Nonlinear Analysis 137 (2016) 222–245

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Two subtle convex nonlocal approximations of the BV-norm

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A R T I C L E I N F O

Article history:
Received 30 January 2016
Accepted 3 February 2016
Communicated by Enzo Mitidieri

For Juan-Luis Vazquez on his 70th birthday, wishing him continued success and inspiration in his wonderful mathematics

Keywords:
Asymptotic behavior
Sequence of functionals
Pointwise convergence

A B S T R A C T

Inspired by the BBM formula and by work of G. Leoni and D. Spector, we analyze the asymptotic behavior of two sequences of convex nonlocal functionals $(\Psi_n(u))$ and $(\Phi_n(u))$ which converge formally to the BV-norm of u. We show that pointwise convergence when u is not smooth can be delicate; by contrast, Γ-convergence to the BV-norm is a robust and very useful mode of convergence.

1. Introduction

Throughout this paper, Ω denotes a smooth bounded open subset of $\mathbb{R}^d$ ($d \geq 1$). We first recall a formula (BBM formula) due to J. Bourgain, H. Brezis, and P. Mironescu [2] (with a refinement by J. Davila [11]). Let (ρ_n) be a sequence of radial mollifiers in the sense that

$$\rho_n \in L^1_{loc}(0, +\infty), \quad \rho_n \geq 0, \tag{1.1}$$

$$\int_0^\infty \rho_n(r) r^{d-1}\, dr = 1 \quad \forall\, n, \tag{1.2}$$

and

$$\lim_{n \to +\infty} \int_\delta^\infty \rho_n(r) r^{d-1}\, dr = 0 \quad \forall\, \delta > 0. \tag{1.3}$$

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http://dx.doi.org/10.1016/j.na.2016.02.005
0362-546X/

Set

$$I_n(u) = \int_\Omega \int_\Omega \frac{|u(x) - u(y)|}{|x - y|} \rho_n(|x - y|)\, dx\, dy \le +\infty, \quad \forall\, u \in L^1(\Omega) \tag{1.4}$$

and

$$I(u) = \begin{cases} \gamma_d \displaystyle\int_\Omega |\nabla u| & \text{if } u \in BV(\Omega), \\ +\infty & \text{if } u \in L^1(\Omega) \setminus BV(\Omega), \end{cases} \tag{1.5}$$

where, for any $e \in \mathbb{S}^{d-1}$,

$$\gamma_d = \int_{\mathbb{S}^{d-1}} |\sigma \cdot e|\, d\sigma = \begin{cases} \dfrac{2}{d-1} |\mathbb{S}^{d-2}| & \text{if } d \ge 3, \\ 4 & \text{if } d = 2, \\ 2 & \text{if } d = 1. \end{cases} \tag{1.6}$$

Then

$$\lim_{n \to +\infty} I_n(u) = I(u) \quad \forall\, u \in L^1(\Omega). \tag{1.7}$$

It has also been established by A. Ponce [23] that $I_n \to I$ as $n \to +\infty$ in the sense of Γ-convergence in $L^1(\Omega)$. For works related to the BBM formula, see [5–7,15,16]. Other functionals converging to the BV-norm are considered in [3,8,9,17–22].

One of the goals of this paper is to analyze the asymptotic behavior of sequences of functionals which "resemble" $I_n(u)$ and converge to $I(u)$ (at least when u is smooth). As we are going to see pointwise convergence of $I_n(u)$ when u is not smooth can be delicate and depends heavily on the specific choice of (ρ_n). By contrast, Γ-convergence to I is a robust concept which is not sensitive to the choice of (ρ_n). We first consider the sequence (Ψ_n) of functionals defined by

$$\Psi_n(u) = \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{1+\varepsilon_n}}{|x - y|^{1+\varepsilon_n}} \rho_n(|x - y|)\, dx\, dy \right)^{\frac{1}{1+\varepsilon_n}} \le +\infty, \quad \forall\, u \in L^1(\Omega), \tag{1.8}$$

where $(\varepsilon_n) \to 0_+$ and (ρ_n) is a sequence of mollifiers as above.

A general result concerning pointwise convergence is the following

Proposition 1. *We have*

$$\lim_{n \to +\infty} \Psi_n(u) = I(u) \quad \forall\, u \in \bigcup_{q > 1} W^{1,q}(\Omega) \tag{1.9}$$

and

$$\liminf_{n \to +\infty} \Psi_n(u) \ge I(u) \quad \forall\, u \in L^1(\Omega). \tag{1.10}$$

By choosing a special sequence of (ρ_n), one may greatly improve the conclusion of Proposition 1:

Proposition 2. *There exists a sequence (ρ_n) and a constant C such that*

$$\Psi_n(u) \le C I(u) \quad \forall\, n, \forall\, u \in L^1(\Omega) \tag{1.11}$$

and

$$\lim_{n \to +\infty} \Psi_n(u) = I(u) \quad \forall\, u \in L^1(\Omega). \tag{1.12}$$

The proof of Propositions 1 and 2 is presented in Section 2.1. By contrast, some sequences (ρ_n) may produce pathologies:

Proposition 3. *Assume $d = 1$. There exists a sequence (ρ_n) and some $v \in W^{1,1}(\Omega)$ such that*

$$\Psi_n(v) = +\infty \quad \forall n \geq 1. \tag{1.13}$$

Proposition 4. *Assume $d = 1$. Given any $M > 1$, there exists a sequence (ρ_n) and a constant C such that*

$$\Psi_n(u) \leq CI(u) \quad \forall n, \forall u \in L^1(\Omega), \tag{1.14}$$

$$\lim_{n\to+\infty} \Psi_n(u) = I(u) \quad \forall u \in W^{1,1}(\Omega), \tag{1.15}$$

and, for some nontrivial $v \in BV(\Omega)$,

$$\lim_{n\to+\infty} \Psi_n(v) = MI(v). \tag{1.16}$$

The proofs of Propositions 3 and 4 are presented in Section 2.2. In Sections 2.3 and 2.4, we return to a general sequence (ρ_n) and we establish the following results:

Proposition 5. *We have*

$$\Psi_n \to I \text{ in the sense of } \Gamma\text{-convergence in } L^1(\Omega), \quad \text{as } n \to +\infty. \tag{1.17}$$

Motivated by Image Processing (see, e.g., [1,12–14,25]), we set

$$E_n(u) = \int_\Omega |u - f|^q + \Psi_n(u) \quad \text{for } u \in L^q(\Omega), \tag{1.18}$$

and

$$E_0(u) = \int_\Omega |u - f|^q + I(u) \quad \text{for } u \in L^q(\Omega), \tag{1.19}$$

where $q > 1$ and $f \in L^q(\Omega)$. Our main result is

Proposition 6. *For each n, there exists a unique $u_n \in L^q(\Omega)$ such that*

$$E_n(u_n) = \min_{u \in L^q(\Omega)} E_n(u).$$

Let v be the unique minimizer of E_0 in $L^q(\Omega) \cap BV(\Omega)$. We have, as $n \to +\infty$,

$$u_n \to v \quad \text{in } L^q(\Omega)$$

and

$$E_n(u_n) \to E_0(v).$$

In Section 3, we investigate similar questions for the sequence (Φ_n) of functionals defined by

$$\Phi_n(u) = \int_\Omega dx \left[\int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^p} \rho_n(|x - y|)\, dy \right]^{1/p} \leq +\infty, \quad \text{for } u \in L^1(\Omega),$$

where $p > 1$. Such functionals were introduced and studied by G. Leoni and D. Spector [15,16] (see also [26]); their motivation came from a paper by G. Gilboa and S. Osher [13] (where $p = 2$) dealing with Image Processing.

2. Asymptotic analysis of the sequence (Ψ_n)

2.1. Some positive facts about the sequence (Ψ_n)

We start with the

Proof of Proposition 1. We first establish (1.10). By Hölder's inequality, we have for every $u \in L^1(\Omega)$

$$I_n(u) \le \Psi_n(u)\Big(\int_\Omega \int_\Omega \rho_n(|x-y|)\,dx\,dy\Big)^{\frac{\varepsilon_n}{1+\varepsilon_n}}. \tag{2.1}$$

From (1.2), we have

$$\int_\Omega \int_\Omega \rho_n(|x-y|)\,dx\,dy \le |\mathbb{S}^{d-1}||\Omega|. \tag{2.2}$$

Note that

$$\lim_{n\to+\infty} \big(|\mathbb{S}^{d-1}||\Omega|\big)^{\frac{\varepsilon_n}{1+\varepsilon_n}} = 1.$$

Inserting (1.7) in (2.1) yields (1.10).

