Two subtle convex nonlocal approximations of the \(BV\)-norm

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For Juan-Luis Vazquez on his 70th birthday, wishing him continued success and inspiration in his wonderful mathematics

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\textbf{A B S T R A C T}

Inspired by the BBM formula and by work of G. Leoni and D. Spector, we analyze the asymptotic behavior of two sequences of convex nonlocal functionals \((\Psi_n(u))\) and \((\Phi_n(u))\) which converge formally to the \(BV\)-norm of \(u\). We show that pointwise convergence when \(u\) is not smooth can be delicate; by contrast, \(\Gamma\)-convergence to the \(BV\)-norm is a robust and very useful mode of convergence.

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1. Introduction

Throughout this paper, \(\Omega\) denotes a smooth bounded open subset of \(\mathbb{R}^d\) \((d \geq 1)\). We first recall a formula (BBM formula) due to J. Bourgain, H. Brezis, and P. Mironescu [2] (with a refinement by J. Davila [11]). Let \((\rho_n)\) be a sequence of radial mollifiers in the sense that

\[
\rho_n \in L^1_{\text{loc}}(0, +\infty), \quad \rho_n \geq 0,
\]

\[
\int_0^\infty \rho_n(r) r^{d-1} \, dr = 1 \quad \forall \, n,
\]

and

\[
\lim_{n \to +\infty} \int_\delta^\infty \rho_n(r) r^{d-1} \, dr = 0 \quad \forall \, \delta > 0.
\]
Set

\[ I_n(u) = \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|}{|x - y|} \rho_n(|x - y|) \, dx \, dy \leq +\infty, \; \forall u \in L^1(\Omega) \]  

(1.4)

and

\[ I(u) = \begin{cases} \gamma d \int_{\Omega} |\nabla u| & \text{if } u \in BV(\Omega), \\ +\infty & \text{if } u \in L^1(\Omega) \setminus BV(\Omega), \end{cases} \]  

(1.5)

where, for any \( e \in S^{d-1} \),

\[ \gamma_d = \int_{S^{d-1}} |\sigma \cdot e| \, d\sigma = \begin{cases} \frac{2}{d-1} |S^{d-2}| & \text{if } d \geq 3, \\ \frac{4}{2} & \text{if } d = 2, \\ \frac{2}{1} & \text{if } d = 1. \end{cases} \]  

(1.6)

Then

\[ \lim_{n \to +\infty} I_n(u) = I(u) \quad \forall u \in L^1(\Omega). \]  

(1.7)

It has also been established by A. Ponce [23] that \( I_n \to I \) as \( n \to +\infty \) in the sense of \( \Gamma \)-convergence in \( L^1(\Omega) \). For works related to the BBM formula, see [5–7,15,16]. Other functionals converging to the BV-norm are considered in [3,8,9,17–22].

One of the goals of this paper is to analyze the asymptotic behavior of sequences of functionals which “resemble” \( I_n(u) \) and converge to \( I(u) \) (at least when \( u \) is smooth). As we are going to see pointwise convergence of \( I_n(u) \) when \( u \) is not smooth can be delicate and depends heavily on the specific choice of \((\rho_n)\). By contrast, \( \Gamma \)-convergence to \( I \) is a robust concept which is not sensitive to the choice of \((\rho_n)\). We first consider the sequence \((\Psi_n)\) of functionals defined by

\[ \Psi_n(u) = \left( \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{1+\varepsilon_n}}{|x - y|^{1+\varepsilon_n}} \rho_n(|x - y|) \, dx \, dy \right)^{\frac{1}{1+\varepsilon_n}} \leq +\infty, \; \forall u \in L^1(\Omega), \]  

(1.8)

where \((\varepsilon_n) \to 0_+\) and \((\rho_n)\) is a sequence of mollifiers as above.

A general result concerning pointwise convergence is the following

**Proposition 1.** We have

\[ \lim_{n \to +\infty} \Psi_n(u) = I(u) \quad \forall u \in L^1(\Omega) \]  

(1.9)

and

\[ \liminf_{n \to +\infty} \Psi_n(u) \geq I(u) \quad \forall u \in L^1(\Omega). \]  

(1.10)

By choosing a special sequence of \((\rho_n)\), one may greatly improve the conclusion of Proposition 1:

**Proposition 2.** There exists a sequence \((\rho_n)\) and a constant \(C\) such that

\[ \Psi_n(u) \leq CI(u) \quad \forall n, \forall u \in L^1(\Omega) \]  

(1.11)

and

\[ \lim_{n \to +\infty} \Psi_n(u) = I(u) \quad \forall u \in L^1(\Omega). \]  

(1.12)

The proof of Propositions 1 and 2 is presented in Section 2.1. By contrast, some sequences \((\rho_n)\) may produce pathologies:
**Proposition 3.** Assume $d = 1$. There exists a sequence $(\rho_n)$ and some $v \in W^{1,1}(\Omega)$ such that
\[ \Psi_n(v) = +\infty \quad \forall n \geq 1. \] (1.13)

**Proposition 4.** Assume $d = 1$. Given any $M > 1$, there exists a sequence $(\rho_n)$ and a constant $C$ such that
\[ \Psi_n(u) \leq CI(u) \quad \forall n, \forall u \in L^1(\Omega), \] (1.14)
\[ \lim_{n \to +\infty} \Psi_n(u) = I(u) \quad \forall u \in W^{1,1}(\Omega), \] (1.15)
and, for some nontrivial $v \in BV(\Omega)$,
\[ \lim_{n \to +\infty} \Psi_n(v) = MI(v). \] (1.16)

The proofs of Propositions 3 and 4 are presented in Section 2.2. In Sections 2.3 and 2.4, we return to a general sequence $(\rho_n)$ and we establish the following results:

**Proposition 5.** We have
\[ \Psi_n \to I \text{ in the sense of } \Gamma\text{-convergence in } L^1(\Omega), \quad \text{as } n \to +\infty. \] (1.17)

Motivated by Image Processing (see, e.g., [1,12–14,25]), we set
\[ E_n(u) = \int_{\Omega} |u - f|^q + \Psi_n(u) \quad \text{for } u \in L^q(\Omega), \] (1.18)
and
\[ E_0(u) = \int_{\Omega} |u - f|^q + I(u) \quad \text{for } u \in L^q(\Omega), \] (1.19)
where $q > 1$ and $f \in L^q(\Omega)$. Our main result is

**Proposition 6.** For each $n$, there exists a unique $u_n \in L^q(\Omega)$ such that
\[ E_n(u_n) = \min_{u \in L^q(\Omega)} E_n(u). \]

Let $v$ be the unique minimizer of $E_0$ in $L^q(\Omega) \cap BV(\Omega)$. We have, as $n \to +\infty$,
\[ u_n \to v \quad \text{in } L^q(\Omega) \]
and
\[ E_n(u_n) \to E_0(v). \]

In Section 3, we investigate similar questions for the sequence $(\Phi_n)$ of functionals defined by
\[ \Phi_n(u) = \int_{\Omega} dx \left[ \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^p} \rho_n(|x - y|) dy \right]^{1/p} \leq +\infty, \quad \text{for } u \in L^1(\Omega), \]
where $p > 1$. Such functionals were introduced and studied by G. Leoni and D. Spector [15,16] (see also [26]); their motivation came from a paper by G. Gilboa and S. Osher [13] (where $p = 2$) dealing with Image Processing.

2. Asymptotic analysis of the sequence $(\Psi_n)$

2.1. Some positive facts about the sequence $(\Psi_n)$

We start with the
**Proof of Proposition 1.** We first establish (1.10). By Hölder’s inequality, we have for every $u \in L^1(\Omega)$

$$I_n(u) \leq \Psi_n(u) \left( \int_{\Omega} \int_{\Omega} \rho_n(|x-y|) \, dx \, dy \right)^{\frac{\varepsilon_n}{1+\varepsilon_n}}. \quad (2.1)$$

From (1.2), we have

$$\int_{\Omega} \int_{\Omega} \rho_n(|x-y|) \, dx \, dy \leq |S^{d-1}| |\Omega|. \quad (2.2)$$

Note that

$$\lim_{n \to +\infty} \left( |S^{d-1}| |\Omega| \right)^{\frac{\varepsilon_n}{1+\varepsilon_n}} = 1.$$

Inserting (1.7) in (2.1) yields (1.10).

We next establish (1.9) for $u \in W^{1,q}(\Omega)$ with $q > 1$. Assuming $n$ sufficiently large so that $1 + \varepsilon_n < q$, we may write using Hölder’s inequality

$$\Psi_n(u) \leq I_n(u)^{a_n} J_{n,q}^{b_n}, \quad (2.3)$$

where

$$J_{n,q} = \left( \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^q}{|x-y|^q} \rho_n(|x-y|) \, dx \, dy \right)^{1/q}, \quad (2.4)$$

$$a_n + b_n = 1 \quad \text{and} \quad a_n + \frac{b_n}{q} = \frac{1}{1 + \varepsilon_n}, \quad (2.5)$$

i.e.,

$$b_n \left(1 - \frac{1}{q}\right) = \frac{\varepsilon_n}{1 + \varepsilon_n} \quad \text{and} \quad a_n = 1 - b_n. \quad (2.6)$$

From [2], we know that

$$J_{n,q} \leq C \|\nabla u\|_{L^q}, \quad \text{with } C \text{ independent of } n. \quad (2.7)$$

Combining (2.3), (2.6), (2.7), and using (1.7), we obtain

$$\limsup_{n \to +\infty} \Psi_n(u) \leq I(u).$$

This proves (1.9) since we already know (1.10).

