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PERIODIC SOLUTIONS OF THE FORCED RELATIVISTIC PENDULUM

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Abstract. The existence of at least one classical T-periodic solution is proved for differential equations of the form

$$(\phi(u'))' - g(x,u) = h(x)$$

when $\phi : (-a,a) \to \mathbb{R}$ is an increasing homeomorphism, g is a Carathéodory function T-periodic with respect to x, 2π-periodic with respect to u, of mean value zero with respect to u, and $h \in L^1_{loc}(R)$ is T-periodic and has mean value zero. The problem is reduced to finding a minimum for the corresponding action integral over a closed convex subset of the space of T-periodic Lipschitz functions, and then to show, using variational inequalities techniques, that such a minimum solves the differential equation. A special case is the "relativistic forced pendulum equation"

$$\Big(\frac{u'}{\sqrt{1-u'^2}}\Big)' + A\sin u = h(x).$$

1. Introduction

The first global result for the existence of periodic solutions of the forced pendulum equation started with the rigorous mathematical study of the equation

$$u'' + A\sin u = h(x) \tag{1.1}$$

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initiated in 1922 by Hamel, in a paper of the special issue of the *Mathematische Annalen* dedicated to Hilbert's sixtieth birthday anniversary [9]. Hamel proved the existence of a 2π-periodic solution of equation (1.1) with $h(x) = B \sin x$, by showing that the corresponding action integral

$$\mathcal{A}(u) := \int_0^{2\pi} \Big[\frac{u'(x)^2}{2} + A\cos u(x) + Bu(x)\sin x\Big]\, dx$$

has a minimum over the space of 2π-periodic C^1-functions, and his argument easily extends to the case where $B \sin x$ is replaced by a continuous 2π-periodic function $h(x)$ with mean value $\overline{h}$ equal to zero.

In the late nineteen seventies, Fučik wrote in his monograph [8] that "the description of the set $\mathcal{P}$ of h for which equation (1.1) has a 2π-periodic solution seems to remain a *terra incognita*." Motivated by this remark, but also unaware of the existence of Hamel's paper, Castro [6] (for $|A| \leq 1$), Dancer [7] and Willem [16], independently (for arbitrary A), reintroduced in the early nineteen eighties the use of the direct method of the calculus of variations, in the setting of Sobolev spaces. One can consult [11] for a survey and a bibliography of the recent developments in this direction.

On the other hand, periodic solutions of differential equations of the form

$$(\phi(u'))' = f(x, u, u'),$$

with $\phi : (-a, a) \to \mathbb{R}$ an increasing homeomorphism satisfying $\phi(0) = 0$, have been recently studied by in [3, 4], using a fixed-point reduction and Leray-Schauder degree. The motivation came from the special case where $\phi(s) = \frac{s}{\sqrt{1-s^2}}$, which occurs in the dynamics of special relativity. Using the Schauder fixed-point theorem, Torres [14, 15] has recently proved, for the relativistic pendulum equation with continuous T-periodic forcing h and arbitrary dissipation f

$$(\phi(u'))' + f(u)u' + A\sin u = h(x),$$

the existence of at least two T-periodic solution when

$$aT < 2\sqrt{3} \quad \textit{and} \qquad |\overline{h}| < A\Big(1 - \frac{aT}{2\sqrt{3}}\Big),$$

and of at least one T-periodic solution when

$$aT = 2\sqrt{3} \quad \textit{and} \quad \overline{h} = 0,$$

where $\overline{h}$ denotes the mean value of h over $[0, T]$. Those assumptions have been respectively improved in [2] to

$$aT < \pi\sqrt{3} \quad and \quad |\overline{h}| < A\cos\Big(\frac{aT}{2\sqrt{3}}\Big),$$

and

$$aT = \pi\sqrt{3} \quad and \quad \overline{h} = 0,$$

using Leray-Schauder degree arguments. It is also proved in [2], using lower and upper solutions, that at least two T-periodic solutions exist when $||h||_\infty < A$, and at least one when $\|h\|_\infty = A$.

