
**Dedicated to the memory of Guido Stampacchia, a beloved mentor and friend**

**Abstract.** — In 1964 J. Serrin proposed the following conjecture. Let \( u \in W^{1,1}_{\text{loc}}(\Omega) \) be a weak solution of the second order elliptic equation (1) below, in divergence form, with Hölder continuous coefficients \( a_{ij}(x) \); then \( u \) is a “classical” solution. We announce a solution of this conjecture assuming only \( u \in BV_{\text{loc}}(\Omega) \) and Dini continuous coefficients.

**Key words:** Divergence elliptic equations; very weak solutions.

**Mathematics Subject Classification (2000):** Primary 35J15.

Let \( \Omega \subset \mathbb{R}^N, N \geq 2 \), be a bounded domain and let \( u \in W^{1,1}_{\text{loc}}(\Omega) \) be a weak solution of the equation

\[
\sum_{i,j} \frac{\partial}{\partial x_j} \left( a_{ij} \frac{\partial u}{\partial x_i} \right) = 0 \quad \text{in } \Omega,
\]

where the coefficients \( a_{ij}(x) \) are bounded measurable and elliptic, i.e.,

\[
\lambda |\xi|^2 \leq \sum_{i,j} a_{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2, \quad x \in \Omega, \ \xi \in \mathbb{R}^N,
\]

with \( 0 < \lambda \leq \Lambda < \infty \). A weak solution \( u \in W^{1,1}_{\text{loc}}(\Omega) \) satisfies, by definition,

\[
\sum_{i,j} \int a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \xi}{\partial x_i} = 0 \quad \forall \xi \in C^1_c(\Omega)
\]

where the subscript \( c \) indicates compact support.

A celebrated result of E. De Giorgi [4] asserts that if \( u \) is a weak solution of (1) and moreover \( u \in H^1_{\text{loc}}(\Omega) \), then \( u \) is locally Hölder continuous, and in particular \( u \in L^\infty_{\text{loc}}(\Omega) \) (see also [13]). Subsequently J. Serrin produced in [11] a striking example showing that the assumption \( u \in H^1_{\text{loc}}(\Omega) \) is essential; more precisely, for every \( p, \ 1 < p < 2 \), and all \( N \geq 2 \), he constructed an equation of the form (1) which has a solution \( u \in W^{1,p}(\Omega) \) and \( u \notin L^\infty_{\text{loc}}(\Omega) \). J. Serrin conjectured in [11] that if the coefficients \( a_{ij} \) are locally Hölder continuous, then any weak solution \( u \in W^{1,1}_{\text{loc}}(\Omega) \) of
must be a “classical” solution, i.e., \( u \in H^{1,\text{loc}}(\Omega) \). Serrin’s conjecture was confirmed by R. A. Hager and J. Ross \([7]\) provided \( u \) is a weak solution of class \( W^{1,p}(\Omega) \) for some \( p \) with \( 1 < p < 2 \).

We announce here the solution of Serrin’s conjecture in full generality, starting with \( u \in W^{1,1}(\Omega) \), or even with \( u \in BV^{\text{loc}}(\Omega) \), i.e., \( u \in L^1_{\text{loc}}(\Omega) \) and its derivatives (in the sense of distributions) are measures.

Our first result is an improvement of the theorem of Hager and Ross: instead of \( a_{ij} \in C^0(\overline{\Omega}) \) for some \( \alpha \in (0,1) \), we assume only \( a_{ij} \in C^0(\overline{\Omega}) \).

**THEOREM 1.** Assume \( a_{ij} \in C^0(\overline{\Omega}) \) and \( u \in W^{1,p}(\Omega) \) for some \( p > 1 \). If \( u \) is a weak solution of (1), then \( u \in W^{1,q}_{\text{loc}}(\Omega) \) for every \( q < \infty \). Moreover,

\[
\|u\|_{W^{1,q}(\omega)} \leq C \|u\|_{W^{1,p}(\Omega)}
\]

for every \( \omega \subset \subset \Omega \), where \( C \) depends only on \( N, \lambda, \Lambda, p, q, \omega, \Omega \), and the modulus of continuity of \( a_{ij} \) on \( \overline{\Omega} \).

**OPEN PROBLEM.** We do not know whether the conclusion of Theorem 1 holds in the two limiting cases: \( p = 1 \) and/or \( q = \infty \). (The answer to both questions is positive if the coefficients \( a_{ij} \) are Dini continuous; see Theorem 2 below.)