We next establish (1.9) for $u \in W^{1,q}(\Omega)$ with $q > 1$. Assuming n sufficiently large so that $1 + \varepsilon_n < q$, we may write using Hölder's inequality

$$\Psi_n(u) \le I_n(u)^{a_n} J_{n,q}^{b_n}, \tag{2.3}$$

where

$$J_{n,q} = \Big(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^q}{|x-y|^q}\rho_n(|x-y|)\,dx\,dy\Big)^{1/q}, \tag{2.4}$$

$$a_n + b_n = 1 \quad \text{and} \quad a_n + \frac{b_n}{q} = \frac{1}{1+\varepsilon_n}, \tag{2.5}$$

i.e.,

$$b_n\Big(1 - \frac{1}{q}\Big) = \frac{\varepsilon_n}{1+\varepsilon_n} \quad \text{and} \quad a_n = 1 - b_n. \tag{2.6}$$

From [2], we know that

$$J_{n,q} \le C\|\nabla u\|_{L^q}, \quad \text{with } C \text{ independent of } n. \tag{2.7}$$

Combining (2.3), (2.6), (2.7), and using (1.7), we obtain

$$\limsup_{n\to+\infty} \Psi_n(u) \le I(u).$$

This proves (1.9) since we already know (1.10). □

Proof of Proposition 2. The sequence (ρ_n) is defined by

$$\rho_n(t) = \frac{1+d+\varepsilon_n}{\delta^{1+d+\varepsilon_n}} t^{1+\varepsilon_n} \mathbb{1}_{(0,\delta_n)}(t), \tag{2.8}$$

where $\mathbb{1}_A$ denotes the characteristic function of the set A, and (δ_n) is a positive sequence converging to 0 and satisfying

$$\lim_{n\to+\infty} \delta_n^{\varepsilon_n} = 1; \tag{2.9}$$

one may take for example

$$\delta_n = e^{-1/\sqrt{\varepsilon_n}}. \tag{2.10}$$

We have

$$\Psi_n^{1+\varepsilon_n}(u) = \frac{1+d+\varepsilon_n}{\delta_n^{1+d+\varepsilon_n}} \int_\Omega \int_{\substack{\Omega \\ |x-y|<\delta_n}} |u(x)-u(y)|^{1+\varepsilon_n}\,dx\,dy. \tag{2.11}$$

From the Sobolev embedding, we know that $BV(\Omega) \subset L^q(\Omega)$ with $q = d/(d-1)$ and moreover,

$$\left(\int_\Omega \int_\Omega |u(x) - u(y)|^q \, dx \, dy \right)^{1/q} \le C I(u) \quad \forall \, u \in L^1(\Omega). \tag{2.12}$$

Applying Hölder's inequality as above, we find

$$\Psi_n(u) \le \left(\frac{1 + d + \varepsilon_n}{\delta_n^{1+d+\varepsilon_n}} \right)^{\frac{1}{1+\varepsilon_n}} X_n^{a_n} Y_n^{b_n}, \tag{2.13}$$

where

$$X_n = \int_\Omega \int_\Omega_{|x-y|<\delta_n} |u(x) - u(y)| \, dx \, dy, \tag{2.14}$$

$$Y_n = \left(\int_\Omega \int_\Omega_{|x-y|<\delta_n} |u(x) - u(y)|^q \, dx \, dy \right)^{1/q}, \tag{2.15}$$

and a_n and b_n are as in (2.5). From [2] (applied with $\rho_n(t) = \frac{1+d}{\delta_n^{1+d}} t \mathbb{1}_{(0,\delta_n)}(t)$), we know that

$$X_n \le C \delta_n^{1+d} I(u). \tag{2.16}$$

Moreover, by (1.7), we have

$$\lim_{n \to +\infty} \frac{1+d}{\delta_n^{1+d}} X_n = I(u). \tag{2.17}$$

On the other hand, by (2.12), we obtain

$$Y_n \le C I(u) := Y. \tag{2.18}$$

Inserting (2.16) and (2.18) in (2.13) gives

$$\Psi_n(u) \le C \frac{1}{\delta_n^{\alpha_n}} I(u), \tag{2.19}$$

where, by (2.6),

$$\begin{aligned} \alpha_n = \frac{1 + d + \varepsilon_n}{1 + \varepsilon_n} - (1 + d) a_n &= \frac{1 + d + \varepsilon_n}{1 + \varepsilon_n} - (1 + d) + \frac{(1 + d) q \varepsilon_n}{(q-1)(1 + \varepsilon_n)} \\ &= - \frac{\varepsilon_n d}{1 + \varepsilon_n} + \frac{(1 + d) q \varepsilon_n}{(q-1)(1 + \varepsilon_n)} = \frac{\varepsilon_n d^2}{1 + \varepsilon_n}. \end{aligned}$$

From (2.19) and (2.9), we obtain (1.11).

We next prove (1.12). In view of (1.10), it suffices to verify that

$$\limsup_{n \to +\infty} \Psi_n(u) \le I(u) \quad \forall \, u \in L^1(\Omega). \tag{2.20}$$

We return to (2.13) and write

$$\Psi_n(u) \le \left(\frac{1 + d + \varepsilon_n}{\delta_n^{1+d+\varepsilon_n}} \right)^{\frac{1}{1+\varepsilon_n}} \left(\frac{\delta_n^{d+1}}{d+1} \right)^{a_n} \left(\frac{(1+d) X_n}{\delta_n^{1+d}} \right)^{a_n} Y^{b_n} = \gamma_n \delta_n^{-\alpha_n} \left(\frac{(1+d) X_n}{\delta_n^{1+d}} \right)^{a_n} Y^{b_n},$$

where $\gamma_n \to 1, a_n \to 1$, and $b_n \to 0$. Using (2.9) and (2.17), we conclude that (2.20) holds. □

2.2. Some sequences (ρ_n) producing pathologies

In this section, we establish Propositions 3 and 4.

Proof of Proposition 3. Take $\Omega = (-1/2, 1/2)$ and $\rho_n(t) = \varepsilon_n t^{\varepsilon_n - 1} \mathbb{1}_{(0,1)}(t)$. Then

$$\Psi_n^{1+\varepsilon_n}(u) \geq \varepsilon_n \int_0^{1/2} dx \int_{-1/2}^{0} \frac{|u(x) - u(y)|^{1+\varepsilon_n}}{|x - y|^2}\, dy.$$

If we assume in addition that $u(y) = 0$ on $(-1/2, 0)$, we obtain

$$\Psi_n^{1+\varepsilon_n}(u) \geq \varepsilon_n \int_0^{1/2} |u(x)|^{1+\varepsilon_n} \Big(\frac{1}{x} - \frac{1}{x + 1/2}\Big)\, dx. \tag{2.21}$$

Choosing, for example,

$$u(x) = \begin{cases} |\ln x|^{-\alpha} & \text{on } 0 < x < 1/2, \\ 0 & \text{on } -1/2 < x \leq 0, \end{cases} \tag{2.22}$$

with $\alpha > 0$, we see that $u \in W^{1,1}(\Omega)$ while the RHS in (2.21) is $+\infty$ when $\alpha(1 + \varepsilon_n) \leq 1$; we might take, for example, $\alpha = \min_n \{1/(1 + \varepsilon_n)\}$. □

Proof of Proposition 4. Take $\Omega = (-1, 1)$ and (ρ_n) as in (2.8) (but do not take δ_n as in (2.9)). Let

$$v(x) = \begin{cases} 0 & \text{for } x \in (-1, 0), \\ 1 & \text{for } x \in (0, 1). \end{cases}$$

Then

$$\Psi_n(v) = \frac{2 + \varepsilon_n}{\delta_n^{2+\varepsilon_n}} \int_0^1 \int_0^1_{x+y<\delta_n} dx\, dy = \frac{2 + \varepsilon_n}{\delta_n^{\varepsilon_n}}.$$

Since $I(v) = 2$ (see (1.5) and (1.6)), we deduce that

$$\Psi_n(v) = \frac{2 + \varepsilon_n}{2\delta_n^{\varepsilon_n}} I(v). \tag{2.23}$$

Given $M > 1$, let $A = \ln M > 0$ and $\delta_n = e^{-A/\varepsilon_n}$. Then

$$\lim_{n \to +\infty} \Psi_n(v) = M I(v).$$

On the other hand, we have, for every $u \in BV(\Omega)$,

$$\Psi_n(u) \leq \frac{2 + \varepsilon_n}{\delta_n^{2+\varepsilon_n}} \int_\Omega \int_\Omega_{|x-y|<\delta_n} |u(x) - u(y)|^{1+\varepsilon_n}\, dx\, dy.$$

As in the proof of Proposition 2 (see (2.19)), we find

$$\Psi_n(u) \leq C \frac{1}{\delta_n^{\alpha_n}} I(u),$$

Since $\delta_n = e^{-A/\varepsilon_n}$, we deduce that (1.14) holds.

In order to obtain (1.15), we recall (see (1.9)) that

$$\lim_{n \to +\infty} \Psi(\tilde{u}) = I(\tilde{u}) \quad \forall\, \tilde{u} \in C^1(\bar{\Omega}). \tag{2.24}$$

For $u \in W^{1,1}(\Omega)$, we write

$$\Psi_n(u) - I(u) = \Psi_n(u) - \Psi_n(\tilde{u}) + \Psi_n(\tilde{u}) - I(\tilde{u}) + I(\tilde{u}) - I(u),$$

and thus by (1.14),

$$|\Psi_n(u) - I(u)| \le CI(u - \tilde{u}) + |\Psi_n(\tilde{u}) - I(\tilde{u})|. \tag{2.25}$$

We conclude that $\lim_{n\to+\infty} |\Psi_n(u) - I(u)| = 0$ using (2.24) and the density of $C^1(\bar{\Omega})$ in $W^{1,1}(\Omega)$.

2.3. *Γ-convergence*

This section is devoted to the proof of Proposition 5 and a slightly stronger variant.