The aim of this paper is to use the direct method of the calculus of variations to prove, for equations of the type

$$(\phi(u'))' + A\sin u = h(x), \tag{1.2}$$

with h locally integrable and T-periodic, the existence of a T-periodic solution under the sole restriction that $\overline{h} = 0$, i.e. to fully extend to the relativistic forced pendulum the result of Hamel-Dancer-Willem mentioned above.

It is straightforward to write the action integral associated to this problem, and quite standard to prove that this integral has a minimum u in the set of T-periodic Lipschitz functions such that $\|u'\|_\infty \leq a$. This is done in Section 2 (Theorem 1). The corresponding variational inequality satisfied by any minimizer is established in Section 3 (Lemma 2). To go from this variational inequality to the Euler-Lagrange differential equation, and so to prove that the minimizer u is a classical solution of equation (1.2) (Theorem 2) requires showing that $\|u'\| < a$. This is done in Section 5, using a preliminary result proved in Section 4 (Lemma 3). Although technically different, the approach is in the spirit of the pioneering paper [5] on the regularity of weak solutions of some elliptic variational inequalities (see also [10]).

2. Minimization problem

Let $a > 0$, $\Phi : [-a, a] \to \mathbb{R}$ satisfy the following conditions:

(H_Φ) *: Φ is continuous on $[-a, a]$, of class C^1 on $(-a, a)$, strictly convex, and $\phi := \Phi' : (-a, a) \to \mathbb{R}$ is a homeomorphism such that $\phi(0) = 0$.*

This easily implies that

$$\phi(s)s > 0 \quad for\ all \quad s \in (-a, a) \setminus \{0\}. \tag{2.1}$$

Let $g : \mathbb{R}^2 \to \mathbb{R}$ satisfy the following conditions:

(H_g) : *g is a Carathéodory function, bounded on $\mathbb{R}^2$, $g(\cdot,u)$ is T-periodic for any $u \in \mathbb{R}$ and some $T > 0$, $g(x,\cdot)$ is 2π-periodic for a.e. $x \in \mathbb{R}$, $G(x,u) := \int_0^u g(x,s)\,ds$ is bounded on $\mathbb{R}^2$, and $G(x,\cdot)$ is 2π-periodic for a.e. $x \in \mathbb{R}$.*

Let $Lip_T(\mathbb{R}) = C_T^{0,1}(\mathbb{R})$ denote the space of functions $u : \mathbb{R} \to \mathbb{R}$ which are T-periodic and Lipschitz with Lipschitz constant

$$[u]_{0,1} := \sup_{x,y\in[0,T],x\neq y} \frac{|u(x)-u(y)|}{|x-y|} < \infty.$$

With the norm

$$\|u\|_{0,1} := \max_{x\in[0,T]} |u(x)| + [u]_{0,1},$$

$Lip_T(\mathbb{R})$ is a Banach space. Any element of $Lip_T(\mathbb{R})$ is almost everywhere differentiable and u' corresponds to the distributional derivative of u.

Given $h \in L^1_T(\mathbb{R})$, where $L^1_T(\mathbb{R})$ denotes the space of locally Lebesgue integrable and T-periodic functions normed by $\|h\|_1 = \int_0^T |h(x)|\,dx$, we write

$$\overline{h} := \frac{1}{T}\int_0^T h(x)\,dx, \quad \widetilde{h} = h - \overline{h},$$

so that

$$\int_0^T \widetilde{h}(x)\,dx = 0.$$

Notice that, if $u \in Lip_T(\mathbb{R})$, then $\widetilde{u}$ vanishes at some $y \in [0,T]$, and hence, for all $x \in [0,T]$ (and consequently all $x \in \mathbb{R}$), we have

$$|\widetilde{u}(x)| = |\widetilde{u}(x) - \widetilde{u}(y)| \leq \int_0^T |u'(t)|\,dt \leq T[u]_{0,1}. \tag{2.2}$$

For $h \in L^\infty(\mathbb{R})$ and T-periodic, we denote the usual norm by $\|h\|_\infty$.