**Proof of Proposition 2.** The sequence $(\rho_n)$ is defined by

$$\rho_n(t) = \frac{1 + d + \varepsilon_n}{\delta^{1+d+\varepsilon_n}} t^{1+\varepsilon_n} 1_{(0,\delta_n)}(t), \quad (2.8)$$

where $1_A$ denotes the characteristic function of the set $A$, and $(\delta_n)$ is a positive sequence converging to 0 and satisfying

$$\lim_{n \to +\infty} \delta_n^{\varepsilon_n} = 1; \quad (2.9)$$

one may take for example

$$\delta_n = e^{-1/\sqrt{\varepsilon_n}}. \quad (2.10)$$

We have

$$\Psi_n^{1+\varepsilon_n}(u) = \frac{1 + d + \varepsilon_n}{\delta_n^{1+d+\varepsilon_n}} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{1+\varepsilon_n}}{|x-y|^{\delta_n}} \, dx \, dy. \quad (2.11)$$
From the Sobolev embedding, we know that \( BV(\Omega) \subset L^q(\Omega) \) with \( q = d/(d-1) \) and moreover, \[
\left( \int_\Omega \int_\Omega |u(x) - u(y)|^q \, dx \, dy \right)^{1/q} \leq CI(u) \quad \forall u \in L^1(\Omega). \tag{2.12}
\]
Applying Hölder’s inequality as above, we find
\[
\Psi_n(u) \leq \left( \frac{1 + d + \varepsilon_n}{\delta_{n}^{d+\varepsilon_n}} \right)^{\frac{1}{1+\varepsilon_n}} X_n^a_n Y_n^b_n, \tag{2.13}
\]
where
\[
X_n = \int_\Omega \int_\Omega |u(x) - u(y)| \, dx \, dy, \quad \text{and} \quad Y_n = \left( \int_\Omega \int_\Omega |u(x) - u(y)|^q \, dx \, dy \right)^{1/q}, \tag{2.14}
\]
and \( a_n \) and \( b_n \) are as in (2.5). From [2] (applied with \( \rho_n(t) = \frac{1+d}{\delta_n} t^{(0,\delta_n)}(t) \)), we know that
\[
X_n \leq C \delta_{n}^{1+d} I(u). \tag{2.16}
\]
Moreover, by (1.7), we have
\[
\lim_{n \to +\infty} \frac{1 + d}{\delta_{n}^{1+d}} X_n = I(u). \tag{2.17}
\]
On the other hand, by (2.12), we obtain
\[
Y_n \leq CI(u) := Y. \tag{2.18}
\]
Inserting (2.16) and (2.18) in (2.13) gives
\[
\Psi_n(u) \leq C \frac{1}{\delta_{n}^{\alpha_n}} I(u), \tag{2.19}
\]
where, by (2.6),
\[
\alpha_n = \frac{1 + d + \varepsilon_n}{1 + \varepsilon_n} - (1 + d) a_n = \frac{1 + d + \varepsilon_n}{1 + \varepsilon_n} - (1 + d) - \frac{(1+d)q\varepsilon_n}{(q-1)(1+\varepsilon_n)} = \frac{\varepsilon_n d(1+d)}{1+\varepsilon_n} + \frac{(1+d)q\varepsilon_n}{(q-1)(1+\varepsilon_n)} = \frac{\varepsilon_n d^2}{(q-1)(1+\varepsilon_n)}.
\]
From (2.19) and (2.9), we obtain (1.11).

We next prove (1.12). In view of (1.10), it suffices to verify that
\[
\limsup_{n \to +\infty} \Psi_n(u) \leq I(u) \quad \forall u \in L^1(\Omega). \tag{2.20}
\]
We return to (2.13) and write
\[
\Psi_n(u) \leq \left( \frac{1 + d + \varepsilon_n}{\delta_{n}^{d+\varepsilon_n}} \right)^{\frac{1}{1+\varepsilon_n}} \left( \frac{\delta_{n}^{d+\varepsilon_n}}{d+1} \right)^{a_n} (1 + d) X_n^{a_n} Y_n^{b_n} = \gamma_n \delta_{n}^{-\alpha_n} \left( \frac{1 + d}{\delta_{n}^{d+1}} \right)^{a_n} X_n^{a_n} Y_n^{b_n},
\]
where \( \gamma_n \to 1, a_n \to 1, \) and \( b_n \to 0. \) Using (2.9) and (2.17), we conclude that (2.20) holds.
2.2. Some sequences \((\rho_n)\) producing pathologies

In this section, we establish Propositions 3 and 4.

**Proof of Proposition 3.** Take \(\Omega = (-1/2, 1/2)\) and \(\rho_n(t) = \varepsilon_n t^{\varepsilon_n - 1} \mathbb{1}_{(0,1)}(t)\). Then

\[
\Psi_n^{1+\varepsilon_n}(u) \geq \varepsilon_n \int_{0}^{1/2} dx \int_{-1/2}^{0} \frac{|u(x) - u(y)|^{1+\varepsilon_n}}{|x - y|^2} dy.
\]

If we assume in addition that \(u(y) = 0\) on \((-1/2, 0)\), we obtain

\[
\Psi_n^{1+\varepsilon_n}(u) \geq \varepsilon_n \int_0^{1/2} |u(x)|^{1+\varepsilon_n} \left(\frac{1}{x} - \frac{1}{x + 1/2}\right) dx.
\]

Choosing, for example,

\[
u(x) = \begin{cases}
|\ln x|^{-\alpha} & \text{on } 0 < x < 1/2, \\
0 & \text{on } -1/2 < x \leq 0,
\end{cases}
\]

with \(\alpha > 0\), we see that \(u \in W^{1,1}(\Omega)\) while the RHS in (2.21) is \(+\infty\) when \(\alpha(1 + \varepsilon_n) \leq 1\); we might take, for example, \(\alpha = \min_n \{1/(1 + \varepsilon_n)\}\). \(\Box\)

**Proof of Proposition 4.** Take \(\Omega = (-1,1)\) and \((\rho_n)\) as in (2.8) (but do not take \(\delta_n\) as in (2.9)). Let

\[
v(x) = \begin{cases}
0 & \text{for } x \in (-1,0), \\
1 & \text{for } x \in (0,1).
\end{cases}
\]

Then

\[
\Psi_n(v) = \frac{2 + \varepsilon_n}{\delta_n^{2+\varepsilon_n}} \int_{x+y<\delta_n} dxdy = \frac{2 + \varepsilon_n}{\delta_n^{\varepsilon_n}}.
\]

Since \(I(v) = 2\) (see (1.5) and (1.6)), we deduce that

\[
\Psi_n(v) = \frac{2 + \varepsilon_n}{2\delta_n^{\varepsilon_n}} I(v).
\]

(2.23)

Given \(M > 1\), let \(A = \ln M > 0\) and \(\delta_n = e^{-A/\varepsilon_n}\). Then

\[
\lim_{n \to +\infty} \Psi_n(v) = MI(v).
\]

On the other hand, we have, for every \(u \in BV(\Omega)\),

\[
\Psi_n(u) \leq \frac{2 + \varepsilon_n}{\delta_n^{2+\varepsilon_n}} \int_{\Omega} \int_{|x-y|<\delta_n} |u(x) - u(y)|^{1+\varepsilon_n} dxdy.
\]

As in the proof of Proposition 2 (see (2.19)), we find

\[
\Psi_n(u) \leq C \frac{1}{\delta_n^{\varepsilon_n}} I(u),
\]

Since \(\delta_n = e^{-A/\varepsilon_n}\), we deduce that (1.14) holds.

In order to obtain (1.15), we recall (see (1.9)) that

\[
\lim_{n \to +\infty} \Psi(\bar{u}) = I(\bar{u}) \quad \forall \bar{u} \in C^1(\bar{\Omega}).
\]

(2.24)

For \(u \in W^{1,1}(\Omega)\), we write

\[
\Psi_n(u) - I(u) = \Psi_n(u) - \Psi_n(\tilde{u}) + \Psi_n(\tilde{u}) - I(\tilde{u}) + I(\tilde{u}) - I(u),
\]
and thus by (1.14),
\[ |\Psi_n(u) - I(u)| \leq CI(u - \tilde{u}) + |\Psi_n(\tilde{u}) - I(\tilde{u})|. \tag{2.25} \]
We conclude that \( \lim_{n \to +\infty} |\Psi_n(u) - I(u)| = 0 \) using (2.24) and the density of \( C^1(\bar{\Omega}) \) in \( W^{1,1}(\Omega) \).

