Where Sobolev interacts with Gagliardo–Nirenberg

Haïm Brezis[a,b,c], Petru Mironescu[d,e,*]

[a] *Department of Mathematics, Rutgers University, Hill Center, Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854, USA*
[b] *Departments of Mathematics and Computer Science, Technion - Israel Institute of Technology, Haifa, 32000, Israel*
[c] *Sorbonne Universités, UPMC Univ. Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions, F–75005, Paris, France*
[d] *Université de Lyon; Université Lyon 1; CNRS UMR 5208 Institut Camille Jordan; 43, boulevard du 11 novembre 1918, F-69622 Villeurbanne Cedex, France*
[e] *Simion Stoilow Institute of Mathematics of the Romanian Academy; Calea Griviţei 21, 010702 Bucureşti, România*

Abstract

We investigate the validity of the fractional Gagliardo-Nirenberg-Sobolev inequality

(1) $\|f\|_{W^{r,q}(\Omega)} \lesssim \|f\|^{\theta}_{W^{s_1,p_1}(\Omega)}\|f\|^{1-\theta}_{W^{s_2,p_2}(\Omega)},\ \forall f\in W^{s_1,p_1}(\Omega)\cap W^{s_2,p_2}(\Omega).$

Here, s_1, s_2, r are non-negative numbers (not necessarily integers), $1\le p_1, p_2, q\le\infty$, and we assume, for some $\theta\in(0,1)$, the standard relations

(2) $r < s := \theta s_1 + (1-\theta)s_2$ and $\dfrac{1}{q} = \left(\dfrac{\theta}{p_1}+\dfrac{1-\theta}{p_2}\right) - \dfrac{s-r}{N}.$

Formally, estimate (1) is obtained by combining the "pure" fractional Gagliardo-Nirenberg style interpolation inequality

(3) $\|f\|_{W^{s,p}(\Omega)} \lesssim \|f\|^{\theta}_{W^{s_1,p_1}(\Omega)}\|f\|^{1-\theta}_{W^{s_2,p_2}(\Omega)}$ (with $1/p := \theta/p_1 + (1-\theta)/p_2$)

with the fractional Sobolev style embedding

(4) $W^{s,p}(\Omega) \hookrightarrow W^{r,q}(\Omega),\ 0\le r<s,\ 1\le p<q\le\infty,\ \dfrac{1}{q}=\dfrac{1}{p}-\dfrac{s-r}{N},\ p(s-r)\le N.$

Estimates (3) and (4) are true "most of the time", but not always; the exact range of validity of (3) and (4) has been known. Combining these results, we infer that (1) is valid "most of the time". However, the validity of (1) when (3) and/or (4) fail was unclear. The goal of this paper is to characterize the values of $s_1, s_2, r, p_1, p_2, q, \theta, N$ such that (1) holds.

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1. Introduction

This is a follow-up of [5], and we use the same notation as in [5]. There, we have investigated the validity of the Gagliardo-Nirenberg (GN) *interpolation* estimate

$$\|f\|_{W^{s,p}(\Omega)} \lesssim \|f\|^{\theta}_{W^{s_1,p_1}(\Omega)}\|f\|^{1-\theta}_{W^{s_2,p_2}(\Omega)},\ \forall f\in W^{s_1,p_1}(\Omega)\cap W^{s_2,p_2}(\Omega). \tag{1.1}$$

*Corresponding author
Email addresses: brezis@math.rutgers.edu (Haïm Brezis), mironescu@math.univ-lyon1.fr (Petru Mironescu)

Here, the real numbers $s_1, s_2, s \geq 0$, $1 \leq p_1, p_2, p \leq \infty$ and $\theta \in (0,1)$ satisfy the relations

$$s_1 \leq s_2,\ s = \theta s_1 + (1-\theta)s_2 \text{ and } \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}. \tag{1.2}$$

We say that Ω is a *standard domain* in $\mathbb{R}^N$ if

$$\Omega \text{ is either } \mathbb{R}^N\text{, or a half space, or a Lipschitz bounded domain in } \mathbb{R}^N. \tag{1.3}$$

When all the smoothness exponents, s_1, s_2 and s, are integers, the validity of (1.1) was established by Gagliardo [8] and Nirenberg [12]. For general non-negative exponents, non necessarily integers, (1.1) may fail. In [5], we gave a necessary and sufficient condition for the validity of (1.1). This involves the following assumption:

$$s_2 \text{ is an integer } \geq 1,\ p_2 = 1 \text{ and } 0 < s_2 - s_1 \leq 1 - \frac{1}{p_1}. \tag{1.4}$$

More specifically, we have proved the following

Theorem A ([5])**.** Let Ω be a standard domain in $\mathbb{R}^N$.

1. Assume that (1.4) *fails*. Then (1.1) *holds* for every $\theta \in (0,1)$, with s and p given by (1.2).
2. Assume that (1.4) *holds*. Then (1.1) *fails* for every $\theta \in (0,1)$, with s and p given by (1.2).

Let us also recall the following well-known Sobolev style embeddings. Let s, r, p, q satisfy

$$0 \leq r < s < \infty,\ 1 \leq p < q \leq \infty,\ r - \frac{N}{q} = s - \frac{N}{p}. \tag{1.5}$$

Then we have "most of the time" $W^{s,p}(\Omega) \hookrightarrow W^{r,q}(\Omega)$. More specifically, we have the following result, well-known to experts.

Theorem B. Let Ω be a standard domain in $\mathbb{R}^N$. Let s, r, p, q, N satisfy (1.5). Then we have

$$W^{s,p}(\Omega) \hookrightarrow W^{r,q}(\Omega) \tag{1.6}$$

with the following exceptions, where (1.6) fails.

1. When

   $$N = 1,\ s \text{ is an integer } \geq 1,\ p = 1, 1 < q < \infty \text{ and } r = s - 1 + 1/q, \tag{1.7}$$

   we have

   $$W^{s,1}(\Omega) \not\hookrightarrow W^{s-1+1/q,q}(\Omega). \tag{1.8}$$

2. When

   $$N \geq 1, 1 < p < \infty, q = \infty \text{ and } s - \frac{N}{p} = r \geq 0 \text{ is an integer}, \tag{1.9}$$

   we have

   $$W^{s,p}(\Omega) \not\hookrightarrow W^{r,\infty}(\Omega). \tag{1.10}$$

For the convenience of the reader, we present in the appendix a proof of some special cases of Theorem B that we could not find in the literature, and give references for the other ones.

The Gagliardo-Nirenberg-Sobolev (GNS) inequalities are inequalities obtained, at least formally, by combining (1.1) with (1.6). They are of the form

$$\|f\|_{W^{r,q}(\Omega)} \lesssim \|f\|^{\theta}_{W^{s_1,p_1}(\Omega)} \|f\|^{1-\theta}_{W^{s_2,p_2}(\Omega)}, \ \forall f \in W^{s_1,p_1}(\Omega) \cap W^{s_2,p_2}(\Omega), \tag{1.11}$$

where

$$\begin{aligned} &0 \le s_1 \le s_2, r \ge 0, 1 \le p_1, p_2, q \le \infty, (s_1,p_1) \ne (s_2,p_2), \theta \in (0,1), \\ &r < s := \theta s_1 + (1-\theta)s_2, \frac{1}{q} = \left(\frac{\theta}{p_1} + \frac{1-\theta}{p_2}\right) - \frac{s-r}{N}. \end{aligned} \tag{1.12}$$

[More specifically, (1.11) can be obtained either by using first (1.1), next (1.6), or by applying first (1.6) in order to obtain the embeddings $W^{s_j,p_j} \hookrightarrow W^{r_j,q_j}$, $j = 1,2$, next by applying (1.1) to the couple $(W^{r_1,q_1}, W^{r_2,q_2})$, with interpolation parameter θ. Both procedures lead to the same family of inequalities.]

The conditions $(s_1,p_1) \ne (s_2,p_2)$ and $r < s$ are imposed in order to exclude from (1.11) the GN interpolation inequalities (1.1) and the Sobolev embeddings (1.6). Indeed, let us note that, when $(s_1,p_1) = (s_2,p_2)$ and $r < s = s_1 = s_2$, estimate (1.11) amounts to (1.6), whose validity is settled by Theorem B. On the other hand, when $r = s$, (1.11) becomes (1.1), and we are in position to apply Theorem A.

We also note that, in (1.12), the parameter q is determined by all the other ones.

Estimate (1.11) is valid in "many cases". Indeed, assuming (1.12), by combining Theorems A and B we obtain a wide range of $s_1, s_2, r, p_1, p_2, q, \theta, N$ such that (1.11) holds. Here are two typical "historical" examples.

Ladyzhenskaya's inequality ([9]). Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Then

$$\|f\|_{L^4} \lesssim \|f\|^{1/2}_{L^2} \|\nabla f\|^{1/2}_{L^2}, \forall f \in W^{1,2}_0(\Omega). \tag{1.13}$$

Inequality (1.13) can be obtained as follows. First, Theorem A with $s_1 = 0$, $s_2 = 1$, $p_1 = 2$, $p_2 = 2$, $\theta = 1/2$ yields

$$\|f\|_{W^{1/2,2}} \lesssim \|f\|^{1/2}_{L^2} \|f\|^{1/2}_{W^{1,2}}, \forall f \in W^{1,2}(\Omega). \tag{1.14}$$

Next, Theorem B with $N = 2$, $s = 1/2$, $p = 2$, $r = 0$, $q = 4$ gives

$$\|f\|_{L^4} \lesssim \|f\|_{W^{1/2,2}}, \forall f \in W^{1/2,2}(\Omega). \tag{1.15}$$

We obtain (1.13) from (1.14)–(1.15). □

Nash's inequality ([11]). Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Then

$$\|f\|_{L^2} \lesssim \|f\|^{1/2}_{L^1} \|\nabla f\|^{1/2}_{L^2}, \ \forall f \in W^{1,2}_0(\Omega). \tag{1.16}$$

In order to obtain (1.16), we start, as above, from the GN interpolation style inequality

$$\|f\|_{W^{1/2,4/3}} \lesssim \|f\|^{1/2}_{L^1} \|f\|^{1/2}_{W^{1,2}}, \ \forall f \in W^{1,2}(\Omega) \tag{1.17}$$

and the Sobolev style inequality

$$\|f\|_{L^2} \lesssim \|f\|_{W^{1/2,4/3}}, \ \forall f \in W^{1/2,4/3}(\Omega). \tag{1.18}$$

We obtain (1.16) from (1.17)–(1.18). □

The above technique works well when estimates (1.1) *and* (1.6) are valid. However, it *may* happen (and it *does* happen) that (1.11) holds despite the fact that one (or both) of the estimates (1.1) or (1.6) fails. Here is such an example.