If K denotes the closed convex subset of $Lip_T(\mathbb{R})$ defined by

$$K := \{u \in Lip_T(\mathbb{R}) : |u'(x)| \leq a \textit{ for a.e. } x \in \mathbb{R}\},$$

then the action integral

$$\mathcal{I}(u) := \int_0^T \{\Phi[u'(x)] + G(x,u(x)) + h(x)u(x)\}\,dx \tag{2.3}$$

is well defined on K. This happens for example when

$$\Phi(s) = -\sqrt{1-s^2}, \quad G(x,v) = A\cos v$$

for some $A > 0$, in which case (2.3) can be seen as the action integral associated to the relativistic forced pendulum.

The following lemma is useful to prove the lower semi-continuity of $\mathcal{I}$.

Lemma 1. *If assumption (H_Φ) holds, then, for any sequence $(u_j)_{j\in\mathbb{N}}$ in K which converges uniformly on $[0,T]$ to some $u \in K$, one has*

$$\liminf_{j\to\infty} \int_0^T \Phi[u_j'(x)]\,dx \geq \int_0^T \Phi[u'(x)]\,dx. \tag{2.4}$$

Proof. For any $\lambda \in (0,1)$, we have, by assumption (H_Φ),

$$\int_0^T \Phi[u_j'(x)]\,dx \geq \int_0^T \Phi[\lambda u'(x)]\,dx + \int_0^T \phi[\lambda u'(x)][u_j'(x) - \lambda u'(x)]\,dx. \tag{2.5}$$

On the other hand, u_j' converges to u' for the w*-topology $\sigma(L^\infty, L^1)$. Since $\phi(\lambda u') \in L^\infty(0,T)$, we deduce from (2.5) that

$$\liminf_{j\to\infty} \int_0^T \Phi[u_j'(x)]\,dx \geq \int_0^T \Phi[\lambda u'(x)]\,dx + (1-\lambda)\int_0^T \phi[\lambda u'(x)]u'(x)\,dx.$$

Applying (2.1) we obtain

$$\liminf_{j\to\infty} \int_0^T \Phi[u_j'(x)]\,dx \geq \int_0^T \Phi[\lambda u'(x)]\,dx,$$

which gives (2.4) by letting $\lambda \to 1$. □

We now prove the existence of a minimum to $\mathcal{I}$ when $\overline{h} = 0$.

Theorem 1. *If assumptions (H_Φ) and (H_g) hold, then, for any $h \in L_T^1$ such that*

$$\overline{h} = 0, \tag{2.6}$$

$\mathcal{I}$ has a minimum over K.

Proof. We first observe that, because of the 2π-periodicity of $G(x,\cdot)$ and condition (2.6), we have, for all $u \in K$,

$$\mathcal{I}(u + 2\pi) = \mathcal{I}(u),$$

so that, if u^* minimizes $\mathcal{I}$ over K, the same is true for $u^* + 2j\pi$ for any integer j. Hence, without loss of generality, we can search for a minimizer u^* such that $\overline{u^*} \in [0,2\pi]$; i.e., we can minimize $\mathcal{I}$ in the convex set

$$\widehat{K} := \{u \in Lip_T(\mathbb{R}) : \overline{u} \in [0,2\pi], |u'(x)| \leq a \text{ for a.e. } x \in \mathbb{R}\}.$$

Now, if $u \in \widehat{K}$, we have for all $x \in \mathbb{R}$, using (2.2),

$$|u(x)| \leq |\overline{u}| + |\widetilde{u}(x)| \leq 2\pi + T[u]_{0,1} = 2\pi + Ta,$$

so that $\widehat{K}$ is a bounded and equicontinuous subset of the space of continuous T-periodic functions. If $(u_j)_{j\in\mathbb{N}}$ is a minimizing sequence for $\mathcal{I}$ in $\widehat{K}$, we can assume, using Arzelá-Ascoli's theorem and going if necessary to a subsequence, that $(u_j)_{j\in\mathbb{N}}$ converges uniformly in $\mathbb{R}$ to some continuous T-periodic u^*. From the relations

$$\frac{|u_j(x) - u_j(y)|}{|x-y|} \leq a \quad (x \neq y, j \in \mathbb{N})$$

we easily get that $u^* \in \widehat{K}$. Consequently, using Lemma 1, we have

$$\inf_{\widehat{K}} \mathcal{I} = \lim_{j\to\infty} \mathcal{I}(u_j) \geq \mathcal{I}(u^*)$$

so that u^* minimizes $\mathcal{I}$ over $\widehat{K}$. □

3. Variational inequality

The following lemma provides the variational inequality satisfied by a minimizer of $\mathcal{I}$.

