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Circulation integrals and critical Sobolev spaces: problems of optimal constants

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Dedicated to Vladimir Maz'ya on the occasion of his 70th birthday, with high esteem and friendship

Abstract. We study various questions related to the best constants in the following inequalities established in [**1, 2, 3**];

$$\Big|\int_\Gamma \vec{\varphi}\cdot\vec{t}\Big| \le C_n \|\nabla\varphi\|_{\mathrm{L}^n}|\Gamma| ,$$

and

$$\Big|\int_{\mathbf{R}^n} \vec{\varphi}\cdot\vec{\mu}\Big| \le C_n \|\nabla\varphi\|_{\mathrm{L}^n}\|\vec{\mu}\| ,$$

where Γ is a closed curve in $\mathbf{R}^n$, $\vec{\varphi}\in C_c^\infty(\mathbf{R}^n;\mathbf{R}^n)$ and $\vec{\mu}$ is a bounded measure on $\mathbf{R}^n$ with values into $\mathbf{R}^n$ such that $\operatorname{div}\vec{\mu}=0$. In 2d the answers are rather complete and closely related to the best constants for Sobolev and isoperimetric inequalities.

1. Estimates for curves and divergence-free vector fields

In general, functions in the Sobolev space $\mathrm{W}^{1,n}(\mathbf{R}^n)$ do not need be bounded continuous functions. However, it was shown recently that in circulation integrals, such functions behave as if they were bounded continuous functions:

Theorem 1 (Bourgain, Brezis and Mironescu [**4**]). *There exists $C>0$ depending only on n such that if Γ is a closed rectifiable oriented curve whose tangent vector is $\vec{t}$, for every $\vec{\varphi}\in C_c^\infty(\mathbf{R}^n;\mathbf{R}^n)$,*

$$\Big|\int_\Gamma \vec{\varphi}\cdot\vec{t}\Big| \le C\|\nabla\vec{\varphi}\|_{\mathrm{L}^n(\mathbf{R}^n)}|\Gamma| , \tag{1.1}$$

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Here and in the sequel, the subscript c in $C_c^\infty(\mathbf{R}^n;\mathbf{R}^n)$ means compact support. The original proof of Theorem 1 relied on a Littlewood–Paley decomposition; an elementary proof was given by the second author [**18**].

In two-dimensional space, this estimate follows easily from the classical isoperimetric theorem. This is not anymore the case in higher dimensions, but the inequality still retains some isoperimetric flavour.

The aim of this paper is to present various problems concerning optimal constants in Theorem 1. A first family of problems consists in fixing the curve Γ and finding the optimal constant A_Γ and the optimal vector field $\vec{\varphi}$. Such a vector field always exists. In two dimensions, the value of the associated optimal constant can even be given in terms of the area enclosed by Γ.

The next problem consists in optimizing A_Γ over all possible curves i.e., studying $A = \sup_\Gamma A_\Gamma$. In $\mathbf{R}^2$, one merely needs to maximize the area under a constraint on the perimeter; the classical isoperimetric theorem then yields a circle as the optimal curve. Similar results in higher dimensions would be desirable. However, none of the two-dimensional arguments work. The question whether the optimal constant is attained is even open.

As explained by Bourgain and Brezis [**2**], the optimal constant A in Theorem 1 is the same as the optimal constant in

Theorem 2 (Bourgain and Brezis [**2, 3**]). *There exists $C > 0$ such that if $\vec{\mu}$ is a divergence-free bounded measure, then for every $\vec{\varphi} \in C_c^\infty(\mathbf{R}^n;\mathbf{R}^n)$,*

$$\left|\int_{\mathbf{R}^n} \vec{\varphi}\cdot\vec{\mu}\right| \le C\|\nabla\vec{\varphi}\|_{\mathrm{L}^n(\mathbf{R}^n)}\|\vec{\mu}\| \,. \tag{1.2}$$

Theorem 2 also has a direct proof by Littlewood–Paley decompostion [**3**], and an elementary proof [**17**]. The latter inequality provides thus a relaxed problem for the optimal constants, which could be more tractable.

This paper is organized as follows. In section 2, we explain how for a given curve, one can find an optimal vector field, how this leads to the optimal problem, how the problem of the optimal constant for divergence-free measures is the same as the problem for closed curves. We end this sections by some problems related to the confinement of the inequalities to domains. In section 3, we show that for a given divergence-free vector measure, one can find an optimal vector field. We then raise various questions concerning the optimal constant in the inequality.

