ESTIMATES ON THE SUPPORT OF SOLUTIONS OF PARABOLIC VARIATIONAL INEQUALITIES

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1. Introduction

Consider a parabolic Cauchy problem

\begin{align}
(1.1) \quad & u_t - \Delta u = f \quad (x \in \mathbb{R}^n, \ 0 < t \leq T), \\
(1.2) \quad & u(x, 0) = u_0(x) \quad (x \in \mathbb{R}^n)
\end{align}

where \( \Delta \) is the Laplace operator. The solution \( u \) does not have compact support in general, even when \( f \equiv 0 \) and \( u_0 \) has compact support. For a parabolic variational inequality consisting of

\begin{align}
(1.3) \quad & u \geq 0, \quad (u_t - \Delta u)(v - u) \geq f(v - u) \text{ a.e., for any } v \geq 0,
\end{align}

and of (1.2), the situation is entirely different: when \( f \) is uniformly negative, \( u(x, t) \) has compact support whenever \( u_0(x) \) has compact support. The object of this paper is to study properties of the support.

In Section 2 we study the variational inequality (1.3), (1.2) when \( u_0 \) is any finite measure. Existence and uniqueness are proved.

In Sections 3–6 it is assumed that \( f \) is bounded and is uniformly negative. In Section 3 we show that if \( u_0(x) \) has compact support then \( u(x, t) \) has compact support. An analogous result for elliptic variational inequalities was proved earlier by Brézis [2] (and then generalized by Redheffer [6]).

In Sections 4 and 5 we study the behavior of the support \( S(t) \) of the function \( x \to u(x, t) \). In Section 4 we consider the case where \( u_0 \) is any function in \( L^\infty(\mathbb{R}^n) \) with compact support \( S = S(0) \); thus \( u_0 \) is not required to vanish on \( \partial S \). It is proved that, for all small times \( t \),

\[ S(t) \subset S + B(c[t\log t]^{1/2}) \]

where \( + \) denotes the vector sum, \( B(\rho) = \{x: |x| \leq \rho\} \), and \( c \) is a positive constant. This result is shown to be sharp.

In Section 5 we assume that \( u_0(x) \) vanishes together with its first derivatives on \( \partial S \). We then prove that

\[ S(t) \subset S + B(C/\sqrt{t}) \]

for some positive constant \( C \).

Received February 24, 1975.

This work was partially supported by a National Science Foundation grant.
In Section 6 we consider the case where \( u_0(x) \) does not have compact support, but \( u_0(x) \to 0 \) as \( |x| \to \infty \). We prove that \( S(t) \) is a compact set for any \( t > 0 \). Thus in sharp contrast with the case of (1.1), the support “shrinks” instantaneously.

2. Existence and uniqueness

Consider the parabolic variational inequality

\begin{equation}
(u_t - \Delta u)(v - u) \geq f(v - u) \text{ a.e.} \quad (x \in \mathbb{R}^n, 0 < t < T)
\end{equation}

for any measurable function \( v, v \geq 0 \),

\begin{equation}
u \geq 0 \quad (x \in \mathbb{R}^n, 0 \leq t \leq T),
\end{equation}

\begin{equation}
u(x, 0) = u_0(x) \quad (x \in \mathbb{R}^n).
\end{equation}

Let \( \mu \) be any positive number and introduce the norm

\[
|g|_{L^p, \mu(\mathbb{R}^n)} = \left\{ \int_{\mathbb{R}^n} e^{-\mu|x|} |g(x)|^p \, dx \right\}^{1/p}
\]

for any \( p > 1 \). If \( |g|_{L^p, \mu(\mathbb{R}^n)} < \infty \) then we say that \( g \in L^p, \mu(\mathbb{R}^n) \). We let

\[
W^{k, p, \mu}(\mathbb{R}^n) = \{ u \in L^p, \mu(\mathbb{R}^n); D^k u \in L^p, \mu(\mathbb{R}^n) \text{ for } |x| \leq k \}.
\]

If \( u, u_t, u_x, u_{xx} \) belong to \( L^{2, \mu}(\mathbb{R}^n) \) for any \( t \in (0, T] \), then we can rewrite (2.1) in the form

\[
\int_{\mathbb{R}^n} e^{-2\mu|x|} u_t(v - u) \, dx + \int_{\mathbb{R}^n} e^{-2\mu|x|} D_x u \cdot D_x(v - u) \, dx
\]

\[
+ \int_{\mathbb{R}^n} D_x u \cdot (D_x e^{-2\mu|x|})(v - u) \, dx \geq \int_{\mathbb{R}^n} e^{-2\mu|x|} f(v - u) \, dx
\]

for \( 0 < t \leq T \), and for any \( v \) such that \( v, v_x \) belong to \( L^{2, \mu}(\mathbb{R}^n) \), \( v \geq 0 \) a.e.

We shall assume:

\begin{equation}
u_0 \text{ is a measure, } u_0 \geq 0, \int_{\mathbb{R}^n} u_0 < \infty,
\end{equation}

\begin{equation}f \in L^\infty(\mathbb{R}^n, x(0, T)), f_t \in L^\infty(\mathbb{R}^n, x(0, T)).
\end{equation}

Denote by \( K(x, t, y) \) the fundamental solution of the heat equation. For any function \( f(y) \), the integral of \( f \) with respect to the measure \( u_0 \) is denoted by \( \int_{\mathbb{R}^n} f(y)u_0(y) \, dy \). The condition (2.3) will be taken, later on, in the sense that

\begin{equation}
u(x, t) = \int_{\mathbb{R}^n} K(x, t, y)u_0(y) \, dy \leq Ct
\end{equation}

where \( C \) is a constant independent of \( x \). (2.7) implies in particular that \( u(x, t) \to u_0(x) \) as \( t \downarrow 0 \) for the weak*-topology on the space of measures.
Theorem 2.1. Let (2.5), (2.6) hold. Then there exists a unique solution of (2.1)–(2.3) such that, for any $\delta > 0$,

$$u \in L^\infty([\delta, T); W^{2,p}(\mathbb{R}^n)]$$
$$u_t \in L^\infty([\delta, T); L^{p}(\mathbb{R}^n)] \text{ for any } 2 \leq p < \infty, \mu > 0;$$

the condition (2.3) is satisfied in the sense of (2.7).