2.3. \( \Gamma \)-convergence

This section is devoted to the proof of Proposition 5 and a slightly stronger variant. Recall that (see, e.g., [4,10]), by definition, the sequence \( (\Psi_n)\) \( \Gamma \)-converges to \( \Psi \) in \( L^1(\Omega) \) as \( n \to \infty \) if the following two properties hold:

(G1) For every \( u \in L^1(\Omega) \) and for every sequence \( (u_n) \subset L^1(\Omega) \) such that \( u_n \to u \) in \( L^1(\Omega) \) as \( n \to \infty \), one has
\[ \liminf_{n \to \infty} \Psi_n(u_n) \geq \Psi(u). \]

(G2) For every \( u \in L^1(\Omega) \), there exists a sequence \( (u_n) \subset L^1(\Omega) \) such that \( u_n \to u \) in \( L^1(\Omega) \) as \( n \to \infty \), and
\[ \limsup_{n \to \infty} \Psi_n(u_n) \leq \Psi(u). \]

Proof of (G1). Going back to (2.1)–(2.3), we have
\[ I_n(u) \leq \beta_n \Psi_n(u) \quad \forall u \in L^1(\Omega), \]
where \( \beta_n \to 1 \). Thus
\[ I_n(u_n) \leq \beta_n \Psi_n(u_n) \quad \forall n, \]
and since \( I_n \to I \) in the sense of \( \Gamma \)-convergence in \( L^1(\Omega) \) (see [23] and also [7]), we conclude that
\[ \liminf_{n \to +\infty} \Psi_n(u_n) \geq I(u). \]

Proof of (G2). Given \( u \in BV(\Omega) \), we will construct a sequence \( (u_n) \) converging to \( u \) in \( L^1(\Omega) \) such that
\[ \limsup_{n \to +\infty} \Psi_n(u_n) \leq I(u). \]

Let \( v_k \in C^1(\bar{\Omega}) \) be such that
\[ v_k \to u \quad \text{in} \quad L^1(\Omega) \quad \text{and} \quad I(v_k) \to I(u). \tag{2.26} \]
For each \( k \), let \( n_k \) be such that
\[ |\Psi_n(v_k) - I(v_k)| \leq 1/k \quad \text{if} \quad n > n_k. \tag{2.27} \]
Without loss of generality, one may assume that \( (n_k) \) is an increasing sequence with respect to \( k \). Define
\[ u_n = v_k \quad \text{if} \quad n_k < n \leq n_{k+1}. \]
Combining (2.26) and (2.27) yields
\[ u_n \to u \quad \text{in} \quad L^1(\Omega) \quad \text{and} \quad \lim_{n \to +\infty} \Psi_n(u_n) = I(u). \quad \square \]

In fact, a property stronger than (G1) holds.
**Proposition 7.** For every $u \in L^1(\Omega)$ and for every sequence $(u_n) \subset L^1(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $L^1(\Omega)$ as $n \to +\infty$, one has

$$\liminf_{n \to +\infty} \Psi_n(u_n) \geq I(u). \quad (2.28)$$

**Proof.** We adapt a suggestion of E. Stein (personal communication to H. Brezis) described in [5]. Let $(\mu_k)$ be a sequence of smooth mollifiers such that $\mu_k \geq 0$ and $\text{supp} \mu_k \subset B_{1/k} = B_{1/k}(0) = B(0,1/k)$. Fix $D$ an arbitrary smooth open subset of $\Omega$ such that $\bar{D} \subset \Omega$ and let $k_0 > 0$ be large enough such that $B(x,1/k_0) \subset \subset \Omega$ for every $x \in D$. Given $v \in L^1(\Omega)$, define in $D$

$$v_k = \mu_k \ast v \quad \text{for } k \geq k_0.$$

We have

$$\int_D \int_D \frac{|v_k(x) - v_k(y)|^{1+\varepsilon_n}}{|x-y|^{1+\varepsilon_n}} \rho_n(|x-y|) \, dx \, dy \leq \int_D \int_D \frac{|\mu_k \ast v(x) - \mu_k \ast v(y)|^{1+\varepsilon_n}}{|x-y|^{1+\varepsilon_n}} \rho_n(|x-y|) \, dx \, dy \leq \int_D \int_D \frac{|v(x) - v(y)|^{1+\varepsilon_n}}{|x-y|^{1+\varepsilon_n}} \rho_n(|x-y|) \, dx \, dy,$$

by H"older's inequality. A change of variables implies, for $k \geq k_0$,

$$\int_D \int_D \frac{|v_k(x) - v_k(y)|^{1+\varepsilon_n}}{|x-y|^{1+\varepsilon_n}} \rho_n(|x-y|) \, dx \, dy \leq \int_D \int_D \frac{|v(x) - v(y)|^{1+\varepsilon_n}}{|x-y|^{1+\varepsilon_n}} \rho_n(|x-y|) \, dx \, dy. \quad (2.29)$$

Applying (2.29) to $v = u_n$ we find

$$\int_D \int_D \frac{|u_{k,n}(x) - u_{k,n}(y)|^{1+\varepsilon_n}}{|x-y|^{1+\varepsilon_n}} \rho_n(|x-y|) \, dx \, dy \leq \int_D \int_D \frac{|u(x) - u(y)|^{1+\varepsilon_n}}{|x-y|^{1+\varepsilon_n}} \rho_n(|x-y|) \, dx \, dy \leq \Psi_n^{1+\varepsilon_n}(u_n), \quad (2.30)$$

where $u_{k,n} = \mu_k \ast u_n$ is defined in $D$ for every $n$ and every $k \geq k_0$. Since $u_n \rightharpoonup u$ weakly in $L^1(\Omega)$ we know that for each fixed $k$,

$$u_{k,n} \rightharpoonup \mu_k \ast u \quad \text{strongly in } L^1(D) \quad \text{as } n \to +\infty.$$

Passing to the limit in (2.29) as $n \to +\infty$ (and fixed $k$) and applying Proposition 5 (Property (G1)) we find that

$$\liminf_{n \to +\infty} \int_D \int_D \frac{|u_{k,n}(x) - u_{k,n}(y)|^{1+\varepsilon_n}}{|x-y|^{1+\varepsilon_n}} \rho_n(|x-y|) \, dx \, dy \geq \gamma_d \int_D |\nabla (\mu_k \ast u)|. \quad (2.31)$$

Combining (2.30) and (2.31) yields

$$\liminf_{n \to +\infty} \Psi_n(u_n) \geq \gamma_d \int_D |\nabla (\mu_k \ast u)| \quad \forall k \geq k_0.$$

Letting $k \to +\infty$, we obtain

$$\liminf_{n \to +\infty} \Psi_n(u_n) \geq \gamma_d \int_D |\nabla u|.$$

Since $D$ is arbitrary, Proposition 7 follows. \qed
2.4. Functionals with roots in image processing

We give here the

Proof of Proposition 6. For each fixed $n$, the functional $E_n$ defined on $L^q(\Omega)$ by (1.18) is convex and lower semicontinuous (l.s.c.) for the strong $L^q$-topology (note that $\Psi_n$ is l.s.c. by Fatou’s lemma). Thus $E_n$ is also l.s.c. for the weak $L^q$-topology. Since $q > 1$, $L^q$ is reflexive and $\inf_{u \in L^q(\Omega)} E_n(u)$ is achieved. Uniqueness of the minimizer follows from strict convexity.

We next establish the second statement. Since $q > 1$, one may assume that $u_n \rightharpoonup u_0$ weakly in $L^q(\Omega)$ for some subsequence $(u_{n_k})$. We claim that $u_0 = v$.  

By Proposition 5 (Property (G2)), there exists $(v_n) \subset L^1(\Omega)$ such that $v_n \rightarrow v$ in $L^1(\Omega)$ and

$$\limsup_{n \rightarrow \infty} \Psi_n(v_n) \leq I(v).$$  

Set, for $A > 0$ and $s \in \mathbb{R}$,

$$T_A(s) = \begin{cases} 
  s & \text{if } |s| \leq A, \\
  A & \text{if } s > A, \\
  -A & \text{if } s < -A.
\end{cases}$$  

We have, since $u_n$ is a minimizer of $E_n$,

$$E_n(u_n) \leq E_n(T_A v_n) = \int_\Omega |T_A u_n - f|^q + \Psi_n(T_A v_n) \leq \int_\Omega |T_A v_n - f|^q + \Psi_n(v_n).$$  

Letting $n \rightarrow \infty$ and using (2.33), we derive

$$\limsup_{n \rightarrow +\infty} E_n(u_n) \leq \int_\Omega |T_A v - f|^q + I(v).$$

This implies, by letting $A \rightarrow +\infty$,

$$\limsup_{n \rightarrow +\infty} E_n(u_n) \leq E_0(v).$$  

On the other hand, we have by Proposition 7,

$$\liminf_{n_k \rightarrow +\infty} \Psi_{n_k}(u_{n_k}) \geq I(v),$$  

and therefore

$$E_0(u_0) \leq \liminf_{n_k \rightarrow +\infty} E_{n_k}(u_{n_k}).$$

From (2.36) and (2.38), we obtain claim (2.32).

Next we write

$$\int_\Omega |u_n - f|^q = E_n(u_n) - \Psi_n(u_n).$$

Combining (2.39) with (2.36) and (2.37) gives

$$\limsup_{n_k \rightarrow +\infty} \int_\Omega |u_{n_k} - f|^q \leq E_0(v) - I(v) = \int_\Omega |v - f|^q.$$
Since we already know that $u_{n_k} \rightharpoonup v$ weakly in $L^q(\Omega)$, we deduce from (2.40) that $u_{n_k} \to v$ strongly in $L^q(\Omega)$. The uniqueness of the limit implies that $u_n \to v$ strongly in $L^q(\Omega)$, so that

$$\liminf_{n \to +\infty} E_n(u_n) \geq \int_\Omega |v - f|^q + I(v) = E_0(v).$$

Returning to (2.36) yields

$$\lim_{n \to +\infty} E_n(u_n) = E_0(v). \quad \square$$

**Remark 1.** There is an alternative proof of Proposition 6 which holds when $d \geq 2$ (and also when $d = 1$ provided that we make a mild additional assumptions on $(\rho_n)$). Instead of Proposition 7, one may rely on a compactness argument based on

**Proposition 8.** Let $(u_n)$ be a bounded sequence in $L^1(\Omega)$ such that

$$\sup_n \Psi_n(u_n) < +\infty. \quad (2.41)$$

When $d = 1$, we also assume that for each $n$ the function $t \mapsto \rho_n(t)$ is non-increasing. Then $(u_n)$ is relatively compact in $L^1(\Omega)$.

**Proof.** From (2.1), (2.2) and (2.41), we have

$$I_n(u_n) \leq C \quad \forall n.$$  

We may now invoke a result of J. Bourgain, H. Brezis, P. Mironescu in [2] when $\rho_n$ is non-increasing. A. Ponce in [24] established that the monotonicity of $\rho_n$ is not necessary when $d \geq 2$.  

**Proof of Proposition 6 revisited.** Using Proposition 8 we can assume that $u_{n_k} \rightharpoonup u_0$ weakly in $L^q(\Omega)$ and strongly in $L^1(\Omega)$. We may then rely on Proposition 5 instead of Proposition 7. The rest is unchanged.  