Example 1. Assume that $N = 1$. We have

$$\|f\|_{W^{3/4,3/2}} \lesssim \|f\|_{W^{2/3,2}}^{1/2} \|f\|_{W^{1,1}}^{1/2}, \ \forall f \in W^{2/3,2}(\Omega) \cap W^{1,1}(\Omega). \tag{1.19}$$

It is natural to try to derive (1.19) by combining the (formal) GN inequality

$$\|f\|_{W^{5/6,4/3}} \lesssim \|f\|_{W^{2/3,2}}^{1/2} \|f\|_{W^{1,1}}^{1/2}, \ \forall f \in W^{2/3,2}(\Omega) \cap W^{1,1}(\Omega) \tag{1.20}$$

with the Sobolev estimate

$$\|f\|_{W^{3/4,3/2}} \lesssim \|f\|_{W^{5/6,4/3}}, \ \forall f \in W^{5/6,4/3}(\Omega). \tag{1.21}$$

Here, (1.20) fails, (1.21) holds and, by Theorem 1 below, (1.19) holds.

On the other hand, it may happen that (1.11) *fails* (despite the fact that (1.12) holds). Here is such an example.

Example 2. Assume that $N = 1$. Then, as a consequence of Theorem 1 below, the following estimate fails.

$$\|f\|_{W^{2/3,3}} \lesssim \|f\|_{W^{1/2,2}}^{1/2} \|f\|_{W^{1,1}}^{1/2}, \ \forall f \in W^{1/2,2}(\Omega) \cap W^{1,1}(\Omega). \tag{1.22}$$

In this case, the analogues of (1.20) and (1.21) are

$$\|f\|_{W^{3/4,4/3}} \lesssim \|f\|_{W^{1/2,2}}^{1/2} \|f\|_{W^{1,1}}^{1/2}, \ \forall f \in W^{1/2,2}(\Omega) \cap W^{1,1}(\Omega), \tag{1.23}$$

respectively

$$\|f\|_{W^{2/3,3}} \lesssim \|f\|_{W^{3/4,4/3}}, \ \forall f \in W^{3/4,4/3}(\Omega). \tag{1.24}$$

This time, (1.23) fails, (1.24) holds, and (1.22) fails.

Our main result provides a complete answer to the question of the validity/failure of (1.11).

Theorem 1. Let Ω be a standard domain in $\mathbb{R}^N$. Let $s_1, s_2, r, p_1, p_2, q, \theta, N$ satisfy (1.12). Then the GNS inequality (1.11) holds with the following exceptions, when it fails.

1. $N = 1$, s_2 is an integer ≥ 1, $1 < p_1 \leq \infty$, $p_2 = 1$, $s_1 = s_2 - 1 + \dfrac{1}{p_1}$, $[1 < p_1 < \infty, r = s_2 - 1]$ or $\left[s_2 + \dfrac{\theta}{p_1} - 1 < r < s_2 + \dfrac{\theta}{p_1} - \theta\right]$.
2. $N \geq 1$, $s_1 < s_2$, $s_1 - \dfrac{N}{p_1} = s_2 - \dfrac{N}{p_2} = r$ is an integer, $q = \infty$, $(p_1, p_2) \neq (\infty, 1)$ (for every $\theta \in (0,1)$).

In the special case where

$$s_1 \leq r \leq s_2, \tag{1.25}$$

which is a traditional assumption, considered for example in the seminal work of Nirenberg [12], Theorem 1 takes the following form.

Corollary 1. Let Ω be a standard domain in $\mathbb{R}^N$. Let $s_1, s_2, r, p_1, p_2, q, \theta, N$ satisfy (1.12) and (1.25). Then the GNS inequality (1.11) holds with the following exceptions, when it fails.

1. $N=1$, s_2 is an integer ≥ 1, $1<p_1<\infty$, $p_2=1$, $s_1=s_2-1+\dfrac{1}{p_1}$, $s_2+\dfrac{\theta}{p_1}-1<r<s_2+\dfrac{\theta}{p_1}-\theta$ and $r\geq s_1$.

2. $N\geq 1$, $p_1=\infty$, $1<p_2<\infty$, $q=\infty$, $s_1=r\geq 0$ is an integer, $s_2=r+\dfrac{N}{p_2}$ (for every $\theta\in(0,1)$).

Remark 1. Assume that $0\leq s_1\leq r\leq s_2$ are integers and that (1.12) holds. By Corollary 1, (1.11) holds except when

$$N\geq 1,\ p_1=\infty,\ 1<p_2<\infty,\ q=\infty,\ s_1=r\geq 0 \text{ is an integer},\ s_2=r+\frac{N}{p_2} \text{ (for every } \theta\in(0,1)). \tag{1.26}$$

This corresponds to the framework of Nirenberg's paper [12]. [As observed by a number of people, the exceptional case (1.26) had been overlooked in [12].]

Remark 2. Let us note a striking phenomenon. Let $N\geq 2$, $s_1=1$, $s_2=N$, $r=0$, $p_1=N$, $p_2=1$, $q=\infty$, $\theta\in(0,1)$. Then $W^{1,N}(\mathbb{R}^N)\cap W^{N,1}(\mathbb{R}^N)\hookrightarrow L^\infty(\mathbb{R}^N)$ (since $W^{N,1}(\mathbb{R}^N)\hookrightarrow L^\infty(\mathbb{R}^N)$). Therefore, we have the *additive* inequality

$$\|f\|_{L^\infty}\lesssim \|f\|_{W^{1,N}}+\|f\|_{W^{N,1}},\ \forall f\in W^{1,N}(\mathbb{R}^N)\cap W^{N,1}(\mathbb{R}^N). \tag{1.27}$$

However, by Theorem 1 item 2, there is no *multiplicative* version of (1.27), i.e., there is no $\theta\in(0,1)$ such that

$$\|f\|_{L^\infty}\lesssim \|f\|^{\theta}_{W^{1,N}}\|f\|^{1-\theta}_{W^{N,1}},\ \forall f\in W^{1,N}(\mathbb{R}^N)\cap W^{N,1}(\mathbb{R}^N) \tag{1.28}$$

(see Case 5.4; for an alternative proof, see [6, Appendix]). This in sharp contrast with the GN situation, where additive and multiplicative versions are equivalent.

Remark 3. As we will see in the course of the proof of Theorem 1, the following condition plays a crucial role in the arguments:

$$s_1-\frac{N}{p_1}=s_2-\frac{N}{p_2}. \tag{1.29}$$

If (1.29) holds, the equality

$$r-\frac{N}{q}=\theta\left(s_1-\frac{N}{p_1}\right)+(1-\theta)\left(s_2-\frac{N}{p_2}\right)$$

holds *for every* $\theta\in(0,1)$. Therefore, in Theorem 1 item 2, every $\theta\in(0,1)$ is admissible, while in item 1, there exists a non-empty open interval of admissible $\theta\in(0,1)$.

Our paper is organized as follows. Section 3 is devoted to the proof of Theorem 1. The proof relies heavily on the identification of most of the Sobolev spaces with Triebel-Lizorkin spaces (see e.g. [17, Section 2.3.5], [14, Section 2.1.2]). This approach turned out also to be effective in the proof of Theorem A in [5], and we refer the reader to [5, Sections 2 and 5] for a collection of properties and tools useful in this context. For the convenience of the reader, an initial Section 2 gathers the minimal material related to Sobolev and Triebel-Lizorkin spaces that we need in order to prove Theorem 1. The appendix is devoted to a proof of Theorem B.

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2. Basic properties of Sobolev spaces

To start with, let us define a convenient norm on the Sobolev space $W^{s,p}(\Omega)$, with $\Omega \subset \mathbb{R}^N$ a standard domain. Given $s > 0$ and $1 \le p \le \infty$, let $m = \lfloor s \rfloor$ be the integer part of s and and set

$$|f|_{W^{s,p}(\Omega)} = \begin{cases} \|D^m f\|_{L^p(\Omega)}, & \text{if } s = m \\ \left(\displaystyle\int_\Omega \int_\Omega \frac{|D^m f(x) - D^m f(y)|^p}{|x-y|^{N+(s-m)p}} \, dx dy \right)^{1/p}, & \text{if } m < s < m+1 \end{cases} \tag{2.1}$$

(with the obvious modification when $p = \infty$). Then (see e.g. [17, Section 2.3.8])

Lemma 1. Let $\Omega \subset \mathbb{R}^N$ be a standard domain. Let $s > 0$ and $1 \le p \le \infty$. Then

$$f \mapsto \|f\|_{W^{s,p}} = \|f\|_{W^{s,p}(\Omega)} := \|f\|_{L^p(\Omega)} + |f|_{W^{s,p}(\Omega)} \tag{2.2}$$

is equivalent to the "usual" norms on $W^{s,p}(\Omega)$.

We endow $W^{s,p}(\Omega)$ with this norm.