Recall that (see, e.g., [4,10]), by definition, the sequence $(\Psi_n)\Gamma$-converges to Ψ in $L^1(\Omega)$ as $n \to \infty$ if the following two properties hold:

(G1) For every $u \in L^1(\Omega)$ and for every sequence $(u_n) \subset L^1(\Omega)$ such that $u_n \to u$ in $L^1(\Omega)$ as $n \to \infty$, one has

$$\liminf_{n\to\infty} \Psi_n(u_n) \ge \Psi(u).$$

(G2) For every $u \in L^1(\Omega)$, there exists a sequence $(u_n) \subset L^1(\Omega)$ such that $u_n \to u$ in $L^1(\Omega)$ as $n \to \infty$, and

$$\limsup_{n\to\infty} \Psi_n(u_n) \le \Psi(u).$$

Proof of (G1). Going back to (2.1)–(2.3), we have

$$I_n(u) \le \beta_n \Psi_n(u) \quad \forall\, u \in L^1(\Omega),$$

where $\beta_n \to 1$. Thus

$$I_n(u_n) \le \beta_n \Psi_n(u_n) \quad \forall\, n,$$

and since $I_n \to I$ in the sense of Γ-convergence in $L^1(\Omega)$ (see [23] and also [7]), we conclude that

$$\liminf_{n\to+\infty} \Psi_n(u_n) \ge I(u).$$

Proof of (G2). Given $u \in BV(\Omega)$, we will construct a sequence (u_n) converging to u in $L^1(\Omega)$ such that

$$\limsup_{n\to+\infty} \Psi_n(u_n) \le I(u).$$

Let $v_k \in C^1(\bar{\Omega})$ be such that

$$v_k \to u \quad \text{in } L^1(\Omega) \quad \text{and} \quad I(v_k) \to I(u). \tag{2.26}$$

For each k, let n_k be such that

$$\Big|\Psi_n(v_k) - I(v_k)\Big| \le 1/k \quad \text{if } n > n_k. \tag{2.27}$$

Without loss of generality, one may assume that (n_k) is an increasing sequence with respect to k. Define

$$u_n = v_k \quad \text{if } n_k < n \le n_{k+1}.$$

Combining (2.26) and (2.27) yields

$$u_n \to u \quad \text{in } L^1(\Omega) \quad \text{and} \quad \lim_{n\to+\infty} \Psi_n(u_n) = I(u). \quad \square$$

In fact, a property stronger than (G1) holds.

Proposition 7. *For every $u \in L^1(\Omega)$ and for every sequence $(u_n) \subset L^1(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $L^1(\Omega)$ as $n \to +\infty$, one has*

$$\liminf_{n\to+\infty} \Psi_n(u_n) \ge I(u). \tag{2.28}$$

Proof. We adapt a suggestion of E. Stein (personal communication to H. Brezis) described in [5]. Let (μ_k) be a sequence of smooth mollifiers such that $\mu_k \ge 0$ and $\operatorname{supp}\mu_k \subset B_{1/k} = B_{1/k}(0) = B(0, 1/k)$. Fix D an arbitrary smooth open subset of Ω such that $\bar{D} \subset \Omega$ and let $k_0 > 0$ be large enough such that $B(x, 1/k_0) \subset\subset \Omega$ for every $x \in D$. Given $v \in L^1(\Omega)$, define in D

$$v_k = \mu_k * v \quad \text{for } k \ge k_0.$$

We have

$$\begin{aligned}
&\int_D \int_D \frac{|v_k(x) - v_k(y)|^{1+\varepsilon_n}}{|x-y|^{1+\varepsilon_n}} \rho_n(|x-y|)\,dx\,dy \\
&\quad = \int_D \int_D \frac{|\mu_k * v(x) - \mu_k * v(y)|^{1+\varepsilon_n}}{|x-y|^{1+\varepsilon_n}} \rho_n(|x-y|)\,dx\,dy \\
&\quad = \int_D \int_D \frac{\left| \int_{B(0,1/k)} \mu_k(z) \Big(v(x-z) - v(y-z) \Big)\,dz \right|^{1+\varepsilon_n}}{|x-y|^{1+\varepsilon_n}}\,dx\,dy \\
&\quad \le \int_D \int_D \frac{\int_{B(0,1/k)} \mu_k(z) \Big| v(x-z) - v(y-z) \Big|^{1+\varepsilon_n} dz}{|x-y|^{1+\varepsilon_n}} \rho_n(|x-y|)\,dx\,dy,
\end{aligned}$$

by Hölder's inequality. A change of variables implies, for $k \ge k_0$,

$$\int_D \int_D \frac{|v_k(x) - v_k(y)|^{1+\varepsilon_n}}{|x-y|^{1+\varepsilon_n}} \rho_n(|x-y|)\,dx\,dy \le \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^{1+\varepsilon_n}}{|x-y|^{1+\varepsilon_n}} \rho_n(|x-y|)\,dx\,dy. \tag{2.29}$$

Applying (2.29) to $v = u_n$ we find

$$\int_D \int_D \frac{|u_{k,n}(x) - u_{k,n}(y)|^{1+\varepsilon_n}}{|x-y|^{1+\varepsilon_n}} \rho_n(|x-y|)\,dx\,dy \le \Psi_n^{1+\varepsilon_n}(u_n), \tag{2.30}$$

where $u_{k,n} = \mu_k * u_n$ is defined in D for every n and every $k \ge k_0$. Since $u_n \rightharpoonup u$ weakly in $L^1(\Omega)$ we know that for each *fixed* k,

$$u_{k,n} \to \mu_k * u \quad \textit{strongly} \ \text{ in } L^1(D) \text{ as } n \to +\infty.$$

Passing to the limit in (2.29) as $n \to +\infty$ (and fixed k) and applying Proposition 5 (Property (G1)) we find that

$$\liminf_{n\to+\infty} \int_D \int_D \frac{|u_{k,n}(x) - u_{k,n}(y)|^{1+\varepsilon_n}}{|x-y|^{1+\varepsilon_n}} \rho_n(|x-y|)\,dx\,dy \ge \gamma_d \int_D |\nabla(\mu_k * u)|. \tag{2.31}$$

Combining (2.30) and (2.31) yields

$$\liminf_{n\to+\infty} \Psi_n(u_n) \ge \gamma_d \int_D |\nabla(\mu_k * u)| \quad \forall\, k \ge k_0.$$

Letting $k \to +\infty$, we obtain

$$\liminf_{n\to+\infty} \Psi_n(u_n) \ge \gamma_d \int_D |\nabla u|.$$

Since D is arbitrary, Proposition 7 follows. □

2.4. Functionals with roots in image processing

We give here the

Proof of Proposition 6. For each fixed n, the functional E_n defined on $L^q(\Omega)$ by (1.18) is convex and lower semicontinuous (l.s.c.) for the strong L^q-topology (note that Ψ_n is l.s.c. by Fatou's lemma). Thus E_n is also l.s.c. for the weak L^q-topology. Since $q > 1, L^q$ is reflexive and $\inf_{u \in L^q(\Omega)} E_n(u)$ is achieved. Uniqueness of the minimizer follows from strict convexity.

We next establish the second statement. Since $q > 1$, one may assume that $u_{n_k} \rightharpoonup u_0$ weakly in $L^q(\Omega)$ for some subsequence (u_{n_k}). We claim that

$$u_0 = v. \tag{2.32}$$

By Proposition 5 (Property (G2)), there exists $(v_n) \subset L^1(\Omega)$ such that $v_n \to v$ in $L^1(\Omega)$ and

$$\limsup_{n\to\infty} \Psi_n(v_n) \le I(v). \tag{2.33}$$

Set, for $A > 0$ and $s \in \mathbb{R}$,

$$T_A(s) = \begin{cases} s & \text{if } |s| \le A, \\ A & \text{if } s > A, \\ -A & \text{if } s < -A. \end{cases} \tag{2.34}$$

We have, since u_n is a minimizer of E_n,

$$E_n(u_n) \le E_n(T_A v_n) = \int_\Omega |T_A v_n - f|^q + \Psi_n(T_A v_n) \le \int_\Omega |T_A v_n - f|^q + \Psi_n(v_n). \tag{2.35}$$

Letting $n \to \infty$ and using (2.33), we derive

$$\limsup_{n\to+\infty} E_n(u_n) \le \int_\Omega |T_A v - f|^q + I(v).$$

This implies, by letting $A \to +\infty$,

$$\limsup_{n\to+\infty} E_n(u_n) \le E_0(v). \tag{2.36}$$

On the other hand, we have by Proposition 7,

$$\liminf_{n_k\to+\infty} \Psi_{n_k}(u_{n_k}) \ge I(v), \tag{2.37}$$

and therefore

$$E_0(u_0) \le \liminf_{n_k\to+\infty} E_{n_k}(u_{n_k}). \tag{2.38}$$

From (2.36) and (2.38), we obtain claim (2.32).

Next we write

$$\int_\Omega |u_n - f|^q = E_n(u_n) - \Psi_n(u_n). \tag{2.39}$$

Combining (2.39) with (2.36) and (2.37) gives

$$\limsup_{n_k\to+\infty} \int_\Omega |u_{n_k} - f|^q \le E_0(v) - I(v) = \int_\Omega |v - f|^q. \tag{2.40}$$

Since we already know that $u_{n_k} \rightharpoonup v$ weakly in $L^q(\Omega)$, we deduce from (2.40) that $u_{n_k} \to v$ strongly in $L^q(\Omega)$. The uniqueness of the limit implies that $u_n \to v$ strongly in $L^q(\Omega)$, so that

$$\liminf_{n\to+\infty} E_n(u_n) \geq \int_\Omega |v-f|^q + I(v) = E_0(v).$$

Returning to (2.36) yields

$$\lim_{n\to+\infty} E_n(u_n) = E_0(v). \quad \square$$

Remark 1. There is an alternative proof of Proposition 6 which holds when $d \geq 2$ (and also when $d = 1$ provided that we make a mild additional assumptions on (ρ_n)). Instead of Proposition 7, one may rely on a compactness argument based on

Proposition 8. *Let (u_n) be a bounded sequence in $L^1(\Omega)$ such that*

$$\sup_n \Psi_n(u_n) < +\infty. \tag{2.41}$$

When $d = 1$, we also assume that for each n the function $t \mapsto \rho_n(t)$ is non-increasing. Then (u_n) is relatively compact in $L^1(\Omega)$.

