Rigidity of optimal bases for signal spaces

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Abstract

We discuss optimal L^2-approximations of functions controlled in the H^1-norm. We prove that the basis of eigenfunctions of the Laplace operator with Dirichlet boundary condition is the only orthonormal basis of L^2 which provides an optimal approximation in the sense of (9). This solves an open problem raised in [1].

Résumé

Rigidité des bases optimales pour les espaces de signaux. On s'intéresse à l'approximation optimale pour la norme L^2 de fonctions contrôlées en norme H^1. On prouve que la base des fonctions propes du Laplacien avec condition de Dirichlet au bord est l'unique base orthonormale de L^2 qui réalise une approximation optimale au sens de (9). Ceci résout un problème ouvert posé dans [1].

1 Introduction and main result

This note is a follow-up of the papers by Y. Aflalo, H. Brezis and R. Kimmel [2] and Y. Aflalo, H. Brezis, A. Bruckstein, R. Kimmel and N. Sochen [1].

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Let $e = (e_i)$ be an orthonormal basis of $L^2(\Omega)$ consisting of the eigenfunctions of the Laplace operator with Dirichlet boundary condition:

$$\begin{cases} -\Delta e_i = \lambda_i e_i & \text{in } \Omega, \\ e_i = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

where $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ is the ordered sequence of eigenvalues repeated according to their multiplicity.

We first recall a very standard result:

Theorem 1.1 *We have, for all $n \geq 1$,*

$$\left\| f - \sum_{i=1}^{n} (f, e_i) e_i \right\|_{L^2}^2 \leq \frac{\|\nabla f\|_{L^2}^2}{\lambda_{n+1}} \qquad \forall f \in H_0^1(\Omega). \tag{2}$$

Here and throughout the rest of this paper $(\cdot, \cdot)$ denotes the scalar product in $L^2(\Omega)$.

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Indeed, we may write

$$\left\| f - \sum_{i=1}^{n} (f, e_i) e_i \right\|_{L^2}^2 = \left\| \sum_{i=n+1}^{+\infty} (f, e_i) e_i \right\|_{L^2}^2 = \sum_{i=n+1}^{+\infty} (f, e_i)^2. \tag{3}$$

On the other hand

$$\|\nabla f\|_{L^2}^2 = \sum_{i=1}^{+\infty} \lambda_i (f, e_i)^2 \geq \sum_{i=n+1}^{+\infty} \lambda_i (f, e_i)^2 \geq \lambda_{n+1} \sum_{i=n+1}^{+\infty} (f, e_i)^2. \tag{4}$$

Combining (3) and (4) yields (2). $\square$

The authors of [2] and [1] have investigated the "optimality" in various directions of the basis (e_i), with respect to inequality (2). Here is one of their results restated in a slightly more general form:

Theorem 1.2 (Theorem 3.1 in [2]) *There is no integer* $n \geq 1$, *no constant* $0 \leq \alpha < 1$ *and no sequence* $(\psi_i)_{1 \leq i \leq n}$ *in* $L^2(\Omega)$ *such that*

$$\left\| f - \sum_{i=1}^{n} (f, \psi_i) \psi_i \right\|_{L^2}^2 \leq \frac{\alpha}{\lambda_{n+1}} \|\nabla f\|_{L^2}^2 \qquad \forall f \in H_0^1(\Omega). \tag{5}$$

The proof in [2] relies in the Fischer-Courant max-min principle (see Remark 3.3 below). For the convenience of the reader we present a very elementary proof based on a simple and efficient device originally due to H. Poincaré [5, p. 249-250] (and later rediscovered by many people, e.g. H. Weyl [7, p. 445] and R. Courant [3, p. 17-18]; see also H. Weinberger [6, p. 56] and P. Lax [4, p. 319]).