Definition 1. Let $\psi \in C_c^\infty(\mathbb{R}^N)$ be such that $\psi = 1$ in $B_1(0)$ and $\operatorname{supp} \psi \subset B_2(0)$. Define $\psi_0 = \psi$ and, for $j \ge 1$, $\psi_j(x) := \psi(x/2^j) - \psi(x/2^{j-1})$. Set $\varphi_j := \mathscr{F}^{-1}\psi_j \in \mathscr{S}$. [Equivalently, we have $\varphi_0 = \mathscr{F}^{-1}\psi$ and, for $j \ge 1$, $\varphi_j(x) = 2^{Nj}\varphi_0(2^j x) - 2^{N(j-1)}\varphi_0(2^{j-1}x)$.] Then for each tempered distribution f in $\mathbb{R}^N$ we have

$$f = \sum_{j \ge 0} f_j \text{ in } \mathscr{S}' = \mathscr{S}'(\mathbb{R}^N), \text{ with } f_j := f * \varphi_j. \tag{2.3}$$

$f = \sum_{j\ge 0} f_j$ is *"the" Littlewood-Paley* decomposition of $f \in \mathscr{S}'$.

Note that $\mathscr{F} f_j = \psi_j \mathscr{F} f$ is compactly supported, and therefore $f_j \in C^\infty$ for each j.

Definition 2. Starting from the Littlewood-Paley decomposition, we define the *Triebel-Lizorkin spaces* $F^s_{p,q} = F^s_{p,q}(\mathbb{R}^N)$ as follows. We let

$$\|f\|_{F^s_{p,q}} := \left\| \left\| \left(2^{js} f_j(x) \right)_{j \ge 0} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{R}^N)}, \; 0 < p \le \infty, 0 < q \le \infty, \tag{2.4}$$

$$F^s_{p,q} := \{ f \in \mathscr{S}'; \|f\|_{F^s_{p,q}} < \infty \}, 0 < p < \infty, 0 < q \le \infty. \tag{2.5}$$

The space $F^s_{\infty,\infty}$ is still defined as in (2.5), with $p = q = \infty$.

Note that we *do not define the spaces* $F^s_{\infty,q}$ *when* $q < \infty$; this is a delicate matter (see [17, Section 2.3.4, p. 50]).

Most of the Sobolev spaces can be identified with Triebel-Lizorkin spaces [17, Section 2.3.5], [14, Section 2.1.2].

Lemma 2. The following equalities of spaces hold, with equivalence of norms:

1. If $s > 0$ is not an integer and $1 \le p \le \infty$, then $W^{s,p}(\mathbb{R}^N) = F^s_{p,p}$.

2. If $s \ge 0$ is an integer and $1 < p < \infty$, then $W^{s,p}(\mathbb{R}^N) = F^s_{p,2}$.

When $s \ge 0$ is an integer and either $p = 1$ or $p = \infty$, the Sobolev space $W^{s,p}$ cannot be identified with a Triebel-Lizorkin space.

Definition 3. A *regular* Sobolev space is a space $W^{s,p} = W^{s,p}(\mathbb{R}^N)$ which can be identified with a Triebel-Lizorkin space. Equivalently, $W^{s,p}$ is regular if and only if either [s is not an integer and $1 \le p \le \infty$] or [s is an integer and $1 < p < \infty$]. The remaining Sobolev spaces, $W^{k,1}$ and $W^{k,\infty}$ with $k \ge 0$ an integer, are *exceptional*.

Lemma 3. Let $0 < r_1 < r_2 < \infty$ and $1 \le p \le q \le \infty$ be fixed. Then for every integer $k \ge 0$, $u \in L^p(\mathbb{R}^N)$ and $R > 0$ we have the *direct Nikolskiĭ's estimates*

$$\text{supp } \widehat{u} \subset B(0,R) \implies \|D^k u\|_{L^q(\mathbb{R}^N)} \lesssim R^{k+N(1/p-1/q)} \|u\|_{L^p(\mathbb{R}^N)} \tag{2.6}$$

and the *reverse Nikolskiĭ's estimates*

$$\text{supp } \widehat{u} \subset B(0,r_2R) \setminus B(0,r_1R) \Rightarrow \|u\|_{L^p(\mathbb{R}^N)} \lesssim R^{-k} \|D^k u\|_{L^p(\mathbb{R}^N)}. \tag{2.7}$$

See e.g. [16, Chapter 5, Lemma 3.14] for the first result, and [7, Lemma 2.1.1] for the second one.

In particular, let f_j be as in the Littlewood-Paley decomposition. Then the direct estimates apply to $u := f_j$, with $j \ge 0$ and $R := 2^{j+1}$. The reverse estimates apply to $u := f_j$ with $j \ge 1$, $R := 2^j$, $r_1 := 1/2$, $r_2 := 2$.

Another useful tool is the following.

Lemma 4. Let $s_1, s_2, s \in \mathbb{R}$, $0 < p_1, p_2, p \le \infty$ and $\theta \in (0,1)$ satisfy $s_1 < s_2$ and (1.2). Then for every $0 < q_1, q_2, q \le \infty$ we have

$$\|f\|_{F^s_{p,q}} \lesssim \|f\|^{\theta}_{F^{s_1}_{p_1,q_1}} \|f\|^{1-\theta}_{F^{s_2}_{p_2,q_2}}, \forall f \in \mathscr{S}'. \tag{2.8}$$

The above result is due to Oru [13] (unpublished); for a proof, see [4, Lemma 3.1 and Section III].

We emphasize the fact that the values of N, q_1, q_2, q are irrelevant for the validity of (2.8), and that the essential assumptions are $s_1 \ne s_2$ and the proportionality relations (1.2).

We next establish various estimates needed in the proof of Theorem 1.

Lemma 5. Let $-\infty < s < \infty$, $1 \le p \le \infty$ and $0 < t \le \infty$. Then

$$\|f\|_{F^{s-N/p}_{\infty,\infty}} \lesssim \|f\|_{F^s_{p,t}}, \ \forall f \in \mathscr{S}'(\mathbb{R}^N). \tag{2.9}$$

Proof. When $p = \infty$, the conclusion is clear. Assume that $p < \infty$. Let $f = \sum_{j \ge 0} f_j$ be the Littlewood-Paley decomposition of $f \in \mathscr{S}'(\mathbb{R}^N)$. We may assume that $\|f\|_{F^s_{p,t}} < \infty$. Set

$$g(x) := \left\| \left(2^{js} f_j(x)\right)_{j\ge 0} \right\|_{\ell^t},$$

so that $\|f\|_{F^s_{p,t}} = \|g\|_{L^p} < \infty$.

We have $2^{js}|f_j(x)| \le g(x)$, $\forall x$, $\forall j$. By the direct Nikolskiĭ's estimates (2.6) (with $q := \infty$), we have

$$\|f_j\|_{L^\infty} \lesssim 2^{jN/p} \|f_j\|_{L^p} \le 2^{jN/p-js} \|g\|_{L^p} = 2^{jN/p-js} \|f\|_{F^s_{p,t}},$$

so that

$$\|f\|_{F^{s-N/p}_{\infty,\infty}} = \sup_j 2^{j(s-N/p)} \|f_j\|_{L^\infty} \lesssim \|f\|_{F^s_{p,t}}. \qquad \square$$

Lemma 6. Let $s \ge 0$ and $1 \le p \le \infty$. Then

$$\|f\|_{F^{s-N/p}_{\infty,\infty}} \lesssim \|f\|_{W^{s,p}}, \ \forall f \in W^{s,p}(\mathbb{R}^N). \tag{2.10}$$

Proof. We start with a preliminary remark. Let $f \in L^p(\mathbb{R}^N)$ and let $f = \sum_{j\ge 0} f_j$ be its Littlewood-Paley decomposition. With φ_j as in Definition 1, we have

$$\|f_j\|_{L^p} \le \|f\|_{L^p} \|\varphi_j\|_{L^1} \le C\|f\|_{L^p}, \tag{2.11}$$

for some $C > 0$ independent of f, p and j.

We now proceed with the proof of the lemma. Its conclusion follows from Lemmas 5 and 2, except when $s \ge 0$ is an integer and $p = 1$ or $p = \infty$. For $s \ge 0$ integer and $p = 1$ or $p = \infty$, let $f \in W^{s,p}(\mathbb{R}^N)$ and let $f = \sum_{j\ge 0} f_j$ be its Littlewood-Paley decomposition.

When $p = \infty$, (2.7) and (2.11) yield, for $j \ge 1$,

$$\|f_j\|_{L^\infty} \lesssim 2^{-sj} \|D^s f_j\|_{L^\infty} = 2^{-sj} \|(D^s f)_j\|_{L^\infty} \lesssim 2^{-sj} \|D^s f\|_{L^\infty} \le 2^{-sj} \|f\|_{W^{s,\infty}}. \tag{2.12}$$

Since, on the other hand, we have, by (2.11), $\|f_0\|_{L^\infty} \lesssim \|f\|_{L^\infty}$, we find that $\|f\|_{F^s_{\infty,\infty}} \lesssim \|f\|_{W^{s,\infty}}$.

Similarly, when $p = 1$, (2.6), (2.7) and (2.11) imply, for $j \ge 1$,

$$\begin{aligned}\|f_j\|_{L^\infty} &\lesssim 2^{Nj} \|f_j\|_{L^1} \lesssim 2^{(N-s)j} \|D^s f_j\|_{L^1} = 2^{(N-s)j} \|(D^s f)_j\|_{L^1} \\ &\lesssim 2^{(N-s)j} \|D^s f\|_{L^1} \le 2^{(N-s)j} \|f\|_{W^{s,1}},\end{aligned} \tag{2.13}$$

while

$$\|f_0\|_{L^\infty} \lesssim \|f_0\|_{L^1} \lesssim \|f\|_{L^1} \le \|f\|_{W^{s,1}}. \tag{2.14}$$

Combining (2.13) with (2.14), we find that $\|f\|_{F^{s-N}_{\infty,\infty}} \lesssim \|f\|_{W^{s,1}}$. □

Lemma 7. Let $N \ge 2$, $s > 0$ and $1 \le p < q \le \infty$. Let $\sigma = \sigma_q \in \mathbb{R}$ be defined by $\sigma - N/q = s - N/p$. Then

$$W^{s,p}(\mathbb{R}^N) \hookrightarrow F^\sigma_{q,\infty}. \tag{2.15}$$

Proof. Lemma 6 shows that (2.15) holds when $q = \infty$.