3. **A second approximation of the $BV$-norm**

Motivated by a suggestion of G. Gilboa and S. Osher in [13], G. Leoni and D. Spector [15,16] studied the following functionals

$$\Phi_n(u) = \int_\Omega dx \left[ \int_\Omega \frac{|u(x) - u(y)|}{|x-y|^p} \rho_n(|x-y|) \, dy \right]^{1/p} \leq +\infty \quad \text{for } u \in L^1(\Omega),$$

where $1 < p < +\infty$ and $(\rho_n)$ satisfies (1.1)–(1.3). In [16], they established that $(\Phi_n)$ converges to $J$ in the sense of $\Gamma$-convergence in $L^1(\Omega)$, where $J$ is defined by

$$J(u) := \begin{cases} 
\gamma_{p,d} \int_\Omega |\nabla u| & \text{if } u \in BV(\Omega), \\
+\infty & \text{if } u \in L^1(\Omega) \setminus BV(\Omega). 
\end{cases} \quad (3.2)$$

Here, for any $e \in S^{d-1}$,

$$\gamma_{p,d} := \left( \int_{S^{d-1}} |\sigma \cdot e|^{p} \, d\sigma \right)^{1/p}. \quad (3.3)$$

In particular,

$$\gamma_{p,1} = 2^{1/p}. \quad (3.4)$$

When there is no confusion, we simply write $\gamma$ instead of $\gamma_{p,d}$. [In fact, G. Leoni and D. Spector considered more general functionals involving a second parameter $1 \leq q < +\infty$ and they prove that it $\Gamma$-converges.
in $L^1(\Omega)$ to $\int_\Omega |\nabla u|^q$ up to a positive constant. Here we are concerned only with the most delicate case $q = 1$ which produces the BV-norm in the asymptotic limit.]

Pointwise convergence of the sequence $(\Phi_n)$ turns out to be quite complex and not yet fully understood (which confirms again the importance of $\Gamma$-convergence). Several claims in [15] concerning the pointwise convergence of $(\Phi_n)$ were not correct as was pointed out in [16]. This section is organized as follows. In Sections 3.1–3.3, we describe various results (both positive and negative) concerning pointwise convergence. The case $d = 1$ is of special interest because the situation there is quite satisfactory (the only remaining open problem appears in Remark 3). Our results for the case $d \geq 2$ are not as complete; see e.g. important open problems mentioned in Remarks 5 and 8. We then present a new proof of $\Gamma$-convergence in Section 3.4; as we already mentioned, this result is due to G. Leoni and D. Spector, but our proof is simpler. Finally, in Section 3.5, we discuss variational problems similar to (1.18) (where $\Psi_n$ is replaced by $\Phi_n$) with roots in Image Processing.

3.1. Some positive facts about the sequence $(\Phi_n)$

A general result concerning the pointwise convergence of $(\Phi_n)$ is the following.

**Proposition 9.** We have

$$\lim_{n \to \infty} \Phi_n(u) = J(u) \quad \forall u \in W^{1,p}(\Omega)$$

(3.5)

and

$$\liminf_{n \to \infty} \Phi_n(u) \geq J(u) \quad \forall u \in L^1(\Omega).$$

(3.6)

**Proof.** The proof is divided into three steps.

Step 1: Proof of (3.5) for $u \in C^2(\bar{\Omega})$. We have

$$|u(x) - u(y) - \nabla u(x) \cdot (x - y)| \leq C|x - y|^2 \quad \forall x, y \in \Omega,$$

for some positive constant $C$ independent of $x$ and $y$. It follows that

$$|u(x) - u(y)| \leq |\nabla u(x) \cdot (x - y)| + C|x - y|^2 \quad \forall x, y \in \Omega$$

(3.7)

and

$$|\nabla u(x) \cdot (x - y)| \leq |u(x) - u(y)| + C|x - y|^2 \quad \forall x, y \in \Omega.$$ 

(3.8)

From (3.7), we derive that

$$\left( \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^p} \rho_n(|x - y|) \, dy \right)^{1/p} \leq \left( \int_\Omega \frac{|\nabla u(x) \cdot (y - x)|^p}{|x - y|^p} \rho_n(|x - y|) \, dy \right)^{1/p}$$

$$+ C \left( \int_\Omega \frac{|x - y|^p \rho_n(|x - y|)}{\rho_n(|x - y|)} \, dy \right)^{1/p};$$

which implies, by (1.2) and (1.3),

$$\left( \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^p} \rho_n(|x - y|) \, dy \right)^{1/p} \leq \gamma |\nabla u(x)| + o(1).$$ 

(3.9)

Here and in what follows in this proof, $o(1)$ denotes a quantity which converges to 0 (independently of $x$) as $n \to +\infty$. We derive that

$$\Phi_n(u) \leq \gamma \int_\Omega |\nabla u(x)| \, dx + o(1).$$

(3.10)
For the reverse inequality, we consider an arbitrary open subset $D$ of $\Omega$ such that $\bar{D} \subset \Omega$. For a fixed $x \in D$, using (1.2), (1.3) and (3.8) one can verify as in (3.9) that

$$\gamma|\nabla u(x)| \leq \left( \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^p} \rho_n(|x - y|) \, dy \right)^{1/p} + o(1).$$

It follows that

$$\gamma \int_D |\nabla u(x)| \, dx \leq \Phi_n(u) + o(1). \quad (3.11)$$

Combining (3.10) and (3.11) yields

$$\gamma \int_D |\nabla u(x)| \, dx \leq \liminf_{n \to +\infty} \Phi_n(u) \leq \limsup_{n \to +\infty} \Phi_n(u) \leq \gamma \int_\Omega |\nabla u(x)| \, dx.$$ 

The conclusion of Step 1 follows since $D$ is arbitrary.

Step 2: Proof of (3.6). We follow the same strategy as in the proof of Proposition 7. Let $(\mu_k)$ be a sequence of smooth mollifiers such that $\mu_k \geq 0$ and $\text{supp} \mu_k \subset B_{1/k}$. Fix $D$ an arbitrary smooth open subset of $\Omega$ such that $\bar{D} \subset \Omega$ and let $k_0 > 0$ be large enough such that $B(x, 1/k_0) \subset \subset \Omega$ for every $x \in D$. Given $u \in L^1(\Omega)$, define in $D$

$$u_k = \mu_k * u \quad \text{for } k \geq k_0.$$ 

We have, for $k \geq k_0$,

$$\int_D \left( \int_D \frac{|u_k(x) - u_k(y)|^p}{|x - y|^p} \rho_n(|x - y|) \, dy \right)^{1/p} \, dx \leq \Phi_n(u) \quad \forall n. \quad (3.12)$$

Letting $n \to +\infty$ (for fixed $k$ and fixed $D$), we find, using Step 1 on $D$, that, for $k \geq k_0$,

$$\lim_{n \to +\infty} \int_D \left( \int_D \frac{|u_k(x) - u_k(y)|^p}{|x - y|^p} \rho_n(|x - y|) \, dy \right)^{1/p} \, dx = \gamma \int_D |\nabla u_k(x)| \, dx.$$ 

We derive from (3.12) that

$$\liminf_{n \to +\infty} \Phi_n(u) \geq \gamma \int_D |\nabla u_k(x)| \, dx, \quad (3.13)$$

for $k \geq k_0$. Letting $k \to +\infty$, we obtain

$$\liminf_{n \to +\infty} \Phi_n(u) \geq \gamma \int_D |\nabla u(x)| \, dx. \quad (3.14)$$

We deduce (3.6) since $D$ is arbitrary.

Step 3: Proof of (3.5) for $u \in W^{1,p}(\Omega)$. By Hölder’s inequality, we have

$$\Phi_n(u) \leq |\Omega|^{1-1/p} \left( \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^p} \rho_n(|x - y|) \, dx \, dy \right)^{1/p}. \quad (3.15)$$

We may then invoke a result of [2] to conclude that

$$\Phi_n(u) \leq C \|\nabla u\|_{L^p(\Omega)} \quad \forall u \in W^{1,p}(\Omega), \quad (3.16)$$

with $C > 0$ independent of $n$. We next write, using triangle inequality,

$$|\Phi_n(u) - \Phi_n(\tilde{u})| \leq \Phi_n(u - \tilde{u}) \leq C \|\nabla (u - \tilde{u})\|_{L^p(\Omega)} \quad \forall u, \tilde{u} \in W^{1,p}(\Omega).$$
This implies
\[ |\Phi_n(u) - J(u)| \leq |\Phi_n(u) - \Phi_n(\tilde{u})| + |\Phi_n(\tilde{u}) - J(\tilde{u})| + |J(\tilde{u}) - J(u)| \]
\[ \leq C\|\nabla(u - \tilde{u})\|_{L^p(\Omega)} + |\Phi_n(\tilde{u}) - J(\tilde{u})|. \]
Using the density of \( C^2(\overline{\Omega}) \) in \( W^{1,p}(\Omega) \), we obtain \( (3.5) \). \( \Box \)

By choosing a special sequence \((\rho_n)\), we may greatly improve the conclusion of Proposition 9. More precisely, let \((\delta_n)\) be a positive sequence converging to 0 and define
\[ \rho_n(t) = \frac{(p + d)}{\delta_n^{p+d}} t^p \mathbf{1}_{(0, \delta_n)}(t). \] (3.17)

