Periodic solutions of the forced relativistic pendulum

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Abstract

The existence of at least one classical T-periodic solutions is proved for differential equations of the form

\[ (\phi(u'))' - g(x, u) = h(x) \]

when \( \phi : (-a, a) \to \mathbb{R} \) is an increasing homeomorphism, \( g \) is a Carathéodory function \( T \)-periodic with respect to \( x \), \( 2\pi \)-periodic with respect to \( u \), of mean value zero with respect to \( u \), and \( h \in L^1_{\text{loc}}(\mathbb{R}) \) is \( T \)-periodic and has mean value zero. The problem is reduced to finding a minimum for the corresponding action integral over a closed convex subset of the space of \( T \)-periodic Lipschitzian functions, and then to show, using variational inequalities techniques, that such a minimum solves the differential equation. A special case if the ‘relativistic forced pendulum equation’

\[ \left( \frac{u'}{\sqrt{1-u'^2}} \right)' + A \sin u = h(x). \]

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1 Introduction

The first global result for the existence of periodic solutions of the forced pendulum equation started with the rigorous mathematical study of equation

\[ u'' + A \sin u = h(x) \]
initiated in 1922 by Hamel, in a paper of the special issue of the *Mathematische Annalen* dedicated to Hilbert’s sixtieth birthday anniversary [9]. Hamel proved the existence of a \(2\pi\)-periodic solution of equation (1) with \(h(x) = B \sin x\), by showing that the corresponding action integral

\[
A(u) := \int_0^{2\pi} \left[ \frac{(u'(x))^2}{2} + A \cos u(x) + Bu(x) \sin x \right] \, dx
\]

has a minimum over the space of \(2\pi\)-periodic \(C^1\)-functions, and his argument easily extends to the case where \(B \sin t\) is replaced by a continuous \(2\pi\)-periodic function \(h(t)\) with mean value \(\bar{h}\) equal to zero.

In the late nineteen seventies, Fučik wrote in his monograph [8] that ‘the description of the set \(P\) of \(h\) for which equation (1) has a \(2\pi\)-periodic solution seems to remain a terra incognita.’ Motivated by this remark, but also unaware of the existence of Hamel’s paper, Castro [6] (for \(|A| \leq 1\), Dancer [7] and Willem [16], independently (for arbitrary \(A\)), reintroduced in the early nineteen eighties the use of the direct method of the calculus of variations, in the setting of Sobolev spaces. One can consult [11] for a survey and a bibliography of the recent developments in this direction.

On the other hand, periodic solutions of differential equations of the form

\[
(\phi(u'))' = f(t, u, u')
\]

with \(\phi : (-a, a) \to \mathbb{R}\) an increasing homeomorphism satisfying \(\phi(0) = 0\), have been recently studied by in [3, 4], using a fixed point reduction and Leray-Schauder degree. The motivation came from the special case where \(\phi(s) = \frac{s}{\sqrt{1-s^2}}\), which occurs in the dynamics of special relativity. Using Schauder fixed point theorem, Torres [14, 15] has recently proved, for the relativistic pendulum equation with continuous \(T\)-periodic forcing \(h\) and arbitrary dissipation \(f\)

\[
(\phi(u'))' + f(u)u' + A \sin u = h(t),
\]

the existence of at least two \(T\)-periodic solution when

\[
aT < 2\sqrt{3} \quad \text{and} \quad |\bar{h}| < A \left(1 - \frac{aT}{2\sqrt{3}}\right),
\]

and of at least one \(T\)-periodic solution when

\[
aT = 2\sqrt{3} \quad \text{and} \quad \bar{h} = 0,
\]

where \(\bar{h}\) denotes the mean value of \(h\) over \([0, T]\). Those assumptions have been respectively improved in [2] to

\[
aT < \pi \sqrt{3} \quad \text{and} \quad |\bar{h}| < A \cos \left(\frac{aT}{2\sqrt{3}}\right)
\]
and

\[ aT = \pi \sqrt{3} \quad \text{and} \quad \text{h} = 0, \]

using Leray-Schauder degree arguments. It is also proved in [2], using lower and upper solutions, that at least two T-periodic solutions exist when \(|h|_\infty < A\), and at least one when \(|h|_\infty = A\).