Suppose not, and set

$$f = c_1 e_1 + c_2 e_2 + \cdots + c_n e_n + c_{n+1} e_{n+1} \tag{6}$$

where $c = (c_1, c_2, \cdots, c_n, c_{n+1}) \in \mathbb{R}^{n+1}$. The under-determined linear system

$$(f, \psi_i) = 0, \qquad \forall i = 1, \cdots, n \tag{7}$$

of n equations with $n+1$ unknowns admits a non-trivial solution. Inserting f into (5) yields

$$\lambda_{n+1}\sum_{i=1}^{n+1} c_i^2 \leq \alpha \sum_{i=1}^{n} \lambda_i c_i^2 \leq \alpha\lambda_{n+1}\sum_{i=1}^{n+1} c_i^2. \tag{8}$$

Therefore $\sum_{i=1}^{n+1} c_i^2 = 0$ and thus $c = 0$. A contradiction. This proves Theorem 1.2. $\square$

The authors of [1] were thus led to investigate the question of whether inequality (2) holds *only* for the orthonormal bases consisting of eigenfunctions corresponding to ordered eigenvalues. They established that a "discrete", i.e. finite-dimensional, version does hold; see [1, Theorem 2.1] and Remark 3.2 below. But their proof of "uniqueness" could not be adapted to the infinite-dimensional case (because it relied on a "descending" induction). It was raised there as an open problem (see [1, p. 1166]). Our next result solves this problem.

Theorem 1.3 *Let (b_i) be an orthonormal basis of $L^2(\Omega)$ such that, for all $n \geq 1$,*

$$\left\| f - \sum_{i=1}^{n}(f,b_i)b_i \right\|_{L^2}^2 \leq \frac{\|\nabla f\|_{L^2}^2}{\lambda_{n+1}} \quad \forall f \in H_0^1(\Omega). \tag{9}$$

Then, (b_i) consists of an orthonormal basis of eigenfunctions of $-\Delta$ with corresponding eigenvalues (λ_i).

2 Proof of Theorem 1.3

A basic ingredient of the argument is the following lemma:

Lemma 2.1 *Assume that (9) holds for all $n \geq 1$ and all $f \in H_0^1(\Omega)$, and that*

$$\lambda_i < \lambda_{i+1} \tag{10}$$

for some $i \geq 1$. Then

$$(b_j, e_k) = 0, \qquad \forall j, k \textit{ such that } 1 \leq j \leq i < k. \tag{11}$$

Proof. Fix $k > i$. Let l be the largest integer $l \leq k-1$ such that

$$\lambda_l < \lambda_{l+1}. \tag{12}$$

Clearly

$$i \leq l \tag{13}$$

and

$$\lambda_{l+1} = \lambda_{l+2} = \cdots = \lambda_k. \tag{14}$$

Applying (9) for $n = l$ we get

$$\left\| f - \sum_{i=1}^{l} (f, b_i) b_i \right\|_{L^2}^2 \leq \frac{\|\nabla f\|_{L^2}^2}{\lambda_{l+1}} \quad \forall f \in H_0^1(\Omega). \tag{15}$$

We use again Poincaré's "magic trick". Take f of the form

$$f = c_1 e_1 + \cdots + c_l e_l + c e_k \tag{16}$$

such that

$$(f, b_j) = 0 \qquad \forall j = 1, \cdots, l. \tag{17}$$

This is a system of l linear equations with $l+1$ unknowns, so that there are nontrivial solutions. We may as well assume that

$$c_1^2 + \cdots + c_l^2 + c^2 = 1. \tag{18}$$

By (15) and (14) we have

$$\lambda_{l+1} \leq \lambda_1 c_1^2 + \cdots + \lambda_l c_l^2 + \lambda_k c^2 = \lambda_1 c_1^2 + \cdots + \lambda_l c_l + \lambda_{l+1} c^2. \tag{19}$$

From (18) we get

$$\lambda_{l+1}(c_1^2 + \cdots + c_l^2) \leq \lambda_1 c_1^2 + \cdots + \lambda_l c_l^2. \tag{20}$$

Thus

$$(\lambda_{l+1} - \lambda_1)c_1^2 + \cdots + (\lambda_{l+1} - \lambda_l)c_l^2 \leq 0. \tag{21}$$