Assume next that $q < \infty$. By Theorem B, we have $W^{s,p}(\mathbb{R}^N) \hookrightarrow W^{\sigma,q}(\mathbb{R}^N)$. [Item 1 (resp. item 2) is ruled out since $N \ge 2$ (resp. $q < \infty$).] On the other hand, for sufficiently small $\varepsilon > 0$ and $p < q \le P = p + \varepsilon$, we are in position to apply Lemma 2 and obtain that

$$W^{s,p}(\mathbb{R}^N) \hookrightarrow W^{\sigma,q}(\mathbb{R}^N) = F^\sigma_{q,q} \hookrightarrow F^\sigma_{q,\infty}. \tag{2.16}$$

Finally, let $P < q < \infty$. Let $\theta = P/q \in (0,1)$, so that

$$\frac{1}{q} = \frac{\theta}{P} + \frac{1-\theta}{\infty} \text{ and } \sigma_q = \theta\,\sigma_P + (1-\theta)\sigma_\infty. \tag{2.17}$$

By (2.17), (2.16) with $q = P$ and Lemma 4, we find that

$$\|f\|_{F^{\sigma_q}_{q,\infty}} \lesssim \|f\|^\theta_{F^{\sigma_P}_{P,\infty}} \|f\|^{1-\theta}_{F^{\sigma_\infty}_{\infty,\infty}} \lesssim \|f\|_{W^{s,p}(\mathbb{R}^N)},\ \forall f \in W^{s,p}(\mathbb{R}^N).$$ □

Lemma 8. We have

$$\|f\|_{L^\infty} \lesssim \|f\|_{F^0_{\infty,1}},\ \forall f \in \mathscr{S}'. \tag{2.18}$$

Proof. Let $f = \sum_{j\ge 0} f_j$ be the Littlewood-Paley decomposition of f. Then

$$\|f\|_{L^\infty} \le \sup_{J\ge 0} \left\| \sum_{j\le J} f_j \right\|_{L^\infty} \le \left\| \sum_j |f_j| \right\|_{L^\infty} = \|f\|_{F^0_{\infty,1}}.$$ □

3. Proof of Theorem 1

Outline of the proof. We investigate the validity of (1.11) by considering a number of cases, which are of interest only when

$$\text{at least one of the conditions (1.4), (1.7) or (1.9) is satisfied.} \tag{3.1}$$

Therefore, even if (3.1) is not explicitly assumed in a case, *we may assume that* (3.1) *holds*.

In the "positive" cases where (1.11) holds, it suffices to establish its validity only when $\Omega = \mathbb{R}^N$. Indeed, combining (1.11) in $\Omega = \mathbb{R}^N$ with the existence of a universal extension operator $P : W^{s,p}(\Omega) \hookrightarrow W^{s,p}(\mathbb{R}^N)$, we obtain the validity of (1.11) in all standard domains.

In the "negative" cases where (1.11) fails, it suffices to prove that (1.11) fails in *some* ball B. Indeed, assuming this fact and using the existence of a universal extension operator $\overline{P}$: $W^{s,p}(B) \hookrightarrow W^{s,p}(\Omega)$ (with $\overline{B} \subset \Omega$), we find that (1.11) fails in any domain Ω.

In view of the above, we will work either in $\mathbb{R}^N$ (in the positive cases) or in a (fixed) ball B (in the negative cases).

It will be convenient to consider not only $s_1, s_2, r, p_1, p_2, q, \theta, N$, but also s and p as in Theorem A, given respectively by

$$s := \theta s_1 + (1-\theta) s_2, \tag{3.2}$$

$$\frac{1}{p} := \frac{\theta}{p_1} + \frac{1-\theta}{p_2}. \tag{3.3}$$

Before proceeding with the proof, let us recall the assumption $s_1 \le s_2$, which is part of (1.12).

The proof is divided into eight cases. We will explain at the end why all situations where (3.1) holds are contained in one of these cases.

Case 1. $q = \infty$, $r \ge 0$ is an integer, $r \le s_1$ and $s_1 - N/p_1 \ne s_2 - N/p_2$
Case 2. $s_1 = s_2$
Case 3. $p = 1$
Case 4. (1.1) holds (i.e., (1.4) fails) and $s_1 - N/p_1 \ne s_2 - N/p_2$
Case 5. $s_1 < s_2$, $q = \infty$ and $s_1 - N/p_1 = s_2 - N/p_2$ is an integer ≥ 0
Case 6. $N = 1$, $s_2 \ge 1$ is an integer, $p_2 = 1$, $1 < p_1 \le \infty$ and $s_1 = s_2 - 1 + 1/p_1$
Case 7. $N = 1$, $s_2 \ge 1$ is an integer, $p_2 = 1$, $1 < p_1 \le \infty$ and $s_2 - 1 + 1/p_1 < s_1 < s_2$
Case 8. $N \ge 2$, $s_2 \ge 1$ is an integer, $p_2 = 1$, $1 < p_1 \le \infty$ and $s_2 - 1 + 1/p_1 \le s_1 < s_2$

Case 1. Assume that $q = \infty$, $r \ge 0$ is an integer, $r \le s_1$ and $s_1 - N/p_1 \ne s_2 - N/p_2$. Then (1.11) holds
Proof. We note that $s_1 - r - N/p_1 \ne s_2 - r - N/p_2$ and that $s_j - r \ge 0$, $j = 1, 2$. We are thus in position to combine Lemmas 4 and 6 and find that

$$\|f\|_{F^0_{\infty,1}} \lesssim \|f\|^{\theta}_{F^{s_1-r-N/p_1}_{\infty,\infty}} \|f\|^{1-\theta}_{F^{s_2-r-N/p_2}_{\infty,\infty}} \lesssim \|f\|^{\theta}_{W^{s_1-r,p_1}} \|f\|^{1-\theta}_{W^{s_2-r,p_2}} \lesssim \|f\|^{\theta}_{W^{s_1,p_1}} \|f\|^{1-\theta}_{W^{s_2,p_2}}. \tag{3.4}$$

Replacing in (3.4) f with $\partial^\alpha f$, with α a multi-index such that $|\alpha| = r$, we find that

$$\|D^r f\|_{F^0_{\infty,1}} \lesssim \|D^r f\|^{\theta}_{W^{s_1-r,p_1}} \|D^r f\|^{1-\theta}_{W^{s_2-r,p_2}} \lesssim \|f\|^{\theta}_{W^{s_1,p_1}} \|f\|^{1-\theta}_{W^{s_2,p_2}}. \tag{3.5}$$

Combining (3.4) and (3.5) with Lemmas 1 and 8, we find that

$$\|f\|_{W^{r,\infty}} = \|f\|_{L^\infty} + \|D^r f\|_{L^\infty} \lesssim \|f\|_{F^0_{\infty,1}} + \|D^r f\|_{F^0_{\infty,1}} \lesssim \|f\|^{\theta}_{W^{s_1,p_1}} \|f\|^{1-\theta}_{W^{s_2,p_2}}. \qquad \square$$

Case 2. (1.11) holds when $s_1 = s_2$
Proof. With no loss of generality, we may assume that $1 \le p_1 < p_2 \le \infty$. [Recall that we have

assumed $(s_1,p_1)\neq(s_2,p_2)$, and thus when $s_1=s_2$ we must have $p_1\neq p_2$.] It follows that $r<s=s_1=s_2$ and $p_1<p<p_2$. Let us note that, in view of the assumption $s_1=s_2$, (1.1) holds. Therefore, (1.11) holds also, possibly except when (1.6) fails. We find that we only have have to investigate the validity of (1.11) when

$$s-\frac{N}{p}=r \text{ is an integer} \geq 0 \text{ and } q=\infty. \tag{3.6}$$

In this case, the validity of (1.11) follows from Case 1. □

Case 3. (1.11) holds when $p=1$

Proof. In this case we have $p=p_1=p_2=1$ and thus $s_1<s_2$. In particular, (1.1) holds. The only possible obstruction for the validity of (1.11) can arise from Theorem B item 1. We thus investigate the case where $N=1$, $s\geq 1$ is an integer, $1<q<\infty$, $r=s-1+1/q$.

We let S_1, S_2 such that:

1. We have $s_1<S_1<s<S_2<s_2$, and S_j is not an integer, $j=1,2$.
2. If we let $R_j:=S_j-1+1/q$, $j=1,2$, then $R_j>0$.

The last condition is satisfied provided S_j, $j=1,2$ are sufficiently close to s (since $s-1+1/q\geq 1/q>0$).

Define $\lambda,\lambda_1,\lambda_2\in(0,1)$ by the relations

$$s=\lambda S_1+(1-\lambda)S_2,\ S_1=\lambda_1 s_1+(1-\lambda_1)s_2,\ S_2=\lambda_2 s_1+(1-\lambda_2)s_2. \tag{3.7}$$

Clearly, we have

$$r=\lambda R_1+(1-\lambda)R_2, \tag{3.8}$$
$$\lambda\,\lambda_1+(1-\lambda)\,\lambda_2=\theta, \tag{3.9}$$
$$\lambda(1-\lambda_1)+(1-\lambda)(1-\lambda_2)=1-\theta. \tag{3.10}$$

Let us note that, since $1<q<\infty$ and S_j is not an integer, $j=1,2$, we have

$$W^{S_j,1}(\mathbb{R})\hookrightarrow W^{R_j,q}(\mathbb{R}),\ j=1,2. \tag{3.11}$$

Using successively: (3.8) and Theorem A, (3.11), (3.9)–(3.10) and Theorem A, we find that

$$\begin{aligned}\|f\|_{W^{r,q}(\mathbb{R})}&\lesssim \|f\|^{\lambda}_{W^{R_1,q}(\mathbb{R})}\|f\|^{1-\lambda}_{W^{R_2,q}(\mathbb{R})}\lesssim \|f\|^{\lambda}_{W^{S_1,1}(\mathbb{R})}\|f\|^{1-\lambda}_{W^{S_2,1}(\mathbb{R})}\\ &\lesssim\left(\|f\|^{\lambda_1}_{W^{s_1,1}(\mathbb{R})}\|f\|^{1-\lambda_1}_{W^{s_2,1}(\mathbb{R})}\right)^{\lambda}\left(\|f\|^{\lambda_2}_{W^{s_1,1}(\mathbb{R})}\|f\|^{1-\lambda_2}_{W^{s_2,1}(\mathbb{R})}\right)^{1-\lambda}=\|f\|^{\theta}_{W^{s_1,1}(\mathbb{R})}\|f\|^{1-\theta}_{W^{s_2,1}(\mathbb{R})}.\end{aligned}$$

This completes Case 3. □

In view of Case 2 and Case 3, *from now on we may assume that*

$$s_1<s_2 \tag{3.12}$$

and

$$1<p<\infty, \tag{3.13}$$

and in particular that (1.7) *fails*. [Note that the value $p=\infty$ is excluded, in view of (1.12).]

