Periodic solutions of the forced relativistic pendulum

Haïm Brezis[*][†] and Jean Mawhin[‡]

February 11, 2019

Abstract

The existence of at least one classical T-periodic solutions is proved for differential equations of the form

$$(\phi(u'))' - g(x,u) = h(x)$$

when $\phi : (-a,a) \to \mathbb{R}$ is an increasing homeomorphism, g is a Carathéodory function T-periodic with respect to x, 2π-periodic with respect to u, of mean value zero with respect to u, and $h \in L^1_{loc}(R)$ is T-periodic and has mean value zero. The problem is reduced to finding a minimum for the corresponding action integral over a closed convex subset of the space of T-periodic Lipschitzian functions, and then to show, using variational inequalities techniques, that such a minimum solves the differential equation. A special case if the 'relativistic forced pendulum equation'

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + A\sin u = h(x).$$

MSC : 34C25; 49J40; 58E30; 58E35

Keywords : periodic solutions; relativistic forced pendulum; ϕ-Laplacian; calculus of variations; variational inequalities

1 Introduction

The first global result for the existence of periodic solutions of the forced pendulum equation started with the rigorous mathematical study of equation

$$u'' + A\sin u = h(x) \tag{1}$$

[*]Rutgers University, Department of Mathematics, Hill Center, Busch Campus, 110 Frelinghuysen Road, Piscataway, NJ 08854, USA, brezis@math.rutgers.edu

[†]Department of Mathematics, Technion, Israel Institute of Technology, 32.000 Haifa, Israel

[‡]Département de Mathématique, Université Catholique de Louvain, Chemin du Cyclotron 2, B-1348 Louvain-la-Neuve, Belgium, jean.mawhin@uclouvain.be

initiated in 1922 by Hamel, in a paper of the special issue of the *Mathematische Annalen* dedicated to Hilbert's sixtieth birthday anniversary [9]. Hamel proved the existence of a 2π-periodic solution of equation (1) with $h(x) = B\sin x$, by showing that the corresponding action integral

$$\mathcal{A}(u) := \int_0^{2\pi} \left[\frac{u'(x)^2}{2} + A\cos u(x) + Bu(x)\sin x \right] dx$$

has a minimum over the space of 2π-periodic C^1-functions, and his argument easily extends to the case where $B\sin t$ is replaced by a continuous 2π-periodic function $h(t)$ with mean value $\overline{h}$ equal to zero.

In the late nineteen seventies, Fučik wrote in his monograph [8] that 'the description of the set P of h for which equation (1) has a 2π-periodic solution seems to remain a *terra incognita*.' Motivated by this remark, but also unaware of the existence of Hamel's paper, Castro [6] (for $|A| \leq 1$), Dancer [7] and Willem [16], independently (for arbitrary A), reintroduced in the early nineteen eighties the use of the direct method of the calculus of variations, in the setting of Sobolev spaces. One can consult [11] for a survey and a bibliography of the recent developments in this direction.

On the other hand, periodic solutions of differential equations of the form

$$(\phi(u'))' = f(t, u, u')$$

with $\phi : (-a, a) \to \mathbb{R}$ an increasing homeomorphism satisfying $\phi(0) = 0$, have been recently studied by in [3, 4], using a fixed point reduction and Leray-Schauder degree. The motivation came from the special case where $\phi(s) = \frac{s}{\sqrt{1-s^2}}$, which occurs in the dynamics of special relativity. Using Schauder fixed point theorem, Torres [14, 15] has recently proved, for the relativistic pendulum equation with continuous T-periodic forcing h and arbitrary dissipation f

$$(\phi(u'))' + f(u)u' + A\sin u = h(t),$$

the existence of at least two T-periodic solution when

$$aT < 2\sqrt{3} \quad and \qquad |\overline{h}| < A\left(1 - \frac{aT}{2\sqrt{3}}\right),$$

and of as least one T-periodic solution when

$$aT = 2\sqrt{3} \quad and \quad \overline{h} = 0,$$

where $\overline{h}$ denotes the mean value of h over $[0, T]$. Those assumptions have been respectively improved in [2] to

$$aT < \pi\sqrt{3} \quad and \quad |\overline{h}| < A\cos\left(\frac{aT}{2\sqrt{3}}\right),$$

and

$$aT = \pi\sqrt{3} \quad and \quad \overline{h} = 0,$$

using Leray-Schauder degree arguments. It is also proved in [2], using lower and upper solutions, that at least two T-periodic solutions exist when $||h||_\infty < A$, and at least one when $\|h\|_\infty = A$.