The aim of this paper is to use the direct method of the calculus of variations to prove, for equations of the type

\[ (\phi(u'))' + A \sin u = h(t) \quad (2) \]

with \( h \) locally integrable and T-periodic, the existence of a T-periodic solution under the sole restriction that \( h = 0 \), i.e. to fully extend to the relativistic forced pendulum the result of Hamel-Dancer-Willem mentioned above.

It is straightforward to write the action integral associated to this problem, and quite standard to prove that this integral has a minimum \( u \) in the set of T-periodic Lipschitzian functions such that \( \|u'\|_\infty \leq a \). This is done in Section 2 (Theorem 1). The corresponding variational inequality satisfied by any maximizer is established in Section 3 (Lemma 2). To go from this variational inequality to the Euler-Lagrange differential equation, and so to prove that the minimizer \( u \) is a classical solution of equ. (2) (Theorem 2) requires to show that \( \|u'\| < a \). This is done in Section 5, using some preliminary result proved in Section 4 (Lemma 3). Although technically different, the approach is in the spirit of the pioneering paper [5] on the regularity of weak solutions of some elliptic variational inequalities (see also [10]).

2 Minimization problem

Let \( a > 0 \), \( \Phi : [-a, a] \to \mathbb{R} \) satisfy the following conditions:

\[ \begin{aligned} 
(H_\Phi) & : \Phi \text{ is continuous on } [-a, a], \text{ of class } C^1 \text{ on } (-a, a), \text{ strictly convex,} \\
& \text{and } \phi := \Phi' : (-a, a) \to \mathbb{R} \text{ is a homeomorphism such that } \phi(0) = 0. 
\end{aligned} \]

This easily implies that

\[ \phi(s)s > 0 \quad \text{for all} \quad s \in (-a, a) \setminus \{0\}. \quad (3) \]

Let \( g : \mathbb{R}^2 \to \mathbb{R} \) satisfy the following conditions:

\[ \begin{aligned} 
(H_g) & : g \text{ is a Carathéodory function, bounded on } \mathbb{R}^2, \text{ } g(\cdot, u) \text{ is } T\text{-periodic for any } u \in \mathbb{R} \text{ and some } T > 0, \text{ } g(x, \cdot) \text{ is } 2\pi\text{-periodic for a.e. } x \in \mathbb{R}, \text{ } G(x, u) := \int_0^u g(x, s) ds \text{ is bounded on } \mathbb{R}^2, \text{ and } G(x, \cdot) \text{ is } 2\pi\text{-periodic for a.e. } x \in \mathbb{R}. 
\end{aligned} \]

Let \( Lip_{T,a}(\mathbb{R}) = C^{0,1}_{T,a}(\mathbb{R}) \) denote the space of functions \( u : \mathbb{R} \to \mathbb{R} \) which are T-periodic and Lipschitzian with Lipschitz constant

\[ [u]_{0,1} := \sup_{x, y \in [0, T], x \neq y} \frac{|u(x) - u(y)|}{|x - y|} \leq a. \]
With the norm
\[ \|u\|_{0,1} := \max_{x \in [0,T]} |u(x)| + [u]_{0,1}, \]
\( \text{Lip}_{T,a}(\mathbb{R}) \) is a Banach space. Any element of \( \text{Lip}_{T,a}(\mathbb{R}) \) is a.e. differentiable and \( u' \) corresponds to the distributional derivative of \( u \).

Given \( h \in L^1_T(\mathbb{R}) \), where \( L^1_T(\mathbb{R}) \) denotes the space of locally Lebesgue integrable and \( T \)-periodic functions normed by \( \|h\|_1 = \int_0^T |h(x)| \, dx \), we write
\[ h := \frac{1}{T} \int_0^T h(x) \, dx, \]
so that
\[ \int_0^T \bar{h}(x) \, dx = 0. \]

Notice that if \( u \in \text{Lip}_{T,a}(\mathbb{R}) \), then \( \bar{u} \) vanishes at some \( y \in [0, T] \), and hence, for all \( x \in [0, T] \) (and consequently all \( x \in \mathbb{R} \), we have
\[ |\bar{u}(x)| = |\bar{u}(x) - \bar{u}(y)| \leq \int_0^T |u'(t)| \, dt \leq T[u]_{0,1}. \] (4)

For \( h \in L^\infty(\mathbb{R}) \) and \( T \)-periodic, we denote the usual norm by \( \|h\|_\infty \).