We have

**Proposition 10.** Let \( d \geq 1 \) and assume that either
\[ 1 < p \leq d/(d - 1) \quad \text{and} \quad d \geq 2, \]
or
\[ 1 < p < +\infty \quad \text{and} \quad d = 1, \]
and let \((\rho_n)\) be defined by (3.17). Then
\[ \Phi_n(u) \leq C \int_\Omega |\nabla u| \quad \forall n, \forall u \in L^1(\Omega), \] (3.18)
for some positive constant \( C \) depending only on \( d, p, \) and \( \Omega \), and
\[ \lim_{n \to +\infty} \Phi_n(u) = J(u) \quad \forall u \in W^{1,1}(\Omega). \] (3.19)
On the other hand, there exists some nontrivial \( v \in BV(\Omega) \) such that
\[ \lim_{n \to +\infty} \Phi_n(v) = \alpha_p J(v) \quad \text{with} \quad \alpha_p > 1. \] (3.20)

**Remark 2.** The restriction \( p \leq d/(d - 1) \) in the case \( d \geq 2 \) is quite natural if the goal is to prove (3.18) since the Sobolev embedding \( W^{1,1}(\Omega) \subset L^{d/(d-1)} \) is sharp. In fact, this requirement is necessary. Let \( d \geq 2 \), fix \( x_0 \in \Omega \), and assume that \( \text{diam}(\Omega) < 1/2 \) for notational ease. Set \( u(x) = |x - x_0|^{1-d} \ln^{-2} |x - x_0| \). One can verify that \( u \in W^{1,1}(\Omega) \) and for \( x \in \Omega \) with \( |x - x_0| < \delta_n/2 \)
\[ \int_{\Omega \setminus B_{\delta_n}} |u(x) - u(y)|^p \, dy = +\infty \]
since \( p > d/(d - 1) \). It follows that
\[ \gamma \int_\Omega |\nabla u(x)| \, dx < +\infty = \Phi_n(u) \quad \forall n. \]

**Remark 3.** We do not know whether it is possible to construct a sequence \((\rho_n)\) such that (3.19) holds for every \( u \in BV(\Omega) \). The problem is open even when \( d = 1 \).

The proof of Proposition 10 relies on the following inequality which is just a rescaled version of the standard Sobolev one. Let \( B_R \) be a ball of radius \( R \), then for any \( p \in [1, d/(d - 1)] \),
\[ (\int_{B_R} |u(y)| \, dy)^{1/p} \leq C R^\alpha \int_{B_R} |\nabla u(z)| \, dz \quad \forall u \in L^1(B_R), \] (3.21)
for some positive constant \( C \) depending only on \( d \) and \( p \), where \( \alpha := (d/p) + 1 - d \geq 0 \).
Proof of Proposition 10. Since \( \Phi_n(u) = \Phi_n(u + c) \) for any constant \( c \), without loss of generality, one may assume that \( \int_{\Omega} u = 0 \). Consider an extension of \( u \) to \( \mathbb{R}^d \) which is still denoted by \( u \) such that

\[
\|u\|_{W^{1,1}(\mathbb{R}^d)} \leq C_{\Omega} \|u\|_{W^{1,1}(\Omega)} \leq C_{\Omega} \|\nabla u\|_{L^1(\Omega)}. \tag{3.22}
\]

In view of (3.17), we have

\[
\Phi_n(u) \leq \frac{(p + d)^{1/p}}{\delta_n^{1+d/p}} \int_{\Omega} \left( \int_{B(x,\delta_n)} |u(x) - u(y)|^p dy \right)^{1/p} dx. \tag{3.23}
\]

We have, for \( y \in B(x, \delta_n) \),

\[
|u(x) - u(y)| \leq |u(x) - \int_{B(x, \delta_n)} u| + |u(y) - \int_{B(x, \delta_n)} u|. \tag{3.24}
\]

It follows from the triangle inequality that

\[
\left( \int_{B(x, \delta_n)} |u(x) - u(y)|^p dy \right)^{1/p} \leq C\delta_n^{d/p} \left| u(x) - \int_{B(x, \delta_n)} u \right| + C\delta_n^\alpha \int_{B(x, \delta_n)} |\nabla u(z)| dz. \tag{3.25}
\]

Here and in what follows in this proof, \( C \) denotes a positive constant depending only on \( d, p \), and \( \Omega \). Inserting (3.21) in (3.25) yields

\[
\left( \int_{B(x, \delta_n)} |u(x) - u(y)|^p dy \right)^{1/p} \leq C\delta_n^{d/p} \left| u(x) - \int_{B(x, \delta_n)} u \right| + C\delta_n^\alpha \int_{B(x, \delta_n)} |\nabla u(z)| dz. \tag{3.26}
\]

We claim that

\[
\int_{\Omega} \left| u(x) - \int_{B(x, \delta_n)} u \right| dx \leq C\delta_n \int_{\Omega} |\nabla u| \tag{3.27}
\]

and

\[
\int_{\Omega} dx \int_{B(x, \delta_n)} |\nabla u(z)| dz \leq C\delta_n^d \int_{\Omega} |\nabla u|. \tag{3.28}
\]

Indeed, we have, for \( R \) large enough,

\[
\int_{\Omega} \left| u(x) - \int_{B(x, \delta_n)} u \right| dx \leq C\delta_n^{-d} \int_{B_R} \int_{\{|x-y| < \delta_n\}} |u(x) - u(y)| dx dy
\]

\[
\leq C\delta_n \int_{B_R} |\nabla u| \leq C\delta_n \int_{\Omega} |\nabla u|,
\]

by the BBM formula applied to \( \rho_n(t) = (d + 1)\delta_n^{-(d+1)} t 1_{(0, \delta_n)} \) and by (3.22). On the other hand,

\[
\int_{\Omega} \int_{B(x, \delta_n)} |\nabla u(z)| dz dx \leq \int_{B_R} \int_{B_R} |\nabla u(z)| dz dx \leq C\delta_n^p \int_{\Omega} |\nabla u(x)| dx,
\]

by (3.22). Combining (3.26)–(3.28) yields

\[
\int_{\Omega} \left( \int_{B(x, \delta_n)} |u(x) - u(y)|^p dy \right)^{1/p} dx \leq C\delta_n^{1+d/p} \int_{\Omega} |\nabla u(z)| dz \tag{3.29}
\]

(recall that \( \alpha + d = 1 + d/p \)). It follows from (3.23) that

\[
\Phi_n(u) \leq C\|\nabla u\|_{L^1(\Omega)};
\]

which is (3.18).

Assertion (3.19) is deduced from (3.18) via a density argument as in the proof of Proposition 9.
Proof. It remains to prove (3.20). For simplicity, take $\Omega = (-1/2, 1/2)$ and consider $v(x) = 1_{(0,1/2)}(x)$. Then, for $n$ sufficiently large,
\[
\Phi_n(v) = 2 \frac{(p+1)^{1/p}}{\delta_n^{1/p}} \int_0^{\delta_n} \left( \int_0^{\delta_n-x} dy \right)^{1/p} dx = 2 \frac{(p+1)^{1/p}}{\delta_n^{1+1/p}} \int_0^{\delta_n} (\delta_n - x)^{1/p} dx
\]
\[
= \frac{2(p+1)^{1/p}}{\delta_n^{1+1/p}} \frac{\delta_n^{1+1/p}}{1+1/p} = \frac{2p}{(p+1)^{1-1/p}} > 2^{1/p} = J(v).
\]
Indeed, since $p + 1 < 2p$, it follows that $(p + 1)^{1-1/p} < (2p)^{1-1/p}$ and thus
\[
\frac{2p}{(p+1)^{1-1/p}} > (2p)^{1/p} > 2^{1/p}. \quad \Box
\]

3.2. More about the pointwise convergence of $(\Phi_n)$ when $d = 1$

In this section, we assume that $d = 1$ and $\Omega = (-1/2, 1/2)$.

**Proposition 11.** Assume that $(\rho_n)$ satisfies (1.1)–(1.3). Then, for every $q > 1$, we have
\[
\Phi_n(u) \leq C_q \|u'\|_{L_q(\Omega)} \quad \forall u \in W^{1,q}(\Omega),
\]
for some positive constant $C_q$ depending only on $q$. Moreover,
\[
\lim_{n \to +\infty} \Phi_n(u) = J(u) \quad \forall u \in \bigcup_{q>1} W^{1,q}(\Omega).
\]

**Proof.** Since $\Phi_n(u) = \Phi_n(u + c)$ for any constant $c$, without loss of generality, one may assume that $\int_{\Omega} u = 0$. Consider an extension of $u$ to $\mathbb{R}$ which is still denoted by $u$, such that
\[
\|u\|_{W^{1,q}(\mathbb{R})} \leq C_q \|u\|_{W^{1,q}(\Omega)} \leq C_q \|u'\|_{L_q(\Omega)}.
\]
Let $M(f)$ denote the maximal function of $f$ defined in $\mathbb{R}$, i.e.,
\[
M(f)(x) := \sup_{r>0} \int_{x-r}^{x+r} |f(s)| ds.
\]
From the definition of $\Phi_n$, we have
\[
\Phi_n(u) \leq C \int_{\Omega} \left( \int_{\Omega} |M(u')(x)|^p \rho_n(\|x-y\|) dy \right)^{1/p} dx \leq C \int_{\Omega} M(u')(x) dx.
\]
The first statement now follows from the fact that $\|M(f)\|_{L_q(\mathbb{R})} \leq C_q \|f\|_{L_q(\mathbb{R})}$ since $q > 1$. The second statement is derived from the first statement via a density argument as in the proof of Proposition 9. \quad \Box

Our next result shows that Proposition 11 is sharp and cannot be extended to $q = 1$ (for a general sequence $(\rho_n)$).