The aim of this paper is to use the direct method of the calculus of variations to prove, for equations of the type

$$(\phi(u'))' + A\sin u = h(t) \tag{2}$$

with h locally integrable and T-periodic, the existence of a T-periodic solution under the sole restriction that $\overline{h} = 0$, i.e. to fully extend to the relativistic forced pendulum the result of Hamel-Dancer-Willem mentioned above.

It is straighforward to write the action integral associated to this problem, and quite standard to prove that this integral has a minimum u in the set of T-periodic Lipschitzian functions such that $\|u'\|_\infty \leq a$. This is done in Section 2 (Theorem 1). The corresponding variational inequality satisfied by any maximizer is established in Section 3 (Lemma 2). To go from this variational inequality to the Euler-Lagrange differential equation, and so to prove that the minimizer u is a classical solution of equ. (2) (Theorem 2) requires to show that $\|u'\| < a$. This is done in Section 5, using some preliminary result proved in Section 4 (Lemma 3). Although technically different, the approach is in the spirit of the pioneering paper [5] on the regularity of weak solutions of some elliptic variational inequalities (see also [10]).

2 Minimization problem

Let $a > 0$, $\Phi : [-a, a] \to \mathbb{R}$ satisfy the following conditions :

(H_Φ) : *Φ is continuous on $[-a, a]$, of class C^1 on $(-a, a)$, strictly convex, and $\phi := \Phi' : (-a, a) \to \mathbb{R}$ is a homeomorphism such that $\phi(0) = 0$.*

This easily implies that

$$\phi(s)s > 0 \quad for\ all \quad s \in (-a, a) \setminus \{0\}. \tag{3}$$

Let $g : \mathbb{R}^2 \to \mathbb{R}$ satisfy the following conditions :

(H_g) : *g is a Carathéodory function, bounded on $\mathbb{R}^2$, $g(\cdot, u)$ is T-periodic for any $u \in \mathbb{R}$ and some $T > 0$, $g(x, \cdot)$ is 2π-periodic for a.e. $x \in \mathbb{R}$, $G(x, u) := \int_0^u g(x, s)\, ds$ is bounded on $\mathbb{R}^2$, and $G(x, \cdot)$ is 2π-periodic for a.e. $x \in \mathbb{R}$.*

Let $Lip_{T,a}(\mathbb{R}) = C^{0,1}_{T,a}(\mathbb{R})$ denote the space of functions $u : \mathbb{R} \to \mathbb{R}$ which are T-periodic and Lipschitzian with Lipschitz constant

$$[u]_{0,1} := \sup_{x,y\in[0,T],x\neq y} \frac{|u(x) - u(y)|}{|x - y|} \leq a.$$

With the norm

$$\|u\|_{0,1} := \max_{x\in[0,T]} |u(x)| + [u]_{0,1},$$

$Lip_{T,a}(\mathbb{R})$ is a Banach space. Any element of $Lip_{T,a}(\mathbb{R})$ is a.e. differentiable and u' corresponds to the distributional derivative of u.

