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Remarks on Nonlinear Ergodic Theory

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The development of general ergodic theory for nonlinear operators was begun by Baillon in [1], where he considered a nonexpansive mapping U of a Hilbert space H into H, formed the Cesaro means

$$S_n(x) = (1/n) \sum_{j=0}^{n-1} U^j(x), \tag{1}$$

and showed that if $S_n(x_0)$ is bounded for a given x_0, then $S_n(x)$ converges weakly as $n \to +\infty$ to a fixed point of U for each x in H. A corresponding result (with a simpler proof) for one-parameter semigroups $\{U(t);\ t \geqslant 0\}$ is given by Baillon [2] and Baillon and Brézis [4]. For the special case in which U is an odd mapping, Baillon showed [3] that $S_n(x)$ converges strongly in H.

In [6], the writers generalized Baillon's results to more general summation methods,

$$T_n(x) = \sum_{j=0}^{\infty} a_{n,j} U^j(x), \tag{2}$$

where $\{a_{n,j}\}$ is any strongly regular summation method,

$$\begin{gathered} a_{n,j} \geqslant 0, \qquad \sum_{j=0}^{\infty} a_{n,j} = 1, \\ a_{n,j} \to 0 \quad \text{as} \quad n \to +\infty \quad \text{for fixed } j, \\ \sigma_n = \sum_{j=0}^{\infty} | a_{n,j+1} - a_{n,j} | \to 0 \quad \text{as} \quad n \to +\infty, \end{gathered} \tag{3}$$

and showed that if U is a nonexpansive self-map of a Hilbert space H with a nontrivial fixed point set, then $T_n(x)$ converges weakly for each x to a fixed point of U. Simple proofs of Baillon's result have also been given by Pazy [10] and Tartar. It was also shown [6] that if U satisfies an inequality of the form

$$|(U(x), U(y)) - (x, y)| \leqslant c\{\| x \|^2 - \| U(x)\|^2 + \| y \|^2 - \| U(y)\|^2\}, \tag{4}$$

which would follow from the assumption that U is odd, then under a further "properness" condition on the summation method $\{a_{n,j}\}$, which holds for

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Cesato means in particular, $T_n(x)$ converges strongly in H. More recently, Bruck [8] has reconstructed the proof of [6] to show that properness is unnecessary in the argument. (Another proof using the results of Lorentz on almost convergence has been given by Reich [11].)

A different extension of Baillon's original result has been given by Beauzamy and Enflo [5] who consider the Banach space l_p, $1 < p < \infty$, with a weakly continuous duality mapping $J = \partial\varphi_p$, where $\varphi_p(x) = \| x \|^p$, for a given p with $1 < p < +\infty$ and show the weak convergence of y_k, where y_k is the minimum point of the function,

$$\sum_{k=0}^{n} \| U^k(x) - y \|^p. \tag{5}$$

In the present paper, we give an extension of all these results based on a simple and transparent argument. To include the various interesting cases (as well as to make the underlying structure of the proof more visible) we state the result in a very general form replacing the semigroups $Z^+ = \{n \mid n \geqslant 0\}$ and $R^+ = \{t \mid t \text{ real}, t \geqslant 0\}$ by a general commutative semigroup.

Let S be a commutative semigroup, i.e., there exists a binary operation on S which is both associative and commutative. (This operation we write in the additive form $(a, b) \to a + b$ for $a, b \in S$.) We assume given a σ-field F on S such that for each A in F, a in S, if

$$A_a = A + a \tag{6}$$

then A_a lies in F. We consider measures μ of total mass 1 on F.

Let X be a Banach space, C a closed convex subset of X. By a representation of S in terms of nonexpansive self-mappings of C we mean a family $\{U_s ; s \in S\}$ where each U_s is a nonexpansive mapping of C into C, i.e.,

$$\| U_s(x) - U_s(u) \| \leqslant \| x - u \| \qquad (x, u \in C), \tag{7}$$

and for each x in C, $U_s(x)$ is a strongly F-measurable function on S, while

$$U_{a+b}(x) = U_a(U_b(x)) \tag{8}$$

for all a, b in S and each x in C.