Proof. From (2.1), (2.2) and (2.41), we have

$$I_n(u_n) \leq C \quad \forall\, n.$$

We may now invoke a result of J. Bourgain, H. Brezis, P. Mironescu in [2] when ρ_n is non-increasing. A. Ponce in [24] established that the monotonicity of ρ_n is not necessary when $d \geq 2$. $\square$

Proof of Proposition 6 revisited. Using Proposition 8 we can assume that $u_{n_k} \rightharpoonup u_0$ weakly in $L^q(\Omega)$ and strongly in $L^1(\Omega)$. We may then rely on Proposition 5 instead of Proposition 7. The rest is unchanged. $\square$

3. A second approximation of the BV-norm

Motivated by a suggestion of G. Gilboa and S. Osher in [13], G. Leoni and D. Spector [15,16] studied the following functionals

$$\Phi_n(u) = \int_\Omega dx \Big[\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^p} \rho_n(|x-y|)\, dy\Big]^{1/p} \leq +\infty \quad \text{for } u \in L^1(\Omega) \tag{3.1}$$

where $1 < p < +\infty$ and (ρ_n) satisfies (1.1)–(1.3). In [16], they established that (Φ_n) converges to J in the sense of Γ-convergence in $L^1(\Omega)$, where J is defined by

$$J(u) := \begin{cases} \gamma_{p,d} \displaystyle\int_\Omega |\nabla u| & \text{if } u \in BV(\Omega), \\ +\infty & \text{if } u \in L^1(\Omega) \setminus BV(\Omega). \end{cases} \tag{3.2}$$

Here, for any $e \in \mathbb{S}^{d-1}$,

$$\gamma_{p,d} := \Big(\int_{\mathbb{S}^{d-1}} |\sigma \cdot e|^p \, d\sigma\Big)^{1/p}. \tag{3.3}$$

In particular,

$$\gamma_{p,1} = 2^{1/p}. \tag{3.4}$$

When there is no confusion, we simply write γ instead of $\gamma_{p,d}$. [In fact, G. Leoni and D. Spector considered more general functionals involving a second parameter $1 \leq q < +\infty$ and they prove that it Γ-converges in $L^1(\Omega)$ to $\int_\Omega |\nabla u|^q$ up to a positive constant. Here we are concerned only with the most delicate case $q = 1$ which produces the BV-norm in the asymptotic limit.]

Pointwise convergence of the sequence (Φ_n) turns out to be quite complex and not yet fully understood (which confirms again the importance of Γ-convergence). Several claims in [15] concerning the pointwise convergence of (Φ_n) were not correct as was pointed out in [16].

This section is organized as follows. In Sections 3.1–3.3, we describe various results (both positive and negative) concerning pointwise convergence. The case $d = 1$ is of special interest because the situation there is quite satisfactory (the only remaining open problem appears in Remark 3). Our results for the case $d \geq 2$ are not as complete; see e.g. important open problems mentioned in Remarks 5 and 8. We then present a new proof of Γ-convergence in Section 3.4; as we already mentioned, this result is due to G. Leoni and D. Spector, but our proof is simpler. Finally, in Section 3.5, we discuss variational problems similar to (1.18) (where Ψ_n is replaced by Φ_n) with roots in Image Processing.

3.1. Some positive facts about the sequence (Φ_n)

A general result concerning the pointwise convergence of (Φ_n) is the following.

Proposition 9. *We have*

$$\lim_{n\to\infty} \Phi_n(u) = J(u) \quad \forall\, u \in W^{1,p}(\Omega) \tag{3.5}$$

and

$$\liminf_{n\to\infty} \Phi_n(u) \geq J(u) \quad \forall\, u \in L^1(\Omega). \tag{3.6}$$

Proof. The proof is divided into three steps.

Step 1: Proof of (3.5) for $u \in C^2(\bar{\Omega})$. We have

$$|u(x) - u(y) - \nabla u(x) \cdot (x - y)| \leq C|x - y|^2 \quad \forall\, x,\, y \in \Omega,$$

for some positive constant C independent of x and y. It follows that

$$|u(x) - u(y)| \leq |\nabla u(x) \cdot (x - y)| + C|x - y|^2 \quad \forall\, x,\, y \in \Omega \tag{3.7}$$

and

$$|\nabla u(x) \cdot (x - y)| \leq |u(x) - u(y)| + C|x - y|^2 \quad \forall\, x,\, y \in \Omega. \tag{3.8}$$

From (3.7), we derive that

$$\Big(\int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^p} \rho_n(|x - y|)\, dy\Big)^{1/p} \leq \Big(\int_\Omega \frac{|\nabla u(x) \cdot (y - x)|^p}{|x - y|^p} \rho_n(|x - y|)\, dy\Big)^{1/p} + C\Big(\int_\Omega |x - y|^p \rho_n(|x - y|)\, dy\Big)^{1/p};$$

which implies, by (1.2) and (1.3),

$$\Big(\int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^p} \rho_n(|x - y|)\, dy\Big)^{1/p} \leq \gamma |\nabla u(x)| + o(1). \tag{3.9}$$

Here and in what follows in this proof, $o(1)$ denotes a quantity which converges to 0 (independently of x) as $n \to +\infty$. We derive that

$$\Phi_n(u) \leq \gamma \int_\Omega |\nabla u(x)|\, dx + o(1). \tag{3.10}$$

For the reverse inequality, we consider an arbitrary open subset D of Ω such that $\bar{D} \subset \Omega$. For a fixed $x \in D$, using (1.2), (1.3) and (3.8) one can verify as in (3.9) that

$$\gamma |\nabla u(x)| \le \Big(\int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^p} \rho_n(|x-y|)\, dy \Big)^{1/p} + o(1).$$

It follows that

$$\gamma \int_D |\nabla u(x)|\, dx \le \Phi_n(u) + o(1). \tag{3.11}$$

Combining (3.10) and (3.11) yields

$$\gamma \int_D |\nabla u(x)|\, dx \le \liminf_{n \to +\infty} \Phi_n(u) \le \limsup_{n \to +\infty} \Phi_n(u) \le \gamma \int_\Omega |\nabla u(x)|\, dx.$$

The conclusion of Step 1 follows since D is arbitrary,

Step 2: Proof of (3.6). We follow the same strategy as in the proof of Proposition 7. Let (μ_k) be a sequence of smooth mollifiers such that $\mu_k \ge 0$ and $\operatorname{supp} \mu_k \subset B_{1/k}$. Fix D an arbitrary smooth open subset of Ω such that $\bar{D} \subset \Omega$ and let $k_0 > 0$ be large enough such that $B(x, 1/k_0) \subset\subset \Omega$ for every $x \in D$. Given $u \in L^1(\Omega)$, define in D

$$u_k = \mu_k * u \quad \text{for } k \ge k_0.$$

We have, for $k \ge k_0$,

$$\int_D \Big(\int_D \frac{|u_k(x) - u_k(y)|^p}{|x-y|^p} \rho_n(|x-y|)\, dy \Big)^{1/p} dx \le \Phi_n(u) \quad \forall\, n. \tag{3.12}$$

Letting $n \to +\infty$ (for fixed k and fixed D), we find, using Step 1 on D, that, for $k \ge k_0$,

$$\lim_{n \to +\infty} \int_D \Big(\int_D \frac{|u_k(x) - u_k(y)|^p}{|x-y|^p} \rho_n(|x-y|)\, dy \Big)^{1/p} dx = \gamma \int_D |\nabla u_k(x)|\, dx.$$

We derive from (3.12) that

$$\liminf_{n \to +\infty} \Phi_n(u) \ge \gamma \int_D |\nabla u_k(x)|\, dx, \tag{3.13}$$

for $k \ge k_0$. Letting $k \to +\infty$, we obtain

$$\liminf_{n \to +\infty} \Phi_n(u) \ge \gamma \int_D |\nabla u(x)|\, dx. \tag{3.14}$$

We deduce (3.6) since D is arbitrary.

Step 3: Proof of (3.5) for $u \in W^{1,p}(\Omega)$. By Hölder's inequality, we have

$$\Phi_n(u) \le |\Omega|^{1-1/p} \Big(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^p} \rho_n(|x-y|)\, dx\, dy \Big)^{1/p}. \tag{3.15}$$

We may then invoke a result of [2] to conclude that

$$\Phi_n(u) \le C \|\nabla u\|_{L^p(\Omega)} \quad \forall\, u \in W^{1,p}(\Omega), \tag{3.16}$$

with $C > 0$ independent of n. We next write, using triangle inequality,

$$|\Phi_n(u) - \Phi_n(\tilde{u})| \le \Phi_n(u - \tilde{u}) \le C \|\nabla (u - \tilde{u})\|_{L^p(\Omega)} \quad \forall\, u, \tilde{u} \in W^{1,p}(\Omega).$$

This implies

$$\begin{aligned}|\Phi_n(u) - J(u)| &\le |\Phi_n(u) - \Phi_n(\tilde{u})| + |\Phi_n(\tilde{u}) - J(\tilde{u})| + |J(\tilde{u}) - J(u)| \\ &\le C\|\nabla(u - \tilde{u})\|_{L^p(\Omega)} + |\Phi_n(\tilde{u}) - J(\tilde{u})|.\end{aligned}$$

Using the density of $C^2(\bar{\Omega})$ in $W^{1,p}(\Omega)$, we obtain (3.5). □

By choosing a *special* sequence (ρ_n), we may greatly improve the conclusion of Proposition 9. More precisely, let (δ_n) be a positive sequence converging to 0 and define

$$\rho_n(t) = \frac{(p+d)}{\delta_n^{p+d}} t^p \mathbb{1}_{(0,\delta_n)}(t). \tag{3.17}$$

We have

Proposition 10. *Let $d \ge 1$ and assume that either*

$$1 < p \le d/(d-1) \quad \textit{and} \quad d \ge 2,$$

or

$$1 < p < +\infty \quad \textit{and} \quad d = 1,$$

and let (ρ_n) be defined by (3.17). Then

$$\Phi_n(u) \le C \int_\Omega |\nabla u| \quad \forall\, n, \forall\, u \in L^1(\Omega), \tag{3.18}$$

for some positive constant C depending only on d, p, and Ω, and

$$\lim_{n \to +\infty} \Phi_n(u) = J(u) \quad \forall\, u \in W^{1,1}(\Omega). \tag{3.19}$$