Lemma 2. *If u minimizes $\mathcal{I}$ over K, then*

$$\int_0^T \left(\Phi[v'(x)] - \Phi[u'(x)] + \{g[x,u(x)] + h(x)\}[v(x) - u(x)]\right) dx \geq 0 \quad \textit{for all} \quad v \in K. \tag{3.1}$$

Proof. Let $v \in K$. By assumption, we have, for all $\lambda \in (0,1]$,

$$\mathcal{I}(u) \leq \mathcal{I}[u + \lambda(v-u)];$$

i.e.,

$$\int_0^T \{\Phi[u'(x) + \lambda(v'(x) - u'(x))] - \Phi[u'(x)] + G[x, u(x) + \lambda(v(x) - u(x))] - G[x,u(x)] + \lambda h(x)[v(x) - u(x)]\}\, dx \geq 0.$$

Applying the convexity of Φ we deduce that

$$\int_0^T \left\{\Phi[v'(x)] - \Phi[u'(x)] + \lambda^{-1}\{G[x, u(x) + \lambda(v(x) - u(x))] - G[x,u(x)]\} + h(x)[v(x) - u(x)]\right\}\, dx \geq 0.$$

By the Lebesgue dominated convergence theorem, we obtain, when $\lambda \searrow 0$,

$$\int_0^T \{\Phi[v'(x)] - \Phi[u'(x)] + g[x, u(x)][v(x) - u(x)] \\ + h(x)[v(x) - u(x)]\} \, dx \geq 0. \qquad \square$$

4. Auxiliary problem

To obtain further information about the minimizer u, let us introduce the auxiliary problem

$$(\phi(u'))' - u = f(x), \quad u \quad is \quad T-periodic, \tag{4.1}$$

where ϕ satisfies Assumption (H_Φ) and $f \in L^1_T(\mathbb{R})$. A (classical) solution of (4.1) is a T-periodic function $u \in C^1(\mathbb{R})$ such that $\phi \circ u'$ is absolutely continuous and (4.1) holds almost everywhere on $\mathbb{R}$.

The existence part of the following lemma and the estimate for u' is essentially a special case of Corollary 3 of [4]. The proof given there for f continuous, based upon a reduction to a fixed-point problem and Leray-Schauder degree, can immediately be adapted to the case where $f \in L^1_T(\mathbb{R})$.

Lemma 3. *For any $f \in L^1_T(\mathbb{R})$, problem* (4.1) *has a unique classical solution u, and $\|u'\|_\infty < a$.*

Proof. It remains to prove the uniqueness. If u and v are two solutions of (4.1), then we obtain

$$\int_0^T \{[\phi(u'(x)) - \phi(v'(x))]'[u(x) - v(x)] - [u(x) - v(x)]^2\} \, dx = 0$$

and hence, integrating the first term by parts and using T-periodicity,

$$\int_0^T \{[\phi(u'(x)) - \phi(v'(x))][u'(x) - v'(x)] + [u(x) - v(x)]^2\} \, dx = 0.$$

The monotonicity of ϕ implies the conclusion. $\square$

Lemma 4. *For any $f \in L^1_T(\mathbb{R})$, the unique solution u of* (4.1) *belongs to K and satisfies the variational inequality*

$$\int_0^T \left\{\Phi[v'(x)] - \Phi[u'(x)] + [u(x) + f(x)][v(x) - u(x)]\right\} \, dx \geq 0 \textit{ for all } v \in K.$$

Proof. We have, using integration by parts and (4.1),

$$\int_0^T \{\Phi[v'(x)] - \Phi[u'(x)]\} \, dx \geq \int_0^T \phi[u'(x)][v'(x) - u'(x)] \, dx$$

$$= -\int_0^T (\phi[u'(x)])'[v(x) - u(x)]\, dx$$
$$= -\int_0^T [u(x) + f(x)][v(x) - u(x)]\, dx. \qquad \square$$

5. Periodic solutions of relativistic pendulum-like equations

We can now combine the results of the previous sections to obtain conditions for the existence of at least one (classical) T-periodic solution for the differential equation

$$(\phi(u'))' - g(x, u) = h(x). \tag{5.1}$$

A classical T-periodic solution of (5.1) is a T-periodic function $u \in C^1(\mathbb{R})$ such that $\phi \circ u'$ is absolutely continuous and (5.1) holds almost everywhere on $\mathbb{R}$.

