{M})\, dx < 0 .$$

Indeed, in this case, $N(A)$ is reduced to constant functions so that Pf is the average of f over Ω.

EXAMPLE 2. Suppose $n = 2$, $a^1 = a^2 = 1$. Then (2) has a solution if and only if there exist $\underline{M}$, $\bar{M}$, $\delta > 0$ such that, for all $r \in [0, 2\pi]$,

$$\int_0^{2\pi} g(r+s, s, \bar{M})\, ds \geqq \delta , \qquad \int_0^{2\pi} g(r+s, s, \underline{M})\, ds \leqq -\delta .$$

In this case, $N(A)$ consists of all functions of the form $\phi(x_1 - x_2)$, $\phi \in L^2$. Here, characteristics of A are closed curves (on the torus) and in fact (2) can be solved along the characteristics.

It was pointed out to us by Jürgen Moser that Theorem 1 yields the existence of a C^∞ invariant n-dimensional torus in $R^{n+1} : (x, y)$, $x \in R^n$, $y \in R^1$,

$$\begin{aligned} \dot{x}^j &= a^j , && j = 1, \cdots, n, \\ \dot{y} &= -g(x, y) , && y \in R^1 , \end{aligned}$$

with g as in the theorem. The torus is given by $y = u(x)$, where u is our solution. Many authors have studied perturbation problems of the following form: Given a flow with an invariant surface, does there exist a neighbouring invariant surface for a slightly perturbed flow? If it does, how smooth is it? (See for instance Fenichel [4].) Our result gives some information in a nonperturbation problem but for a very special flow.

Remark. Since the operators A and $Bu = g(x, u)$ are monotone, one could try to use the results of [2], [3]. They lead indeed to the fact that $\mathrm{Int}_{L^2} R(A+B) = \mathrm{Int}_{L^2} [R(A) + R(B)]$. However, in most cases, $R(A) + R(B)$ has empty interior in L^2. Consider Example 2 with g independent of x, continuous, nondecreasing, $|g|$ bounded; set $g_\pm = \lim_{u \to \pm\infty} g(u)$. Suppose that

$f(x) \in R(A+B)$; then one has necessarily, for every $r \in [0, 2\pi]$,

$$g_- \leqq \frac{1}{2\pi} \int_0^{2\pi} f(r+s, s)\, ds \leqq g_+ .$$

But the set of all such f has empty interior in L^2.

2. Generalizations

1. It is clear from the proof that Theorem 1 is valid if we replace $A = \sum a^i \partial/\partial x_i$ by any constant coefficient operator

$$A = -\sum a^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum a^i \frac{\partial}{\partial x_i}$$

with

$$\sum a^{ij} \xi_i \xi_j \geqq 0 \quad \text{for all} \quad \xi \in R^n .$$

2. Instead of assuming g is C^∞ and $g_u > 0$, assume that g is continuous in (x, u) and nondecreasing in u.

THEOREM 2. *Assume g is continuous in (x, u) and nondecreasing in u. If* (3) *holds, then* (2) *has at least one solution $u \in L^\infty$.*

By a solution of (2) we now mean a function $u \in L^\infty$ such that $Au \in L^\infty$ (in the distribution sense) and (2) holds a.e.

Indeed, we solve (2_ε) as above (here $u_\varepsilon \in W^{2,p}$ for every $p < \infty$) and we get the bound $|u_\varepsilon| \leqq M$ with the aid of Stampacchia's form of the maximum principle (cf. [5], Chapter 8). Thus we can find a sequence $\varepsilon_j \to 0$ such that $u_{\varepsilon_j} \rightharpoonup u$ in weak* L^∞. Multiplying (2_ε) by Au_ε and integrating over Ω leads to

$$\int_\Omega (|Au_\varepsilon|^2 + g(x, u_\varepsilon) \cdot Au_\varepsilon)\, dx = 0 .$$

Therefore, Au_ε remains bounded in L^2 and $Au_{\varepsilon_j} \rightharpoonup Au$ weakly in L^2.

Now we pass to the limit in (2_ε) using Minty's device (cf. [1]): Let us denote by B_ε the operator

$$B_\varepsilon v = -\varepsilon \Delta v + \varepsilon v + Av + g(x, v) .$$

For $v \in C^2$, we have

$$\int_\Omega (B_\varepsilon u_\varepsilon - B_\varepsilon v)(u_\varepsilon - v)\, dx \geqq 0 ,$$

that is

$$\int_\Omega B_\varepsilon v (u_\varepsilon - v)\, dx \leqq 0 .$$

As $\varepsilon \to 0$ we have for every $v \in C^2$

$$\int_\Omega (Av + g(x, v))(u - v)\, dx \leqq 0\,. \tag{4}$$

By a density argument (via convolution by smooth functions) we see that (4) holds for every $v \in L^\infty$ such that $Av \in L^2$. In particular, we can insert $v = u - tw$ in (4) where $t > 0$ and $w \in C^2$; dividing by t and letting $t \to 0$ we find

$$\int_\Omega (Au + g(x, u))w\, dx \leqq 0 \quad \text{for} \quad \text{all} \quad w \in C^2\,,$$

and consequently $Au + g(x, u) = 0$.