Definition 1. Let $\{\mu_\lambda ; \lambda \in \Lambda\}$ be a family of positive measures of mass 1 on F indexed on the partially ordered set Λ. Then the family is said to be strongly regular if

(1) For each a in S, $\mu_\lambda(S_a) \to 1$.

(2) For each a in S, if we set $\mu_{\lambda,a}(A) = \mu_\lambda(A_a)$ for each A in F, then

$$| \mu_{\lambda,a}(A) - \mu_\lambda(A) | \to 0.$$

We use $\rightharpoonup$ to denote weak convergence, $\to$ for strong convergence.

THEOREM 1. *Let X be a uniformly convex Banach space. Suppose that there exists a strictly convex function $\gamma: R^+ \to R^+$ with $\gamma(0) = 0$, $\gamma(r) \to +\infty$ as $r \to +\infty$ such that if $g(x) = \gamma(\| x \|)$, then the subgradient T_γ of g is continuous on bounded subsets of X from the weak topology of X to the weak topology of its conjugate space X^*.*

Let S be a commutative semigroup, $\{U_s\,;\ s \in S\}$ a representation of S by nonexpansive self-mappings of X such that $\{U_s(x_0)\}$ is bounded for some x_0 in X. Let x be a given element of X, and let

$$\rho_\lambda(y) = \int_S \gamma(\| U_s(x) - y \|)\, \mu_\lambda(ds).$$

Then

(1) *For each λ, there exists a unique element y_λ of X for which*

$$\rho_\lambda(y_\lambda) = \min\{\rho_\lambda(y); y \in X\}.$$

(2) *$y_\lambda \rightharpoonup z$, where z is the asymptotic center of the family $\{U_s(x); s \in S\}$ and is a fixed point of each U_a, $a \in S$.*

Before proceeding to the proof of Theorem 1, let us introduce some basic considerations which allow us to specify the meaning of the terms used in Theorem 1. First of all, we introduce an ordering on S by saying that $b \geqslant a$ if and only if $b \in S_a$ (or in other words, considering the family $\{S_a\}$ as the base of a filter on S). For each a in S, we set

$$\sigma_a(y) = \sup_{s \in S_a} \gamma(\| U_s(x) - y \|).$$

Since $\{\| U_s(x_0)\|\}$ is bounded and each U_s is nonexpansive, the family $\{U_s(x)\}$ is bounded for x, $\| U_s(x)\| \leqslant M$ $(s \in S)$. Moreover, since the space X is uniformly convex, the family $\{U_s\}$ has a stationary point, i.e., a point x_0 for which $U_s(x_0) = x_0$. (For the basic fixed point theory of nonexpansive maps on uniformly convex spaces, we refer to the detailed discussion in [7].)

Each σ_a is a convex locally Lipschitzian function on X with $\sigma_a(y) \geqslant \gamma(\| y \| - M)$. For $b \geqslant a$, $0 \leqslant \sigma_b(y) \leqslant \sigma_a(y)$. Hence

$$\sigma(y) = \lim_a \sigma_a(y)$$

exists and is a convex continuous function on X with $\sigma(y) \geqslant \gamma(\| y \| - M)$. Since X is reflexive, the set

$$\{y_0 \mid \sigma(y_0) = \min_{y \in X} \sigma(y)\}$$

is nonempty. We assert that this set consists of a single element z, which we call the asymptotic center of $\{U_s(x)\}$. To prove this, we use the following elementary lemma.

LEMMA 1. *Let X be uniformly convex, γ a strictly convex function on X with $\gamma(0) = 0$, $M_0 > 0$. Then for each $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that for all u and v in X with $\| u \|, \| v \| \leqslant M_0$, and $\| u - v \| \geqslant \epsilon$,*

$$\gamma(\| \tfrac{1}{2}(u + v)\|) \leqslant \tfrac{1}{2}\gamma(\| u \|) + \tfrac{1}{2}\gamma(\| v \|) - \delta_\epsilon .$$