By (12) the coefficients $\lambda_{l+1} - \lambda_i$ are positive for every $i = 1, \cdots, l$. Therefore

$$c_1 = \cdots = c_l = 0. \tag{22}$$

Hence $c = \pm 1$ so that $f = \pm e_k$ and by (17)

$$(b_j, e_k) = 0 \qquad \forall j = 1, \cdots, l. \tag{23}$$

The conclusion follows from (23) and (13). $\square$

Before we present the proof in the general case, for the convenience of the reader, we start with the case of simple eigenvalues. Since $\lambda_1 < \lambda_2$ then, by the lemma,

$$(b_1, e_k) = 0 \qquad \forall k \geq 2. \tag{24}$$

Thus $b_1 = \pm e_1$. Next we apply the lemma with $\lambda_2 < \lambda_3$. We have that

$$(b_2, e_k) = 0 \qquad \forall k \geq 3. \tag{25}$$

Also, we have that

$$(b_2, e_1) = \pm(b_2, b_1) = 0. \tag{26}$$

Therefore $b_2 = \pm e_2$. Similarly, we have that $b_i = \pm e_i$ for $i \geq 3$.

We now turn to the general case:

Proof of Theorem 1.3 Consider the first index $i \geq 1$ such that $\lambda_i < \lambda_{i+1}$. Call it i_1. From the lemma we have that

$$(b_j, e_k) \qquad \forall j, k \text{ such that } 1 \leq j \leq i_1 < k. \tag{27}$$

Therefore $b_1, \cdots, b_{i_1} \in \operatorname{span}(e_1, \cdots, e_{i_1})$. Hence, each b_j with $1 \leq j \leq i_1$ is an eigenfunction of $-\Delta$ with corresponding eigenvalue $\lambda = \lambda_1 = \cdots = \lambda_{i_1}$. Therefore, due to dimensions, $b_1, \cdots, b_{i_1}$ is an orthonormal basis of

$$\operatorname{span}(b_1, \cdots, b_{i_1}) = \operatorname{span}(e_1, \cdots, e_{i_1}) = \ker(-\Delta - \lambda_{i_1} I); \tag{28}$$

in particular each

$$e_k \in \operatorname{span}(b_1, \cdots, b_{i_1}) \qquad k = 1, \cdots, i_1. \tag{29}$$

Consider the next block

$$\lambda = \lambda_{i_1+1} = \cdots = \lambda_{i_2} < \lambda_{i_2+1}. \tag{30}$$

From the lemma we have that

$$(b_j, e_k) = 0 \qquad \forall j, k \text{ such that } 1 \leq j \leq i_2 < k. \tag{31}$$

We also know that for $j \geq i_1 + 1$,

$$(b_j, e_k) = 0 \qquad k = 1, \cdots, i_1 \tag{32}$$

because of (29). Combining (31) and (32) yields

$$(b_j)_{i_1+1 \leq j \leq i_2} \in \operatorname{span}(e_j)_{i_1+1 \leq j \leq i_2}. \tag{33}$$

As above we conclude, using (30), that $b_{i_1+1}, \cdots, b_{i_2}$ is an orthonormal basis of

$$\operatorname{span}(b_j)_{i_1+1 \leq j \leq i_2} = \operatorname{span}(e_j)_{i_1+1 \leq j \leq i_2} = \ker(-\Delta - \lambda_{i_2} I). \tag{34}$$

Similarly for the next blocks. $\square$

3 Final remarks

Remark 3.1 We call the attention of the reader to the fact that the functions b_i are only assumed to be in $L^2(\Omega)$ and we deduce from Theorem 1.3 that (surprisingly) they belong to $H_0^1(\Omega) \cap \mathcal{C}^\infty(\Omega)$.