In general, uniqueness does not hold; however if u and $\hat{u}$ are two solutions of (2), we have $u - \hat{u} \in N(A)$ and $g(x, u) = g(x, \hat{u})$. This is because $\int_\Omega (g(x, u) - g(x, \hat{u}))(u - \hat{u})\, dx = 0$ — which implies $g(x, u) = g(x, \hat{u})$ a.e.

3. We can even solve (2) for some functions g which are discontinuous in u. For simplicity we consider only equations of the form

$$Au + \beta(u) \ni f(x)\,, \tag{5}$$

where β is a maximal monotone graph in R and $f \in C$. By a solution of (5) we mean a function $u \in L^\infty$ such that $Au \in L^\infty$ and (5) holds a.e. Let $R(\beta)$ be the range of β and set $\beta_- = \inf R(\beta)$ (possibly $\beta_- = -\infty$), $\beta_+ = \sup R(\beta)$ (possibly $\beta_+ = +\infty$).

THEOREM 3. *Suppose $f \in C$ and assume there is $\delta > 0$ such that*

$$\beta_- + \delta \leqq Pf \leqq \beta_+ - \delta\,. \tag{6}$$

Then (5) has at least one solution. Furthermore the difference of two solutions lies in $N(A)$.

Proof: *Existence.* Let β_λ be the Yosida approximation of β (see for example Proposition 2.6 in [1]). Since β_λ is continuous, we apply Theorem 2 to $g(x, u) = \beta_\lambda(u) - f$. This is because $R(\beta_\lambda) = R(\beta)$, so that $\lim_{u \to \pm\infty} \beta_\lambda(u) = \beta_\pm$, and thus condition (3) follows from (6) for suitable constants $\underline{M}$, $\bar{M}$. Therefore there exists a solution $u_\lambda \in L^\infty$ of

$$Au_\lambda + \beta_\lambda(u_\lambda) = f\,. \tag{7}$$

We note that

$$|\beta_\lambda(u_\lambda)| \leqq \|f\|_{L^\infty}\,. \tag{8}$$

Indeed if u_λ were a C^1 function, we could apply the maximum principle to (7): at a point where u_λ achieves its maximum, $\beta_\lambda(u_\lambda)$ also achieves its maximum and there we have $\beta_\lambda(u_\lambda)=f$. this argument can be made rigorous by approximating β_λ by smooth functions to which Theorem 1 applies; passing to the limit we obtain a solution of (7) satisfying (8). Thus any solution of (7) satisfies (8) in virtue of the fact that the difference of two solutions lies in $N(A)$.

CLAIM. $|u_\lambda| \leqq C$ independent of λ.

We shall obtain such an upper bound for u_λ—a similar argument yields a lower bound.

Case I. $\beta_+=+\infty$. It can be easily seen that there exist C and $\lambda_0>0$ such that $\beta_\lambda(u)>\max|f|$ for $u>C$ and $0<\lambda<\lambda_0$. Hence $u_\lambda \leqq C$ for $0<\lambda<\lambda_0$.

Case II. $\beta_+<\infty$. It can be easily seen that there exist C and $\lambda_0>0$ such that $\beta_\lambda(u)>\beta_+-\frac{1}{2}\delta$ for $u>C$ and $0<\lambda<\lambda_0$. By Lemma 1 we write $f=Av+\zeta+R$ with $v,\zeta\in C^\infty$, $\zeta\in N(A)$ and $|R|<\frac{1}{4}\delta$. From (6) we find $\zeta+PR\leqq\beta_+-\delta$ so that $\zeta\geqq\beta_+-\frac{3}{4}\delta$. By (7) we obtain

$$A(u_\lambda-v)+\beta_\lambda(u_\lambda)=\zeta+R\leqq\beta_+-\tfrac{1}{2}\delta\,.$$

Again with the aid of the maximum principle—which can be justified as above—we find that $u_\lambda\leqq|C|+2\max|v|$ for $0<\lambda<\lambda_0$.

PASSAGE TO THE LIMIT. We make use of the following lemma (an easy consequence of Proposition 2.5 in [1]).