Theorem 2. *If assumptions* (H_Φ) *and* (H_g) *hold, then, for any* $h \in L^1_T$ *such that* $\overline{h} = 0$, *equation* (5.1) *has at least one* T*-periodic solution which minimizes* $\mathcal{I}$ *over* K.

Proof. Let u be a minimizer of $\mathcal{I}$ over K. Then, by Lemma 2, u satisfies the variational inequality (3.1). This variational inequality can be written

$$\int_0^T \{\Phi[v'(x)] - \Phi[u'(x)] + u(x)[v(x) - u(x)]$$
$$+[g[x, u(x)] + h(x) - u(x)][v(x) - u(x)]\}\, dx \geq 0 \quad \textit{for all} \quad v \in K,$$

so that u is a solution of the variational inequality

$$\int_0^T \{\Phi[v'(x)] - \Phi[u'(x)] + u(x)[v(x) - u(x)]$$
$$+f_u(x)[v(x) - u(x)]\}\, dx \geq 0 \quad \textit{for all} \quad v \in K, \tag{5.2}$$

where

$$f_u = g[\cdot, u(\cdot)] + h - u \in L^1_T(\mathbb{R}).$$

Now, given any $w \in K$, the unique solution $\widehat{u}_w$ of problem (4.1) with $f = f_w$ satisfies, by Lemma 4,

$$\int_0^T \{\Phi[v'(x)] - \Phi[\widehat{u}_w'(x)] + \widehat{u}_w(x)[v(x) - \widehat{u}_w(x)]$$
$$+f_w(x)[v(x) - \widehat{u}_w(x)]\}\, dx \geq 0 \quad \textit{for all} \quad v \in K. \tag{5.3}$$

Choosing $v = \widehat{u}_u$ in (5.2), $w = v = u$ (u the minimizer of $\mathcal{I}$ over K) in (5.3), and adding the resulting inequalities, we obtain

$$\int_0^T [u(x) - \widehat{u}_u(x)]^2 \, dx \leq 0. \tag{5.4}$$

It follows from (5.4) that $u = \widehat{u}_u$ and hence that $\|u'\|_\infty = \|(\widehat{u}_u)'\|_\infty < a$. Moreover, u is a classical T-periodic solution of (5.1), since $\widehat{u}_u$ is a classical T-periodic solution of (4.1) with $f = f_u$. $\square$

Corollary 1. *For any $T > 0$, $A \in \mathbb{R}$, and $h \in L^1_T(\mathbb{R})$ such that $\overline{h} = 0$, the relativistic pendulum equation*

$$\Big(\frac{u'}{\sqrt{1-u'^2}}\Big)' + A \sin u = h(x)$$

has at least one classical T-periodic solution.

Proof. It suffices to take $\Phi(s) = -\sqrt{1-s^2}$, so that $\phi(s) = \frac{s}{\sqrt{1-s^2}}$, and $G(x,u) = A\cos u$, so that $g(x,u) = -A \sin u$. Assumptions H_Φ with $a = 1$ and H_g hold. $\square$

Remark 1. It would be interesting to investigate similar questions in higher dimensions, for example

$$\mathrm{Min}_{u \in K} \int_{\mathbb{T}^N} \Big\{ -\sqrt{1 - |\nabla u|^2} + G[x, u(x)] + h(x)u(x) \Big\} \, dx,$$

where $\mathbb{T}^N$ denotes the N-dimensional torus,

$$K := \{u \in Lip(\mathbb{T}^N) : |\nabla u(x)| \leq 1 \; for \; a.e. \; x \in \mathbb{T}^N\}.$$

Bartnik and Simon [1] have studied related questions under Dirichlet boundary conditions.

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