If \( K \) denotes the closed convex subset of \( \text{Lip}_{T,a}(\mathbb{R}) \) defined by
\[ K := \{ u \in \text{Lip}_{T,a}(\mathbb{R}) : |u'(x)| \leq a \text{ for a.e. } x \in \mathbb{R} \}, \]
then the action integral
\[ I(u) := \int_0^T \{ \Phi[u'(x)] + G(x, u(x)) + h(x)u(x) \} \, dx \] (5)
is well defined on \( K \). This happens for example when
\[ \Phi(s) = -\sqrt{1 - s^2}, \quad G(x, v) = A \cos v \]
for some \( A > 0 \), in which case (5) can be seen as the action integral associated to the relativistic forced pendulum.

The following lemma is useful to prove the lower semi-continuity of \( I \).

**Lemma 1** If assumption \((H_\Phi)\) holds, then, for any sequence \( (u_j)_{j \in \mathbb{N}} \) in \( K \) which converges uniformly on \([0, T]\) to some \( u \in K \), one has
\[ \liminf_{j \to \infty} \int_0^T \Phi[u'_j(x)] \, dx \geq \int_0^T \Phi[u'(x)] \, dx. \] (6)
Proof. For any \( \lambda \in (0, 1) \), we have, by assumption \((H_\Phi)\),
\[
\int_0^T \Phi[u'_j(x)] \, dx \geq \int_0^T \Phi[\lambda u'(x)] \, dx + \int_0^T \phi(\lambda u'(x)) \, dx.
\]
(7)

On the other hand, \( u'_j \) converges to \( u' \) for the \( w^* \)-topology \( \sigma(L^\infty, L^1) \). Since \( \phi(\lambda u') \in L^\infty(0, T) \), we deduce from (7) that
\[
\liminf_{j \to \infty} \int_0^T \Phi[u'_j(x)] \, dx \geq \int_0^T \Phi[\lambda u'(x)] \, dx + (1 - \lambda) \int_0^T \phi(\lambda u'(x)) u'(x) \, dx.
\]
Applying (3) we obtain
\[
\liminf_{j \to \infty} \int_0^T \Phi[u'_j(x)] \, dx \geq \int_0^T \Phi[\lambda u'(x)] \, dx,
\]
which gives (6) by letting \( \lambda \to 1 \).

We now prove the existence of a minimum to \( \mathcal{I} \) when \( \overline{h} = 0 \).

**Theorem 1** If assumptions \((H_\Phi)\) and \((H_g)\) hold, then, for any \( h \in L^1_T \) such that \( h = 0 \),
\[
\mathcal{I} \text{ has a minimum over } K.
\]

**Proof.** We first observe that, because of the \( 2\pi \)-periodicity of \( G(x, \cdot) \) and condition (8), we have, for all \( u \in K \),
\[
\mathcal{I}(u + 2\pi) = \mathcal{I}(u),
\]
so that, if \( u^* \) minimizes \( \mathcal{I} \) over \( K \), the same if true for \( u^* + 2j\pi \) for any integer \( j \). Hence, without loss of generality, we can search for a minimizer \( u^* \) such that \( u^* \in [0, 2\pi] \), i.e. we can minimize \( \mathcal{I} \) in the convex set
\[
\hat{K} := \{ u \in Lip_{T,a}(\mathbb{R}) : u \in [0, 2\pi], |u'(x)| \leq a \text{ for a.e. } x \in \mathbb{R} \}.
\]
Now, if \( u \in \hat{K} \), we have for all \( x \in \mathbb{R} \), using (4),
\[
|u(x)| \leq |\overline{u}| + |\tilde{u}(x)| \leq 2\pi + T|u|_{0,1} = 2\pi + Ta,
\]
so that \( \hat{K} \) is a bounded and equicontinuous subset of the space of continuous \( T \)-periodic functions. If \( (u_j)_{j \in \mathbb{N}} \) is a minimizing sequence for \( \mathcal{I} \) in \( \hat{K} \), we can assume, using Arzelá-Ascoli’s theorem and going if necessary to a subsequence, that \( (u_j)_{j \in \mathbb{N}} \) converges uniformly in \( \mathbb{R} \) to some continuous \( T \)-periodic \( u^* \). From the relations
\[
\frac{|u_j(x) - u_j(y)|}{|x - y|} \leq a \quad (x \neq y, j \in \mathbb{N})
\]
we easily get that \( u^* \in \hat{K} \). Consequently, using Lemma 1, we have
\[
\inf_{\hat{K}} \mathcal{I} = \lim_{j \to \infty} \mathcal{I}(u_j) \geq \mathcal{I}(u^*)
\]
so that \( u^* \) minimizes \( \mathcal{I} \) over \( \hat{K} \).
3 Variational inequality

The following lemma provides the variational inequality satisfied by a minimizer of $I$.