Notice that, by the Sobolev inequalities, $u$ is a continuous function for $0 < t \leq T$.

Proof. Let $Q_R = \{x; |x| < R\}, \varepsilon > 0$, and consider the “truncated problem”

$$u_t - \Delta u + \beta_\varepsilon(u) = f \text{ if } x \in Q_R, 0 < t < T,$$
$$u(x, 0) = u_0(x) \text{ if } x \in Q_R,$$
$$u(x, t) = 0 \text{ if } x \in \partial Q_R, t > 0.$$

Here the $\beta_\varepsilon(u)$ are $C^\infty$ functions of $u$, defined for $\varepsilon > 0, u \in \mathbb{R}^1$, and satisfying:

$$\beta_\varepsilon(u) = 0 \text{ if } u > 0,$$
$$\beta_\varepsilon(u) \to -\infty \text{ if } u < 0, \varepsilon \downarrow 0,$$
$$\beta_\varepsilon'(u) > 0 \text{ if } u < 0.$$

Denote the solution of (2.9)–(2.11) by $u_{R,\varepsilon}$. We claim that

$$\min \{\inf f, 0\} \leq \beta_\varepsilon(u_{R,\varepsilon}) \leq 0.$$

To prove this as well as the existence of $u_{R,\varepsilon}$ it suffices to consider the case where $u_0(x)$ is a (nonnegative) continuous function; for then we can use approximation to handle the general case where $u_0$ is a measure.

The function $\beta_\varepsilon(u_{R,\varepsilon})$ takes its minimum in $\overline{Q}_R \times [0, T]$ at some point $(\bar{x}, \bar{i})$. If $u_{R,\varepsilon}(\bar{x}, \bar{i}) < 0$ then $u_{R,\varepsilon}$ also takes its minimum at $(\bar{x}, \bar{i})$, since $\beta_\varepsilon'(u) > 0$ if $u < 0$. Hence, if $(\bar{x}, \bar{i})$ does not lie on the parabolic boundary, then (2.9) yields

$$\beta_\varepsilon(u_{R,\varepsilon}) \geq f \text{ at } (\bar{x}, \bar{i}), \text{ provided } u_{R,\varepsilon}(\bar{x}, \bar{i}) < 0.$$

If $(\bar{x}, \bar{i})$ lies on the parabolic boundary, then

$$\beta_\varepsilon(u_{R,\varepsilon}) = 0 \text{ at } (\bar{x}, \bar{i}).$$

We have thus proved that if $u_{R,\varepsilon}(\bar{x}, \bar{i}) < 0$ then

$$\beta_\varepsilon(u_{R,\varepsilon}(\bar{x}, \bar{i})) \geq \min (0, \inf f).$$

If $u_{R,\varepsilon}(\bar{x}, \bar{i}) \geq 0$ then this inequality is also (trivially) true. This completes the proof of (2.12).

From (2.9), (2.12) we see that $u = u_{R,\varepsilon}$ satisfies

$$u_t - \Delta u = f - \beta_\varepsilon(u) \in L^\infty(Q_R).$$
Denote by \( K_R(x, t, y) \) the Green function of the heat operator in the cylinder \( Q_n \times (0, T) \). By the maximum principle,

\[
0 \leq K_R(x, t, y) \leq K(x, t, y).
\]

Using the construction of \( K_R \) as \( K + h_R \) with a suitable \( h_R \) (see [4]), recalling the standard estimates on \( D_x K \), and estimating \( D_x h_R \) by the interior Schauder estimates (for instance), we conclude that

\[
|D_x K_R(x, t, y)| \leq \frac{C}{t^{(n+1)/2}} \exp \left[ -\frac{|x-y|^2}{2t} \right] \quad \text{if} \ |x| < R - 1,
\]

where \( C \) is a constant independent of \( R \).

We can represent \( u \in U_R \) as follows:

\[
\begin{align*}
|u_1(x, t)|_{W^{1, \infty}(Q_{R-1})} &\leq \frac{C}{t^{(n+1)/2}}, \quad |u_2(x, t)|_{W^{1, \infty}(Q_{R-1})} \leq C t^{1/2} \\
\end{align*}
\]

where \( C \) is a constant independent of \( R \) and \( t \). Hence

\[
|e^{-\mu|x|}u_1(x, t)|_{W^{1, r}(Q_{R-1})} \leq \frac{C}{t^{(n+1)/2}}, \quad |e^{-\mu|x|}u_2(x, t)|_{W^{1, r}(Q_{R-1})} \leq C t^{1/2}
\]

for any \( \mu > 0 \), where \( C \) is a constant independent of \( R, t \).

Next, from the \( L^p \) estimates of [3], [7], for any \( \delta > 0 \),

\[
\begin{align*}
\int_\delta^T \int_Q e^{-\mu|x|} \left( \left| \frac{\partial}{\partial t} u_2 \right|^p + |D_x u_2|^p + |D_x^2 u_2|^p \right) dx \, dt \leq C(\delta)
\end{align*}
\]

where \( C(\delta) \) a constant independent of \( R \). Indeed, we write down (2.17) for \( u_2 \xi_i \), where \( \{\xi_i\} \) is a suitable partition of unity for \( \bar{Q}_R \), and sum over \( i \); then, using (2.16), we obtain (2.17) with a constant independent of \( R \) (cf. [1], [5]).