Given $h \in L^1_T(\mathbb{R})$, where $L^1_T(\mathbb{R})$ denotes the space of locally Lebesgue integrable and T-periodic functions normed by $\|h\|_1 = \int_0^T |h(x)|\,dx$, we write

$$\overline{h} := \frac{1}{T}\int_0^T h(x)\,dx, \quad \widetilde{h} = h - \overline{h},$$

so that

$$\int_0^T \widetilde{h}(x)\,dx = 0.$$

Notice that if $u \in Lip_{T,a}(\mathbb{R})$, then $\widetilde{u}$ vanishes at some $y \in [0,T]$, and hence, for all $x \in [0,T]$ (and consequently all $x \in \mathbb{R}$), we have

$$|\widetilde{u}(x)| = |\widetilde{u}(x) - \widetilde{u}(y)| \leq \int_0^T |u'(t)|\,dt \leq T[u]_{0,1}. \tag{4}$$

For $h \in L^\infty(\mathbb{R})$ and T-periodic, we denote the usual norm by $\|h\|_\infty$.

If K denotes the closed convex subset of $Lip_{T,a}(\mathbb{R})$ defined by

$$K := \{u \in Lip_{T,a}(\mathbb{R}) : |u'(x)| \leq a \textit{ for a.e. } x \in \mathbb{R}\},$$

then the action integral

$$\mathcal{I}(u) := \int_0^T \{\Phi[u'(x)] + G(x,u(x)) + h(x)u(x)\}\,dx \tag{5}$$

is well defined on K. This happens for example when

$$\Phi(s) = -\sqrt{1-s^2}, \quad G(x,v) = A\cos v$$

for some $A > 0$, in which case (5) can be seen as the action integral associated to the relativistic forced pendulum.

The following lemma is useful to prove the lower semi-continuity of $\mathcal{I}$.

Lemma 1 *If assumption (H_Φ) holds, then, for any sequence $(u_j)_{j\in\mathbb{N}}$ in K which converges uniformly on $[0,T]$ to some $u \in K$, one has*

$$\liminf_{j\to\infty} \int_0^T \Phi[u_j'(x)]\,dx \geq \int_0^T \Phi[u'(x)]\,dx. \tag{6}$$

Proof. For any $\lambda \in (0,1)$, we have, by assumption (H_Φ),

$$\int_0^T \Phi[u_j'(x)]\,dx \geq \int_0^T \Phi[\lambda u'(x)]\,dx + \int_0^T \phi[\lambda u'(x)][u_j'(x) - \lambda u'(x)]\,dx. \qquad (7)$$

On the other hand, u_j' converges to u' for the w*-topology $\sigma(L^\infty, L^1)$. Since $\phi(\lambda u') \in L^\infty(0,T)$, we deduce from (7) that

$$\liminf_{j\to\infty} \int_0^T \Phi[u_j'(x)]\,dx \geq \int_0^T \Phi[\lambda u'(x)]\,dx + (1-\lambda)\int_0^T \phi[\lambda u'(x)]u'(x)\,dx.$$

Applying (3) we obtain

$$\liminf_{j\to\infty} \int_0^T \Phi[u_j'(x)]\,dx \geq \int_0^T \Phi[\lambda u'(x)]\,dx,$$

which gives (6) by letting $\lambda \to 1$. ∎

We now prove the existence of a minimum to $\mathcal{I}$ when $\overline{h} = 0$.

Theorem 1 *If assumptions (H_Φ) and (H_g) hold, then, for any $h \in L_T^1$ such that*

$$\overline{h} = 0, \qquad (8)$$

$\mathcal{I}$ has a minimum over K.

Proof. We first observe that, because of the 2π-periodicity of $G(x,\cdot)$ and condition (8), we have, for all $u \in K$,

$$\mathcal{I}(u + 2\pi) = \mathcal{I}(u),$$

so that, if u^* minimizes $\mathcal{I}$ over K, the same if true for $u^* + 2j\pi$ for any integer j. Hence, without loss of generality, we can search for a minimizer u^* such that $\overline{u^*} \in [0, 2\pi]$, i.e. we can minimize $\mathcal{I}$ in the convex set

$$\widehat{K} := \{u \in Lip_{T,a}(\mathbb{R}) : \overline{u} \in [0,2\pi], |u'(x)| \leq a \text{ } for \text{ } a.e. \text{ } x \in \mathbb{R}\}.$$