**Proposition 12.** For every $p > 1$, there exist a sequence $(\rho_n)$ satisfying (1.1)–(1.3) and some function $v \in W^{1,1}(\Omega)$ such that
\[
\Phi_n(v) = +\infty \quad \forall n.
\]

**Proof.** Fix $\alpha > 0$ and $\beta > 1$ such that
\[
\alpha + \beta/p < 1. \quad (3.30)
\]
Since \( p > 1 \) such \( \alpha \) and \( \beta \) exist. Let \( (\delta_n) \) be a sequence of positive numbers converging to 0 and consider

\[
\rho_n(t) := A_n \frac{1}{|t|^{\alpha} \ln t^{\beta} \mathbb{1}_{(0, \delta_n)}},
\]

Here \( A_n \) is chosen in such a way that \( (1.3) \) holds, i.e., \( A_n \int_0^{\delta_n} \frac{dt}{|t|^{\alpha} \ln t^{\beta}} = 1 \). Set

\[
v(x) = \begin{cases} 
0 & \text{if } -1/2 < x < 0, \\
|\ln x|^{-\alpha} & \text{if } 0 < x < 1/2.
\end{cases}
\]

Clearly, \( v \in W^{1,1}(\Omega) \). We have

\[
\Phi_n(v) = \int_{-1/2}^{1/2} \left( \int_{-1/2}^{1/2} \frac{|v(x) - v(y)|^p}{|x-y|^p} \rho_n(|x-y|) \, dy \right)^{1/p} \, dx \\
\geq \int_0^{\delta_n} A_n^{1/p} |v(x)| \left( \int_0^{\delta_n-x} \frac{1}{|x+y|^p} \rho_n(x+y) \, dy \right)^{1/p} \, dx.
\]

(3.31)

We have, for \( 0 < x < \delta_n/2 \),

\[
\int_0^{\delta_n-x} \frac{1}{|x+y|^p} \rho_n(x+y) \, dy \geq \int_x^{\delta_n} \frac{dt}{t^{p+1} |\ln t|^{\beta}} \geq \int_x^{2x} \frac{dt}{t^{p+1} |\ln t|^{\beta}} \geq \frac{C_{p,\beta}}{x^{p \beta}}
\]

and thus

\[
\left( \int_0^{\delta_n-x} \frac{1}{|x+y|^p} \rho_n(x+y) \, dy \right)^{1/p} \geq \frac{C_{p,\beta}}{x^{1\beta/p}}.
\]

(3.32)

Since, by (3.30),

\[
\int_0^{\delta_n/2} \frac{1}{x |\ln x|^{\beta/p+\alpha}} \, dx = +\infty,
\]

it follows from (3.31) and (3.32) that

\[
\Phi_n(v) = +\infty \quad \forall \, n. \quad \square
\]

**Remark 4.** D. Spector [26] has noticed that the sequence \((\rho_n)\) and the function \( v \) constructed by A. Ponce (presented in [17]) satisfy (1.1)–(1.3), \( v \in W^{1,1}(\Omega) \), \( \Phi_n(v) < +\infty \) for all \( n \), and \( \lim_{n \to +\infty} \Phi_n(v) = +\infty \). In our construction, the pathology is even more dramatic since \( \Phi_n(v) = +\infty \) for all \( n \).

### 3.3. More about the pointwise convergence of \( \Phi_n \) when \( d \geq 2 \)

In this section, we present two “improvements” of (3.5) concerning the (pointwise) convergence of \( \Phi_n(u) \) to \( J(u) \). In the first one (Proposition 13) \((\rho_n)\) is a general sequence (satisfying (1.1)–(1.3)), but the assumption on \( u \) is quite restrictive: \( u \in W^{1,q}(\Omega) \) with \( q > q_0 \) where \( q_0 \) is defined in (3.33). In the second one (Proposition 14) there is an additional assumption on \((\rho_n)\), but pointwise convergence holds for a large (more natural) class of \( u \)’s: \( u \in W^{1,q}(\Omega) \) with \( q > q_1 \) where \( q_1 < q_0 \) is defined in (3.44).

**Proposition 13.** Let \( p > 1 \) and assume that \((\rho_n)\) satisfies (1.1)–(1.3). Set

\[
q_0 := pd/(d+p-1),
\]

(3.33)

so that \( 1 < q_0 < p \). Then

\[
\Phi_n(u) \leq C \|\nabla u\|_{L^q} \quad \forall \, u \in W^{1,q}(\Omega) \text{ with } q > q_0,
\]

(3.34)
for some positive constant $C = C_{p,q,\Omega}$ depending only on $p$, $q$, and $\Omega$. Moreover,

$$\lim_{n \to +\infty} \Phi_n(u) = J(u) \quad \forall u \in W^{1,q}(\Omega) \text{ with } q > q_0.$$  \hspace{1cm} (3.35)

**Proof.** Since $\Phi_n(u) = \Phi_n(u+c)$ for any constant $c$, without loss of generality, one may assume that $\int_\Omega u = 0$. Consider an extension of $u$ to $\mathbb{R}^d$ which is still denoted by $u$, such that

$$\|u\|_{W^{1,q}(\mathbb{R}^d)} \leq C_q,\Omega \|u\|_{W^{1,q}(\Omega)} \leq C_q,\Omega \|\nabla u\|_{L^q(\Omega)}.$$  

For simplicity of notation, we assume that $\text{diam}(\Omega) \leq 1/2$. Then

$$\Phi_n(u) \leq \int_\Omega \left[ \int_0^1 \int_{S^{d-1}} |u(x + r\sigma) - u(x)|^p \rho_n(r)r^{d-1}drd\sigma \right]^{1/p} dx.$$  

We have

$$|u(x + r\sigma) - u(x)| \leq |u(x + r\sigma) - \int_{S^{d-1}} u(x + r\sigma')d\sigma'| + |u(x) - \int_{S^{d-1}} u(x + r\sigma')d\sigma'|$$

$$\leq \int_{S^{d-1}} |u(x + r\sigma) - u(x + r\sigma')|d\sigma' + \int_{S^{d-1}} |u(x) - u(x + r\sigma')|d\sigma'.$$

It follows that

$$\Phi_n(u) \lesssim T_1 + T_2,$$  \hspace{1cm} (3.36)

where

$$T_1 = \int_\Omega \left[ \int_0^1 \int_{S^{d-1}} \int_{S^{d-1}} |u(x + r\sigma) - u(x + r\sigma')|^p d\sigma' d\sigma \rho_n(r)r^{d-1-p} dr \right]^{1/p} dx$$

and

$$T_2 = \int_\Omega \left[ \int_0^1 \left( \int_{S^{d-1}} |u(x) - u(x + r\sigma')|d\sigma' \right)^p \rho_n(r)r^{d-1-p} dr \right]^{1/p} dx.$$  

In this proof the notation $a \lesssim b$ means that $a \leq Cb$ for some positive constant $C$ depending only on $p$, $q$, and $\Omega$.

We first estimate $T_1$. Let $B_1$ denotes the open unit ball of $\mathbb{R}^d$. By (3.33) we know that the trace mapping $u \mapsto u|_{\partial B_1}$ is continuous from $W^{1,q_0}(B_1)$ into $L^q(\partial B_1)$. It follows that

$$\int_{S^{d-1}} \int_{S^{d-1}} |u(x + r\sigma) - u(x + r\sigma')|^p d\sigma' d\sigma \lesssim \|\nabla u(x + r\cdot)|^p_{L^{q_0}(B_1)} \lesssim r^p M^{p/q_0}(\|\nabla u\|^{q_0})(x)$$

(recall that $M(f)$ denotes the maximal function of a function $f$ defined in $\mathbb{R}^d$). Using (1.2), we derive that

$$T_1 \lesssim \int_\Omega \left[ \int_0^1 M^{p/q_0}(\|\nabla u\|^{q_0})(x) \rho_n(r)r^{d-1}dr \right]^{1/p} dx \lesssim \int_\Omega M^{1/q_0}(\|\nabla u\|^{q_0})(x) dx.$$  \hspace{1cm} (3.37)

Since $q > q_0$, it follows from the theory of maximal functions that

$$\int_\Omega M^{1/q_0}(\|\nabla u\|^{q_0})(x) dx \lesssim \|\nabla u\|_{L^q(\Omega)}.$$  \hspace{1cm} (3.38)

Combining (3.37) and (3.38) yields

$$T_1 \lesssim \|\nabla u\|_{L^q(\Omega)}.$$  \hspace{1cm} (3.39)

We next estimate $T_2$. We have

$$\int_{S^{d-1}} |u(x) - u(x + r\sigma')|d\sigma' \leq \int_{S^{d-1}} \int_0^r |\nabla u(x + s\sigma')|dsd\sigma'.$$
Applying Lemma 1, we obtain, for $0 < r < 1$ and $x \in \Omega$,
\[ \int_{S^{d-1}} \int_0^r |\nabla u(x + s\sigma')| \, ds \, d\sigma' \leq CrM(|\nabla u|(x)). \tag{3.40} \]
We derive that
\[ T_2 \lesssim \int_{\Omega} M(|\nabla u|(x)) \, dx \lesssim \|\nabla u\|_{L^q} \tag{3.41} \]
by the theory of maximal functions since $q > 1$. Combining (3.36), (3.39) and (3.41) yields (3.44).

Assertion (3.35) follows from (3.44) via a density argument as in the proof of Proposition 9. \qed

In the proof of Proposition 13, we used the following elementary.

**Lemma 1.** Let $d \geq 1$, $r > 0$, $x \in \mathbb{R}^d$, and $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. We have
\[ \int_{S^{d-1}} \int_0^r |f(x + s\sigma)| \, ds \, d\sigma \leq C_d r M(f)(x), \tag{3.42} \]
for some positive constant $C_d$ depending only on $d$.