The inequality (2.17) can be verified directly for \( u_1 \). Since \( u = u_1 + u_2 \), we deduce that

\[
\begin{align*}
\int_\delta^T \int_Q e^{-\mu|x|} |u_1|^p \, dx \, dt \leq C.
\end{align*}
\]

Let \( \xi(t) \) be a \( C^\infty \) nonnegative function, \( \xi(t) = 0 \) if \( t < \delta/2 \), \( \xi(t) = 1 \) if \( t > \delta \). Differentiating (2.9) with respect to \( t \), we get

\[
u_t - \Delta u_t + \beta'_\xi(u) u_t = f_t.
\]
Multiplying both sides by $\exp(-p\mu|x|)\xi|u_t|^p u_t$ and integrating over $Q_R(x(0, T), we find that we have

$$\frac{1}{p} \int_{Q_R} |e^{-p\mu|x|}u_t(x, T)|^p \, dx + \int_0^T \int_{Q_R} \sum_i u_{txi} \frac{\partial}{\partial x_i} (\exp(-p\mu|x|)) \cdot \xi|u_t|^{p-2}u_t \, dx \, dt$$

$$\leq \int_0^T \int_{Q_R} |f_i| e^{-p\mu|x|} \xi|u_t|^{p-1} \, dx \, dt + \int_0^T \int_{Q_R} \frac{1}{p} |u_t|^p \xi e^{-\mu p|x|} \, dx \, dt$$

But

$$u_{txi}|u_t|^{p-2}u_t = \frac{1}{p} \frac{\partial}{\partial x_i} |u_t|^p$$

so that

$$\int_0^T \int_{Q_R} \sum_i u_{txi} \frac{\partial}{\partial x_i} (e^{-p\mu|x|}) \cdot \xi|u_t|^{p-2}u_t \, dx \, dt$$

$$= \int_0^T \int_{Q_R} \left( \frac{1}{p} \right) \xi|u_t|^p - \Delta(e^{-p\mu|x|}) \, dx \, dt.$$

However

$$\Delta e^{-p\mu|x|} = \left( p^2 \mu^2 - \frac{(n-1)p\mu}{|x|} \right) e^{-p\mu|x|}$$

$$\leq p^2 \mu^2 e^{-p\mu|x|}.$$

Hence we conclude that

$$\frac{1}{p} \int_{Q_R} |e^{-p\mu|x|} u_t(x, T)|^p \, dx \leq \int_0^T \int_{Q_R} |f_i| e^{-p\mu|x|} \xi|u_t|^{p-1} \, dx \, dt$$

$$+ \int_0^T \int_{Q_R} \frac{1}{p} |u_t|^p \xi e^{-\mu p|x|} \, dx \, dt$$

$$+ \int_0^T \int_{Q_R} \mu^2 \xi|u_t|^p e^{-\mu p|x|} \, dx \, dt.$$

Recalling (2.18), we conclude that, for any $\delta > 0$,

$$(2.19) \int_{Q_R} |e^{-\mu|x|} u_t(x, t)|^p \, dx \leq C$$

where $C$ is a constant independent of $R$. From (2.9), (2.12) we then also have

$$(2.20) \int_{Q_R} |e^{-\mu|x|} \Delta u(x, t)|^p \, dx \leq C$$

with another constant $C$, independent of $R$.

We extend the definition of $u = u_{R, \epsilon}$ into $R^n x [0, T]$ in such a way that (2.19), (2.20) remain valid with $Q_R$ replaced by $R^n$, and the $u_{R, \epsilon}$ remain uniformly bounded.
Using the standard $L^p$ estimates for $\Delta$, we can then choose a sequence $u = u_{R, \varepsilon} (R \to \infty, \varepsilon \to 0)$ which is convergent uniformly in compact subsets to a function $u$, such that

$$\frac{\partial}{\partial t} u_{R, \varepsilon} \to \frac{\partial}{\partial t} u, \quad D_x^2 u_{R, \varepsilon} \to D_x^2 u \quad (1 \leq |x| \leq 2)$$

weakly in the weak star topology of $L^p((\delta_0, T); L^p(R^n))$ for any $\delta_0 > 0$, $2 \leq p < \infty$. Thus, $u$ satisfies (2.8).

The fact that $u$ is a solution of the variational inequality (2.1), (2.2) follows by a standard argument. Next, from (2.15) we obtain

$$u_{R, \varepsilon}(x, t) \leq \int_{Q_R} K_R(x, t, y) u_0(y) \, dy \quad (2.21)$$

where, by (2.13) and the boundedness of $f - \beta_\varepsilon$, $|u_2(x, t)| \leq Ct$, $C$ a constant independent of $R$, $\varepsilon$. Going to the limit in (2.21), we obtain the inequality (2.7).

This completes the proof of existence. The proof of uniqueness follows by a standard argument: One writes (2.4) for $u$ and $v = \hat{u}$ and then for $\hat{u}$ and $v = u$, where $u, \hat{u}$ are two solutions. Then, by adding the inequalities, one gets, after some simple manipulations,

$$\frac{d}{dt} \int_{R^n} |e^{-\mu|x|}(\hat{u} - u)(x, t)|^2 \, dx \leq C \int_{R^n} |e^{-\mu|x|}(\hat{u} - u)|^2 \, dx;$$

hence $\hat{u} - u \equiv 0$ by (2.7).

3. Compact support for the solution

We shall now assume that

$$f \in L^\infty(R^n x(0, T)), \quad f_t \in L^\infty(R^n x(0, T)) \quad (3.1)$$

By Theorem 2.1, the variational inequality (3.1)-(3.3) has a unique solution $u(x, t)$ in $R^n x(0, \infty)$ (satisfying (2.8) for any $0 < \delta < T < \infty$). The object of the remaining part of this paper is to study the support of $u$. We shall henceforth need the condition:

$$f \leq -v \quad (3.2) \quad (v \text{ positive constant}).$$

**Theorem 3.1.** Let (2.5), (3.1), (3.2) hold. Then there is a positive number $T_0$ such that $u(x, t) \equiv 0$ if $t \geq T_0$.