Case 4. Assume that (1.1) holds (i.e., that (1.4) fails) and that $s_1-N/p_1\neq s_2-N/p_2$. Then (1.11) holds

Proof. We may assume that (3.12) and (3.13) hold. It suffices to investigate the cases where (1.6) fails. In view of Theorem B and of the assumption (3.13), we thus have that $r \geq 0$ is an integer, $q = \infty$, $1 < p < \infty$ and $s = r + N/p$.

It will be convenient to rely on geometric interpretations of the conditions (1.2) and (1.4) with $s_1 < s_2$. Condition (1.2) asserts that the point $(s, 1/p)$ belongs to the open line segment $I = I(s_1, s_2, p_1, p_2)$ determined by its endpoints $(s_1, 1/p_1)$ and $(s_2, 1/p_2)$. On the other hand, condition (1.4) is equivalent to the fact that the right endpoint of I, i.e., $(s_2, 1/p_2)$, is of the form $(k, 1)$, with k positive integer, and that in addition the slope of I is ≤ -1. Therefore, given s_1, s_2, p_1, p_2, if (1.4) is satisfied for *some* couple (s, p) with $(s, 1/p) \in I$, then it is satisfied by *every* such couple. Equivalently, given I, if (1.1) holds for *some* couple (s, p) with $(s, 1/p) \in I$, then (1.1) holds for *every* such couple.

Using these considerations, (3.12) and the assumption that (1.1) is satisfied by (s, p), we obtain the following fact (which can also be checked analytically). Let $s_1 < S_1 < s < S_2 < s_2$ and define P_1, P_2 such that the points $(S_j, 1/P_j)$, $j = 1, 2$, belong to I. Define λ, λ_1, λ_2 as in (3.7). Then

$$\frac{1}{P_j} = \frac{\lambda_j}{p_1} + \frac{1-\lambda_j}{p_2},\ j = 1, 2,\ \frac{1}{p} = \frac{\lambda}{P_1} + \frac{1-\lambda}{P_2} \tag{3.14}$$

and

$$\|f\|_{W^{S_j,P_j}} \lesssim \|f\|_{W^{s_1,p_1}}^{\lambda_j} \|f\|_{W^{s_2,p_2}}^{1-\lambda_j},\ \forall f \in W^{s_1,p_1}(\mathbb{R}^N) \cap W^{s_2,p_2}(\mathbb{R}^N),\ \forall j = 1, 2. \tag{3.15}$$

We choose S_j such that $S_j - r > 0$, $j = 1, 2$; this is possible since $s - r = N/p > 0$.

We next note that, under the assumption $s_1 - N/p_1 \neq s_2 - N/p_2$, the function $I \ni (s, 1/p) \mapsto s - N/p$ is strictly monotone, and thus in particular $S_1 - N/P_1 \neq S_2 - N/P_2$. Since $r < S_1$, by Case 1 we have

$$\|f\|_{W^{r,\infty}} \lesssim \|f\|_{W^{S_1,P_1}}^{\lambda} \|f\|_{W^{S_2,P_2}}^{1-\lambda}. \tag{3.16}$$

We complete Case 4 by combining (3.16), (3.15) and (3.9)–(3.10). □

In Case 5 below, we assume (3.12), i.e., $s_1 < s_2$.

Case 5. Assume that $q = \infty$ and that $s_1 - N/p_1 = s_2 - N/p_2$ is an integer ≥ 0. Then (1.11) fails except in the trivial case where $p_1 = \infty$, $p_2 = 1$

Proof. Let us note that we have $p_2 < p_1$ and $r = s - N/p = s_1 - N/p_1 = s_2 - N/p_2 \geq 0$ is an integer.

Case 5.1. $p_1 = \infty$ and $p_2 = 1$

In this case, we have $s_1 = r$, $s_2 = r + N$, and thus $W^{s_1,p_1} = W^{r,\infty}$ and $W^{s_2,p_2} \hookrightarrow W^{r,\infty}$, whence (1.11).

Case 5.2. $p_2 > 1$ and $p_1 < \infty$

We have $W^{s_2,p_2} \hookrightarrow W^{s_1,p_1}$, which implies $W^{s_1,p_1} \cap W^{s_2,p_2} = W^{s_1,p_1}$. However, we have $W^{s_1,p_1} \not\hookrightarrow W^{r,\infty}$, so that (1.11) fails.

Case 5.3. $p_2 > 1$ and $p_1 = \infty$ (so that $s_1 = r$)

In this case, (1.11) becomes $\|f\|_{W^{r,\infty}} \lesssim \|f\|_{W^{r,\infty}}^{\theta} \|f\|_{W^{s_2,p_2}}^{1-\theta}$, which fails since $W^{s_2,p_2} \not\hookrightarrow W^{r,\infty}$.

Case 5.4. $p_2 = 1$ and $1 < p_1 < \infty$

This is a more delicate case. We want to prove that the estimate

$$\|f\|_{W^{r,\infty}} \lesssim \|f\|_{W^{s_1,p_1}}^{\theta} \|f\|_{W^{r+N,1}}^{1-\theta},\ \forall f \in W^{s_1,p_1}(B) \cap W^{r+N,1}(B) \tag{3.17}$$

fails in the unit ball B.

When $r = 0$, this is an immediate consequence of the analysis in [6, Appendix]. We present a a proof valid for all integers $r \geq 0$.

Fix some function $\varphi \in C_c^\infty(B)$ such that $\varphi(x) = x_1^r/r!$ near the origin. For such φ, we have $0 < C_j < \infty$, $j = 1,2,3$, where

$$C_1 := |\varphi|_{W^{s_1,p_1}(\mathbb{R}^N)},\ C_2 := |\varphi|_{W^{r+N,1}(\mathbb{R}^N)},\ C_3 := |\varphi|_{W^{r,\infty}(\mathbb{R}^N)}.$$

Set

$$\varphi^\lambda(x) = \lambda^{-r}\varphi(\lambda x),\ \forall\, \lambda > 1,$$

so that $\varphi^\lambda \in C_c^\infty(B)$.

A simple scaling argument shows that

$$|\varphi^\lambda|_{W^{s_1,p_1}(B)} \to C_1,\ |\varphi^\lambda|_{W^{r+N,1}(B)} \to C_2 \text{ as } \lambda \to \infty,\ |\varphi^\lambda|_{W^{r,\infty}(B)} = C_3, \tag{3.18}$$

$$\|\varphi^\lambda\|_{L^t(B)} \to 0 \text{ as } \lambda \to \infty,\ \forall\, 1 \le t \le \infty \tag{3.19}$$

and

$$D^m\varphi^\lambda \to 0 \text{ a.e. in } B \text{ as } \lambda \to \infty,\ \forall\, m \ge 0 \text{ integer.} \tag{3.20}$$

In view of (3.18)–(3.20), of Lemma 1 and of the Brezis-Lieb lemma ([3]), for every fixed function $f \in C_c^\infty(B)$ and for every fixed number $\beta > 0$ we have

$$\lim_{\lambda\to\infty} \|f + \beta\varphi^\lambda\|_{W^{s_1,p_1}(B)}^{p_1} = \|f\|_{W^{s_1,p_1}(B)}^{p_1} + (C_1)^{p_1}\beta^{p_1} \tag{3.21}$$

and

$$\lim_{\lambda\to\infty} \|f + \beta\varphi^\lambda\|_{W^{r+N,1}(B)} = \|f\|_{W^{r+N,1}(B)} + C_2\beta. \tag{3.22}$$

Using (3.21)–(3.22) and a straightforward induction argument, for every sequence (β_j) of positive numbers we may choose a sequence (λ_j) such that

$$\frac{J+1}{4J}(C_1)^{p_1}\sum_{j=1}^J(\beta_j)^{p_1} \le \left\|\sum_{j=1}^J \beta_j\varphi^{\lambda_j}\right\|_{W^{s_1,p_1}(B)}^{p_1} \le \frac{4J}{J+1}(C_1)^{p_1}\sum_{j=1}^J(\beta_j)^{p_1},\ \forall\, J \ge 1 \tag{3.23}$$

and

$$\frac{J+1}{4J}C_2\sum_{j=1}^J\beta_j \le \left\|\sum_{j=1}^J \beta_j\varphi^{\lambda_j}\right\|_{W^{r+N,1}(B)} \le \frac{4J}{J+1}C_2\sum_{j=1}^J\beta_j,\ \forall\, J \ge 1 \tag{3.24}$$

(see [10] for a similar construction).

On the other hand, we have

$$\left\|\sum_{j=1}^J \beta_j\varphi^{\lambda_j}\right\|_{W^{r,\infty}(B)} \ge \left|\frac{\partial^r}{\partial x_1^r}\left(\sum_{j=1}^J\beta_j\varphi^{\lambda_j}\right)(0)\right| = \sum_{j=1}^J\beta_j. \tag{3.25}$$

We consider a sequence (β_j) of positive numbers such that

$$\sum_{j\ge1}\beta_j = \infty \text{ and } \sum_{j\ge1}(\beta_j)^{p_1} < \infty \tag{3.26}$$

(note that this is possible, since $p_1 > p_2 = 1$).

We now argue by contradiction and assume that (3.17) holds. We obtain a contradiction (via (3.23)–(3.26)) by testing (3.17) on $f_J := \sum_{j=1}^J \beta_j\varphi^{\lambda_j}$ and letting $J \to \infty$.