Proof of Lemma 1. Since $\| u - v \| \geqslant \epsilon$, at least one of the pair $\| u \|, \| v \|$ must be at least $\epsilon/2$. Hence we may assume that $\| v \| \leqslant \| u \|$, $\epsilon/2 \leqslant \| u \| \leqslant M_0$. By the uniform convexity of X, there exists $\delta_1(\epsilon) > 0$ such that if

$$\| u \| \geqslant \| v \| \geqslant \| u \| - \delta_1(\epsilon)$$

then

$$\tfrac{1}{2} \| u + v \| \leqslant \tfrac{1}{2} \| u \| + \tfrac{1}{2} \| v \| - \delta_1(\epsilon).$$

Then

$$\gamma(\tfrac{1}{2} \| u + v \|) \leqslant \tfrac{1}{2}\gamma(\| u \|) + \tfrac{1}{2}\gamma(\| v \|) - \delta_2(\epsilon),$$

for a suitable $\delta_2(\epsilon) > 0$.

On the other hand, if $\| v \| \leqslant \| u \| - \delta_1(\epsilon)$, then setting $r = \| u \|$, $s = \| v \|$, we have $s \leqslant r - \delta_1(\epsilon)$, and by the strict convexity of γ

$$\gamma(\tfrac{1}{2} \| u + v \|) \leqslant \gamma(\tfrac{1}{2}(r + s)) \leqslant \tfrac{1}{2}\gamma(r) + \tfrac{1}{2}\gamma(s) - \delta_3(\epsilon). \qquad \text{Q.E.D.}$$

If we assume that for two distinct points y_0 and y_1,

$$\sigma(y_1) = \sigma(y_0) = \min_y \sigma(y),$$

and set $\epsilon = \| y_1 - y_0 \| \geqslant 0$, then for $y' = \tfrac{1}{2}(y_0 + y_1)$ and any s in S,

$$\begin{aligned}\gamma(\| U_s - y' \|) &= \gamma(\tfrac{1}{2} \| (U_s - y_0) + (U_s - y_1)\|) \\ &\leqslant \tfrac{1}{2}\gamma(\| U_s - y_0 \|) + \tfrac{1}{2}\gamma(\| U_s - y_1 \|) - \delta_\epsilon .\end{aligned}$$

For $s \geqslant a_0$, we may ensure that

$$\gamma(\| U_s - y_0 \|) \leqslant \sigma(y_1) + \delta_\epsilon/2, \qquad \gamma(\| U_s - y_1 \|) \leqslant \sigma(y_1) + \tfrac{1}{2}\delta_\epsilon .$$

Hence for such s,

$$\gamma(\| U_s - y' \|) \leqslant \min_y \sigma(y) + \tfrac{1}{2}\delta_\epsilon - \delta_\epsilon < \min_y \delta(y) - \tfrac{1}{2}\delta_\epsilon .$$

Hence $\sigma(y') \leqslant \min_y \sigma(y) - \tfrac{1}{2}\delta_\epsilon < \min \sigma(y)$, which is a contradiction. Thus the asymptotic center z is well defined.

LEMMA 2. *Let X be a uniformly convex Banach space, $\{y_\lambda\}$ a bounded set in X indexed on Λ. Suppose that U is a nonexpansive self-map of X and that $\|(I - U)y_\lambda\| \to 0$. Then every weak accumulation point of $\{y_\lambda\}$ is a fixed point of U.*

Proof of Lemma 2. This is a basic property of nonexpansive maps jn uniformly convex spaces (cf. [7, Theorem 8.4]).

LEMMA 3. *Let X be a uniformly convex B-space. For each λ, there exists one and only one point y_λ of X such that*

$$\rho_\lambda(y_\lambda) = \min\{\rho_\lambda(y); y \in X\}.$$

Proof of Lemma 3. Since ρ_λ is a continuous convex function of y in X with

$$\rho_\lambda(y) \geqslant \int_S \gamma(\|y\| - M)\,\mu_\lambda(ds) = \gamma(\|y\| - M),$$

it follows that ρ_λ has at least one minimum point for each λ. Moreover, the set of possible minimum points $\{y_\lambda\}$ is uniformly bounded since

$$\rho_\lambda(0) \leqslant \int_S \gamma(M)\,\mu_\lambda(ds) = \gamma(M),$$

and for $\|y\| > 2M$, $\gamma(\|y\| - M) > \rho_\lambda(0)$.