Remark 3.2 Theorem 1.3 holds in a more general setting. Let V and H be Hilbert spaces such that $V \subset H$ with compact and dense inclusion ($\dim H \leq +\infty$). Let $a : V \times V \to \mathbb{R}$ be a continuous bilinear symmetric form for which there exist constants $C, \alpha > 0$ such that, for all $v \in V$,

$$a(v,v) \geq 0,$$
$$a(v,v) + C|v|_H^2 \geq \alpha \|v\|_V^2.$$

Let $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$ be the sequence of eigenvalues associated with the orthonormal (in H) eigenfunctions $e_1, e_2, \cdots \in V$, i.e.,

$$a(e_i, v) = \lambda_i (e_i, v) \qquad \forall v \in V,$$

where $(\cdot,\cdot)$ denotes the scalar product in H. We point that, in this general setting, it may happen that $\lambda_1 = 0$ (e.g. $-\Delta$ with Neumann boundary conditions); and λ_1 may have multiplicity > 1. Recall that, for every $n \geq 1$ and $f \in V$:

$$\lambda_{n+1} \left| f - \sum_{i=1}^{n} (e_i, f) e_i \right|_H^2 \leq a(f,f). \tag{35}$$

Let (b_i) be an orthonormal basis of H such that for all $n \geq 1$ and $f \in V$

$$\lambda_{n+1} \left| f - \sum_{i=1}^{n} (b_i, f) b_i \right|_H^2 \leq a(f,f). \tag{36}$$

Then, (b_i) consists of an orthonormal basis of eigenfunctions of a with corresponding eigenvalues (λ_i). The proof is identical to the one above.

When $\dim H < +\infty$ and $V = H$ this result is originally due to [1]. The proof of rigidity was quite different and could not be adapted to the infinite dimensional case. It was raised there as an open problem.

Remark 3.3 Recall that the usual Fischer-Courant max-min principle asserts that for every $n \geq 1$ we have

$$\lambda_{n+1} = \max_{\substack{M \subset L^2(\Omega) \\ M \text{ linear space} \\ \dim M = n}} \min_{\substack{0 \neq f \in H_0^1(\Omega) \\ f \in M^\perp}} \frac{\|\nabla f\|_{L^2}^2}{\|f\|_{L^2}^2}, \tag{37}$$

(see, e.g., [4] or [6]). Our technique sheds some light about the structure of the maximizers in (37). Let (b_i) be an orthonormal sequence in $L^2(\Omega)$ such that, for every $n \geq 1$,

$$\lambda_{n+1} = \min_{\substack{0 \neq f \in H_0^1(\Omega) \\ f \in M_n^\perp}} \frac{\|\nabla f\|_{L^2}^2}{\|f\|_{L^2}^2} \qquad \text{where } M_n = \operatorname{span}(b_1, b_2, \cdots, b_n). \tag{38}$$

Then, each b_i is an eigenfunction associated to λ_i. This is an easy consequence of the proof of Theorem 1.3.

Remark 3.4 (Rigidity of the tail) Assume that (9) holds only for $n = k, k+1, \cdots$. Let the eigenvalues be simple. Applying the same reasoning as in our proof gives

$$\operatorname{span}(b_1, \cdots, b_n) = \operatorname{span}(e_1, \cdots, e_n) \qquad n = k, k+1, \cdots \tag{39}$$

The same argument as before yields $b_i = \pm e_i$ for $i = k+1, k+2, \cdots$. Concerning the b_i's for $i \leq k$ we only know that $b_1, \cdots, b_k \in \operatorname{span}(e_1, \cdots, e_k)$ and therefore they are smooth. A similar result holds if the eigenvalues are not simple.

Remark 3.5 We now turn to the reverse situation, i.e., we assume that (9) *holds only for* $1 \leq n \leq k$*. In this case* (9) *yields very little information on the* b_i*'s. Consider for example the case* $n = k = 1$*. In other words, assume that* $b = b_1 \in L^2(\Omega)$ *is such that* $\|b\|_{L^2} = 1$ *and*

$$\|f - (f, b)b\|_{L^2}^2 \leq \frac{1}{\lambda_2} \|\nabla f\|_{L^2}^2 \qquad \forall f \in H_0^1(\Omega). \tag{40}$$