On the other hand, there exists some nontrivial $v \in BV(\Omega)$ such that

$$\lim_{n \to +\infty} \Phi_n(v) = \alpha_p J(v) \quad \textit{with } \alpha_p > 1. \tag{3.20}$$

Remark 2. The restriction $p \le d/(d-1)$ in the case $d \ge 2$ is quite natural if the goal is to prove (3.18) since the Sobolev embedding $W^{1,1}(\Omega) \subset L^{d/(d-1)}$ is sharp. In fact, this requirement is necessary. Let $d \ge 2$, fix $x_0 \in \Omega$, and assume that $\operatorname{diam}(\Omega) < 1/2$ for notational ease. Set $u(x) = |x - x_0|^{1-d} \ln^{-2} |x - x_0|$. One can verify that $u \in W^{1,1}(\Omega)$ and for $x \in \Omega$ with $|x - x_0| < \delta_n/2$

$$\int_{\substack{\Omega \\ |x-y|<\delta_n}} |u(x) - u(y)|^p \, dy = +\infty$$

since $p > d/(d-1)$. It follows that

$$\gamma \int_\Omega |\nabla u(x)| \, dx < +\infty = \Phi_n(u) \quad \forall\, n.$$

Remark 3. We do not know whether it is possible to construct a sequence (ρ_n) such that (3.19) holds for every $u \in BV(\Omega)$. The problem is open even when $d = 1$.

The proof of Proposition 10 relies on the following inequality which is just a rescaled version of the standard Sobolev one. Let B_R be a ball of radius R, then for any $p \in [1, d/(d-1)]$,

$$\left(\int_{B_R} \left| u(y) - \fint_{B_R} u \right|^p dy \right)^{1/p} \le C R^\alpha \int_{B_R} |\nabla u(z)| \, dz \quad \forall\, u \in L^1(B_R), \tag{3.21}$$

for some positive constant C depending only on d and p, where $\alpha := (d/p) + 1 - d \ge 0$.

Proof of Proposition 10. Since $\Phi_n(u) = \Phi_n(u+c)$ for any constant c, without loss of generality, one may assume that $\int_\Omega u = 0$. Consider an extension of u to $\mathbb{R}^d$ which is still denoted by u such that

$$\|u\|_{W^{1,1}(\mathbb{R}^d)} \le C_\Omega \|u\|_{W^{1,1}(\Omega)} \le C_\Omega \|\nabla u\|_{L^1(\Omega)}. \tag{3.22}$$

In view of (3.17), we have

$$\Phi_n(u) \le \frac{(p+d)^{1/p}}{\delta_n^{1+d/p}} \int_\Omega \Big(\int_{B(x,\delta_n)} |u(x)-u(y)|^p \, dy \Big)^{1/p} dx. \tag{3.23}$$

We have, for $y \in B(x, \delta_n)$,

$$|u(x)-u(y)| \le \Big| u(x) - \fint_{B(x,\delta_n)} u \Big| + \Big| u(y) - \fint_{B(x,\delta_n)} u \Big|. \tag{3.24}$$

It follows from the triangle inequality that

$$\Big(\int_{B(x,\delta_n)} |u(x)-u(y)|^p \, dy \Big)^{1/p} \le C\delta_n^{d/p} \Big| u(x) - \fint_{B(x,\delta_n)} u \Big| + \Big(\int_{B(x,\delta_n)} \Big| u(y) - \fint_{B(x,\delta_n)} u \Big|^p dy \Big)^{1/p}. \tag{3.25}$$

Here and in what follows in this proof, C denotes a positive constant depending only on d, p, and Ω. Inserting (3.21) in (3.25) yields

$$\Big(\int_{B(x,\delta_n)} |u(x)-u(y)|^p \, dy \Big)^{1/p} \le C\delta_n^{d/p} \Big| u(x) - \fint_{B(x,\delta_n)} u \Big| + C\delta_n^{\alpha} \int_{B(x,\delta_n)} |\nabla u(z)| \, dz. \tag{3.26}$$

We claim that

$$\int_\Omega \Big| u(x) - \fint_{B(x,\delta_n)} u \Big| \, dx \le C\delta_n \int_\Omega |\nabla u| \tag{3.27}$$

and

$$\int_\Omega dx \int_{B(x,\delta_n)} |\nabla u(z)| \, dz \le C\delta_n^d \int_\Omega |\nabla u|. \tag{3.28}$$

Indeed, we have, for R large enough,

$$\begin{aligned} \int_\Omega \Big| u(x) - \fint_{B(x,\delta_n)} u \Big| \, dx &\le C\delta_n^{-d} \int_{B_R} \int_{\substack{B_R \\ |x-y|<\delta_n}} |u(x)-u(y)| \, dx \, dy \\ &\le C\delta_n \int_{B_R} |\nabla u| \le C\delta_n \int_\Omega |\nabla u|, \end{aligned}$$

by the BBM formula applied to $\rho_n(t) = (d+1)\delta_n^{-(d+1)} t \mathbb{1}_{(0,\delta_n)}$ and by (3.22). On the other hand,

$$\int_\Omega \int_{B(x,\delta_n)} |\nabla u(z)| \, dz \, dx \le \int_{B_R} \int_{\substack{B_R \\ |x-z|<\delta_n}} |\nabla u(z)| \, dz \, dx \le C\delta_n^p \int_\Omega |\nabla u(x)| \, dx,$$

by (3.22). Combining (3.26)–(3.28) yields

$$\int_\Omega \Big(\int_{B(x,\delta_n)} |u(x)-u(y)|^p \, dy \Big)^{1/p} dx \le C\delta_n^{1+d/p} \int_\Omega |\nabla u(z)| \, dz \tag{3.29}$$

(recall that $\alpha + d = 1 + d/p$). It follows from (3.23) that

$$\Phi_n(u) \le C \|\nabla u\|_{L^1(\Omega)};$$

which is (3.18).

Assertion (3.19) is deduced from (3.18) via a density argument as in the proof of Proposition 9.

It remains to prove (3.20). For simplicity, take $\Omega = (-1/2, 1/2)$ and consider $v(x) = \mathbb{1}_{(0,1/2)}(x)$. Then, for n sufficiently large,

$$\begin{aligned}\Phi_n(v) &= 2\frac{(p+1)^{1/p}}{\delta_n^{1/p}} \int_0^{\delta_n} \Big(\int_0^{\delta_n - x} dy \Big)^{1/p} dx = \frac{2(p+1)^{1/p}}{\delta_n^{1+1/p}} \int_0^{\delta_n} (\delta_n - x)^{1/p} \, dx \\ &= \frac{2(p+1)^{1/p}}{\delta_n^{1+1/p}} \frac{\delta_n^{1+1/p}}{1+1/p} = \frac{2p}{(p+1)^{1-1/p}} > 2^{1/p} = J(v).\end{aligned}$$

Indeed, since $p+1 < 2p$, it follows that $(p+1)^{1-1/p} < (2p)^{1-1/p}$ and thus

$$\frac{2p}{(p+1)^{1-1/p}} > (2p)^{1/p} > 2^{1/p}. \quad \square$$

3.2. *More about the pointwise convergence of (Φ_n) when $d = 1$*

In this section, we assume that $d = 1$ and $\Omega = (-1/2, 1/2)$.

Proposition 11. *Assume that (ρ_n) satisfies (1.1)–(1.3). Then, for every $q > 1$, we have*

$$\Phi_n(u) \le C_q \|u'\|_{L^q(\Omega)} \quad \forall\, u \in W^{1,q}(\Omega),$$

for some positive constant C_q depending only on q. Moreover,

$$\lim_{n\to+\infty} \Phi_n(u) = J(u) \quad \forall\, u \in \bigcup_{q>1} W^{1,q}(\Omega).$$

Proof. Since $\Phi_n(u) = \Phi_n(u+c)$ for any constant c, without loss of generality, one may assume that $\int_\Omega u = 0$. Consider an extension of u to $\mathbb{R}$ which is still denoted by u, such that

$$\|u\|_{W^{1,q}(\mathbb{R})} \le C_q \|u\|_{W^{1,q}(\Omega)} \le C_q \|u'\|_{L^q(\Omega)}.$$

Let $M(f)$ denote the maximal function of f defined in $\mathbb{R}$, i.e.,

$$M(f)(x) := \sup_{r>0} \fint_{x-r}^{x+r} |f(s)| \, ds.$$

From the definition of Φ_n, we have

$$\Phi_n(u) \le C \int_\Omega \Big(\int_\Omega |M(u')(x)|^p \rho_n(|x-y|) \, dy \Big)^{1/p} dx \le C \int_\Omega M(u')(x) \, dx.$$

The first statement now follows from the fact that $\|M(f)\|_{L^q(\mathbb{R})} \le C_q \|f\|_{L^q(\mathbb{R})}$ since $q > 1$. The second statement is derived from the first statement via a density argument as in the proof of Proposition 9. $\square$

Our next result shows that Proposition 11 is sharp and cannot be extended to $q = 1$ (for a general sequence (ρ_n)).

