



Mathematical analysis/Partial differential equations

Non-convex, non-local functionals converging to the total variation



Convergence de fonctionnelles non convexes et non locales vers la variation totale

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ABSTRACT

We present new results concerning the approximation of the total variation, $\int_{\Omega} |\nabla u|$, of a function u by non-local, non-convex functionals of the form

$$\Lambda_{\delta}(u) = \int_{\Omega} \int_{\Omega} \frac{\delta \varphi(|u(x) - u(y)|/\delta)}{|x - y|^{d+1}} dx dy,$$

as $\delta \rightarrow 0$, where Ω is a domain in \mathbb{R}^d and $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ is a non-decreasing function satisfying some appropriate conditions. The mode of convergence is extremely delicate, and numerous problems remain open. The original motivation of our work comes from Image Processing.

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RÉSUMÉ

Nous présentons des résultats nouveaux concernant l'approximation de la variation totale $\int_{\Omega} |\nabla u|$ d'une fonction u par des fonctionnelles non convexes et non locales de la forme

$$\Lambda_{\delta}(u) = \int_{\Omega} \int_{\Omega} \frac{\delta \varphi(|u(x) - u(y)|/\delta)}{|x - y|^{d+1}} dx dy,$$

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quand $\delta \rightarrow 0$, où Ω est un domaine de \mathbb{R}^d et $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ est une fonction croissante vérifiant certaines hypothèses. Le mode de convergence est extrêmement délicat et de nombreux problèmes restent ouverts. La motivation provient du traitement d'images.

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1. Introduction

Let $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ be non-decreasing, and continuous on $[0, +\infty)$ except at a finite number of points in $(0, +\infty)$. Assume that $\varphi(0) = 0$ and that $\varphi(t) = \varphi(t_-)$ for all $t > 0$. Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain of \mathbb{R}^d . Given a measurable function u on Ω , and $\delta > 0$, we define the following non-local functionals:

$$\Lambda(u) := \int_{\Omega} \int_{\Omega} \frac{\varphi(|u(x) - u(y)|)}{|x - y|^{d+1}} dx dy \leq +\infty \quad \text{and} \quad \Lambda_{\delta}(u) := \delta \Lambda(u/\delta).$$

We make the following three basic assumptions on φ :

$$\varphi(t) \leq at^2 \text{ in } [0, 1] \text{ for some positive constant } a, \tag{1}$$

$$\varphi(t) \leq b \text{ in } \mathbb{R}_+ \text{ for some positive constant } b, \tag{2}$$

and

$$\gamma_d \int_0^{\infty} \varphi(t)t^{-2} dt = 1, \text{ where } \gamma_d := 2|B^{d-1}|; \tag{3}$$

here B^{d-1} denotes the unit ball in \mathbb{R}^{d-1} and $|B^{d-1}|$ denotes its $(d - 1)$ -Hausdorff measure (with $\gamma_d = 2$ when $d = 1$). Condition (3) is a normalization condition prescribed in order to have (7) below with constant 1 in front of $\int_{\Omega} |\nabla u|$. Denote

$$\mathbf{A} = \{\varphi; \varphi \text{ satisfies (1)–(3)}\}. \tag{4}$$

Note that Λ is **never convex** when $\varphi \in \mathbf{A}$.

Here are three examples of functions φ that we have in mind. They all satisfy (1) and (2). In order to achieve (3), we choose $\varphi = c_i \tilde{\varphi}_i$, where $\tilde{\varphi}_i$ is taken from the list below and c_i is an appropriate constant:

$$\tilde{\varphi}_1(t) = \begin{cases} 0 & \text{if } t \leq 1 \\ 1 & \text{if } t > 1, \end{cases} \quad \tilde{\varphi}_2(t) = \begin{cases} t^2 & \text{if } t \leq 1 \\ 1 & \text{if } t > 1, \end{cases} \quad \text{and} \quad \tilde{\varphi}_3(t) = 1 - e^{-t^2}.$$

Example 1 is extensively studied in [3,6,10–14] (see also [5,15]). Examples 2 and 3 are motivated by Image Processing (see [8,17]).