**Lemma 2** If $u$ minimizes $I$ over $K$, then

$$
\int_0^T \left( \Phi[u'(x)] - \Phi[u(x)] + \{g[x, u(x)] + h(x)\}[v(x) - u(x)] \right) \, dx \\
\geq 0 \quad \text{for all } v \in K.
$$

(9)

**Proof.** Let $v \in K$. By assumption, we have, for all $\lambda \in (0, 1]$,

$$
I(u) \leq I[u + \lambda(v - u)],
$$

i.e.

$$
\int_0^T \left( \Phi[u'(x) + \lambda(v'(x) - u'(x))] - \Phi[u'(x)] + G[x, u(x) + \lambda(v(x) - u(x))] \\
- G[x, u(x)] + \lambda h(x)[v(x) - u(x)] \right) \, dx \geq 0.
$$

Applying the convexity of $\Phi$ we deduce that

$$
\int_0^T \left( \Phi[v'(x)] - \Phi[u'(x)] + \lambda^{-1}\{G[x, u(x) + \lambda(v(x) - u(x))] - G[x, u(x)]\} \\
+ h(x)[v(x) - u(x)] \right) \, dx \geq 0.
$$

By Lebesgue dominated convergence theorem, we obtain, when $\lambda \searrow 0$,

$$
\int_0^T \left( \Phi[v'(x)] - \Phi[u'(x)] + g[x, u(x)][v(x) - u(x)] \\
+ h(x)[v(x) - u(x)] \right) \, dx \geq 0.
$$

4 Auxiliary problem

To obtain further information about the minimizer $u$, let us introduce the auxiliary problem

$$
(\phi(u'))' - u = f(x), \quad u \text{ is } T - \text{periodic},
$$

(10)

where $\phi$ satisfies Assumption $(H_\phi)$ and $f \in L_T^1(\mathbb{R})$. A (classical) solution of (10) is a T-periodic function $u \in C^1(\mathbb{R})$ such that $\phi \circ u'$ is absolutely continuous and (10) holds a.e. on $\mathbb{R}$.

The existence part of the following Lemma and the estimate for $u'$ is essentially a special case of Corollary 3 of [4]. The proof given there for $f$ continuous, based upon a reduction to a fixed point problem and Leray-Schauder degree, can immediately be adapted to the case where $f \in L_T^1(\mathbb{R})$. 
Lemma 3 For any $f \in L^1_T(\mathbb{R})$, problem (10) has a unique classical solution $u$, and $\|u'\|_{\infty} < 1$.

Proof. It remains to prove the uniqueness. If $u$ and $v$ are two solutions of (10), then we obtain

$$\int_0^T \left( [\phi(u'(x)) - \phi(v'(x))][u(x) - v(x)] - [u(x) - v(x)]^2 \right) dx = 0$$

and hence, integrating the first term by parts and using T-periodicity,

$$\int_0^T \left( [\phi(u'(x)) - \phi(v'(x))][u'(x) - v'(x)] + [u(x) - v(x)]^2 \right) dx = 0.$$

The monotonicity of $\phi$ implies the conclusion. □

Lemma 4 For any $f \in L^1_T(\mathbb{R})$, the unique solution $u$ of (10) belongs to $K$ and verifies the variational inequality

$$\int_0^T \left\{ \Phi[u'(x)] - \Phi[u'(x)] + [u(x) + f(x)][v(x) - u(x)] \right\} dx \geq 0 \quad \text{for all} \quad v \in K.$$

Proof. We have, using integration by parts and (10),

$$\int_0^T \left\{ \Phi[u'(x)] - \Phi[u'(x)] \right\} dx \geq \int_0^T \phi[u'(x)][v'(x) - u'(x)] dx
= -\int_0^T (\phi[u'(x)]')[v(x) - u(x)] dx = -\int_0^T [u(x) + f(x)][v(x) - u(x)] dx.$$

□

5 Periodic solutions of relativistic pendulum-like equations

We can now combine the results of the previous sections to obtain conditions for the existence of at least one (classical) T-periodic solution for the differential equation

$$(\phi(u'))' - g(x, u) = h(x). \quad (11)$$

A classical T-periodic solution of (11) is a T-periodic function $u \in C^1(\mathbb{R})$ such that $\phi \circ u'$ is absolutely continuous and (11) holds a.e. on $\mathbb{R}$.