LEMMA. *Suppose* $u_{\lambda_j}\rightharpoonup u$ *weakly in* L^2, $\beta_{\lambda_j}(u_{\lambda_j})\rightharpoonup h$ *weakly in* L^2 *and* $\overline{\lim_{j\to\infty}}\int_\Omega \beta_{\lambda_j}(u_{\lambda_j})\cdot(u_{\lambda_j}-u)\,dx\leqq 0$. *Then* $h\in\beta(u)$ *a.e.*

In our case there is some sequence $\lambda_j\to 0$ such that $u_{\lambda_j}\rightharpoonup u$ in weak* L^∞, $Au_{\lambda_j}\rightharpoonup Au$ in weak* L^∞, and thus $\beta_{\lambda_j}(u_{\lambda_j})\rightharpoonup f-Au$ in weak* L^∞. On the other hand,

$$\begin{aligned}\int_\Omega \beta_{\lambda_j}(u_{\lambda_j})(u_{\lambda_j}-u)\,dx &= \int_\Omega (f-Au_{\lambda_j})(u_{\lambda_j}-u)\,dx\\ &= \int_\Omega (f-Au)(u_{\lambda_j}-u)\,dx\to 0\,.\end{aligned}$$

Therefore the lemma applies and $f-Au\in\beta(u)$.

UNIQUENESS MODULO $N(A)$. We shall use the following

LEMMA 2. *Let $u \in L^2$ be such that $Au \in L^2$. Then $Au = 0$ a.e. on the set where $u = 0$.*

Proof: We may work locally and thus consider the case where $A = \partial/\partial x_1$. For almost all values of $x' = (x_2, \cdots, x_n)$ it reduces by Fubini to the one-dimensional case—which is well known (see for example [5], Appendix I).

Suppose u and $\hat{u}$ are two solutions of (5). The monotonicity of β implies that $[(f - Au) - (f - A\hat{u})](u - \hat{u}) \geqq 0$ a.e., that is $(Au - A\hat{u})(u - \hat{u}) \leqq 0$ a.e. Since, on the other hand,

$$\int_\Omega (Au - A\hat{u})(u - \hat{u})\, dx = 0$$

we conclude that $(Au - A\hat{u})(u - \hat{u}) = 0$ a.e. By Lemma 2 it follows that $Au = A\hat{u}$ a.e. Theorem 3 is proved.

We illustrate Theorem 3 by two examples.

EXAMPLE 1. Let β be a C^∞ function on (a, b) with $\beta' > 0$, $\lim_{t\downarrow a} \beta(t) = -\infty$, and $\lim_{t\uparrow b} \beta(t) = +\infty$. Then, for every $f \in C^\infty$, there is a unique C^∞ solution of

$$Au + \beta(u) = f.$$

Indeed, existence and uniqueness of an L^∞ solution follows from Theorem 3. Since $|\beta(u)| \leqq \max |f|$, we may apply Theorem 1 to a truncation of β and infer that $u \in C^\infty$.

EXAMPLE 2. This is concerned with solving a variational inequality.

COROLLARY. *Assume $f \in C$ and that a^j are linearly independent over the rationals. If $\int_\Omega f\, dx < 0$, there exists a unique solution $u \in L^\infty$ of the variational inequality*

$$(9) \qquad u \geqq 0, \qquad Au - f \geqq 0, \qquad u(Au - f) = 0 \quad \text{a.e. on} \quad \Omega,$$

with $Au \in L^\infty$.

Proof: Observe that (9) is equivalent to the equation

$$Au + \beta(u) \ni f,$$

where β is the graph given by

$$\beta(u)=\begin{cases}0 & \text{if} \quad u>0\,,\\ (-\infty,0] & \text{if} \quad u=0\,.\end{cases}$$

Existence follows from Theorem 3. To prove uniqueness, suppose u and $\hat{u}$ are solutions. By Theorem 3, $u-\hat{u}$ is a constant c. If $c\neq 0$ say $c>0$, we have $u\geqq c$, and therefore $Au=f$—which contradicts the assumption $\int f<0$. Thus $c=0$.

Remark. The condition $\int_\Omega f\,dx<0$ cannot be dropped in general. It is easy to find a continuous function f such that $\int_\Omega f\,dx=0$ and for which (9) has no solution $u\in L^\infty$ with $Au\in L^\infty$. If a solution did exist, we would have $0=\int_\Omega (Au-f)\,dx$ and hence $Au=f$ a.e. We now describe a continuous function f for which $Au=f$ has no solution $u\in L^\infty$: Let $\{k^j\}$ be a sequence of multi-indices $k^j=(k_1^j,\cdots,k_n^j)$ of integers such that

$$|a\cdot k^j|\leqq\frac{1}{j^3}, \qquad j=1,2,\cdots.$$

Set

$$f=\sum_j j(a\cdot k^j)e^{ik^j\cdot x}.$$

Since

$$\sum j\,|a\cdot k^j|<\infty,$$

the function f is continuous. On the other hand, the distribution solution (unique up to an additive constant) of $Au=f$ is

$$u=\frac{1}{i}\sum_j je^{ik^j\cdot x}$$

which is not in L^∞.