Case 5 is complete. □

Case 6. Assume that $N=1$, $s_2\geq 1$ is an integer, $p_2=1$, $1<p_1\leq\infty$ and $s_1=s_2-1+1/p_1$. Then (1.11) holds if and only if: $[1<p_1<\infty$ and $p_1/\theta\leq q<\infty]$ or $[p_1=\infty$ and $q=\infty]$

Proof. We first assume that $s_2=1$ (and thus $s_1=1/p_1$ and $r=1/q$); as we will see below, the case where $s_2\geq 2$ easily reduces to this special case.

Case 6.1. $s_2=1$ and $p_1=\infty$

By Theorem B, when $1<q<\infty$ we have

$$W^{1,1}(\Omega)\cap L^\infty(\Omega)=W^{1,1}(\Omega)\not\hookrightarrow W^{1/q,q}(\Omega),$$

and thus (1.11) cannot hold.

On the other hand, when $q=\infty$, the estimate $\|f\|_{L^\infty}\lesssim\|f\|_{L^\infty}^\theta\|f\|_{W^{1,1}}^{1-\theta}$ holds for every $\theta\in(0,1)$, in view of the embedding $W^{1,1}(\Omega)\hookrightarrow L^\infty(\Omega)$.

Case 6.2. $s_2=1$, $1<p_1<\infty$ and $q=p_1/\theta$ (and thus $r=\theta/p_1$)

By Theorem A and the embedding $W^{1,1}(\mathbb{R})\hookrightarrow L^\infty(\mathbb{R})$, we have

$$\|f\|_{W^{r,q}}=\|f\|_{W^{\theta/p_1,p_1/\theta}}\lesssim\|f\|_{W^{1/p_1,p_1}}^\theta\|f\|_{L^\infty}^{1-\theta}\leq\|f\|_{W^{1/p_1,p_1}}^\theta\|f\|_{W^{1,1}}^{1-\theta},$$

whence (1.11).

Case 6.3. $s_2=1$, $1<p_1<\infty$ and $p_1/\theta<q<\infty$

By Theorem B and the previous case, we have

$$\|f\|_{W^{r,q}}=\|f\|_{W^{1/q,q}}\lesssim\|f\|_{W^{\theta/p_1,p_1/\theta}}\lesssim\|f\|_{W^{1/p_1,p_1}}^\theta\|f\|_{W^{1,1}}^{1-\theta}.$$

Case 6.4. $s_2=1$, $1<p_1<\infty$ and $q<p_1/\theta$

Note that we must have $q>1$. We will prove that, for every $1<q<p_1/\theta$, the estimate

$$\|f\|_{W^{1/q,q}}\lesssim\|f\|_{W^{1/p_1,p_1}}^\theta\|f\|_{W^{1,1}}^{1-\theta},\ \forall f\in W_c^{1/p_1,p_1}(I)\cap W^{1,1}(I)\tag{3.27}$$

fails in the interval $I=(-2,2)$.

For $0<\varepsilon<1/2$, $a>0$, $b>0$, we consider the function $v=v_{\varepsilon,a,b}:(-2a,2a)\to\mathbb{R}$ given by:

$$v(x):=\begin{cases}0, & \text{if } |x|>(1+\varepsilon)a\\ b, & \text{if } |x|<a\\ \text{affine}, & \text{in } (a,(1+\varepsilon)a) \text{ and in } (-(1+\varepsilon)a,-a)\end{cases}.\tag{3.28}$$

By straightforward calculations, we have

$$|v|_{W^{1,1}(\mathbb{R})}=2b\tag{3.29}$$

and

$$C_tb^t|\ln\varepsilon|\leq|v|_{W^{1/t,t}((-2a,2a))}^t\leq C_t'b^t|\ln\varepsilon|,\ \forall 1<t<\infty\ (\text{with } 0<C_t,C_t'<\infty).\tag{3.30}$$

Arguing by contradiction and assuming the validity of (3.27), we obtain a contradiction (via (3.29)–(3.30)) by testing (3.27) with $v_{\varepsilon,1,1}$ and letting $\varepsilon\to 0$.

Case 6.5. $s_2=1$, $1<p_1<\infty$ and $q=\infty$

This is a sub-case of Case 5.4.

Case 6.6. $s_2\geq 2$

Let us note that s_1,s_2,r,p_1,p_2,q are in a positive case if and only of $s_1-s_2+1,1,r-s_2+1,p_1,p_2,q$ are in a positive case.

Therefore, in a positive case we are in position to apply (1.11) with $s_2 = 1$ and find that

$$\begin{aligned}\|f^{(s_2-1)}\|_{W^{r-s_2+1,q}} &\lesssim \|f^{(s_2-1)}\|^{\theta}_{W^{1/p_1,p_1}} \|f^{(s_2-1)}\|^{1-\theta}_{W^{1,1}} \\ &\lesssim \|f\|^{\theta}_{W^{s_1,p_1}} \|f\|^{1-\theta}_{W^{s_2,1}}, \forall f \in W^{s_1,p_1}(\mathbb{R}) \cap W^{s_2,1}(\mathbb{R}),\end{aligned} \tag{3.31}$$

$$\begin{aligned}\|f\|_{L^q} \le \|f\|_{W^{r-s_2+1,q}} &\lesssim \|f\|^{\theta}_{W^{1/p_1,p_1}} \|f\|^{1-\theta}_{W^{1,1}} \\ &\lesssim \|f\|^{\theta}_{W^{s_1,p_1}} \|f\|^{1-\theta}_{W^{s_2,1}}, \forall f \in W^{s_1,p_1}(\mathbb{R}) \cap W^{s_2,1}(\mathbb{R}).\end{aligned} \tag{3.32}$$

We obtain (1.11) from (3.31), (3.32) and Lemma 1.

If we are in a negative case, then there exists a sequence

$$(f_j)_{j\ge 1} \subset (W^{s_1-s_2+1,p_1}(I) \cap W^{1,1}(I)) \setminus \{0\}$$

such that

$$\|f_j\|_{W^{r-s_2+1,q}} \ge j\|f_j\|^{\theta}_{W^{s_1-s_2+1,p_1}} \|f_j\|^{1-\theta}_{W^{1,1}}, \ \forall j \ge 1. \tag{3.33}$$

We consider some finite length open interval J such that $\overline{I} \subset J$. By Lemma 1 and the existence of extension operators, there exist functions $g_j : J \to \mathbb{R}$, $\forall j \ge 1$, such that

1. $g_j^{(s_2-1)} = f_j$ in I.
2. $\|g_j\|_{W^{s_1,p_1}(J)} \approx \|f_j\|_{W^{s_1-s_2+1,p_1}(I)}$, $\|g_j\|_{W^{s_2,1}(J)} \approx \|f_j\|_{W^{1,1}(I)}$, $\|g_j\|_{W^{r,q}(J)} \approx \|f_j\|_{W^{r-s_2+1,q}(I)}$, $\forall j \ge 1$.

Using (3.33) and the above properties of g_j, we find that (1.11) fails.

Case 6 is complete. □

Case 7. Assume that $N = 1$, $s_2 \ge 1$ is an integer, $p_2 = 1$, $1 < p_1 \le \infty$ and $s_2 - 1 + 1/p_1 < s_1 < s_2$. Then (1.11) holds

Proof. As explained in Case 6, *we may assume that* $s_2 = 1$, and thus $1/p_1 < s_1 < 1$.

Case 7.1. $s_2 = 1$ and $1 < q < p_1$

Let $f \in W^{s_1,p_1}(\mathbb{R}) \cap W^{1,1}(\mathbb{R})$. Set $A := \|f\|_{W^{s_1,p_1}}$ and $B := \|f\|_{W^{1,1}}$. We may assume that $A > 0$ and $B > 0$. We want to prove the estimate

$$\|f\|^q_{W^{r,q}} \lesssim A^{\theta q} B^{(1-\theta)q}. \tag{3.34}$$

Let $f = \sum_{j\ge 0} f_j$ be the Littlewood-Paley decomposition of f. In view of Lemma 2, (3.34) amounts to

$$\sum_{j\ge 0} 2^{rjq} \|f_j\|^q_{L^q} \lesssim A^{\theta q} B^{(1-\theta)q}. \tag{3.35}$$

We now note that the following estimates hold:

$$\|f_j\|_{L^1} \lesssim 2^{-j}\|f_j'\|_{L^1} = 2^{-j}\|(f')_j\|_{L^1} \lesssim 2^{-j}\|f'\|_{L^1} \le 2^{-j}B, \ \forall j \ge 1, \tag{3.36}$$

$$\|f_0\|_{L^q} \lesssim \|f_0\|_{L^1} \lesssim \|f\|_{L^1} \le B, \tag{3.37}$$

$$\|f_j\|_{L^q} \lesssim 2^{j(1-1/q)}\|f_j\|_{L^1} \lesssim 2^{-j/q}B, \ \forall j \ge 1 \tag{3.38}$$

and

$$\|f_j\|_{L^{p_1}} \le 2^{-s_1 j}\|f\|_{F^{s_1}_{p_1,p_1}} \approx 2^{-s_1 j}\|f\|_{W^{s_1,p_1}} = 2^{-s_1 j}A. \tag{3.39}$$

Indeed:

1. (3.36) follows from (2.7) and (2.11);
2. (3.37) is a special case of (2.6);
3. (3.38) is a consequence of (2.6) and (3.36);
4. (3.39) is an immediate consequence of the formula of $\|f\|_{F^{s_1}_{p_1,p_1}}$ combined with Lemma 2.