Proposition 12. *For every $p > 1$, there exist a sequence (ρ_n) satisfying (1.1)–(1.3) and some function $v \in W^{1,1}(\Omega)$ such that*

$$\Phi_n(v) = +\infty \quad \forall\, n.$$

Proof. Fix $\alpha > 0$ and $\beta > 1$ such that

$$\alpha + \beta/p < 1. \tag{3.30}$$

Since $p > 1$ such α and β exist. Let (δ_n) be a sequence of positive numbers converging to 0 and consider

$$\rho_n(t) := A_n \frac{1}{t|\ln t|^\beta} \mathbb{1}_{(0,\delta_n)}.$$

Here A_n is chosen in such a way that (1.3) holds, i.e., $A_n \int_0^{\delta_n} \frac{dt}{t|\ln t|^\beta} = 1$. Set

$$v(x) = \begin{cases} 0 & \text{if } -1/2 < x < 0, \\ |\ln x|^{-\alpha} & \text{if } 0 < x < 1/2. \end{cases}$$

Clearly, $v \in W^{1,1}(\Omega)$. We have

$$\begin{aligned} \Phi_n(v) &= \int_{-1/2}^{1/2} \Big(\int_{-1/2}^{1/2} \frac{|v(x) - v(y)|^p}{|x-y|^p} \rho_n(|x-y|)\, dy \Big)^{1/p} dx \\ &\geq \int_0^{\delta_n} A_n^{1/p} |v(x)| \Big(\int_0^{\delta_n - x} \frac{1}{|x+y|^p} \rho_n(x+y)\, dy \Big)^{1/p} dx. \end{aligned} \tag{3.31}$$

We have, for $0 < x < \delta_n/2$,

$$\int_0^{\delta_n - x} \frac{1}{|x+y|^p} \rho_n(x+y)\, dy \geq \int_x^{\delta_n} \frac{dt}{t^{p+1}|\ln t|^\beta} \geq \int_x^{2x} \frac{dt}{t^{p+1}|\ln t|^\beta} \geq \frac{C_{p,\beta}}{x^p |\ln x|^\beta};$$

and thus

$$\Big(\int_0^{\delta_n - x} \frac{1}{|x+y|^p} \rho_n(x+y)\, dy \Big)^{1/p} \geq \frac{C_{p,\beta}}{x|\ln x|^{\beta/p}}. \tag{3.32}$$

Since, by (3.30),

$$\int_0^{\delta_n/2} \frac{1}{x|\ln x|^{\beta/p+\alpha}}\, dx = +\infty,$$

it follows from (3.31) and (3.32) that

$$\Phi_n(v) = +\infty \quad \forall\, n. \quad \square$$

Remark 4. D. Spector [26] has noticed that the sequence (ρ_n) and the function v constructed by A. Ponce (presented in [17]) satisfy (1.1)–(1.3), $v \in W^{1,1}(\Omega)$, $\Phi_n(v) < +\infty$ for all n, and $\lim_{n\to+\infty} \Phi_n(v) = +\infty$. In our construction, the pathology is even more dramatic since $\Phi_n(v) = +\infty$ for all n.

3.3. More about the pointwise convergence of (Φ_n) when $d \geq 2$

In this section, we present two "improvements" of (3.5) concerning the (pointwise) convergence of $\Phi_n(u)$ to $J(u)$. In the first one (Proposition 13) (ρ_n) is a general sequence (satisfying (1.1)–(1.3)), but the assumption on u is quite restrictive: $u \in W^{1,q}(\Omega)$ with $q > q_0$ where q_0 is defined in (3.33). In the second one (Proposition 14) there is an additional assumption on (ρ_n), but pointwise convergence holds for a large (more natural) class of u's: $u \in W^{1,q}(\Omega)$ with $q > q_1$ where $q_1 < q_0$ is defined in (3.44).

Proposition 13. *Let $p > 1$ and assume that (ρ_n) satisfies (1.1)–(1.3). Set*

$$q_0 := pd/(d + p - 1), \tag{3.33}$$

so that $1 < q_0 < p$. Then

$$\Phi_n(u) \leq C \|\nabla u\|_{L^q} \quad \forall\, u \in W^{1,q}(\Omega) \text{ with } q > q_0, \tag{3.34}$$

for some positive constant $C = C_{p,q,\Omega}$ depending only on p, q, and Ω. Moreover,

$$\lim_{n\to+\infty} \Phi_n(u) = J(u) \quad \forall\, u \in W^{1,q}(\Omega) \text{ with } q > q_0. \tag{3.35}$$

Proof. Since $\Phi_n(u) = \Phi_n(u+c)$ for any constant c, without loss of generality, one may assume that $\int_\Omega u = 0$. Consider an extension of u to $\mathbb{R}^d$ which is still denoted by u, such that

$$\|u\|_{W^{1,q}(\mathbb{R}^d)} \le C_{q,\Omega}\|u\|_{W^{1,q}(\Omega)} \le C_{q,\Omega}\|\nabla u\|_{L^q(\Omega)}.$$

For simplicity of notation, we assume that $\operatorname{diam}(\Omega) \le 1/2$. Then

$$\Phi_n(u) \le \int_\Omega \Big[\int_{\mathbb{S}^{d-1}}\int_0^1 \frac{|u(x+r\sigma)-u(x)|^p}{r^p}\rho_n(r)r^{d-1}\,dr\,d\sigma\Big]^{1/p}dx.$$

We have

$$\begin{aligned}|u(x+r\sigma)-u(x)| &\le \Big|u(x+r\sigma) - \fint_{\mathbb{S}^{d-1}} u(x+r\sigma')\,d\sigma'\Big| + \Big|u(x) - \fint_{\mathbb{S}^{d-1}} u(x+r\sigma')\,d\sigma'\Big| \\ &\le \fint_{\mathbb{S}^{d-1}} |u(x+r\sigma)-u(x+r\sigma')|\,d\sigma' + \fint_{\mathbb{S}^{d-1}} |u(x)-u(x+r\sigma')|\,d\sigma'.\end{aligned}$$

It follows that

$$\Phi_n(u) \lesssim T_1 + T_2, \tag{3.36}$$

where

$$T_1 = \int_\Omega \Big[\int_0^1 \int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}} |u(x+r\sigma)-u(x+r\sigma')|^p\,d\sigma'\,d\sigma\,\rho_n(r)r^{d-1-p}\,dr\Big]^{1/p}dx$$

and

$$T_2 = \int_\Omega \Big[\int_0^1 \Big(\int_{\mathbb{S}^{d-1}} |u(x)-u(x+r\sigma')|\,d\sigma'\Big)^p \rho_n(r)r^{d-1-p}\,dr\Big]^{1/p}dx.$$

In this proof the notation $a \lesssim b$ means that $a \le Cb$ for some positive constant C depending only on p, q, and Ω.

We first estimate T_1. Let B_1 denotes the open unit ball of $\mathbb{R}^d$. By (3.33) we know that the trace mapping $u \mapsto u_{|\partial B_1}$ is continuous from $W^{1,q_0}(B_1)$ into $L^q(\partial B_1)$. It follows that

$$\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}} |u(x+r\sigma)-u(x+r\sigma')|^p\,d\sigma'\,d\sigma \lesssim \|\nabla u(x+r\cdot)\|^p_{L^{q_0}(B_1)} \lesssim r^p M^{p/q_0}(|\nabla u|^{q_0})(x)$$

(recall that $M(f)$ denotes the maximal function of a function f defined in $\mathbb{R}^d$). Using (1.2), we derive that

$$T_1 \lesssim \int_\Omega \Big[\int_0^1 M^{p/q_0}(|\nabla u|^{q_0})(x)\rho_n(r)r^{d-1}\,dr\Big]^{1/p}dx \lesssim \int_\Omega M^{1/q_0}(|\nabla u|^{q_0})(x)\,dx. \tag{3.37}$$

Since $q > q_0$, it follows from the theory of maximal functions that

$$\int_\Omega M^{1/q_0}(|\nabla u|^{q_0})(x)\,dx \lesssim \|\nabla u\|_{L^q(\Omega)}. \tag{3.38}$$

Combining (3.37) and (3.38) yields

$$T_1 \lesssim \|\nabla u\|_{L^q(\Omega)}. \tag{3.39}$$

We next estimate T_2. We have

$$\int_{\mathbb{S}^{d-1}} |u(x)-u(x+r\sigma')|\,d\sigma' \le \int_{\mathbb{S}^{d-1}}\int_0^r |\nabla u(x+s\sigma')|\,ds\,d\sigma'.$$

Applying Lemma 1, we obtain, for $0 < r < 1$ and $x \in \Omega$,

$$\int_{\mathbb{S}^{d-1}} \int_0^r |\nabla u(x + s\sigma')| \, ds \, d\sigma' \le C r M(|\nabla u|)(x). \tag{3.40}$$

We derive that

$$T_2 \lesssim \int_\Omega M(|\nabla u|)(x) \, dx \lesssim \|\nabla u\|_{L^q} \tag{3.41}$$

by the theory of maximal functions since $q > 1$. Combining (3.36), (3.39) and (3.41) yields (3.34).

Assertion (3.35) follows from (3.34) via a density argument as in the proof of Proposition 9. □

In the proof of Proposition 13, we used the following elementary.

Lemma 1. *Let $d \ge 1, r > 0, x \in \mathbb{R}^d$, and $f \in L^1_{loc}(\mathbb{R}^d)$. We have*

$$\int_{\mathbb{S}^{d-1}} \int_0^r |f(x + s\sigma)| \, ds \, d\sigma \le C_d r M(f)(x), \tag{3.42}$$

for some positive constant C_d depending only on d.