**Proof.** From the proof of Theorem 2.1 we infer that $u_{R, \varepsilon}(x, 1) \leq M$ where $M$ is a positive constant independent of $R$, $\varepsilon$. Set $T_0 = 1 + M/v$ and consider the function

$$w(x, t) = M - v(t - 1) \quad (x \in R^n, 1 \leq t \leq T_0).$$

Observe that $w > 0$ if $1 \leq t \leq T_0$, $w(x, T_0) = 0$, and

$$w_t - \Delta w + \beta_\varepsilon(w) = -v \quad \text{if } x \in R^n, 1 \leq t \leq T_0.$$
We can apply the maximum principle to \( w - u_{R, \varepsilon} \) in the strip \( 1 \leq t \leq T_0 \), and thus conclude that \( w - u_{R, \varepsilon} \geq 0 \) in this strip. In particular,

\[
u_{R, \varepsilon}(x, T_0) \leq 0.
\]

Taking \( R \to \infty, \varepsilon \to 0 \), we conclude that \( u(x, T_0) \equiv 0 \). By uniqueness, \( u(x, t) \equiv 0 \) if \( t \geq T_0 \).

**Theorem 3.2.** Let the conditions (2.5), (3.1), (3.2) hold and suppose that \( u_0 \) has compact support. Then there is a positive constant \( R_0 \) such that \( u(x, t) \equiv 0 \) if \( |x| > R_0 \).

**Proof.** Let \( \rho \) be a positive number such that \( \text{supp} u_0 \subset \{x; |x| < \rho\} \). From the proof of Theorem 2.1 we infer that

\[
|u_{R, \varepsilon}(x, T)| \leq N \quad \text{if } x \in \mathbb{R}^n, \rho \leq |x| \leq R, 0 < t < T_0.
\]

Consider the function

\[
w(x) = \begin{cases} \mu(R_0 - r)^2 & \text{if } 0 < r < R_0, \\ 0 & \text{if } r > R_0 \end{cases}
\]

where \( r = |x| \) and \( \mu, R_0 \) are positive constants. Choosing \( \mu, R_0 \) such that \( 2\mu \leq v, \mu(R_0 - \rho)^2 \geq N \), we find that

\[
w_t - \Delta w + \beta_\varepsilon(w) = -\Delta w - v \quad \text{if } |x| > \rho,
\]

\[
w \geq N \quad \text{if } |x| = \rho.
\]

We can now apply the maximum principle to \( w - u_{R, \varepsilon} \) and conclude that \( w - u_{R, \varepsilon} \geq 0 \) if \( \rho < |x| < R, 0 < t < T_0 \). In particular,

\[
u_{R, \varepsilon}(x, t) \equiv 0 \quad \text{if } R_0 \leq |x| \leq R, 0 \leq t \leq T_0.
\]

Taking \( R \to \infty, \varepsilon \to 0 \) the assertion of the theorem follows.

We conclude this section with a standard comparison lemma that will be needed in the following sections.

**Lemma 3.3.** Denote by \( u \) and \( \hat{u} \) two functions satisfying (2.1) and (2.2) with

\[
u, \hat{u} \in L^\infty(\delta, T; W^{2,2,\mu}(\mathbb{R}^n)), \quad u_t, \hat{u}_t \in L^\infty(\delta, T; L^{2,1,\mu}(\mathbb{R}^n))
\]

for some \( \mu \) and any \( \delta > 0 \). Assume \( u(\cdot, t) \to u_0(\cdot) \) and \( \hat{u}(\cdot, t) \to \hat{u}_0(\cdot) \) in \( L^{2,\mu}(\mathbb{R}^n) \) as \( t \to 0 \). If \( u_0 \leq \hat{u}_0 \) a.e. on \( \mathbb{R}^n \) and \( f \leq \hat{f} \) a.e. on \( \mathbb{R}^n \times (0, T) \), then \( u \leq \hat{u} \) a.e. in \( \mathbb{R}^n \times (0, T) \).

**Proof.** Let \( w = (u - \hat{u}) \); substituting \( v = \text{Min} \{u, \hat{u}\} \) and then \( \hat{v} = \text{Max} \{u, \hat{u}\} \) in (2.1) we obtain after addition:

\[
w_t w^+ - \Delta w^+ w^+ \leq (f - \hat{f})w^+ \leq 0 \text{ a.e.}
\]
Multiplying through by $e^{-2\mu|x|}$ and integrating by parts, we get, after some simple calculations,

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} e^{-2\mu|x|}|w^+(x, t)|^2 \, dx \leq \int_{\mathbb{R}^n} \frac{1}{2}|w^+(x, t)|^2 \Delta(e^{-2\mu|x|}) \, dx$$

$$\leq 2\mu^2 \int_{\mathbb{R}^n} |w^+(x, t)|^2 e^{-2\mu|x|} \, dx.$$

On the other hand $w^+(\cdot, t) \to 0$ in $L^2, u_0(\mathbb{R}^n)$ as $t \to 0$. We conclude that $w^+ = 0$ and thus $u = \hat{u}$ a.e. as $\mathbb{R}^n \times (0, T)$.

4. Estimates on the support

In what follows we use the notation $B(\rho) = \{x; |x| < \rho\}$. If $A$ and $B$ are sets in $\mathbb{R}^n$, we denote by $A + B$ their vector sum.

We shall denote by $S(t)$ the support of the function $x \mapsto u(x, t)$, and write $S = S(0)$, i.e., $S$ is the support of the measure $u_0$ ($S$ is a closed set).

**Theorem 4.1.** Let $f$ satisfy (3.1), (3.2) and let $u_0(x) \geq 0$ be a function in $L^\infty(\mathbb{R}^n)$. Assume that the support $S$ of $u_0$ consists of a finite union of disjoint bounded closed domains, with $C^1$ boundary. Then, there is a positive constant $c$ such that

$$S(t) \subset S + B(c/\sqrt{t \log t})$$

if $t$ is sufficiently small.