Combining (3.36)–(3.37) with (3.39) we find, via Hölder's inequality, that

$$\|f_j\|_{L^q} \lesssim 2^{-(\lambda s_1+1-\lambda)j} A^\lambda B^{1-\lambda}; \tag{3.40}$$

here, the number $\lambda \in (0,1)$ is defined by the equation

$$\frac{1}{q} = \frac{\lambda}{p_1} + \frac{1-\lambda}{1}. \tag{3.41}$$

[The fact that $0 < \lambda < 1$ follows from the assumption $1 < q < p_1$.]
From (3.37)–(3.38) and (3.40), we obtain, with $x := A/B > 0$,

$$\begin{aligned}\|f_j\|_{L^q} &\lesssim \min\left\{2^{-j/q}B, 2^{-(\lambda s_1+1-\lambda)j} A^\lambda B^{1-\lambda}\right\} \\ &= A^\theta B^{1-\theta} \min\left\{2^{-j/q} x^{-\theta}, 2^{-(\lambda s_1+1-\lambda)j} x^{\lambda-\theta}\right\}.\end{aligned} \tag{3.42}$$

In view of (3.42) and of the desired conclusion (3.35), it thus suffices to prove that

$$\sum_{j\geq 0} \min\left\{2^{(rq-1)j} x^{-\theta q}, 2^{(r-\lambda s_1-1+\lambda)jq} x^{(\lambda-\theta)q}\right\} \lesssim 1. \tag{3.43}$$

We now invoke the following result, whose proof is postponed.

Lemma 9. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ be such that $\alpha\delta = \beta\gamma$ and $\alpha, \beta > 0$. Then there exist $0 < C_1 < C_2 < \infty$ such that

$$C_1 \leq \sum_{j=-\infty}^{\infty} \min\left\{2^{-\alpha j} x^\gamma, 2^{\beta j} x^{-\delta}\right\} \leq C_2, \ \forall x > 0. \tag{3.44}$$

In order to obtain (3.43), it suffices thus to be in position to apply Lemma 9 with

$$\alpha := (1-\lambda+\lambda s_1 - r)q, \beta := rq-1, \gamma := (\lambda-\theta)q, \delta := \theta q.$$

We start by checking the identity

$$\alpha\delta = \beta\gamma, \tag{3.45}$$

which is equivalent to

$$\theta(1-\lambda+\lambda s_1 - r) = (\lambda-\theta)\left(r - \frac{1}{q}\right). \tag{3.46}$$

On the other hand, we have, by (1.11), $r = 1/q + \theta(s_1 - 1/p_1)$. Plugging this value of r into (3.46) shows that (3.46) reduces to (3.41), and thus (3.45) holds.

We next prove that $\alpha, \beta > 0$. We clearly have $\delta > 0$. In view of (3.45), it suffices to prove that $\beta > 0$ and $\gamma > 0$.

The inequality $\beta > 0$ follows from $r - 1/q = \theta(s_1 - 1/p_1)$.

Finally, $\gamma > 0$ is equivalent to $\lambda > \theta$, that we obtain as follows: we have $q > p$, and thus

$$\frac{\lambda}{p_1} + \frac{1-\lambda}{1} = \frac{1}{q} < \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{1},$$

so that $\lambda > \theta$, as claimed.

Case 7.1 is complete.

Case 7.2. $s_2 = 1$ and $q \ge p_1$

Let, for sufficiently small $\varepsilon \ge 0$, $Q := p_1 - \varepsilon$ and define R by

$$R - 1/Q = r - 1/q = s - 1/p = \theta(s_1 - 1/p_1) > 0.$$

Since for $\varepsilon = 0$ we have $0 < R < s_1$, we find that, for small $\varepsilon > 0$, we have $0 < R < s_1 < s$, while $1 < Q < p_1$. By Case 7.1, we have

$$\|f\|_{W^{R,Q}} \lesssim \|f\|_{W^{s_1,p_1}}^{\theta} \|f\|_{W^{1,1}}^{1-\theta}, \ \forall f \in W^{s_1,p_1}(\mathbb{R}) \cap W^{1,1}(\mathbb{R}). \tag{3.47}$$

On the other hand, we have, by Theorem B,

$$W^{R,Q}(\mathbb{R}) \hookrightarrow W^{r,q}(\mathbb{R}). \tag{3.48}$$

Combining (3.47) and (3.48), we find that (1.11) holds.

Case 7 is complete. □

Case 8. Assume that $N \ge 2$, $s_2 \ge 1$ is an integer, $p_2 = 1$, $1 < p_1 \le \infty$ and $s_2 - 1 + 1/p_1 \le s_1 < s_2$. Then (1.11) holds

Proof. We consider several sub-cases.

Case 8.1. $[p_1 < q < \infty]$ or $[p_1 < q = \infty$ and r is not a non-negative integer$]$

By Lemma 2, we have $W^{r,q} = F^r_{q,t}$ for some t. Let r_j, $j = 1,2$, be given by Lemma 7, such that $W^{s_j,p_j}(\mathbb{R}^N) \hookrightarrow F^{r_j}_{q,\infty}$, $j = 1,2$. It is easy to see that $r_1 > r_2$ and $\theta r_1 + (1-\theta) r_2 = r$. Therefore, we are in position to apply Lemma 4 and find that

$$\|f\|_{W^{r,q}} \approx \|f\|_{F^r_{q,t}} \lesssim \|f\|_{F^{r_1}_{q,\infty}}^{\theta} \|f\|_{F^{r_2}_{q,\infty}}^{1-\theta} \lesssim \|f\|_{W^{s_1,p_1}}^{\theta} \|f\|_{W^{s_2,1}}^{1-\theta}, \ \forall f \in W^{s_1,p_1}(\mathbb{R}^N) \cap W^{s_2,1}(\mathbb{R}^N). \tag{3.49}$$

Case 8.2. $p_1 < q = \infty$ and $r \ge 0$ is an integer

Since $q = \infty$, (1.12) yields

$$r = \theta\left(s_1 - \frac{N}{p_1}\right) + (1-\theta)\left(s_2 - \frac{N}{1}\right),$$

and thus

$$r \le \max\left\{s_1 - \frac{N}{p_1}, s_2 - \frac{N}{1}\right\} = s_1 - \frac{N}{p_1} \le s_1.$$

Arguing as in the proof of (3.49), but using Lemmas 6 and 8 instead of Lemmas 7 and 2, we find that

$$\begin{aligned} \|f\|_{L^\infty} &\lesssim \|f\|_{F^0_{\infty,1}} \lesssim \|f\|_{F^{r_1-r}_{\infty,\infty}}^{\theta} \|f\|_{F^{r_2-r}_{\infty,\infty}}^{1-\theta} \\ &\lesssim \|f\|_{W^{s_1-r,p_1}}^{\theta} \|f\|_{W^{s_2-r,1}}^{1-\theta}, \ \forall f \in W^{s_1,p_1}(\mathbb{R}^N) \cap W^{s_2,1}(\mathbb{R}^N). \end{aligned} \tag{3.50}$$

Applying (3.50) to $\partial^\alpha f$, with $|\alpha| = r$, we obtain

$$\begin{aligned} \|D^r f\|_{L^\infty} &\lesssim \|D^r f\|_{W^{s_1-r,p_1}}^{\theta} \|D^r f\|_{W^{s_2-r,1}}^{1-\theta} \\ &\lesssim \|f\|_{W^{s_1,p_1}}^{\theta} \|f\|_{W^{s_2,1}}^{1-\theta}, \ \forall f \in W^{s_1,p_1}(\mathbb{R}^N) \cap W^{s_2,1}(\mathbb{R}^N). \end{aligned} \tag{3.51}$$

We complete the analysis of Case 8.2 by combining (3.50) and (3.51) with Lemma 1.

We have thus settled all the cases where $q > p_1$.

Assume next that $q \le p_1$. We define $\sigma \in \mathbb{R}$ and t by

$$r = \theta s_1 + (1-\theta)\sigma \text{ and } \frac{1}{q} = \frac{\theta}{p_1} + \frac{1-\theta}{t} \ge \frac{1}{p_1}. \tag{3.52}$$

Since, by (1.12), we have

$$r < \theta s_1 + (1-\theta) s_2 \text{ and } \frac{1}{q} < \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{1}, \tag{3.53}$$

we find from (3.52) and (3.53) that $\sigma < s_2$ and $1 < t \le p_1$.

It also follows from (1.12) and (3.52) that

$$\theta\left(s_1 - \frac{N}{p_1}\right) + (1-\theta)\left(s_2 - \frac{N}{1}\right) = r - \frac{N}{q} = \theta\left(s_1 - \frac{N}{p_1}\right) + (1-\theta)\left(\sigma - \frac{N}{t}\right), \tag{3.54}$$

so that

$$\sigma - \frac{N}{t} = s_2 - \frac{N}{1}. \tag{3.55}$$

Case 8.3. $\sigma \ge 0$

In this case, Theorem B and (3.55) imply that $W^{s_2,1} \hookrightarrow W^{\sigma,t}$. Since $p_1, t > 1$ and (3.52) holds, we are in position to apply Theorem A and find that

$$\|f\|_{W^{r,q}} \lesssim \|f\|^{\theta}_{W^{s_1,p_1}} \|f\|^{1-\theta}_{W^{\sigma,t}} \lesssim \|f\|^{\theta}_{W^{s_1,p_1}} \|f\|^{1-\theta}_{W^{s_2,1}}, \ \forall f \in W^{s_1,p_1}(\mathbb{R}^N) \cap W^{s_2,1}(\mathbb{R}^N).$$

Case 8.4. $\sigma < 0$ and $p_1 < \infty$

In this case, we have $\sigma \ne s_1$ and $1 < q < \infty$. By (3.52), (3.55) and Lemmas 2, 4 and 7, we find that, for some appropriate τ, we have

$$\|f\|_{W^{r,q}} \approx \|f\|_{F^r_{q,\tau}} \lesssim \|f\|^{\theta}_{F^{s_1}_{p_1,p_1}} \|f\|^{1-\theta}_{F^{\sigma}_{t,\infty}} \lesssim \|f\|^{\theta}_{W^{s_1,p_1}} \|f\|^{1-\theta}_{W^{s_2,1}}, \ \forall f \in W^{s_1,p_1}(\mathbb{R}^N) \cap W^{s_2,1}(\mathbb{R}^N).$$

Case 8.5. $\sigma < 0$, $p_1 = \infty$ and $[q < \infty]$ or $[q = \infty$ and r is not a non-negative integer$]$