We now turn to Serrin’s conjecture. Here we assume that the coefficients \( a_{ij} \) are Dini continuous in \( \overline{\Omega} \), i.e., \( a_{ij} \in C^0(\overline{\Omega}) \) and

\[
A(r) = \sum_{i,j} \sup_{x,y \in \Omega, |x-y| < r} |a_{ij}(x) - a_{ij}(y)|, \quad r > 0,
\]

satisfies

\[
\int_0^1 \frac{A(r)}{r} \, dr < \infty.
\]

**THEOREM 2.** Assume that the coefficients \( a_{ij} \) are Dini continuous in \( \overline{\Omega} \), and let \( u \in BV(\Omega) \) be a weak solution of (1). Then \( u \in H^{1,\text{loc}}(\Omega) \). Moreover,

\[
\|u\|_{H^1(\omega)} \leq C \|u\|_{BV(\Omega)}
\]

for every \( \omega \subset \subset \Omega \), where \( C \) depends only on \( N, \lambda, \Lambda, \omega, \Omega \), and the modulus of continuity of \( a_{ij} \) on \( \overline{\Omega} \).

**REMARK 1.** Surprisingly, the constant \( C \) in (5) depends only on the modulus of continuity of \( a_{ij} \) in \( \overline{\Omega} \), and not on the Dini modulus of continuity of \( a_{ij} \) in \( \overline{\Omega} \). This suggests that Serrin’s conjecture might be true assuming only continuity of \( a_{ij} \) in \( \overline{\Omega} \) (this corresponds to the open problem mentioned above with \( p = 1 \)).

**REMARK 2.** Using Lemma 2 below we may assert that, under the assumptions of Theorem 2, \( u \in C^1(\Omega) \). If the coefficients \( a_{ij} \) belong to \( C^{0,\alpha}(\overline{\Omega}) \), \( 0 < \alpha < 1 \),
one can further improve the conclusion of Theorem 2, namely $u \in C^{1,\alpha}(\bar{\omega})$ for every $\omega \subset\subset \Omega$. This is a consequence of the standard Schauder regularity theory for elliptic equations in divergence form with $C^{0,\alpha}$ coefficients (see e.g. [10, Theorem 5.5.3(b)], [6, Theorem 3.7], [5, Theorem 3.5], or [3, Theorem 2.6 in Chapter 9]). All the above results extend to elliptic systems.

The proofs use a duality technique in conjunction with a bootstrap argument and the following tools:

a) Standard $L^p$-regularity theory for elliptic equations in divergence form:

**Lemma 1** (see e.g. [10, Theorem 5.5.3(a)], or [3, Theorem 2.2 in Chapter 10]). Assume $a_{ij} \in C^0(\bar{\Omega})$ and $u \in H^1(\Omega)$ is a weak solution of

\[
\sum_{i,j} \frac{\partial}{\partial x_j} \left( a_{ij} \frac{\partial u}{\partial x_i} \right) = \sum_j \frac{\partial}{\partial x_j} f_j
\]

with $f_j \in L^p(\Omega)$ for all $j$, and $p \in (2, \infty)$. Then $u \in W^{1,p}_{\text{loc}}(\Omega)$, and for $\omega \subset\subset \Omega$,

\[
\|u\|_{W^{1,p}(\omega)} \leq C \left( \|u\|_{H^1(\Omega)} + \sum_j \|f_j\|_{L^p(\Omega)} \right)
\]

where $C$ depends on $N$, $\lambda$, $\Lambda$, $p$, $\omega$, $\Omega$, and the modulus of continuity of $a_{ij}$.

b) Schauder regularity theory for elliptic equations in divergence form with Dini continuous coefficients:

**Lemma 2.** Assume $a_{ij} \in C^0(\bar{\Omega})$ satisfy (3), (4), and let $u \in H^1(\Omega)$ be a weak solution of (6) with $f_j \in C^{\infty}_c(\Omega)$ for all $j$. Then $u \in C^1(\Omega)$.

The conclusion of Lemma 2 comes with an estimate of the Dini modulus of continuity of $Du$ involving the Dini modulus of continuity of $a_{ij}$. However, we do not need such an estimate—we use only the qualitative form of Lemma 2; this explains Remark 1. It is not easy to find an early reference for Lemma 2. According to the experts (I am quoting M. Giaquinta), it was common knowledge in Pisa in the late 60’s—the proof being based on Campanato’s approach to Schauder estimates (as presented in [6] or [3]), combined with a result of S. Spanne (Corollary 1 in [12]). A complete proof may be found e.g. in [9, Theorem 5.1]. Y. Li ([8]) has obtained a similar conclusion (also valid for systems) under weaker assumptions on the coefficients $a_{ij}$.

The detailed proofs of Theorems 1 and 2 are presented as an Appendix in [1]. Theorem 2 is used in the paper of A. Ancona [1], who solved some open problems raised in [2], and called my attention to Serrin’s conjecture.

Acknowledgements. I am grateful to Alano Ancona and Yanyan Li for very fruitful discussions. I thank L. Boccardo, M. Giaquinta, G. Mingione, L. Orsina and J. Serrin for useful information regarding references. The author is partially supported by NSF Grant DMS-0802958.
REFERENCES


Received 17 November 2008,
and in revised form 19 November 2008.

Department of Mathematics
Rutgers University
Hill Center, Busch Campus
110 Frelinghuysen Road
PISCATAWAY, NJ 08854, USA
brezis@math.rutgers.edu
and
Department of Mathematics
Technion
32.000 HAIFA, Israel