Theorem 2 If assumptions $(H_\phi)$ and $(H_g)$ hold, then, for any $h \in L^1_T$ such that $\bar{h} = 0$, equation (11) has at least one T-periodic solution which minimizes $\mathcal{I}$ over $K.$
Proof. Let \( u \) be a minimizer of \( I \) over \( K \). Then, by Lemma 2, \( u \) satisfies the variational inequality (9). This variational inequality can be written

\[
\int_0^T \left\{ \Phi'[v'(x)] - \Phi'[u'(x)] + u(x)[v(x) - u(x)] 
+ [g[x, u(x)] + h(x) - u(x)][v(x) - u(x)] \right\} \, dx \geq 0 \quad \text{for all } \, v \in K,
\]

so that \( u \) is a solution of the variational inequality

\[
\int_0^T \left\{ \Phi'[v'(x)] - \Phi'[\hat{u}'_w(x)] + \hat{u}_w(x)[v(x) - \hat{u}_w(x)] 
+ f_w(x)[v(x) - \hat{u}_w(x)] \right\} \, dx \geq 0 \quad \text{for all } \, v \in K,
\]

where

\[ f_u = g[\cdot, u(\cdot)] + h - u \in L^1_T(\mathbb{R}). \]

Now, given any \( w \in K \), the unique solution \( \hat{u}_w \) of problem (10) with \( f = f_w \) satisfies, by Lemma 4,

\[
\int_0^T \left\{ \Phi'[v'(x)] - \Phi'[\hat{u}'_w(x)] + \hat{u}_w(x)[v(x) - \hat{u}_w(x)] 
+ f_w(x)[v(x) - \hat{u}_w(x)] \right\} \, dx \geq 0 \quad \text{for all } \, v \in K.
\]

Chosing \( v = \hat{u}_u \) in (12), \( w = v = u \) (\( u \) the minimizer of \( I \) over \( K \)) in (13), and adding the resulting inequalities, we obtain

\[
\int_0^T [u(x) - \hat{u}_u(x)]^2 \, dx \leq 0.
\]

It follows from (14) that \( u = \hat{u}_u \) and hence that \( \|u'\|_{\infty} = \|u_x\|_{\infty} < a \). Moreover \( u \) is a classical \( T \)-periodic solution of (11), since \( \hat{u}_u \) is a classical \( T \)-periodic solution of (10) with \( f = f_u \).

**Corollary 1** For any \( T > 0 \), \( A \in \mathbb{R} \), and \( h \in L^1_T(\mathbb{R}) \) such that \( \overline{h} = 0 \), the relativistic pendulum equation

\[
\left( \frac{u'}{\sqrt{1 - u'^2}} \right)' + A \sin u = h(x)
\]

has at least one classical \( T \)-periodic solution.

**Proof.** It suffices to take \( \Phi(s) = -\sqrt{1 - s^2} \), so that \( \phi(s) = \frac{s}{\sqrt{1 - s^2}} \), and \( G(x, u) = A \cos u \), so that \( g(x, u) = -A \sin u \). Assumptions \( H_\Phi \) with \( a = 1 \) and \( H_g \) hold. \( \blacksquare \)
Remark 1 It would be interesting to investigate similar questions in higher dimensions, for example

\[ \min_{u \in K} \int_{\mathbb{T}^N} \left\{ -\sqrt{1-|\nabla u|^2} + G[x, u(x)] + h(x)u(x) \right\} \, dx, \]

where \( \mathbb{T}^N \) denotes the N-dimensional torus,

\[ K := \{ u \in \text{Lip}(\mathbb{T}^N) : |\nabla u(x)| \leq 1 \text{ for a.e. } x \in \mathbb{T}^N \}. \]

Bartnik and Simon [1] have studied related questions under Dirichlet boundary conditions.

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References