The argument is almost identical to the one used in Case 8.4. Using, in addition, Lemma 6, we find that

$$\|f\|_{W^{r,q}} \approx \|f\|_{F^r_{q,\tau}} \lesssim \|f\|^{\theta}_{F^{s_1}_{\infty,\infty}} \|f\|^{1-\theta}_{F^{\sigma}_{t,\infty}} \lesssim \|f\|^{\theta}_{W^{s_1,\infty}} \|f\|^{1-\theta}_{W^{s_2,1}}, \ \forall f \in W^{s_1,\infty}(\mathbb{R}^N) \cap W^{s_2,1}(\mathbb{R}^N).$$

Case 8.6. $\sigma < 0$, $q = \infty$ and $r \ge 0$ is an integer

By (3.52), we have $p_1 = t = \infty$ and, by (3.55), $\sigma = s_2 - N < 0$. Going back to (3.52), we find that $r < s_1$, and thus we also have $r < s_2$. We also note that $0 = \theta(s_1 - r) + (1-\theta)(s_2 - r - N)$. Using Lemma 8 and arguing as above, we have

$$\begin{aligned}\|f\|_{L^\infty} &\le \|f\|_{F^0_{\infty,1}} \lesssim \|f\|^{\theta}_{F^{s_1-r}_{\infty,\infty}} \|f\|^{1-\theta}_{F^{s_2-r-N}_{\infty,\infty}} \\ &\lesssim \|f\|^{\theta}_{W^{s_1-r,\infty}} \|f\|^{1-\theta}_{W^{s_2-r,1}} \lesssim \|f\|^{\theta}_{W^{s_1,\infty}} \|f\|^{1-\theta}_{W^{s_2,1}}, \ \forall f \in W^{s_1,\infty}(\mathbb{R}^N) \cap W^{s_2,1}(\mathbb{R}^N).\end{aligned} \tag{3.56}$$

Applying (3.56) to $\partial^\alpha f$, with $|\alpha| = r$, we obtain

$$\begin{aligned}\|D^r f\|_{L^\infty} &\lesssim \|D^r f\|^{\theta}_{W^{s_1-r,\infty}} \|D^r f\|^{1-\theta}_{W^{s_2-r,1}} \\ &\lesssim \|f\|^{\theta}_{W^{s_1,\infty}} \|f\|^{1-\theta}_{W^{s_2,1}}, \ \forall f \in W^{s_1,\infty}(\mathbb{R}^N) \cap W^{s_2,1}(\mathbb{R}^N).\end{aligned} \tag{3.57}$$

We complete this case by combining (3.56) and (3.57) with Lemma 1.

Case 8 is complete. □

Proof of Theorem 1 completed. As explained at the beginning of the proof, we have to investigate the cases where (3.1) is satisfied, i.e., at least one of (1.4), (1.7) or (1.9) holds.

1. Case 1 was a sort of preliminary case, allowing us to rule out some limiting situations (where $q = \infty$ and $r \geq 0$ is an integer).

2. The cases where $s_1 = s_2$ have been investigated in Case 2, and in the other cases we could assume, in addition to (3.1), that $s_1 < s_2$.

3. The cases where (1.7) holds form a sub-case of Case 3.

4. The cases where (1.9) holds are sub-cases of Cases 4 and 5.

5. The cases where $N = 1$ and (1.4) holds were treated in Cases 6 and 7.

6. The cases where $N \geq 2$ and (1.4) holds were investigated in Case 8.

The proof of Theorem 1 is complete. □

Proof of Lemma 9. Let $J = J(x) \in \mathbb{Z}$ be the (unique) integer such that

$$2^{-\alpha j} x^{\gamma} < 2^{\beta j} x^{-\delta} \text{ if } j > J \text{ and } 2^{-\alpha j} x^{\gamma} \geq 2^{\beta j} x^{-\delta} \text{ if } j \leq J. \tag{3.58}$$

It follows from (3.58) that

$$\frac{x^{(\gamma+\delta)/(\alpha+\beta)}}{2} < 2^J \leq x^{(\gamma+\delta)/(\alpha+\beta)}. \tag{3.59}$$

On the other hand, the proportionality relation $\alpha\delta = \beta\gamma$ implies

$$\beta\frac{\gamma+\delta}{\alpha+\beta} = \delta \text{ and } \alpha\frac{\gamma+\delta}{\alpha+\beta} = \gamma. \tag{3.60}$$

Using (3.59) and (3.60), we obtain

$$\begin{aligned}\sum_{j=-\infty}^{\infty} \min\left\{2^{-\alpha j} x^{\gamma}, 2^{\beta j} x^{-\delta}\right\} &= \sum_{j \leq J} 2^{\beta j} x^{-\delta} + \sum_{j>J} 2^{-\alpha j} x^{\gamma} \approx 2^{\beta J} x^{-\delta} + 2^{-\alpha J} x^{\gamma} \\ &\approx x^{\beta(\gamma+\delta)/(\alpha+\beta)-\delta} + x^{-\alpha(\gamma+\delta)/(\alpha+\beta)+\gamma} = 2,\end{aligned}$$

whence (3.44). □

Appendix. Proof of Theorem B

As explained at the beginning of Section 3, in view of the arguments we present it suffices to work in $\Omega = \mathbb{R}^N$ or in a ball. The proof consists of three cases.

Case 1. "Ordinary" cases
The conclusion of the theorem is well-known when both s and r are integers; see e.g. [2, Section 9.3]. Similarly, for the case where both $W^{s,p}$ and $W^{r,q}$ are regular spaces (in the sense of Definition 3); see e.g. [14, Section 2.2.3].

By the above, it remains to consider the case where exactly one of the spaces $W^{s,p}$, $W^{r,q}$ is exceptional, while the other one is of fractional order.

Case 2. $W^{s,p}$ is of fractional order, while $W^{r,q}$ is exceptional
Thus $q = \infty$ and $r \geq 0$ is an integer. We must have $p > 1$, for otherwise, by (1.5), s is an integer, and thus $W^{s,1}$ is exceptional.

The sequence (f_J) constructed in Case 5.4 in the proof of Theorem 1 satisfies $\|f_J\|_{W^{s,p}} \lesssim 1$, while $\|f_J\|_{W^{r,\infty}} \to \infty$ as $J \to \infty$. We find that (1.6) fails.

Case 3. $W^{s,p}$ is exceptional, while $W^{r,q}$ is of fractional order
Thus $s \geq 1$ is an integer, $p = 1$, and $1 < q < \infty$. [Indeed, if $q = \infty$ then r is an integer.] We consider several sub-cases.

Case 3.1. $N \geq 2$ and $r < s-1$
In this case, we have $W^{s,1} \hookrightarrow W^{s-1,N/(N-1)}$, by Case 1. By the same case, we have $W^{s-1,N/(N-1)} \hookrightarrow W^{r,q}$, and thus $W^{s,1} \hookrightarrow W^{r,q}$.

Case 3.2. $N \geq 2$ and $s = 1$
In this case, the embedding $W^{1,1} \hookrightarrow W^{r,q}$ has been established by Solonnikov [15]. Another proof of this embedding can be found in [1, Appendix D]. The proof there is presented only for $N = 2$, but a similar argument holds for every $N \geq 2$; see also the references therein.

Case 3.3. $N \geq 2$, $s \geq 2$ and $s-1 < r < s$
By the previous case, we have

$$u \in W^{s,1} \implies D^{s-1}u \in W^{1,1} \implies D^{s-1}u \in W^{r-s+1,q}. \tag{4.1}$$

On the other hand, we clearly have $1 < q < N/(N-1)$. By the Sobolev embedding $W^{1,1} \hookrightarrow L^{N/(N-1)}$, we find that

$$u \in L^1 \cap L^{N/(N-1)} \hookrightarrow L^q. \tag{4.2}$$

From (4.1) and (4.2), we obtain that $W^{s,1} \hookrightarrow W^{r,q}$.

Case 3.4. $N = 1$ and $s = 1$
In this case, it is possible to construct a function $u : \mathbb{R} \to \mathbb{R}$ such that $\operatorname{supp} u \subset (0,1)$ and $u \in W^{1,1}(\mathbb{R})$, but $u \notin W^{1/q,q}((0,1))$, $\forall\, q > 1$ (see Lemma 10 below). Thus the embedding $W^{1,1} \hookrightarrow W^{1/q,q}$ fails.

Case 3.5. $N = 1$, $s \geq 2$ and $s-1 < r < s$
By Case 3.4, there exists some $u : \mathbb{R} \to \mathbb{R}$ such that $\operatorname{supp} u \subset (0,1)$, $u \in W^{1,1} \setminus W^{r-s+1,q}_{loc}$. Let $v \in W^{s,1}((0,1))$ be such that $v^{(s-1)} = u$. Then we have $v \in W^{s,1} \setminus W^{r,q}$.

The proof of Theorem B is complete. □

Lemma 10. There exists a function $u : \mathbb{R} \to \mathbb{R}$, with $\operatorname{supp} u \subset (0,1)$, such that:

1. $u \in W^{1,1}(\mathbb{R})$.
2. For every $1 < q < \infty$, $u \notin W^{1/q,q}((0,1))$.

Proof. Let $v = v_{\varepsilon,a,b}$ be as in (3.28). Consider a sequence $u_j := v_{\varepsilon_j,a_j,b_j}(\cdot - d_j)$, where $b_j := 1/j^2$, $\varepsilon_j := e^{-e^j}$, and a_j and d_j are chosen such that the intervals $I_j := (d_j - 2a_j, d_j + 2a_j)$ have mutually disjoint supports contained in $(1/3, 2/3)$. Let $u := \sum u_j$. Clearly, $\operatorname{supp} u \subset (0,1)$ and $u \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$. By (3.29)–(3.30), we have $u \in W^{1,1}(\mathbb{R})$ and, for $1 < q < \infty$,

$$|u|^q_{W^{1/q,q}((0,1))} \geq \sum_j |u_j|^q_{W^{1/q,q}(I_j)} \geq C_q \sum_j \frac{1}{j^{2q}} e^j = \infty.$$ □

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