Proof. Set $\varphi(s) = \int_{\mathbb{S}^{d-1}} |f(x + s\sigma)| \, d\sigma$, so that, by the definition of $M(f)(x)$, we have

$$\fint_{B_r(x)} |f(y)| \, dy \le M(f)(x) \quad \forall r > 0,$$

and thus

$$H(r) := \int_0^r \varphi(s) s^{d-1} \, ds \le |B_1| r^d M(f)(x) \quad \forall r > 0. \tag{3.43}$$

Then $H'(r) = \varphi(r) r^{d-1}$, so that

$$\int_0^r \varphi(s) \, ds = \int_0^r \frac{H'(s)}{s^{d-1}} \, ds = \frac{H(r)}{r^{d-1}} + (d-1) \int_0^r \frac{H(s)}{s^d} \, ds \le C_d r M(f)(x),$$

by (3.43); which is precisely (3.42). (The integration by parts can be easily justified by approximation.) □

Under the assumption that ρ_n is non-increasing for every n, one can replace the condition $q > q_0$ in Proposition 13 by the weaker condition $q > q_1$, where

$$q_1 := \max\{pd/(p + d), 1\}, \tag{3.44}$$

so that $1 \le q_1 < q_0$. It is worth noting that the embedding $W^{1,q_1}(\Omega) \subset L^p(\Omega)$ is sharp and therefore q_1 is a natural lower bound for q (see Remark 2). In fact, we prove a slightly more general result:

Proposition 14. *Let $p > 1$ and assume that (ρ_n) satisfies (1.1)–(1.3). Suppose in addition that there exist $\Lambda > 0$ and a sequence of non-increasing functions $(\hat{\rho}_n) \subset L^1_{loc}(0, +\infty)$ such that*

$$\rho_n \le \hat{\rho}_n \quad \text{and} \quad \int_0^\infty \hat{\rho}_n(t) t^{d-1} \, dt \le \Lambda \quad \forall n. \tag{3.45}$$

Then

$$\Phi_n(u) \le C \|\nabla u\|_{L^q} \quad \forall u \in W^{1,q}(\Omega) \text{ with } q > q_1, \tag{3.46}$$

for some positive constant $C = C(p, q, \Lambda, \Omega)$ depending only on p, q, Λ, and Ω. Moreover,

$$\lim_{n \to +\infty} \Phi_n(u) = J(u) \quad \forall u \in W^{1,q}(\Omega) \text{ with } q > q_1. \tag{3.47}$$

Remark 5. We do not know whether the conclusions of Proposition 14 hold without assuming the existence of Λ and $(\hat{\rho}_n)$. Equivalently, we do not know whether the conclusions of Proposition 13 hold under the weaker condition $q > q_1$.

Proof. For simplicity of notation, we assume that ρ_n is non-increasing for all n and work directly with ρ_n instead of $\hat{\rho}_n$. We first prove (3.46). As in the proof of Proposition 13, one may assume that $\int_\Omega u = 0$. Consider an extension of u to $\mathbb{R}^d$ which is still denoted by u such that

$$\|u\|_{W^{1,q}(\mathbb{R}^d)} \le C_{q,\Omega}\|u\|_{W^{1,q}(\Omega)} \le C_{q,\Omega}\|\nabla u\|_{L^q(\Omega)}.$$

For simplicity of notation, we also assume that $\operatorname{diam}(\Omega) \le 1/2$. Then

$$\Phi_n(u) \le \int_\Omega \Big[\int_{\mathbb{S}^{d-1}}\int_0^1 \frac{|u(x+r\sigma)-u(x)|^p}{r^p}\rho_n(r)r^{d-1}\,dr\,d\sigma\Big]^{1/p}dx.$$

We claim that for a.e. $x \in \Omega$,

$$Z(x) = \Big[\int_{\mathbb{S}^{d-1}}\int_0^1 \frac{|u(x+r\sigma)-u(x)|^p}{r^p}\rho_n(r)r^{d-1}\,dr\,d\sigma\Big]^{1/p} \le CM^{1/q_1}(|\nabla u|^{q_1})(x). \tag{3.48}$$

Here and in what follows, C denotes a positive constant depending only on p, d, and Λ.

From (3.48), we deduce (3.46) via the theory of maximal functions since $q > q_1$. Assertion (3.47) follows from (3.46) by density as in the proof of Proposition 9.

It remains to prove (3.48). Without loss of generality we establish (3.48) for $x = 0$. The proof relies heavily on two inequalities valid for all $R > 0$:

$$\Big[\fint_{B_R}\Big|u(\xi) - \fint_{B_R} u\Big|^p d\xi\Big]^{1/p} \le CRM^{1/q_1}(|\nabla u|^{q_1})(0) \tag{3.49}$$

and

$$\fint_{B_R}|u(\xi)-u(0)|\,d\xi \le CRM^{1/q_1}(|\nabla u|^{q_1})(0), \tag{3.50}$$

where $B_R = B_R(0)$.

Inequality (3.49) is simply a rescaled version of the Sobolev inequality

$$\Big\|u - \fint_{B_1} u\Big\|_{L^p(B_1)} \le C\|\nabla u\|_{L^{q_1}(B_1)},$$

which implies that

$$\Big[\fint_{B_R}\Big|u(\xi) - \fint_{B_R} u\Big|^p d\xi\Big]^{1/p} \le CR\Big[\fint_{B_R}|\nabla u|^{q_1}\Big]^{1/q_1} \le CRM^{1/q_1}(|\nabla u|^{q_1})(0).$$

To prove (3.50), we write

$$\begin{aligned}\int_{B_R}|u(\xi)-u(0)|\,d\xi &= \int_0^R\int_{\mathbb{S}^{d-1}}|u(r\sigma)-u(0)|r^{d-1}\,dr\,d\sigma\\ &\le C\int_0^R r^{d-1}\,dr\int_{\mathbb{S}^{d-1}}\int_0^r|\nabla u(s\sigma)|\,ds\,d\sigma\\ &\le C\int_0^R r^d M(|\nabla u|)(0) \quad \text{by Lemma 1}.\end{aligned}$$

Thus

$$\fint_{B_R}|u(\xi)-u(0)|\,d\xi \le CRM(|\nabla u|)(0) \le CRM^{1/q_1}(|\nabla u|^{q_1})(0).$$

From (3.48), we obtain

$$Z(0)^p = \sum_{i=0}^{\infty} \int_{\mathbb{S}^{d-1}} \int_{2^{-(i+1)}}^{2^{-i}} |u(r\sigma) - u(0)|^p \rho_n(r) r^{d-1-p} \, dr \, d\sigma,$$

so that

$$Z(0)^p \le C \sum_{i=0}^{\infty} \rho_n(2^{-(i+1)}) 2^{ip} \int_{\mathbb{S}^{d-1}} \int_{2^{-(i+1)}}^{2^{-i}} |u(r\sigma) - u(0)|^p r^{d-1} \, dr \, d\sigma. \tag{3.51}$$

We have

$$|u(r\sigma) - u(0)| \le \Big| u(r\sigma) - \fint_{B_{2^{-i}}} u \Big| + \Big| \fint_{B_{2^{-i}}} u - u(0) \Big|. \tag{3.52}$$

Inserting (3.52) into (3.51) yields

$$Z(0)^p \le C \sum_{i=0}^{\infty} (U_i + V_i), \tag{3.53}$$

where

$$U_i = \rho_n(2^{-(i+1)}) 2^{ip} \int_{\mathbb{S}^{d-1}} \int_{2^{-(i+1)}}^{2^{-i}} \Big| u(r\sigma) - \fint_{B_{2^{-i}}} u \Big|^p r^{d-1} \, dr \, d\sigma$$

and

$$V_i = \rho_n(2^{-(i+1)}) 2^{ip} \int_{\mathbb{S}^{d-1}} \int_{2^{-(i+1)}}^{2^{-i}} \Big| \fint_{B_{2^{-i}}} u - u(0) \Big|^p r^{d-1} \, dr \, d\sigma.$$

Clearly,

$$U_i \le \rho_n(2^{-(i+1)}) 2^{ip} \int_{\mathbb{S}^{d-1}} \int_{0}^{2^{-i}} \Big| u(r\sigma) - \fint_{B_{2^{-i}}} u \Big|^p r^{d-1} \, dr \, d\sigma \le \rho_n(2^{-(i+1)}) 2^{-id} A, \tag{3.54}$$

by (3.49), where $A = M^{p/q_1}(|\nabla u|^{q_1})(0)$. On the other hand,

$$V_i \le \rho_n(2^{-(i+1)}) 2^{ip} \Big[\fint_{B_{2^{-i}}} |u(\xi) - u(0)| \, d\xi \Big]^p 2^{-id} \le C \rho_n(2^{-(i+1)}) 2^{-id} A \quad \text{by (3.50)}. \tag{3.55}$$

Combining (3.53)–(3.55), we obtain

$$Z(0)^p \le C \sum_{i=0}^{\infty} \rho_n(2^{-(i+1)}) 2^{-id} A.$$

Finally, we observe that

$$\int_0^1 \rho_n(r) r^{d-1} \, dr \ge \sum_{i=0}^{\infty} \int_{2^{-(i+2)}}^{2^{-(i+1)}} \rho_n(r) r^{d-1} \, dr \ge C \sum_{i=0}^{\infty} \rho_n(2^{-(i+1)}) 2^{-id}$$

and thus

$$Z(0)^p \le C M^{p/q_1}(|\nabla u|^{q_1})(0) \int_0^1 \rho_n(r) r^{d-1} \, dr \le C M^{p/q_1}(|\nabla u|^{q_1})(0). \quad \square$$

Remark 6. Assumption (3.45) holds e.g. for the sequence (ρ_n) defined in (3.17), i.e.,

$$\rho_n(t) = \frac{p+d}{\delta_n^{p+d}} t^p \mathbb{1}_{(0,\delta_n)}(t).$$

Indeed, we may choose

$$\hat{\rho}_n(t) = \frac{p+d}{\delta_n^d} \mathbb{1}_{(0,\delta_n)}(t).$$

Applying Proposition 14 we recover Proposition 10 since $q_1 = 1$ (note that $pd \le p + d$ when $d = 1$ and also when $d \ge 2$ provided that $p \le d/(d-1)$). Note, however that in Proposition 14 we must take $q > q_1 = 1$, while $q = 1$ was allowed in Proposition 10. This discrepancy is related to our next remark.