The proof of Theorem 4.1 relies on the following lemmas.

**Lemma 4.2.** There exists a function $w(x, t)$, $x \in \mathbb{R}, t \in (0, 1)$ such that

$$w \in L^\infty(\mathbb{R} \times (0, 1),$$

$$w_t, w_x, w_{xx} \in L^\infty(\mathbb{R} \times (0, 1)) \text{ for each } 0 < \delta < 1,$$

$$w \geq 0 \text{ as } \mathbb{R} \times (0, 1),$$

$$w(x, t) = 0 \text{ for } x > \sqrt{6t \log t} \text{ and } t \in (0, 1),$$

$$|w_t - w_{xx}| \leq k t^{1/2} |\log t|^{3/2} \text{ for } x \in \mathbb{R}, t \in (0, 1) \text{ and } k \text{ some constant},$$

$$w(x, t) = 0 \text{ for } x > \sqrt{6t |\log t|} \text{ and } t \in (0, 1).$$

**Proof of Lemma 4.2.** Let $s(t) = \sqrt{6t |\log t|}$ and define for $x \in \mathbb{R}, t \in (0, 1)$:

$$v(x, t) = \begin{cases} A x^2 + B t + C t \log t + \frac{D}{\sqrt{t}} e^{-x^2/4t} & \text{when } |x| < s(t), \\ 0 & \text{when } |x| > s(t). \end{cases}$$

We determine the constants $A, B, C,$ and $D$ in such a way that

$$v(s(t), t) = 0, \ v_x(s(t), t) = 0 \text{ for } t \in (0, 1).$$
Therefore it is required that
\[-6At \log t + Bt + Ct \log t + Dt = 0\]
and
\[2s(t)(A - D/4) = 0,\]
i.e.
\[(4.10) \quad A = D/4, B = -D, C = 3D/2.\]

It is easy to verify that when \(D > 0\), then \(v > 0\). Define now for \(x \in \mathbb{R}\) and \(t \in (0, 1)\),
\[(4.11) \quad w(x, t) = \int_{x}^{s(t)} v(\xi, t) \, d\xi,
\]
so that \(w(x, t) = 0\) when \(x > s(t)\) and hence \(w(x, 0) = 0\) for \(x > 0\). Next let \(x < 0\); if \(t\) is small enough to insure \(s(t) < -x\), then
\[w(x, t) = \int_{-s(t)}^{+s(t)} v(\xi, t) \, d\xi.
\]
Therefore
\[w(x, t) = 2 \int_{0}^{s(t)} \left( A\xi^2 + Bt + Dt \log t + \frac{D}{\sqrt{t}} e^{-\xi^2/4t} \right) d\xi = \frac{2}{3} As^3(t) + 2(Bt + Dt \log t)s(t) + \frac{2D}{\sqrt{t}} \int_{0}^{s(t)} e^{-\xi^2/4t} d\xi.
\]

The last term equals
\[2D \int_{0}^{\sqrt{6} \log t} e^{-\eta^2/4} d\eta,
\]
and thus as \(t \to 0\) we see that, for \(x < 0\),
\[w(x, t) \to 2D \int_{0}^{+\infty} e^{-\eta^2/4} d\eta.
\]

We fix now \(D\) in such a way that
\[2D \int_{0}^{+\infty} e^{-\eta^2/4} d\eta = 1
\]
and next \(A, B,\) and \(C\) are determined by \((4.10)\).

In order to compute \(Lw = w_t - w_{xx}\) we distinguish three regions.

**Region I.** \(x > s(t)\), where \(w = 0\) and so \(Lw = 0\).

**Region II.** \(x < -s(t)\) where
\[w(x, t) = \int_{-s(t)}^{+s(t)} v(\xi, t) \, d\xi,
\]

\[w_t = v(s(t), t)s'(t) + v(-s(t), t)s'(t) + \int_{-s(t)}^{+s(t)} v_t(\xi, t) \, d\xi,
\]
\[w_{xx} = 0.
\]
By (4.9) we get
\[ w_t(x, t) = 2 \int_0^{s(t)} \left( B + C \log t + C \right) \, d\xi + 2D \int_0^{s(t)} \xi_t(x, t) \, d\xi \]
where
\[ \xi(x, t) = \frac{1}{\sqrt{t}} e^{-x^2/4t}. \]

Since \( \xi_t = \xi_{xx} \) we have
\[ \int_0^{s(t)} \xi_t(x, t) \, d\xi = \int_0^{s(t)} \xi_{xx}(x, t) \, d\xi = \xi_x(s(t), t) - \xi_x(0, t) = -\frac{s(t)}{2}. \]

Finally
\[ Lw = w_t(x, t) = 2s(t)(B + C \log t + C) - Ds(t) = 3Ds(t) \log t. \]

Region III. \(-s(t) < x < +s(t)\) where \( w_t(x, t) = \int_x^{s(t)} v(\xi, t) \, d\xi \). Thus
\[ w_t(x, t) = v(x, t), \quad w_{xx}(x, t) = -v(x, t) = \int_x^{s(t)} v_{xx}(\xi, t) \, d\xi. \]

Consequently
\[ Lw = \int_x^{s(t)} (v_t - v_{xx})(\xi, t) \, d\xi \]
\[ = \int_x^{s(t)} (B + C \log t + 2A) \, d\xi \]
\[ = (s(t) - x)C \log t. \]

In the three regions we conclude that \( |Lw| \leq 3Ds(t)|\log t|. \)