Of course, (40) *holds with* $b = e_1$*. From Lemma 2.1 we know that* (40) *implies that*

$$(e_2, b) = 0. \tag{41}$$

Clearly, (41) *is not sufficient. Indeed, take* $b = e_3$*. Then,* (41) *holds but* (40) *fails for* $f = e_1$*. We do not have a simple characterization of the functions* b *satisfying* (40)*. But we can construct a large family of functions* b *(which need not be smooth) such that* (40) *holds. Assume that* $0 < \lambda_1 \leq \lambda_2 < \lambda_3$*. Let* $\chi \in L^2(\Omega)$ *be any function such that*

$$(e_1, \chi) = 0, \tag{42}$$

$$(e_2, \chi) = 0, \tag{43}$$

$$\|\chi\|_{L^2}^2 = 1. \tag{44}$$

Set

$$b = \alpha e_1 + \varepsilon \chi \qquad \alpha^2 + \varepsilon^2 = 1, \text{ with } 0 < \varepsilon < 1. \tag{45}$$

Claim: there exists $\varepsilon_0 > 0$*, depending on* $(\lambda_i)_{1 \leq i \leq 3}$*, such that for every* $0 < \varepsilon < \varepsilon_0$ (40) *holds. We have, for* $f \in H_0^1(\Omega)$*, and with* $c_i = (f, e_i)$*,*

$$\frac{1}{\lambda_2}\|\nabla f\|_{L^2}^2 - \|f - (f,b)b\|_{L^2}^2 = \frac{1}{\lambda_2}\|\nabla f\|_{L^2}^2 - \left(\|f\|_{L^2}^2 - (f,b)^2\right) \tag{46}$$

$$= \sum_{i=1}^{+\infty} \frac{\lambda_i}{\lambda_2} c_i^2 - \sum_{i=1}^{+\infty} c_i^2 + (f,b)^2. \tag{47}$$

On the other hand

$$
\begin{aligned}
(f, b)^2 &= (\alpha(f, e_1) + \varepsilon(f, \chi))^2 && (48)\\
&= \alpha^2 c_1^2 + 2\alpha\varepsilon(f, e_1)(f, \chi) + \varepsilon^2(f, \chi)^2 && (49)\\
&= \alpha^2 c_1^2 + 2\alpha\varepsilon(f - c_2e_2, e_1)(f - c_2e_2, \chi) + \varepsilon^2(f, \chi)^2 && (50)\\
&\geq \alpha^2 c_1^2 - 2\varepsilon\|f - c_2e_2\|_{L^2}^2 && (51)\\
&= \alpha^2 c_1^2 - 2\varepsilon \sum_{i\neq 2} c_i^2. && (52)
\end{aligned}
$$

Going back to (47)*, using* (45) *and choosing* $\varepsilon < \varepsilon_0$ *small enough, yields*

$$
\begin{aligned}
&\frac{1}{\lambda_2}\|\nabla f\|_{L^2}^2 - \|f - (f, b)b\|_{L^2}^2\\
&\qquad \geq \left(\frac{\lambda_1}{\lambda_2} - 2\varepsilon - \varepsilon^2\right) c_1^2 + \sum_{i=3}^{+\infty}\left(\frac{\lambda_i}{\lambda_2} - 1 - 2\varepsilon\right) c_i^2 && (53)\\
&\qquad \geq 0. && (54)
\end{aligned}
$$

Remark 3.6 In the general setting of Remark 3.2 it may happen that $0 = \lambda_1 < \lambda_2$*. Suppose now that* $b \in H$ *is such that* $\|b\|_H = 1$ *and*

$$
\|f - (f, b)b\|_H^2 \leq \frac{1}{\lambda_2} a(f, f) \qquad \forall f \in V. \tag{55}
$$

Claim: we have $b = \pm e_1$*. Indeed, let* $f = e_1$ *in* (55) *we have that*

$$
\|e_1 - (e_1, b)b\|_H^2 \leq \frac{\lambda_1}{\lambda_2} = 0, \tag{56}
$$

Therefore $b = \pm e_1$*.*

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