**Proof.** Set $\varphi(s) = \int_{S^{d-1}} |f(x + s\sigma)| \, d\sigma$, so that, by the definition of $M(f)(x)$, we have
\[ \int_{B_r(x)} |f(y)| \, dy \leq M(f)(x) \quad \forall \, r > 0, \]
and thus
\[ H(r) := \int_0^r \varphi(s) s^{d-1} \, ds \leq |B_1| r^d M(f)(x) \quad \forall \, r > 0. \tag{3.43} \]
Then $H'(r) = \varphi(r) r^{d-1}$, so that
\[ \int_0^r \varphi(s) \, ds = \int_0^r \frac{H'(s)}{s^{d-1}} \, ds = \frac{H(r)}{r^{d-1}} + (d-1) \int_0^r \frac{H(s)}{s^d} \, ds \leq C_d r M(f)(x), \]
by (3.43); which is precisely (3.42). (The integration by parts can be easily justified by approximation.) \qed

Under the assumption that $\rho_n$ is non-increasing for every $n$, one can replace the condition $q > q_0$ in Proposition 13 by the weaker condition $q > q_1$, where
\[ q_1 := \max\{pd/(p + d), 1\}, \tag{3.44} \]
so that $1 \leq q_1 < q_0$. It is worth noting that the embedding $W^{1,q_1}(\Omega) \subset L^p(\Omega)$ is sharp and therefore $q_1$ is a natural lower bound for $q$ (see Remark 2). In fact, we prove a slightly more general result:

**Proposition 14.** Let $p > 1$ and assume that $(\rho_n)$ satisfies (1.1)–(1.3). Suppose in addition that there exist $\Lambda > 0$ and a sequence of non-increasing functions $(\hat{\rho}_n) \subset L^1_{\text{loc}}(0, +\infty)$ such that
\[ \rho_n \leq \hat{\rho}_n \quad \text{and} \quad \int_0^\infty \hat{\rho}_n(t)t^{d-1} \, dt \leq \Lambda \quad \forall \, n. \tag{3.45} \]
Then
\[ \Phi_n(u) \leq C \|\nabla u\|_{L^q} \quad \forall \, u \in W^{1,q}(\Omega) \text{ with } q > q_1, \tag{3.46} \]
for some positive constant $C = C(p, q, \Lambda, \Omega)$ depending only on $p, q, \Lambda,$ and $\Omega$. Moreover,
\[ \lim_{n \to +\infty} \Phi_n(u) = J(u) \quad \forall \, u \in W^{1,q}(\Omega) \text{ with } q > q_1. \tag{3.47} \]
Remark 5. We do not know whether the conclusions of Proposition 14 hold without assuming the existence of $\Lambda$ and $(\hat{\rho}_n)$. Equivalently, we do not know whether the conclusions of Proposition 13 hold under the weaker condition $q > q_1$.

Proof. For simplicity of notation, we assume that $\rho_n$ is non-increasing for all $n$ and work directly with $\rho_n$ instead of $\hat{\rho}_n$. We first prove (3.46). As in the proof of Proposition 13, one may assume that $\int_{\Omega} u = 0$. Consider an extension of $u$ to $\mathbb{R}^d$ which is still denoted by $u$ such that

$$
\|u\|_{W^{1,q}(\mathbb{R}^d)} \leq C_q,\Omega \|u\|_{W^{1,q}(\Omega)} \leq C_{q,\Omega} \|\nabla u\|_{L^q(\Omega)}.
$$

For simplicity of notation, we also assume that $\text{diam}(\Omega) \leq 1/2$. Then

$$
\Phi_n(u) \leq \int_{\Omega} \left[ \int_{S^{d-1}} \int_0^1 \frac{|u(x + r\sigma) - u(x)|^p}{r^p} \rho_n(r)r^{d-1} \, dr \, d\sigma \right]^{1/p} \, dx.
$$

We claim that for a.e. $x \in \Omega$,

$$
Z(x) = \left[ \int_{S^{d-1}} \int_0^1 \frac{|u(x + r\sigma) - u(x)|^p}{r^p} \rho_n(r)r^{d-1} \, dr \, d\sigma \right]^{1/p} \leq CM^{1/q_1}(\|\nabla u\|_{q_1})(x). \tag{3.48}
$$

Here and in what follows, $C$ denotes a positive constant depending only on $p, d,$ and $\Lambda$.

From (3.48), we deduce (3.46) via the theory of maximal functions since $q > q_1$. Assertion (3.47) follows from (3.46) by density as in the proof of Proposition 9.

It remains to prove (3.48). Without loss of generality we establish (3.48) for $x = 0$. The proof relies heavily on two inequalities valid for all $R > 0$:

$$
\left[ \int_{B_R} \left| u(\xi) - \int_{B_R} u \right|^p d\xi \right]^{1/p} \leq CRM^{1/q_1}(\|\nabla u\|_{q_1})(0) \tag{3.49}
$$

and

$$
\int_{B_R} |u(\xi) - u(0)| \, d\xi \leq CRM^{1/q_1}(\|\nabla u\|_{q_1})(0), \tag{3.50}
$$

where $B_R = B_R(0)$.

Inequality (3.49) is simply a rescaled version of the Sobolev inequality

$$
\left\| u - \int_{B_1} u \right\|_{L^p(B_1)} \leq C \|\nabla u\|_{L^{q_1}(B_1)},
$$

which implies that

$$
\left[ \int_{B_R} \left| u(\xi) - \int_{B_R} u \right|^p d\xi \right]^{1/p} \leq CR \left[ \int_{B_R} |\nabla u|_{q_1} \right]^{1/q_1} \leq CRM^{1/q_1}(\|\nabla u\|_{q_1})(0).
$$

To prove (3.50), we write

$$
\int_{B_R} |u(\xi) - u(0)| \, d\xi = \int_0^R \int_{S^{d-1}} |u(r\sigma) - u(0)|r^{d-1} \, dr \, d\sigma
$$

$$
\leq C \int_0^R r^{d-1} \int_{S^{d-1}} \int_0^r |\nabla u(s\sigma)| \, ds \, d\sigma
$$

$$
\leq C \int_0^R r^{d}M(\|\nabla u\|)(0) \quad \text{by Lemma 1}.
$$

Thus

$$
\int_{B_R} |u(\xi) - u(0)| \, d\xi \leq CRM(\|\nabla u\|)(0) \leq CRM^{1/q_1}(\|\nabla u\|_{q_1})(0).
$$
From (3.48), we obtain

$$Z(0)^p = \sum_{i=0}^{\infty} \int_{S^{d-1}} \int_{2^{-(i+1)}} |u(r\sigma) - u(0)|^p \rho_n(r)d^{d-1}r \ d\sigma,$$

so that

$$Z(0)^p \leq C \sum_{i=0}^{\infty} \rho_n(2^{-i}) 2^{ip} \int_{S^{d-1}} \int_{2^{-(i+1)}} |u(r\sigma) - u(0)|^p r \ d\sigma \ dr \ d\sigma.$$  (3.51)

We have

$$|u(r\sigma) - u(0)| \leq |u(r\sigma) - \int_{B_{2^{-i}}} u| + \int_{B_{2^{-i}}} u - u(0)|.$$  (3.52)

Inserting (3.52) into (3.51) yields

$$Z(0)^p \leq C \sum_{i=0}^{\infty} (U_i + V_i),$$

where

$$U_i = \rho_n(2^{-(i+1)}) 2^{ip} \int_{S^{d-1}} \int_{2^{-(i+1)}} |u(r\sigma) - \int_{B_{2^{-i}}} u|^p r \ d\sigma \ dr \ d\sigma$$

and

$$V_i = \rho_n(2^{-(i+1)}) 2^{ip} \int_{S^{d-1}} \int_{2^{-(i+1)}} \int_{B_{2^{-i}}} u - u(0)|^p r \ d\sigma \ dr \ d\sigma.$$  (3.53)

Clearly,

$$U_i \leq \rho_n(2^{-(i+1)}) 2^{ip} \int_{S^{d-1}} \int_{0}^{2^{-i}} |u(r\sigma) - \int_{B_{2^{-i}}} u|^p r \ d\sigma \ dr \ d\sigma \leq \rho_n(2^{-(i+1)}) 2^{-id} A,$$  (3.54)

by (3.49), where $A = M^{p/q}(|\nabla u|^q)(0)$. On the other hand,

$$V_i \leq \rho_n(2^{-(i+1)}) 2^{ip} \left[ \int_{B_{2^{-i}}} |u(\xi) - u(0)| \ d\xi \right]^p 2^{-id} \leq C \rho_n(2^{-(i+1)}) 2^{-id} A \ by \ (3.50).$$  (3.55)

Combining (3.53)–(3.55), we obtain

$$Z(0)^p \leq C \sum_{i=0}^{\infty} \rho_n(2^{-(i+1)}) 2^{-id} A.$$  (3.56)

Finally, we observe that

$$\int_0^1 \rho_n(r) r^{d-1} dr \geq \sum_{i=0}^{\infty} \int_{2^{-(i+2)}}^{2^{-(i+1)}} \rho_n(r)r^{d-1} dr \geq C \sum_{i=0}^{\infty} \rho_n(2^{-(i+1)}) 2^{-id}$$

and thus

$$Z(0)^p \leq CM^{p/q}(|\nabla u|^q)(0) \int_0^1 \rho_n(r)r^{d-1} dr \leq CM^{p/q}(|\nabla u|^q)(0). \ \ \square$$

**Remark 6.** Assumption (3.45) holds e.g. for the sequence $(\rho_n)$ defined in (3.17), i.e.,

$$\rho_n(t) = \frac{p + d}{\delta_n^{p+d}} t_p \mathbb{1}_{(0,\delta_n)}(t).$$
Indeed, we may choose

\[ \hat{\rho}_n(t) = \frac{p + d}{\delta_n^\alpha} 1_{(0, \delta_n)}(t). \]

Applying Proposition 14 we recover Proposition 10 since \( q_1 = 1 \) (note that \( pd \leq p + d \) when \( d = 1 \) and also when \( d \geq 2 \) provided that \( p \leq d/(d - 1) \)). Note, however that in Proposition 14 we must take \( q > q_1 = 1 \), while \( q = 1 \) was allowed in Proposition 10. This discrepancy is related to our next remark.