**Lemma 4.3.** Let \( C \) be the cube \((-\theta, +\theta)^n (\theta > 0)\). There exists a function \( z(x, t), x \in R^n, t \in (0, 1) \) such that

\[ z \in L^\infty(R^n \times (0, 1)), \]
\[ z_t, z_{xx}, z_{xixj} \in L^2(R^n \times (\delta, 1)) \text{ for each } 0 < \delta < 1, \]
\[ z \geq 0 \text{ on } R^n \times (0, 1), \]
\[ as t \to 0, z(x, t) \to 0 \text{ for } x \in C, \text{ and } z(x, t) \to \text{ limit } \geq 1 \text{ for } x \notin \bar{C}, \]
\[ |z_t - \Delta z| \leq k't^{1/2}|\log t|^{3/2} \text{ for } x \in R^n, t \in (0, 1) \text{ and } k \text{ is some constant,} \]
\[ z(x, t) = 0 \text{ for Max}_{1 \leq i \leq n} |x_i| < \theta - \sqrt{6t|\log t|} \text{ (t small).} \]

**Proof.** It is clear from Lemma 4.2 that the function
\[ z(x, t) = \sum_{i=1}^{n} [w(x_i + \theta, t) + w(\theta - x_i, t)] \]
satisfies all the required properties.
Proof of Theorem 4.1. Let \( \alpha = \text{ess sup}_S u_0 \). We denote by \( v(x_0) \) the unit outward normal at every point \( x_0 \in \partial S \) and by \( C(x_0, 2\theta) \) an open cube centered on \( v(x_0) \) whose side has length \( 2\theta \) and such that \( x_0 \) is one of the vertices.

Since \( \partial S \) is \( C^1 \), there exists a fixed \( \theta > 0 \), independent of \( x_0 \), sufficiently small such that \( C(x_0, 2\theta) \cap S = \emptyset \) for every \( x_0 \in \partial S \). By shifting the origin we can always assume that \( C(x_0, 2\theta) \) is centered at the origin and has the form \((-\theta, +\theta)\). It follows from the comparison Lemma 3.3 that \( u \leq \alpha x \) on \( \mathbb{R}^n x(0, t_0) \) where \( t_0 \) is small enough to insure that

\[
kt^{1/2}|\log t|^{3/2} \leq v \quad \text{for} \quad 0 < t < t_0.
\]

Therefore we conclude that \( u(x, t) = 0 \) for \( t \) small enough and for \( x \) of the form \( x = x_0 + \lambda v(x_0), \sqrt{6n|\log t|} < \lambda < \theta \).

The conclusion of the theorem follows.

Remark. The proof of Theorem 4.1 applies also in cases where \( \partial S \) is not in \( C^1 \); for instance in case \( S \) is a convex set.

Let \( S \) be a closed set in \( \mathbb{R}^n \). Suppose for any \( x \in \partial S \) there exists a cone \( V_x \) with vertex \( x \) and with opening \( \sigma \) and height \( h \) independent of \( x \) such that \( V_x \subset S \); then we say that \( S \) satisfies the uniform cone property.

In the next theorem we derive a lower bound on \( S(t) \).

THEOREM 4.2. Let \( f \) satisfy (3.1), and let \( f \geq -v_0 > 0, v_0 \) constant. Let \( U_0 \) be a bounded measurable function whose support \( S \) satisfies the uniform cone property. If there is a positive constant \( \beta \) such that \( U_0(x) \geq \beta \) for \( x \in S \), then there is a positive constant \( c \) such that

\[
S(t) \geq S + B(c\sqrt{\log t}) \quad \text{for all} \quad t \text{ sufficiently small.}
\]

Proof. Consider the function

\[
w(x, t) = \frac{\beta}{(2\pi t)^{n/2}} \int_S \exp \left[ -\frac{|x - \xi|^2}{4t} \right] \, d\xi - v_0 t.
\]

It satisfies \( \partial_t w - \Delta w = -v_0, w(x, 0) \leq u_0(x) \). Since \( \partial_t \geq f \geq -v_0 \), the maximum principle can be applied to \( u - w \). It gives

\[
u(x, t) \geq w(x, t).
\]

Denote by \( d(y) \) the distance of a point \( y \) to \( S \). If we can prove that

\[
w(y, t) > 0 \quad \text{whenever} \quad y \notin S, d(y) \leq c\sqrt{\log t},
\]

then, by (4.14), also \( u(y, t) > 0 \) and, consequently, the assertion (4.12) would follow.

In order to prove (4.15), let \( x_0 \) be a point on \( \partial S \) such that \( d(y) = |y - x_0| \). Integrating in (4.13) only over the cone with vertex \( x_0 \), opening \( \sigma \), and height \( \eta \) \((0 < \eta < h)\) which lies in \( S \), we find that

\[
w(y, t) > \beta_0 \frac{\eta^n}{t^{n/2}} \exp \left[ -\frac{\mu}{t} d^2(y) - \frac{\mu \eta^2}{t} \right] - v_0 t.
\]
for any $0 < \eta < h$, where $\beta_0, \mu$ are positive constants. If $t$ is sufficiently small then we can take $\eta = \sqrt{t}$. Hence, $w(y, t) > 0$ if

$$\beta_1 \exp \left[ -\frac{\mu d^2(y)}{t} \right] \geq v_0 t$$

where $\beta_1$ is a positive constant. Taking the logarithm we see that $w(y, t) > 0$ if

$$\frac{\mu d^2(y)}{t} \leq |\log t| + \text{const.}$$

This gives (4.12) with $c < 1/\sqrt{\mu}$.

5. Estimates on the support (continued)

**Theorem 5.1.** Let (3.1), (3.2) hold and let $S = \text{supp } u_0$ be a finite disjoint union of bounded closed domains with $C^2$ boundary. Assume that

(5.1) $u_0 \in C^2(S), \quad u_0 = 0, \quad D_i u_0 = 0$ on $\partial S$.

Then there exists a positive constant $\alpha$, depending only on the data, such that

(5.2) $S(t) \subset S + B(\alpha \sqrt{t})$ for all $t \geq 0$.