Now, if $u \in \widehat{K}$, we have for all $x \in \mathbb{R}$, using (4),

$$|u(x)| \leq |\overline{u}| + |\widetilde{u}(x)| \leq 2\pi + T[u]_{0,1} = 2\pi + Ta,$$

so that $\widehat{K}$ is a bounded and equicontinuous subset of the space of continuous T-periodic functions. If $(u_j)_{j\in\mathbb{N}}$ is a minimizing sequence for $\mathcal{I}$ in $\widehat{K}$, we can assume, using Arzelá-Ascoli's theorem and going if necessary to a subsequence, that $(u_j)_{j\in\mathbb{N}}$ converges uniformly in $\mathbb{R}$ to some continuous T-periodic u^*. From the relations

$$\frac{|u_j(x) - u_j(y)|}{|x-y|} \leq a \quad (x \neq y, j \in \mathbb{N})$$

we easily get that $u^* \in \widehat{K}$. Consequently, using Lemma 1, we have

$$\inf_{\widehat{K}} \mathcal{I} = \lim_{j\to\infty} \mathcal{I}(u_j) \geq \mathcal{I}(u^*)$$

so that u^* minimizes $\mathcal{I}$ over $\widehat{K}$. ∎

3 Variational inequality

The following lemma provides the variational inequality satisfied by a minimizer of $\mathcal{I}$.

Lemma 2 *If u minimizes $\mathcal{I}$ overs K, then*

$$\int_0^T (\Phi[v'(x)] - \Phi[u'(x)] + \{g[x,u(x)] + h(x)\}[v(x) - u(x)])\ dx \geq 0 \quad for\ all \quad v \in K. \tag{9}$$

Proof. Let $v \in K$. By assumption, we have, for all $\lambda \in (0,1]$,

$$\mathcal{I}(u) \leq \mathcal{I}[u + \lambda(v-u)],$$

i.e.

$$\int_0^T \{\Phi[u'(x) + \lambda(v'(x) - u'(x))] - \Phi[u'(x)] + G[x, u(x) + \lambda(v(x) - u(x))] - G[x,u(x)] + \lambda h(x)[v(x) - u(x)]\}\, dx \geq 0.$$

Applying the convexity of Φ we deduce that

$$\int_0^T \{\Phi[v'(x)] - \Phi[u'(x)] + \lambda^{-1}\{G[x, u(x) + \lambda(v(x) - u(x))] - G[x,u(x)]\} + h(x)[v(x) - u(x)]\}\ dx \geq 0.$$

By Lebesgue dominated convergence theorem, we obtain, when $\lambda \searrow 0$,

$$\int_0^T \{\Phi[v'(x)] - \Phi[u'(x)] + g[x,u(x)][v(x) - u(x)] + h(x)[v(x) - u(x)]\}\, dx \geq 0.$$

∎

4 Auxiliary problem

To obtain further information about the minimizer u, let us introduce the auxiliary problem

$$(\phi(u'))' - u = f(x), \quad u \quad is \quad T-periodic, \tag{10}$$

where ϕ satisfies Assumption (H_Φ) and $f \in L^1_T(\mathbb{R})$. A (classical) solution of (10) is a T-periodic function $u \in C^1(\mathbb{R})$ such that $\phi \circ u'$ is absolutely continuous and (10) holds a.e. on $\mathbb{R}$.

The existence part of the following Lemma and the estimate for u' is essentially a special case of Corollary 3 of [4]. The proof given there for f continuous, based upon a reduction to a fixed point problem and Leray-Schauder degree, can immediately be adapted to the case where $f \in L^1_T(\mathbb{R})$.