To prove the uniqueness of y_λ, suppose for a given λ that for $t_0 \neq y_1$, $\|y_1 - y_0\| \| \epsilon > 0$,

$$\rho_\lambda(y_0) = \rho_\lambda(y_1) = \min_y \rho_\lambda(y).$$

Set $y' = \frac{1}{2}(y_0 + y_1)$. Then

$$\begin{aligned}\rho_\lambda(y') = \int_S \gamma(\|U_s(x) - y'\|)\,\mu_\lambda\,ds &\leqslant \tfrac{1}{2}\rho_\lambda(y_0) + \tfrac{1}{2}\rho_\lambda(y_1) - \delta_\epsilon \\ &\leqslant \min \rho_\lambda(y) - \delta_\epsilon,\end{aligned}$$

which is a contradiction. Thus y_λ is unique. Q.E.D.

LEMMA 4. *Let X be a Banach space, $\{U_s\,; s \in S\}$ a representation of S by nonexpansive maps of X into X, and let $\{\mu_\lambda\,; \lambda \in \Lambda\}$ be a strongly regular family of positive measures of mass* 1. *Then for each a in S, x in X,*

(a) $\overline{\lim}\{\rho_\lambda(U_a(y_\lambda)) - \rho_\lambda(y_\lambda)\} \leqslant 0$.

(b) $\|y_\lambda - U_a y_\lambda\| \to 0$.

Proof of Lemma 4. Proof of (a): We observe that

$$\begin{aligned}
\rho_\lambda(U_a(y_\lambda)) &= \int_S \gamma(\| U_s(x) - U_a(y_\lambda)\| \mu_\lambda(ds) \\
&\leqslant \int_{S_a} \gamma(\| U_s(x) - U_a(y_\lambda)\| \mu_\lambda(ds) + \int_{S\setminus S_a} \gamma(\| U_s(x) - U_a(y_\lambda)\|) \mu_\lambda(ds) \\
&= I_1 + I_2 .
\end{aligned}$$

The first integral may be written in the form

$$\begin{aligned}
I_1 &= \int_S \gamma(\| U_a(U_t(x)) - U_a(y_\lambda)\|) \mu_{\lambda,a}(dt) \\
&\leqslant \int_S \gamma(\| U_t(x) - y_\lambda \|) \mu_{\lambda,a}(dt) \\
&\leqslant \int_S \gamma(\| U_t(x) - y_\lambda \|) \mu_\lambda(dt) + M_1 \mid \mu_{\lambda,a} - \mu_\lambda \mid (S),
\end{aligned}$$

where $M_1 = \sup_s \gamma(\| U_s(x) - y_\lambda \|)$.

For the second integral,

$$I_2 \leqslant M_2 \mu_\lambda(S\setminus S_a) \to 0,$$

where $M_2 = \sup_s \gamma(\| U_s(x) - U_a(y_\lambda)\|)$.

Hence

$$\rho_\lambda(U_a(y_\lambda)) - \rho_\lambda(y_\lambda) \leqslant M_1 \mid \mu_{\lambda,a} - \mu_\lambda \mid (S) + M_2 \mu_\lambda(S\setminus S_a),$$

from which it follows that

$$\overline{\lim}\{\rho_\lambda(U_a(y_\lambda) - \rho_\lambda(y_\lambda)\} \leqslant 0.$$

Proof of (b). Consider a value of λ such that $\| y_\lambda - U_a(y_\lambda)\| \geqslant \epsilon$. Set $w = \frac{1}{2}(y_\lambda + U_a(y_\lambda))$. Then

$$\rho_\lambda(w) \leqslant \tfrac{1}{2}\rho_\lambda(y_\lambda) + \tfrac{1}{2}\rho_\lambda(U_a(y_\lambda)) - \delta_\epsilon .$$

For $\lambda \geqslant \lambda_\epsilon$,

$$\rho_\lambda(U_a(y_\lambda)) \leqslant \rho_\lambda(y_\lambda) + \delta_\epsilon .$$

Hence for such λ,

$$\rho_\lambda(w) \leqslant \rho_\lambda(y_\lambda) + \tfrac{1}{2}\delta_\epsilon - \delta_\epsilon = \rho_\lambda(y_\lambda) - \tfrac{1}{2}\delta_\epsilon .$$

This is impossible. Hence for $\lambda \geqslant \lambda_\epsilon$, $\| y_\lambda - U_a(y_\lambda)\| < \epsilon$. Q.E.D.