**Proof.** Let $y$ be any point outside $S$. Let $\delta = \text{dist.} (y, S)$. For simplicity we take $y = 0$.

Using (5.1) we find that, for any $x \in S$,

(5.3) $u_0(x) = \left| u_0(x) - u_0(x') \right| \leq C_0 |x - x'|^2 \leq C_0 (|x| - \delta)^2$

where $x'$ is the first point where the ray from $x$ to $y$ intersects $\partial S$.

Setting $r = |x|, \lambda = (r - \delta) / \sqrt{t}$, we shall construct a comparison function

$$w(x, t) = \begin{cases} t F(\lambda) & \text{if } \delta - \alpha \sqrt{t} \leq r < \infty, \\ 0 & \text{if } r < \delta - \alpha \sqrt{t} \end{cases}$$

for $0 < t < \sigma, \sigma$ sufficiently small, where $F$ is a nonnegative function defined on $[-\alpha, +\infty)$. We require that

(5.4) $F(-\alpha) = 0, \quad F'(\alpha) = 0$,

so that $w$ is continuously differentiable across $r = \delta - \alpha \sqrt{t}$. We also require that $w(x, 0) \geq u_0(x)$. In view of (5.3), the last inequality holds if

(5.5) $\lim_{\lambda \to +\infty} \frac{F(\lambda)}{\lambda^2} \geq C_0$.

Finally, we require that $w_t - \Delta w \geq -v$; in terms of $F$ this means that

(5.6) $F - \frac{1}{2} \lambda F' - \frac{n - 1}{r} \sqrt{t} F' - F'' \geq -v$

where the argument in $F, F', F''$ is $\lambda$. 
We seek $F(\lambda)$ of the form

$$F(\lambda) = \begin{cases} 
\mu(\lambda + \alpha)^2 & \text{if } -\alpha < \lambda < 0, \\
A\lambda^2 + B\lambda + C & \text{if } \lambda > 0.
\end{cases}$$

Then $w$ is continuously differentiable across $\lambda = 0$ if

$$\mu x^2 = C, \, 2\mu x = B.$$ 

If we take

$$A \geq C_0$$

then (5.5) holds. The conditions in (5.4) are clearly satisfied.

We now turn to verifying the inequality (5.6). In the region where $-\alpha < \lambda < 0$, (5.6) reduces to

$$\mu(\lambda + \alpha)^2 - \mu\lambda(\lambda + \alpha) - \mu \frac{2(n-1)}{r} \sqrt{t(\lambda + \alpha) - 2}\mu \geq -v.$$ 

If $t$ is sufficiently small then $\delta/2 < r < \delta$; the last inequality is then a consequence of

$$\mu \left[ \lambda x + \alpha^2 - \frac{4(n-1)}{\delta} \sqrt{t(\lambda + \alpha) - 2} \right] \geq -v,$$

or, a consequence of

$$\mu \left( 2 + \frac{4(n-1)}{\delta} \sqrt{\sigma x - \alpha^2} \right) \leq v \quad (0 < t \leq \sigma).$$

In the region where $\lambda > 0$, (5.6) holds if

$$A\lambda^2 + B\lambda + C - \frac{1}{2}\lambda(2A\lambda + B) - \frac{n-1}{r} \sqrt{t(2A\lambda + B) - 2A} \geq -v.$$ 

Since $r \geq \delta$, this inequality holds, for all $0 < t < \sigma$, if

$$\frac{B}{2} - 2 \frac{n-1}{\delta} \sqrt{\sigma A} \geq 0,$$

$$C - \frac{n-1}{\delta} \sqrt{\sigma B - 2A} \geq -v.$$ 

From (5.7) we find that

$$\alpha = 2C/B, \quad \mu = B^2/4C.$$ 

Taking $C \geq 2A - v + 1$ we see that (5.11) holds if $\sigma$ is sufficiently small. If we further choose $B$, $C$ to be positive, then $\alpha$ and $\mu$ are positive. If we also take $C/B$ to be sufficiently large, then $\alpha$ becomes so large that the left-hand side of (5.9) is negative. Thus (5.9) is satisfied. Notice that also (5.10) is satisfied if $\sigma$ is sufficiently small.

Thus, with the above choice of $B$, $C$, and $A$, and with the definitions of $\alpha$, $\mu$
by (5.12), we have established that the function $w$ is a comparison function, i.e., it satisfies the conditions of Lemma 3.3. Consequently, $u(x, t) \leq w(x, t)$ in $\mathbb{R}^n \times (0, \sigma)$. The conclusion of Theorem 5.1 follows.

**Remark 1.** Theorem 5.1 extends to the case where $S$ consists of a finite disjoint union of closed convex domains with $C^1$ boundary.

**Remark 2.** If $f_t \leq 0$ and
\[
f + \Delta u_0 < 0 \quad \text{in } S
\]
then one can show that $u_t \leq 0$. Consequently, $S(t) \subset S(t')$ if $t > t' > 0$.

6. Instantaneous shrinking of the support

In this section we consider the case where $u_0$ need not have compact support, but $u_0(x) \to 0$ if $|x| \to \infty$. We shall show that the support $S(t)$ of $x \to u(x, t)$ is compact, for any $t > 0$.

**THEOREM 6.1.** Let $f$ satisfy (3.1), (3.2) and assume that
\[
u_0 \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n), \quad u_0(x) \to 0 \text{ if } |x| \to \infty.
\]
Then $S(t)$ is a compact set, for any $t > 0$.