However if $f\in C^\infty$, and the coefficients a^j satisfy, for some positive constants C, σ,

$$|a\cdot k|\geqq C\,|k|^{-\sigma} \quad \text{for} \quad k\neq 0,$$

then the necessary and sufficient condition for (9) to have a solution is $\int f\,dx \leqq 0$. Necessity is clear; for sufficiency the only case that has to be considered is the case $\int f\,dx = 0$. In this case, (9) has a (nonunique) solution in C^∞, namely, writing f as a Fourier series

$$f = \sum_{k \neq 0} c_k e^{ik \cdot x},$$

the function

$$u = \text{large constant} + \frac{1}{i} \sum_{k \neq 0} \frac{c_k}{a \cdot k} e^{ik \cdot x}$$

is such a solution.

4. (*Added in proofs*). We now present an existence theorem for

$$Au + g(u) = f(x), \qquad A = \sum a^j \frac{\partial}{\partial x^j}, \tag{2'}$$

in which g is not required to be monotone. We wish to express our thanks to H. Amann for useful discussions.

THEOREM 2′. *Let $g \in C(R)$ be locally of bounded variation; let $f \in C(\Omega)$. Assume that, for some $\delta > 0$,*

$$\overline{\lim_{u \to -\infty}}\, g(u) + \delta \leqq Pf \leqq \underline{\lim}_{u \to +\infty}\, g(u) - \delta.$$

Then there exists a solution $u \in L^\infty$ of (2′).

Proof: We first construct super and sub solutions $\bar{u} \geqq \underline{u}$, i.e., functions satisfying

$$A\bar{u} + g(\bar{u}) - f \geqq 0 \geqq A\underline{u} + g(\underline{u}) - f.$$

To this end let M be such that

$$g(u) \geqq \underline{\lim}_{u \to +\infty}\, g(u) - \tfrac{1}{4}\delta \quad \text{for} \quad u \geqq M.$$

By Lemma 1 we may write

$$f = A\bar{u} + \zeta + R \quad \text{for} \quad \bar{u} \in C^\infty,\ \zeta \in C^\infty \cap N(A),\ |R| < \tfrac{1}{4}\delta.$$

By adding a large constant to $\bar{u}$ we may always suppose that $\bar{u} \geqq M$. We have $Pf = P\zeta + PR = \zeta + PR$ and consequently $\zeta \leqq \underline{\lim}_{u \to +\infty} g(u) - \frac{3}{4}\delta$. Hence $A\bar{u} + g(\bar{u}) - f = -\zeta - R + g(\bar{u}) \geqq 0$. Similarly we construct $\underline{u}$ which we may always take to satisfy $\underline{u} \leqq \bar{u}$.

Now we use a monotone iteration scheme. Let $g_1, g_2 \in C(R)$ be nondecreasing, bounded functions such that $g = g_1 - g_2$ on $[\min \underline{u}, \max \bar{u}]$. We shall solve

$$u = (I + g_1 + A)^{-1}(f + u + g_2(u)) \equiv Tu$$

using the fact that T is order preserving (note that $I + g_1 + A$ is invertible in L^2: A is maximal monotone, g_1 is monotone continuous and thus $A + g_1$ is maximal monotone). Since $\underline{u}$, $\bar{u}$ are sub and super solutions we see easily that $\underline{u} \leqq T\underline{u}$, $\bar{u} \geqq T\bar{u}$. Consequently the sequences $u_n = T^n(\underline{u})$, $v_n = T^n(\bar{u})$ satisfy $u_n \leqq v_n$, $u_n \nearrow$, $v_n \searrow$. Hence both sequences u_n, v_n converge to functions $\underset{\sim}{u}$, $\tilde{u}$ in $\underline{u} \leqq \underset{\sim}{u} \leqq \tilde{u} \leqq \bar{u}$ and $\underset{\sim}{u}$ and $\tilde{u}$ are generalized solutions of (2′).

The argument may be extended also to $g(x, u)$ under appropriate assumptions.

Remark. In the proof of Theorem 1, in particular in our derivation of an upper bound for the solution of (2_ε), we decomposed $g(x, \bar{M}) = -Av + \zeta + R$. For ε sufficiently small, the function

$$\bar{u} = \bar{M} + v + \max |v|$$

is in fact a super solution of (2_ε).

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Received October, 1976.