Proof of Theorem 1 *concluded.* Since y_λ is the minimum point of p_λ, we know that for each v in X,

$$0 = (d/dt)|_{t=0}\, \rho_\lambda(y_\lambda + tv) = -\int_S (J_\gamma(U_s(x) - y_\lambda), v)\, \mu_\lambda(ds).$$

Thus y_λ satisfies the implicit equation

$$\int_S J_\gamma(U_s(x) - y_\lambda)\, \mu_\lambda(ds) = 0. \tag{9}$$

(If J_γ is the identity map of a Hilbert space, this gives us the simpler explicit formula

$$y_\lambda = \int_S U_s(x)\, \mu_\lambda(ds), \tag{10}$$

which generalizes the special case of the strongly regular averaging process.)

To show that the bounded directed set $\{y_\lambda\}$ in X converges weakly to z, it suffices to show that each weak accumulation point w of $\{y_\lambda\}$ must coincide with z. By Lemma 4, $\| y_\lambda - U_a(y_\lambda)\| \to 0$. Hence by Lemma 2, w must be a fixed point of each U_a $(a \in S)$. It follows that for $t \geqslant s$, $t = s + a$,

$$\begin{aligned}\gamma(\| U_t(x) - w \|) &= \gamma(\| U_a U_s(x) - U_a w \|)\\ &\leqslant \gamma(\| U_s(x) - w \|).\end{aligned}$$

Hence

$$\sigma(w) = \lim_s \gamma(\| U_s(x) - w \|).$$

If $w \neq z$, then $\sigma(w) - \sigma(z) = \epsilon > 0$. For $\lambda \geqslant \lambda_\delta$,

$$\rho_\lambda(w) = \int_S \gamma(\| U_s(x) - w \|)\, \mu_\lambda(ds) \geqslant \sigma(w) - \delta.$$

On the other hand

$$\begin{aligned}\rho_\lambda(z) &= \int_S \gamma(\| U_s(x) - z \|)\, \mu_\lambda(ds)\\ &\leqslant \int_{S_a} \gamma(\| U_s(x) - z \|)\, \mu_\lambda(ds) + M_1 \mu_\lambda(S \backslash S_a) \leqslant \sigma(z) + \delta\end{aligned}$$

for $\lambda \geqslant \lambda_\delta$. Thus for such λ

$$\sigma(w) \leqslant \rho_\lambda(w) + \delta \leqslant \rho_\lambda(w) - \rho_\lambda(z) + \sigma(z) + 2\delta,$$

i.e.,

$$\epsilon = \sigma(w) - \sigma(z) \leqslant \rho_\lambda(w) - \rho_\lambda(z) + 2\delta.$$

Suppose $\xi = \epsilon - 2\delta > 0$. Then for $\lambda \geqslant \lambda_\delta$

$$\xi \leqslant \rho_\lambda(w) - \rho_\lambda(z).$$

However,

$$\rho_\lambda(w) - \rho_\lambda(z) = \int_S \{\gamma(\| U_s(x) - w \|) - \gamma(\| U_s(x) - z \|\} \mu_\lambda(ds),$$

where

$$\gamma(\| U_s(x) - w \|) - \gamma(\| U_s(x) - z \|) \leqslant (J_\gamma(U_s(x) - w), z - w),$$

hence

$$\xi \leqslant \int_S (J(U_s(x) - w), z - w) \mu_\lambda(ds).$$

For λ in a cofinal set Λ_0 of Λ, $\lambda_\lambda - w \rightharpoonup 0$. Hence $U_s(x) - y_\lambda \rightharpoonup U_s(x) - w$ uniformly in s. Since J_γ is uniformly continuous on bounded sets from the weak topology of X to the weak topology of X^*, it follows that

$$(J(U_s x - y_\lambda), z - w) \to (J(U_s(x) - w), z - w)$$

uniformly in s. Therefore,

$$0 = \underline{\lim} \int_S (J(U_s(x) - y_\lambda , z - w) \mu_\lambda(ds) \geqslant \xi > 0,$$

which is a contradiction.