Lemma 3 *For any $f \in L^1_T(\mathbb{R})$, problem (10) has a unique classical solution u, and $\|u'\|_\infty < 1$.*

Proof. It remains to prove the uniqueness. If u and v are two solutions of (10), then we obtain

$$\int_0^T \{[\phi(u'(x)) - \phi(v'(x))]'[u(x) - v(x)] - [u(x) - v(x)]^2\} \, dx = 0$$

and hence, integrating the first term by parts and using T-periodicity,

$$\int_0^T \{[\phi(u'(x)) - \phi(v'(x))][u'(x) - v'(x)] + [u(x) - v(x)]^2\} \, dx = 0.$$

The monotonicity of ϕ implies the conclusion. ∎

Lemma 4 *For any $f \in L^1_T(\mathbb{R})$, the unique solution u of (10) belongs to K and verifies the variational inequality*

$$\int_0^T \{\Phi[v'(x)] - \Phi[u'(x)] + [u(x) + f(x)][v(x) - u(x)]\} \, dx$$
$$\geq 0 \quad for\ all \quad v \in K.$$

Proof. We have, using integration by parts and (10),

$$\begin{aligned} \int_0^T \{\Phi[v'(x)] - \Phi[u'(x)]\} \, dx &\geq \int_0^T \phi[u'(x)][v'(x) - u'(x)] \, dx \\ = -\int_0^T (\phi[u'(x)])'[v(x) - u(x)] \, dx &= -\int_0^T [u(x) + f(x)][v(x) - u(x)] \, dx. \end{aligned}$$

∎

5 Periodic solutions of relativistic pendulum-like equations

We can now combine the results of the previous sections to obtain conditions for the existence of at least one (classical) T-periodic solution for the differential equation

$$(\phi(u'))' - g(x, u) = h(x). \tag{11}$$

A classical T-periodic solution of (11) is a T-periodic function $u \in C^1(\mathbb{R})$ such that $\phi \circ u'$ is absolutely continuous and (11) holds a.e. on $\mathbb{R}$.

Theorem 2 *If assumptions (H_Φ) and (H_g) hold, then, for any $h \in L^1_T$ such that $\overline{h} = 0$, equation (11) has at least one T-periodic solution which minimizes $\mathcal{I}$ over K.*

Proof. Let u be a minimizer of $\mathcal{I}$ over K. Then, by Lemma 2, u satisfies the variational inequality (9). This variational inequality can be written

$$\int_0^T \{\Phi[v'(x)] - \Phi[u'(x)] + u(x)[v(x) - u(x)]$$
$$+[g[x, u(x)] + h(x) - u(x)][v(x) - u(x)]\}\, dx \geq 0 \quad for\ all \quad v \in K,$$

so that u is a solution of the variational inequality

$$\int_0^T \{\Phi[v'(x)] - \Phi[u'(x)] + u(x)[v(x) - u(x)]$$
$$+f_u(x)[v(x) - u(x)]\}\, dx \geq 0 \quad for\ all \quad v \in K, \tag{12}$$

where

$$f_u = g[\cdot, u(\cdot)] + h - u \in L^1_T(\mathbb{R}).$$

Now, given any $w \in K$, the unique solution $\widehat{u}_w$ of problem (10) with $f = f_w$ satisfies, by Lemma 4,

$$\int_0^T \{\Phi[v'(x)] - \Phi[\widehat{u}'_w(x)] + \widehat{u}_w(x)[v(x) - \widehat{u}_w(x)]$$
$$+f_w(x)[v(x) - \widehat{u}_w(x)]\}\, dx \geq 0 \quad for\ all \quad v \in K. \tag{13}$$

Chosing $v = \widehat{u}_u$ in (12), $w = v = u$ (u the minimizer of $\mathcal{I}$ over K) in (13), and adding the resulting inequalities, we obtain

$$\int_0^T [u(x) - \widehat{u}_u(x)]^2\, dx \leq 0. \tag{14}$$

It follows from (14) that $u = \widehat{u}_u$ and hence that $\|u'\|_\infty = \|(\widehat{u}_u)'\|_\infty < a$. Moreover u is a classical T-periodic solution of (11), since $\widehat{u}_u$ is a classical T-periodic solution of (10) with $f = f_u$. ■

Corollary 1 *For any $T > 0$, $A \in \mathbb{R}$, and $h \in L^1_T(\mathbb{R})$ such that $\overline{h} = 0$, the relativistic pendulum equation*

$$\left(\frac{u'}{\sqrt{1 - u'^2}}\right)' + A \sin u = h(x)$$

has at least one classical T-periodic solution.