Remark 7. Assume that $d \ge 2$ and $1 < p \le d/(d-1)$, so that $q_1 = 1$. The conclusion of Proposition 14 fails in the borderline case $q = q_1 = 1$. More precisely, for every $p \in \big(1, d/(d-1)\big]$, there exist a sequence (ρ_n) satisfying (1.1)–(1.3) and (3.45), and a function $v \in W^{1,1}(\Omega)$ such that $\Phi_n(v) = +\infty$ for all n. The construction is similar to the one presented in the proof of Proposition 10. Indeed, let $\Omega = B_{1/2}(0)$. Fix $\alpha > 0$ and $\beta > 1$ such that

$$\alpha + \beta/p < 1. \tag{3.56}$$

Since $p > 1$ such α and β exist. Let (δ_n) be a sequence of positive numbers converging to 0 and consider

$$\rho_n(t) := A_n \frac{1}{t^d |\ln t|^\beta} \mathbb{1}_{(0,\delta_n)}.$$

Note that the functions $t \mapsto \rho_n(t)$ are non-increasing. Here A_n is chosen in such a way that (1.3) holds, i.e., $A_n \int_0^{\delta_n} \frac{dt}{t|\ln t|^\beta} = 1$. Set

$$V(x) = v(x_1) := \begin{cases} 0 & \text{if } -1/2 < x_1 < 0, \\ |\ln x_1|^{-\alpha} & \text{if } 0 < x_1 < 1/2. \end{cases}$$

Clearly, $V \in W^{1,1}(\Omega)$. We have

$$\begin{aligned} \Phi_n(V) &= \int_\Omega \Big(\int_\Omega \frac{|V(x) - V(y)|^p}{|x-y|^p} \rho_n(|x-y|)\, dy \Big)^{1/p} dx \\ &\gtrsim \int_{\substack{B_{1/4}(0) \\ 0<x_1<\delta_n/4}} A_n^{1/p} \Big(\int_{|y-x|\le\delta_n} \frac{|v(x_1) - v(y_1)|^p}{|x-y|^{p+d} \big|\ln|x-y|\big|^\beta}\, dy \Big)^{1/p} dx. \end{aligned}$$

Note that, for $0 < x_1 < \delta_n/4$,

$$\begin{aligned} \int_{|y-x|\le\delta_n} \frac{|v(x_1) - v(y_1)|^p\, dy}{|x-y|^{p+d}\big|\ln|x-y|\big|^\beta} &\gtrsim \int_{\substack{|y_1-x_1|\le\delta_n/4 \\ |x'-y'|\le\delta_n/4}} \frac{|v(x_1) - v(y_1)|^p\, dy'\, dy_1}{\big(|x_1-y_1|^{p+d} + |x'-y'|^{p+d}\big)\big|\ln|x_1-y_1|\big|^\beta} \\ &\gtrsim \int_{|y_1-x_1|\le\delta_n/4} \frac{|v(x_1) - v(y_1)|^p\, dy_1}{|x_1-y_1|^{p+1}\big|\ln|x_1-y_1|\big|^\beta}. \end{aligned}$$

We derive as in the proof of Proposition 12 that

$$\int_{|y-x|\le\delta_n} \frac{|v(x_1) - v(y_1)|^p\, dy}{|x-y|^{p+d}\big|\ln|x-y|\big|^\beta} \gtrsim \frac{v(x_1)^p}{x_1^p |\ln x_1|^\beta}.$$

It follows that

$$\Phi_n(V) \gtrsim \int_{\substack{\Omega \\ 0<x_1<\delta_n/4}} A_n^{1/p} \frac{v(x_1)}{x_1 |\ln x_1|^{\beta/p}}\, dx = \int_{\substack{\Omega \\ 0<x_1<\delta_n/4}} A_n^{1/p} \frac{1}{x_1 |\ln x_1|^{\alpha+\beta/p}}\, dx = +\infty,$$

(by (3.56)).

Remark 8. Assume that $d \ge 2$ and $p > d/(d-1)$, so that $q_1 = pd/(p+d) > 1$. It is not known whether the conclusions of Proposition 14 hold in the borderline case $q = q_1$. More precisely, assume that

$d \ge 2, p > d/(d-1)$, and that (ρ_n) satisfying (1.1)–(1.3) and (3.45). Is it true that $\lim_{n\to+\infty} \Phi_n(u) = J(u)$ for all $u \in W^{1,q_1}(\Omega)$? Take for example $d = 2$ and $p = 3$ so that $q_1 = 6/5$.

Remark 9. The technique we use in the proof of Proposition 14 is somewhat similar to the one used by D. Spector [26] (see e.g. the proof of his Theorem 1.8). However, the results are quite different in nature.

3.4. Γ-convergence

Concerning the Γ-convergence of Φ_n, G. Leoni and D. Spector proved in [16].

Proposition 15. *For every $p > 1$ we have*

$$\Phi_n \xrightarrow{\Gamma} \Phi_0(\cdot) := \gamma \int_\Omega |\nabla \cdot| \quad \text{in } L^1(\Omega),$$

where γ is given in (3.3).

Their proof is quite involved. Here is a simpler proof.

Proof. For D an open subset of Ω such that $\bar{D} \subset \Omega$, set

$$\Phi_n(u, D) = \int_D dx \Big[\int_D \frac{|u(x)-u(y)|^p}{|x-y|^p} \rho_n(|x-y|)\, dy\Big]^{1/p} \quad \text{for } u \in L^1(D).$$

Let $u \in L^1(\Omega)$ and $(u_n) \subset L^1(\Omega)$ be such that $u_n \to u$ in $L^1(\Omega)$. We must prove that

$$\liminf_{n\to\infty} \Phi_n(u_n) \ge \gamma \int_\Omega |\nabla u|.$$

Let (μ_k) be a sequence of smooth mollifiers such that $\operatorname{supp} \mu_k \subset B_{1/k}$. Let D be a smooth open subset of Ω such that $\bar{D} \subset \Omega$ and fix k_0 such that $D + B_{1/k_0} \subset \Omega$. We have as in (2.29), for $k \ge k_0$,

$$\Phi_n(\mu_k * u_n, D) \le \Phi_n(u_n). \tag{3.57}$$

Using the fact that

$$|\Phi_n(u, D) - \Phi_n(v, D)| \le C_D \|u - v\|_{W^{1,\infty}(D)} \quad \forall\, u, v \in W^{1,\infty}(D),$$

we obtain

$$|\Phi_n(\mu_k * u_n, D) - \Phi_n(\mu_k * u, D)| \le C_{k,D} \|u_n - u\|_{L^1(\Omega)}.$$

Hence

$$\Phi_n(\mu_k * u, D) \le \Phi_n(\mu_k * u_n, D) + C_{k,D} \|u_n - u\|_{L^1(\Omega)}. \tag{3.58}$$

Combining (3.57) and (3.58) yields

$$\gamma \int_D |\nabla(\mu_k * u)| \le \liminf_{n\to+\infty} \Phi_n(u_n).$$

Letting $k \to \infty$, we reach

$$\gamma \int_D |\nabla u| \le \liminf_{n\to+\infty} \Phi_n(u_n).$$

Since $D \subset\subset \Omega$ is arbitrary, we derive that

$$\gamma \int_\Omega |\nabla u| \le \liminf_{n\to+\infty} \Phi_n(u_n).$$

We next fix $u \in BV(\Omega)$ and construct a sequence (u_n) converging to u in $L^1(\Omega)$ such that

$$\limsup_{n\to+\infty} \Phi_n(u_n) \le \gamma \int_\Omega |\nabla u|.$$

Let $v_k \in C^1(\bar{\Omega})$ be such that

$$v_k \to u \quad \text{in } L^1(\Omega) \quad \text{and} \quad \int_\Omega |\nabla v_k| \to \int_\Omega |\nabla u|. \tag{3.59}$$

For each k, let n_k be such that

$$\left| \Phi_n(v_k) - \gamma \int_\Omega |\nabla v_k| \right| \le 1/k \quad \text{if } n > n_k. \tag{3.60}$$

Without loss of generality, one may assume that (n_k) is an increasing sequence with respect to k. Define

$$u_n = v_k \quad \text{if } n_k < n \le n_{k+1}.$$

We derive from (3.59) and (3.60) that

$$u_n \to u \quad \text{in } L^1(\Omega) \quad \text{and} \quad \lim_{n\to+\infty} \Phi_n(u_n) = \gamma \int_\Omega |\nabla u|.$$

The proof is complete. □

3.5. Functionals with roots in image processing

Set

$$\hat{E}_n(u) := \int_\Omega |u - f|^q + \Phi_n(u),$$

and

$$\hat{E}_0(u) := \int_\Omega |u - f|^q + \gamma \int_\Omega |\nabla u|,$$

where $q > 1$ and $f \in L^q(\Omega)$ is a given function. Motivated by Image Processing, we study variational problems related to $\hat{E}_n$. More precisely, we establish

Proposition 16. *For every n, there exists a unique $u_n \in L^q(\Omega)$ such that*

$$\hat{E}_n(u_n) = \min_{u\in L^q(\Omega)} \hat{E}_n(u).$$

Let u_0 be the unique minimizer of $\hat{E}_0$. We have, as $n \to +\infty$,

$$u_n \to u_0 \quad \text{in } L^q(\Omega)$$

and

$$\hat{E}_n(u_n) \to \hat{E}_0(u_0).$$

Proof. The proof is similar to the one of Proposition 6. The details are left to the reader. □

Acknowledgments

The research of first author was partially supported by NSF grant DMS-1207793 and by ITN “FIRST” of the European Commission, Grant Number PITN-GA-2009-238702.

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