Thus $w = z$ and the proof of Theorem 1 is complete. Q.E.D.

Remark. The outstanding class of uniformly convex Banach spaces X for which the hypotheses of Theorem 1 are valid are the l_p-spaces with $1 < p < +\infty$. For each such space, the function $\gamma_p(r) = r^p$ has the corresponding subgradient $J_{\gamma_p} = \partial g_p$, with $g_p(x) = \gamma_p(\| x \|)$ given by

$$x = \{x_1, x_2, \ldots\} \in l_p$$

$$J_{\gamma_p}(x) = \{p \mid x_j \mid^{p-1} \operatorname{sgn}(x_j)\} \in l_p$$

and J_{γ_p} is uniformly continuous on bounded sets from the weak topology of l_p to the weak topology of l_p.

Our basic result on strong convergence is established only in Hilbert spaces. We give it only in the formally simplest case of $S = Z^+$ where it sharpens the result of [8].

THEOREM 2. *Let H be a Hilbert space, $\{x_j\}$ a sequence in H such that for each m, (x_j, x_{n+j}) converges as $j \to +\infty$, the convergence being uniform for $m \geqslant 0$. Let $\{a_{nj}\}$ be a strongly regular summation method.*

If $y_n = \sum_{j=0}^{\infty} a_{nj} x_j$, *then* y_n *converges strongly in* H *to the asymptotic center of the sequence* $\{x_j\}$.

Proof of Theorem 2. Let z be the asymptotic center of the sequence $\{x_j\}$. We divide the proof into two stages: (a) $y_n \rightharpoonup z$; (b) $y_n \to z$.

Proof of (a). Let $\epsilon > 0$ be given. By hypothesis, for each $m \geqslant 0$, there exist $\xi(m)$ and an integer j_ϵ such that for $j \geqslant j_\epsilon$,

$$|(x_j, x_{j+m}) - \xi(m)| < \epsilon.$$

Fix one such value of j. Then for each n

$$(x_j, y_n) = \sum_{k=0}^{j-1} (x_j, a_{nk} x_k) + \sum_{k=0}^{\infty} a_{n,j+k}(x_j, x_{j+k})$$

implies that

$$\left| (x_j, y_n) - \sum_{k=0}^{\infty} a_{n,j+k} \xi(k) \right| \leqslant M \sum_{k=0}^{j-1} a_{n,k} + \epsilon.$$

Furthermore,

$$\left| (x_j, y_n) - \sum_{k=0}^{\infty} a_{n,k} \xi(k) \right| \leqslant M \sum_{k=0}^{j-1} a_{n,k} + \epsilon + \sum_{k=0}^{\infty} | a_{n,j+k} - a_{n,k} |,$$

where the right-hand side approaches ϵ as $n \to +\infty$. Thus

$$\overline{\lim_{n}} \left| (x_j, y_n) - \sum_{k=0}^{\infty} a_{n,k} \xi(k) \right| \leqslant \epsilon.$$

Let $\{y_{n_k}\}$ be a weakly convergent subsequence of $\{y_n\}$ with weak limit w. Then for large j, j'

$$|(x_j, w) - (x_{j'}, w)| \leqslant 2\epsilon,$$

i.e.,

$$(x_j, w) \to \tau \qquad (j \to +\infty).$$

Hence

$$(y_m, w) = \sum_{j=0}^{\infty} a_{m,j}(x_j, w) \to \tau \qquad (m \to +\infty).$$

Let $\{y_{m_j}\}$ converge weakly in H to w_1. Then

$$(w_1, w) = \tau = (w, w) = (w_1, w_1).$$

Hence $\| w \| = \| w_1 \|$, $(w_1, w) = \| w \| \cdot \| w_1 \|$, and by the sharp form of the