In this note, we are concerned with modes of convergence of Λ_{δ} to the total variation as $\delta \rightarrow 0$. The convergence to the total variation of a sequence of **convex** non-local functionals J_{ε} , defined by

$$J_{\varepsilon}(u) = \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|}{|x - y|} \rho_{\varepsilon}(|x - y|) dx dy, \tag{5}$$

where ρ_{ε} is a sequence of radial mollifiers, was originally analyzed by J. Bourgain, H. Brezis and P. Mironescu and thoroughly investigated in [1,2,4,9].

The asymptotic analysis of Λ_{δ} is **much more delicate** than the one of J_{ε} , because two basic properties satisfied by J_{ε} (which played an important role in [1]) are **not** fulfilled by Λ_{δ} :

i) there is **no** constant C such that

$$\Lambda_{\delta}(u) \leq C \int_{\Omega} |\nabla u| \quad \forall u \in C^1(\bar{\Omega}), \forall \delta > 0, \tag{6}$$

ii) $\Lambda_{\delta}(u)$ is **not** a convex functional.

2. Statement of the main results

Concerning the pointwise limit of Λ_δ as $\delta \rightarrow 0$, i.e. the convergence of $\Lambda_\delta(u)$ for fixed u , we prove that, for every $\varphi \in \mathbf{A}$,

$$\Lambda_\delta(u) \text{ converges, as } \delta \rightarrow 0, \text{ to } TV(u) = \int_{\Omega} |\nabla u| \quad \forall u \in \bigcup_{p>1} W^{1,p}(\Omega). \quad (7)$$

If $u \in W^{1,1}(\Omega)$, we can only assert that, for every $\varphi \in \mathbf{A}$,

$$\liminf_{\delta \rightarrow 0} \Lambda_\delta(u) \geq \int_{\Omega} |\nabla u|.$$

Surprisingly, for every $d \geq 1$ and for every $\varphi \in \mathbf{A}$, one can construct a function $u \in W^{1,1}(\Omega)$ such that

$$\lim_{\delta \rightarrow 0} \Lambda_\delta(u) = +\infty.$$

This kind of “pathology” was originally discovered by A. Ponce and presented in [10] for $\varphi = c_1 \tilde{\varphi}_1$ (for another example, see [7]). One may also construct (see [7]) functions $u \in W^{1,1}(\Omega)$ such that

$$\liminf_{\delta \rightarrow 0} \Lambda_\delta(u) = \int_{\Omega} |\nabla u| \quad \text{and} \quad \limsup_{\delta \rightarrow 0} \Lambda_\delta(u) = +\infty.$$

When dealing with functions $u \in BV(\Omega)$, the situation becomes even more intricate. It may happen, for some $\varphi \in \mathbf{A}$ and some $u \in BV(\Omega)$, that

$$\liminf_{\delta \rightarrow 0} \Lambda_\delta(u) < \int_{\Omega} |\nabla u|.$$

All these facts suggest that the mode of convergence of Λ_δ to TV as $\delta \rightarrow 0$ is delicate and that a theory of pointwise convergence is out of reach. It turns out that Γ -convergence (in the sense of E. De Giorgi) is the appropriate framework to analyze the asymptotic behavior of Λ_δ as $\delta \rightarrow 0$.

Our main result is the following.

Theorem 1. *For every $\varphi \in \mathbf{A}$, there exists a constant $K = K(\varphi) \in (0, 1]$, which is independent of Ω , such that, as $\delta \rightarrow 0$,*

$$\Lambda_\delta \text{ } \Gamma\text{-converges to } \Lambda_0 \text{ in } L^1(\Omega), \quad (8)$$

where Λ_0 is defined on $L^1(\Omega)$ by

$$\Lambda_0(u) = K \int_{\Omega} |\nabla u| \text{ for } u \in BV(\Omega), \text{ and } +\infty \text{ otherwise.}$$

The proof of [Theorem 1](#) is extremely involved and it would be desirable to simplify it. When $\varphi = c_1 \tilde{\varphi}_1$ and $\Omega = \mathbb{R}^d$, [Theorem 1](#) is originally due to H.-M. Nguyen [11,13]. One of the key ingredients was the following earlier result, basically due to J. Bourgain and H.-M. Nguyen [3, Lemma 2].