Proof. It suffices to take $\Phi(s) = -\sqrt{1 - s^2}$, so that $\phi(s) = \frac{s}{\sqrt{1-s^2}}$, and $G(x, u) = A \cos u$, so that $g(x, u) = -A \sin u$. Assumptions H_Φ with $a = 1$ and H_g hold. ■

Remark 1 It would be interesting to investigate similar questions in higher dimensions, for example

$$\mathrm{Min}_{u\in K}\int_{\mathbb{T}^N}\left\{-\sqrt{1-|\nabla u|^2}+G[x,u(x)]+h(x)u(x)\right\}dx,$$

where $\mathbb{T}^N$ denotes the N-dimensional torus,

$$K := \{u \in Lip(\mathbb{T}^N) : |\nabla u(x)| \leq 1 \ for\ a.e.\ x \in \mathbb{T}^N\}.$$

Bartnik and Simon [1] have studied related questions under Dirichlet boundary conditions.

Acknowledgments. Part of this work was done when the second author visited Rutgers University; he thanks its Mathematics Department for the hospitality. The first author is partially supported by NSF Grant DMS-0802958.

References

[1] R. Bartnik and L. Simon, *Spacelike hypersurfaces with prescribed boundary values and mean curvature,* Comm. Math. Phys. 87 (1982), 131-152.

[2] C. Bereanu, P. Jebelean and J. Mawhin, *Periodic solutions of pendulum-like perturbations of singular and bounded ϕ-Laplacians*, preprint.

[3] C. Bereanu and J. Mawhin, *Periodic solutions of nonlinear perturbations of ϕ-laplacians with possibly bounded ϕ*, Nonlinear Anal. 68 (2008), 1668-1681

[4] C. Bereanu and J. Mawhin, *Existence and multiplicity results for some nonlinear problems with singular ϕ-Laplacian,* J. Differential Equations 243 (2007), 536-557

[5] H. Brezis and G. Stampacchia, *Sur la régularité de la solution d'inéquations elliptiques*, Bull. Soc. Math. France 96 (1968), 153-180

[6] A. Castro, *Periodic solutions of the forced pendulum equation,* in Differential Equations, Ahmad, Keener, Lazer ed., Academic Press, New York, 1980 149-160

[7] E.N. Dancer, *On the use of asymptotics in nonlinear boundary value problems,* Ann. Mat. Pura Appl. (4) 131 (1982), 167-185

[8] S. Fučik, *Solvability of Nonlinear Equations and Boundary value Problems,* Reidel, Dordrecht, 1980

[9] G. Hamel, *Ueber erzwungene Schingungen bei endlischen Amplituden,* Math. Ann. 86 (1922), 1-13

[10] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, 1980

[11] J. Mawhin, *Global results for the forced pendulum equation*, in Handbook of Differential Equations : Ordinary Differential Equations, vol. 1, A. Canada, P. Drábek, A. Fonda ed., Elsevier, Amsterdam, 2004, 533-590

[12] J. Mawhin, and M. Willem, *Multiple solutions of the periodic boundary value problem for some forced pendulum-type equations,* J. Differential Equations 52 (1984), 264-287

[13] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems,* Springer, New York, 1989

[14] P.J. Torres, *Periodic oscillations of the relativistic pendulum with friction*, Physics Letters A 372 (2008), 6386-6387.

[15] P.J. Torres, *Nondegeneracy of the periodically forced Liénard differential equation with ϕ-Laplacian*, Commun. Contemporary Math., to appear

[16] M. Willem, *Oscillations forcées de l'équation du pendule,* Pub. IRMA Lille, 3 (1981), V-1-V-3