Schwarz inequality, $w = w_1$. Hence y_n converges weakly to an element w of H. Moreover, $(x_j, w) \to \| w \|^2$ as $j \to +\infty$. Hence

$$\| x_j - w \|^2 = \| x_j \|^2 - 2(x_j, w) + \| w \|^2 \to \xi(0) - \| w \|^2$$

as $j \to +\infty$. For any u in H,

$$\| x_j - u \|^2 = \| x_j - w \|^2 + \| w - u \|^2 + 2(x_j - w, w - u).$$

Thus

$$\sum_{j=0}^{\infty} a_{n,j} \| x_j - u \|^2 = \sum_{j=0}^{\infty} a_{n,j}\{\| x_j - w \|^2 + \| w - u \|^2\} + 2(y_n - w, w - u).$$

Since $\sigma(u) \geqslant \underline{\lim} \sum_{j=0}^{\infty} a_{n,j} \| x_j - u \|^2$, it follows that

$$\sigma(u) \geqslant \sigma(w) + \| w - u \|^2,$$

since $(y_n - w, w - u) \to 0$. Thus w is the asymptotic center of the sequence $\{x_j\}$.

To prove strong convergence we introduce the summation method $\{b_{n,r}\}$ defined by

$$b_{n,0} = \sum_{j=0}^{\infty} (a_{n,j})^2,$$

$$b_{n,r} = 2 \sum_{j=0}^{\infty} a_{n,j} a_{n,j+r} \qquad \text{for} \quad r \geqslant 1.$$

We assert that $\{b_{n,r}\}$ is a strongly regular summation method when $\{a_{n,j}\}$ is. Indeed

$$\sum b_{n,r} = \left(\sum_{j=0}^{\infty} a_{n,j} \right)^2 = 1,$$

$$b_{n,0} \leqslant \sigma_n + a_{n,0},$$

since

$$a_{n,j} \leqslant \sigma_n + a_{n,0} \qquad \left(\sigma_n = \sum_{j=0}^{\infty} | a_{n,j+1} - a_{n,j} | \right),$$

$$b_{n,r} \leqslant 2(\sigma_n + a_{n,0}) \qquad \text{for} \quad r \geqslant 1,$$

and finally

$$\begin{aligned} \sum_{r=0}^{\infty} | b_{n,r+1} - b_{n,r} | &\leqslant \sum_{r=1}^{\infty} | b_{n,r+1} - b_{n,r} | + | b_{n,1} - b_{n,0} | \\ &\leqslant 2 \sum_{r=1}^{\infty} \sum_{j=0}^{\infty} a_{n,j} | a_{n,j+r+1} - a_{n,j+r} | + | b_{n,1} - b_{n,0} | \\ &\leqslant 2\sigma_n + | b_{n,1} - b_{n,0} | \to 0 \qquad \text{as} \quad n \to \infty. \end{aligned}$$

We note that

$$\| y_n \|^2 = \sum_{j,k=0}^{\infty} a_{n,j} a_{n,k} (x_j , x_k)$$

$$= 2 \sum_{j=0}^{\infty} \sum_{r=1}^{\infty} a_{n,j} a_{n,j+r} (x_j , x_{j+r}) + \sum_{j=0}^{\infty} a_{n,j}^2 \| x_j \|^2 .$$

Let $\epsilon > 0$ be given; we choose m so that $(x_m , w) \leqslant \| w \|^2 + \epsilon$, and also

$$|(x_j , x_{j+r}) - \xi(r)| < \epsilon$$

for $j \geqslant m$ and all $r \geqslant 0$. In particular, for $j \geqslant m$ and all $r \geqslant 0$,