Lemma 1. *Let $\Omega = (0, 1)$, $\varphi = c_1 \tilde{\varphi}_1$. There exists a constant $k > 0$ such that*

$$\liminf_{\delta \rightarrow 0} \Lambda_\delta(u) \geq k|u(t_2) - u(t_1)|,$$

for every $u \in L^1(\Omega)$, and for all Lebesgue points $t_1, t_2 \in (0, 1)$ of u .

Furthermore, one can show that

$$\inf_{\varphi \in \mathbf{A}} K(\varphi) > 0.$$

One of the most intriguing remaining questions is

Open Problem 1. *Is it true that for every $\varphi \in \mathbf{A}$, $K(\varphi) < 1$ in [Theorem 1](#)?*

It has been proved in [11] (see also [7]) that $K(c_1\tilde{\varphi}_1) < 1$. However, the answer to Open Problem 1 is **not** known for $\varphi = c_2\tilde{\varphi}_2$ and $\varphi = c_3\tilde{\varphi}_3$, even when $d = 1$.

Motivated by questions arising in Image Processing (see, e.g., [7,8,16,17]), we consider the problem

$$\inf_{u \in L^q(\Omega)} E_\delta(u), \tag{9}$$

where

$$E_\delta(u) = \lambda \int_{\Omega} |u - f|^q + \Lambda_\delta(u), \tag{10}$$

$q \geq 1$, $f \in L^q(\Omega)$ is given, and λ is a fixed positive constant. Our goal is twofold: investigate the existence of minimizers for E_δ (for fixed δ) and analyze their behavior as $\delta \rightarrow 0$. The existence of a minimizer in (9) is not obvious since Λ_δ is **not convex** and one cannot invoke the standard tools of Functional Analysis. Our main result in this direction is the following.

Theorem 2. Assume that $\varphi \in \mathbf{A}$ and $\varphi(t) > 0$ for all $t > 0$. Let $q \geq 1$ and $f \in L^q(\Omega)$. For each $\delta > 0$, there exists a minimizer u_δ of (9). Moreover, $u_\delta \rightarrow u_0$ in $L^q(\Omega)$ as $\delta \rightarrow 0$, where u_0 is the unique minimizer of the functional E_0 defined on $L^q(\Omega) \cap BV(\Omega)$ by

$$E_0(u) := \lambda \int_{\Omega} |u - f|^q + K \int_{\Omega} |\nabla u|,$$

and $0 < K \leq 1$ is the constant coming from Theorem 1.

Note that the minimizers u_δ of (9) need not be unique, but the convergence assertion in Theorem 2 holds for any choice of minimizers. The proof of the existence of a minimizer for (9) relies on the following compactness lemma for **fixed** δ , e.g., with $\delta = 1$.

Lemma 2. Let $\varphi \in \mathbf{A}$ be such that $\varphi(t) > 0$ for all $t > 0$, and let (u_n) be a bounded sequence in $L^1(\Omega)$ such that

$$\sup_n \Lambda(u_n) < +\infty. \tag{11}$$

There exists a subsequence (u_{n_k}) of (u_n) and $u \in L^1(\Omega)$ such that (u_{n_k}) converges to u in $L^1(\Omega)$.

The proof of the convergence as $\delta \rightarrow 0$ in Theorem 2 relies heavily on the Γ -convergence of Λ_δ (Theorem 1), and also on the following compactness lemma (with roots in H.-M. Nguyen [14]).

Lemma 3. Let $\varphi \in \mathbf{A}$, $(\delta_n) \rightarrow 0$, and let (u_n) be a bounded sequence in $L^1(\Omega)$ such that

$$\sup_n \Lambda_{\delta_n}(u_n) < +\infty. \tag{12}$$

There exists a subsequence (u_{n_k}) of (u_n) and $u \in L^1(\Omega)$ such that (u_{n_k}) converges to u in $L^1(\Omega)$.

The proofs of the results announced in this note are given in [7].

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