**Proof.** The assertion of the theorem follows from the assertion that there exists a function $\phi(r)$ and a positive number $R$, such that
\[
\phi(r) > 0, \quad \phi(r) \downarrow 0 \text{ if } r \uparrow \infty,
\]
\[
u(x, t) = 0 \text{ if } t > \phi(|x|), \quad |x| > R.
\]

In view of Theorem 3.2 it suffices to prove (6.3) just for $t < t_0$, where $t_0$ is a sufficiently small positive number. We first establish that
\[
u(x, t) \to 0 \text{ if } |x| \to \infty, \quad \text{uniformly in } t.
\]

Let $z$ be the bounded solution of
\[
z_t - \Delta z = 0 \quad (x \in \mathbb{R}^n, \quad t > 0), \quad z(x, 0) = u_0(x) \quad (x \in \mathbb{R}^n).
\]
Representing $z$ in terms of the fundamental solution and using (6.1), we find that for any $T > 0$,
\[
z(x, t) \to 0 \text{ if } |x| \to \infty, \quad \text{uniformly in } t, \quad 0 \leq t \leq T.
\]
Since $z \geq 0$, we can verify that the function $\hat{u} = z$ satisfies (2.1) with $\hat{f} = 0$. Noting that $\hat{f} \geq \hat{f}$, we can apply Lemma 3.3 to conclude that $z \geq u$. But then (6.4) is a consequence of (6.5).

Let $\eta$ be any small positive number. By (6.4), there is an $R > 0$ sufficiently large such that
\[
u(x, t) < \eta \quad \text{if } |x| > R, \quad 0 \leq t \leq T.
\]
We shall estimate $u(x, t)$ more precisely in a region

$$\text{(6.7)} \quad |x| > R, \ 0 < t < t_0$$

where $t_0$ is a sufficiently small positive number.

Let $r = |x|$ and

$$w(x, t) = \begin{cases} \frac{((\phi(r) - t)^2}{r} & \text{if } |x| > R, \ 0 < t < \phi(r), \\ 0 & \text{if } |x| > R, \ t > \phi(r). \end{cases}$$

Then $w$ satisfies $w \geq u$ if $|x| = R, \ 0 < t < t_0$, or if $|x| > R, \ t = 0$ provided

$$\text{(6.8)} \quad (\phi(R) - t_0)^2 \geq \eta,$$

$$\text{(6.9)} \quad \phi^2(r) \geq u_0(x) \ (r = |x| > R).$$

Also $w$ satisfies the variational inequality (2.1) on $|x| > R, \ 0 < t < t_0$ with $f \geq -\gamma$ provided

$$-2(\phi(r) - t) - 2(\phi(r) - t)\phi''(r) - 2(\phi'(r))^2$$

$$\text{(6.10)} \quad -2 \frac{n-1}{r} (\phi(r) - t)\phi'(r) \geq -\gamma \ (r > R, \ 0 < t < \phi(r)).$$

Since the last inequality is linear in $t$, it suffices to verify it at $t = \phi(r)$ and $t = 0$, i.e.,

$$\text{(6.11)} \quad (\phi'(r))^2 \leq \frac{1}{2}\gamma \ (r > R),$$

$$\text{(6.12)} \quad \phi(r) + \phi(r)\phi''(r) + (\phi'(r))^2 + \frac{n-1}{r} \phi(r)\phi'(r) \leq \frac{1}{2}\gamma.$$

We shall now construct a function $\phi$ satisfying (6.8), (6.9), (6.11), and (6.12). Since $u_0(x) \to 0$ as $|x| \to \infty$, we can find an increasing sequence $(a_n)$ such that

$$a_1 = R$$

and

$$\sqrt{u_0(x)} \leq \sqrt{\eta} \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \chi_{[a_n, a_{n+1}]}(|x|) \quad \text{for } |x| > R$$

where $\chi_{[a, b]}$ is the characteristic function of the interval $[a, b]$. We can always assume that $a_{n+1} - a_n \geq 1$ for $n \geq 1$.

Let $\zeta(t), \ t \in R^1$, be a smooth function with compact support such that $\zeta \geq 0$, $\zeta(t) = 1$ for $0 \leq t \leq 1$, $|\zeta(t)| \leq 1$, $|\zeta'(t)| \leq 1$ and $|\zeta''(t)| \leq 1$ for $t \in R^1$.

Define

$$\phi(r) = \rho \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \zeta \left( \frac{r - a_n}{a_{n+1} - a_n} \right), \ \rho > 0.$$  

Clearly $\phi$ is a smooth function and $\phi(r) \to 0$ as $r \to +\infty$; we are going to see that for $\eta$ small enough, it is possible to choose $\rho$ in such a way that $\phi$ satisfies (6.8), (6.9), (6.11), and (6.12).
Since \( \phi(R) \geq \rho \), the conditions

\[
\rho \geq 2\sqrt{\eta} \quad \text{and} \quad t_0 \leq \sqrt{\eta}
\]

imply (6.8).

We have, for \( |x| > R \),

\[
\sqrt{u_0(x)} \leq \frac{\sqrt{\eta}}{\rho} \phi(|x|)
\]

and therefore (6.13) also implies (6.9).

On the other hand \( |\phi'(r)| \leq 2\rho \) and \( |\phi''(r)| \leq 2\rho \) for \( r \in R \). Thus (6.11) and (6.12) are consequences of the following

\[
4\rho^2 \leq \frac{1}{2}\gamma
\]

\[
2\rho + 4\rho^2 + 4\rho^2 + 4\rho^2 \leq \frac{1}{2}\gamma.
\]

Conclusion: we first choose a \( \rho > 0 \) satisfying (6.14), (6.15); next \( \eta \) and \( t_0 \) are obtained from (6.13). Finally, we choose \( R, \{a_n\} \) and construct \( \phi \).

From a variant of Lemma 3.3 we deduce that

\[
u(x, t) \leq w(x, t) \quad \text{if} \quad |x| > R, \quad 0 < t < t_0
\]

and the assertion (6.3) follows.

Remark. If \( u_0 \) has compact support, then in the above proof we can take \( \phi(r) \) to vanish if \( r \) is sufficiently large. Thus \( u(x, t) \) will vanish if \( |x| \geq R_0 \) for some \( R_0 \) sufficiently large. This gives another proof of Theorem 3.2.

References