$$|(x_j , x_{j+r}) - (x_m , x_{m+r})| < 2\epsilon .$$

We find with $M = \operatorname{Sup} | x_i |$,

$$\| y_n \|^2 \leqslant 2 \sum_{j \geqslant m} \sum_{r=1}^{\infty} a_{n,j} a_{n,j+r} (x_m , x_{m+r}) + 2M^2 \sum_{j<m} \sum_{r=1}^{\infty} a_{n,j} a_{n,f+r}$$
$$+ \sum_{j \geqslant m} a_{n,j}^2 \| x_m \|^2 + M^2 \sum_{j<m} a_{n,j}^2 + \epsilon$$
$$\leqslant 2 \sum_{j=0}^{\infty} \sum_{r=1}^{\infty} a_{n,j} a_{n,j+r} (x_m , x_{m+r})$$
$$+ \sum_{j=0}^{\infty} a_{n,j}^2 \| x_m \|^2 + 4M^2 \sum_{j<m} \sum_{r=1} a_{n,j} a_{n,j+r} + 2M^2 \sum_{j<m} a_{n,j}^2 + \epsilon$$
$$= \left(x_m , \sum_{r=0}^{\infty} b_{n,r} x_{m+r} \right) + 4M^2 \sum_{j<m} \sum_{r=1} a_{n,j} a_{n,j+r} + 2M^2 \sum_{j<m} a_{n,j}^2 .$$

Clearly the last two terms approach zero as $n \rightarrow +\infty$ (for fixed m).

We apply now the result of the previous discussion to the strongly regular summation method $\{b_{n,r}\}$ and we deduce that for each m

$$\left(x_m , \sum_{r=0}^{\infty} b_{n,r} x_{m+r} \right) \rightarrow (x_m , w) \qquad \text{as} \quad n \rightarrow +\infty .$$

(We use here the shifted sequence $\{x_{m+r}\}$ instead of $\{x_r\}$; is satisfies the same property as $\{x_r\}$ and has the same asymptotic center.)

Finally, we conclude that

$$\overline{\lim} \| y_n \|^2 \leqslant (x_m , w) + \epsilon$$
$$\leqslant \| w \|^2 + 2\epsilon .$$

Since $\epsilon > 0$ is arbitrary, and since $y_n \rightharpoonup w$, it follows that $y_n \rightarrow w$. Q.E.D.

To apply Theorem 2, we note that if U is a nonexpansive mapping on H which satisfies the inequality (4) with $U(0) = 0$ and if we set $x_j = U^j(x)$ for a fixed x in H, then

$$|(x_{j+r+1}, x_{j+1}) - (x_{j+r}, x_j)| \leqslant c\{\| x_j \|^2 - \| x_{j+1} \|^2 + \| x_{j+r} \|^2 - \| x_{j+r+1} \|^2\}.$$

Summing in j from k to $s - 1$, $k < s$, we see that

$$|(x_{s+r}, x_s) - (x_{k+r}, x_k)| \leqslant c\{\| x_k \|^2 - \| x_s \|^2 + \| x_{k+r} \|^2 - \| x_{s+r} \|^2\}.$$

Since $\| x_j \|^2$ decreases to a limit ξ_0 as j increases, it follows that

$$|(x_{s+r}, x_s) - (x_{k,r}, x_k)| \leqslant 2c\{\| x_k \|^2 - \xi_0\} \to 0$$

as $k \to +\infty$ uniformly in s and r. Hence (x_j, x_{j+r}) converges uniformly in r and Theorem 2 is applicable.

COROLLARY. *Suppose U is a nonexpansive map of H into H with $U(0) = 0$ such that* (4) *holds. Then for any strongly regular $\{a_{n,j}\}$,*

$$\sum_{j=0}^{\infty} a_{n,j} U^j(x) \to w$$

where w is the asymptotic center of $\{U^j(x)\}$.

If U is odd and nonexpansive,

$$\| U(u) - U(v)\|^2 \leqslant \| u - v \|^2$$
$$\| U(u) + U(v)\|^2 \leqslant \| u + v \|^2,$$

i.e.,

$$(U(u), U(v) - (u, v)) \leqslant \| u \|^2 - \| U(u)\|^2 + \| v \|^2 - \| U(v)\|^2,$$
$$(u, v) - (U(u), U(v)) \leqslant \| u \|^2 - \| U(u)\|^2 + \| v \|^2 - \| U(v)\|^2.$$

Hence the inequality (4) holds and the corollary is applicable to every odd U.

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