**Remark 7.** Assume that \( d \geq 2 \) and \( 1 < p \leq d/(d - 1) \), so that \( q_1 = 1 \). The conclusion of Proposition 14 fails in the borderline case \( q = q_1 = 1 \). More precisely, for every \( p \in (1, d/(d - 1)] \), there exist a sequence \( (\rho_n) \) satisfying (1.1)–(1.3) and (3.45), and a function \( v \in W^{1,1}(\Omega) \) such that \( \Phi_n(v) = +\infty \) for all \( n \). The construction is similar to the one presented in the proof of Proposition 10. Indeed, let \( \Omega = B_{1/2}(0) \). Fix \( \alpha > 0 \) and \( \beta > 1 \) such that

\[ \alpha + \beta/p < 1. \] (3.56)

Since \( p > 1 \) such \( \alpha \) and \( \beta \) exist. Let \( (\delta_n) \) be a sequence of positive numbers converging to 0 and consider

\[ \rho_n(t) := A_n \frac{1}{t^\alpha |\ln t|^\beta} 1_{(0, \delta_n)}. \]

Note that the functions \( t \mapsto \rho_n(t) \) are non-increasing. Here \( A_n \) is chosen in such a way that (1.3) holds, i.e., \( A_n \int_0^{\delta_n} \frac{dt}{t^\alpha |\ln t|^\beta} = 1 \). Set

\[ V(x) = v(x_1) := \begin{cases} 0 & \text{if } -1/2 < x_1 < 0, \\ |\ln x_1|^{-\alpha} & \text{if } 0 < x_1 < 1/2. \end{cases} \]

Clearly, \( V \in W^{1,1}(\Omega) \). We have

\[ \Phi_n(V) = \int_\Omega \left( \int_\Omega \frac{|V(x) - V(y)|^p}{|x - y|^p} \rho_n(|x - y|) dy \right)^{1/p} dx \]

\[ \geq \int_{B_{1/4}(0)} A_n^{1/p} \left( \int_{0 < x_1 < \delta_n/4} \frac{|v(x_1) - v(y_1)|^p}{|x - y|^p + |\ln |x - y||^\beta} dy \right)^{1/p} dx. \]

Note that, for \( 0 < x_1 < \delta_n/4 \),

\[ \int_{|y - x| \leq \delta_n/4} \frac{|v(x_1) - v(y_1)|^p dy |x - y|^{-p'd}}{|\ln |x - y||^\beta} \geq \int_{|y - x| \leq \delta_n/4} \frac{|v(x_1) - v(y_1)|^p dy'}{|x_1 - y_1|^p + |\ln |x_1 - y_1||^\beta}. \]

We derive as in the proof of Proposition 12 that

\[ \int_{|y - x| \leq \delta_n} \frac{|v(x_1) - v(y_1)|^p dy |x - y|^{-p'd}}{|\ln |x - y||^\beta} \geq \frac{v(x_1)^p}{x_1^{p'd} |\ln |x_1||^\beta}. \]

It follows that

\[ \Phi_n(V) \geq \int_{0 < x_1 < \delta_n/4} A_n^{1/p} \frac{v(x_1)}{x_1 |\ln x_1|^{\beta/p}} dx = \int_{0 < x_1 < \delta_n/4} A_n^{1/p} \frac{1}{x_1 |\ln x_1|^{\alpha + \beta/p}} dx = +\infty, \]

(by (3.56)).

**Remark 8.** Assume that \( d \geq 2 \) and \( p > d/(d - 1) \), so that \( q_1 = pd/(p + d) > 1 \). It is not known whether the conclusions of Proposition 14 hold in the borderline case \( q = q_1 \). More precisely, assume that
\( d \geq 2, p > d/(d-1) \), and that \((\rho_n)\) satisfying (1.1)–(1.3) and (3.45). Is it true that \( \lim_{n \to +\infty} \Phi_n(u) = J(u) \) for all \( u \in W^{1,q_1}(\Omega) \)? Take for example \( d = 2 \) and \( p = 3 \) so that \( q_1 = 6/5 \).

**Remark 9.** The technique we use in the proof of Proposition 14 is somewhat similar to the one used by D. Spector [26] (see e.g. the proof of his Theorem 1.8). However, the results are quite different in nature.

### 3.4. \( \Gamma \)-convergence

Concerning the \( \Gamma \)-convergence of \( \Phi_n \), G. Leoni and D. Spector proved in [16].

**Proposition 15.** For every \( p > 1 \) we have

\[
\Phi_n \rightharpoonup_{\Gamma} \Phi_0(\cdot) := \gamma \int_{\Omega} |\nabla \cdot | \quad \text{in} \quad L^1(\Omega),
\]

where \( \gamma \) is given in (3.3).

Their proof is quite involved. Here is a simpler proof.

**Proof.** For \( D \) an open subset of \( \Omega \) such that \( \bar{D} \subset \Omega \), set

\[
\Phi_n(u, D) = \int_D dx \left[ \int_D \frac{|u(x) - u(y)|^p}{|x - y|^p} \rho_n(|x - y|) dy \right]^{1/p} \quad \text{for} \quad u \in L^1(D).
\]

Let \( u \in L^1(\Omega) \) and \( (u_n) \subset L^1(\Omega) \) be such that \( u_n \to u \) in \( L^1(\Omega) \). We must prove that

\[
\liminf_{n \to \infty} \Phi_n(u_n) \geq \gamma \int_{\Omega} |\nabla u|.
\]

Let \( (\mu_k) \) be a sequence of smooth mollifiers such that \( \text{supp} \mu_k \subset B_{1/k} \). Let \( D \) be a smooth open subset of \( \Omega \) such that \( \bar{D} \subset \Omega \) and fix \( k_0 \) such that \( D + B_{1/k_0} \subset \Omega \). We have as in (2.29), for \( k \geq k_0 \),

\[
\Phi_n(\mu_k * u_n, D) \leq \Phi_n(u_n). \quad \text{(3.57)}
\]

Using the fact that

\[
|\Phi_n(u, D) - \Phi_n(v, D)| \leq C_D \|u - v\|_{W^{1,\infty}(D)} \quad \forall u, v \in W^{1,\infty}(D),
\]

we obtain

\[
|\Phi_n(\mu_k * u_n, D) - \Phi_n(\mu_k * u, D)| \leq C_{k,D} \|u_n - u\|_{L^1(\Omega)}.
\]

Hence

\[
\Phi_n(\mu_k * u, D) \leq \Phi_n(\mu_k * u_n, D) + C_{k,D} \|u_n - u\|_{L^1(\Omega)}. \quad \text{(3.58)}
\]

Combining (3.57) and (3.58) yields

\[
\gamma \int_D |\nabla (\mu_k * u)| \leq \liminf_{n \to +\infty} \Phi_n(u_n).
\]

Letting \( k \to \infty \), we reach

\[
\gamma \int_{\Omega} |\nabla u| \leq \liminf_{n \to +\infty} \Phi_n(u_n).
\]

Since \( D \subset \subset \Omega \) is arbitrary, we derive that

\[
\gamma \int_{\Omega} |\nabla u| \leq \liminf_{n \to +\infty} \Phi_n(u_n).
\]
We next fix \( u \in BV(\Omega) \) and construct a sequence \((u_n)\) converging to \( u \) in \( L^1(\Omega) \) such that
\[
\limsup_{n \to +\infty} \Phi_n(u_n) \leq \gamma \int_\Omega |\nabla u|.
\]
Let \( v_k \in C^1(\bar{\Omega}) \) be such that
\[
v_k \to u \quad \text{in} \quad L^1(\Omega) \quad \text{and} \quad \int_\Omega |\nabla v_k| \to \int_\Omega |\nabla u|.
\] (3.59)
For each \( k \), let \( n_k \) be such that
\[
\|\Phi_n(v_k) - \gamma \int_\Omega |\nabla v_k|\| \leq 1/k \quad \text{if} \quad n > n_k.
\] (3.60)
Without loss of generality, one may assume that \((n_k)\) is an increasing sequence with respect to \( k \). Define
\[
u_n = v_k \quad \text{if} \quad n_k < n \leq n_{k+1}.
\]
We derive from (3.59) and (3.60) that
\[
u_n \to u \quad \text{in} \quad L^1(\Omega) \quad \text{and} \quad \lim_{n \to +\infty} \Phi_n(u_n) = \gamma \int_\Omega |\nabla u|.
\]
The proof is complete. \( \Box \)

### 3.5. Functionals with roots in image processing

Set
\[
\hat{E}_n(u) := \int_\Omega |u - f|^q + \Phi_n(u),
\]
and
\[
\hat{E}_0(u) := \int_\Omega |u - f|^q + \gamma \int_\Omega |\nabla u|,
\]
where \( q > 1 \) and \( f \in L^q(\Omega) \) is a given function. Motivated by Image Processing, we study variational problems related to \( \hat{E}_n \). More precisely, we establish

**Proposition 16.** For every \( n \), there exists a unique \( u_n \in L^q(\Omega) \) such that
\[
\hat{E}_n(u_n) = \min_{u \in L^q(\Omega)} \hat{E}_n(u).
\]
Let \( u_0 \) be the unique minimizer of \( \hat{E}_0 \). We have, as \( n \to +\infty \),
\[
u_n \to u_0 \quad \text{in} \quad L^q(\Omega)
\]
and
\[
\hat{E}_n(u_n) \to \hat{E}_0(u_0).
\]

**Proof.** The proof is similar to the one of **Proposition 6**. The details are left to the reader. \( \Box \)

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