2. The two-dimensional case

2.1. Curves in 2d. Throughout this section, we assume that Γ is a simple, closed, smooth, oriented curve with tangent vector $\vec{t}$. The optimal constant for the inequality of Theorem 1 is given by

$$A_\Gamma = \sup\Big\{\int_\Gamma \vec{\varphi}\cdot\vec{t}\ :\ \vec{\varphi}\in C_c^\infty(\mathbf{R}^2;\mathbf{R}^2) \text{ and } \int_{\mathbf{R}^2}|\nabla\vec{\varphi}|^2 \le 1\Big\}\,,$$

where $|\cdot|$ denotes the Euclidean norm (of a vector or a matrix). By Theorem 1, this can be rewritten as

$$A_\Gamma = \sup\Big\{\int_\Gamma \vec{\varphi}\cdot\vec{t}\ :\ \vec{\varphi}\in \dot{\mathrm{H}}^1(\mathbf{R}^2;\mathbf{R}^2) \text{ and } \int_{\mathbf{R}^2}|\nabla\vec{\varphi}|^2 \le 1\Big\}\,,$$

where

$$\dot{\mathrm{H}}^1(\mathbf{R}^2;\mathbf{R}^2) = \Big\{\vec{\varphi}\in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbf{R}^2;\mathbf{R}^2)\ :\ \nabla\vec{\varphi}\in \mathrm{L}^2(\mathbf{R}^2;\mathbf{R}^{2\times 2})\Big\}$$

is the completion of $C_c^\infty(\mathbf{R}^2;\mathbf{R}^2)$ equipped with the norm $\|\nabla\vec\varphi\|_{\mathrm{L}^2}$. Note that there are two possible definitions for $\int_\Gamma \vec\varphi\cdot\vec t$ when $\vec\varphi\in\dot{\mathrm{H}}^1(\mathbf{R}^2;\mathbf{R}^2)$. The first one is in the sense of traces; secondly, starting from (1.1), one can define $\int_\Gamma \vec\varphi\cdot\vec t$ abstractly via density. Both definitions coincide.

Since the space $\dot{\mathrm{H}}^1(\mathbf{R}^2;\mathbf{R}^2)$ modulo constants is a Hilbert space, one has

PROPOSITION 2.1. *There exists a unique $\vec\varphi\in\dot{\mathrm{H}}^1(\mathbf{R}^2;\mathbf{R}^2)$ (modulo constants) such that*

$$\int_\Gamma \vec\varphi\cdot\vec t = A_\Gamma\ , \tag{2.1}$$

$$\int_{\mathbf{R}^2}|\nabla\vec\varphi|^2 = 1\ . \tag{2.2}$$

The vector field $\vec\varphi$ satisfies the equation,

$$-\Delta\vec\varphi = \frac{\vec t}{A_\Gamma}\mathcal{H}^1\llcorner\Gamma \qquad \textit{in } \mathbf{R}^2\ , \tag{2.3}$$

$$\operatorname{div}\vec\varphi = 0 \qquad \textit{in } \mathbf{R}^2\ , \tag{2.4}$$

$$\vec\varphi\in(\mathrm{L}^\infty\cap C)(\mathbf{R}^2;\mathbf{R}^2)\ , \tag{2.5}$$

where $\mathcal{H}^1\llcorner\Gamma$ denotes the 1–dimensional Hausdorff measure restricted to Γ.

Note that the additional property (2.5) allows to give a classical meaning to $\int_\Gamma\vec\varphi\cdot\vec t$.

PROOF OF PROPOSITION 2.1. Since $\dot{\mathrm{H}}^1(\mathbf{R}^2;\mathbf{R}^2)$ is a Hilbert space, there exists a unique $\vec\varphi$ such that

$$\int_\Gamma\vec\varphi\cdot\vec t = A_\Gamma \qquad\text{and}\qquad \int_{\mathbf{R}^2}|\nabla\vec\varphi|^2 = 1\ .$$

The vector field $\vec\varphi$ satisfies the Euler equation

$$-\lambda\Delta\vec\varphi = \vec t\mathcal{H}^1\llcorner\Gamma$$

for some Lagrange multiplier $\lambda\in\mathbf{R}$. Obviously $\lambda = A_\Gamma$. In particular, $\operatorname{div}\vec\varphi$ satisfies

$$-\Delta\operatorname{div}\vec\varphi = 0\ .$$

Since $\operatorname{div}\vec\varphi\in\mathrm{L}^2(\mathbf{R}^2)$, one has necessarily $\operatorname{div}\vec\varphi = 0$. Finally, since $-\Delta\vec\varphi$ is a divergence-free bounded measure, the boundedness and the continuity of $\vec\varphi$ follow from the results of Bourgain and Brezis (see [**2**, Remark 5] and [**3**, Theorem 3]). $\square$

The constant A_Γ can be determined explicitely:

PROPOSITION 2.2. *If $\Gamma=\partial\Sigma$, then*

$$A_\Gamma = |\Sigma|^{\frac12}\ .$$

PROOF. To fix the ideas, we shall assume that Γ is positively oriented. Applying successively Stokes' theorem and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\int_\Gamma\vec\varphi\cdot\vec t &= \int_\Sigma\nabla\wedge\vec\varphi \le |\Sigma|^{1/2}\Big(\int_{\mathbf{R}^2}|\nabla\wedge\vec\varphi|^2\Big)^{\frac12}\\ &\le |\Sigma|^{1/2}\Big(\int_{\mathbf{R}^2}|\nabla\wedge\vec\varphi|^2+|\operatorname{div}\vec\varphi|^2\Big)^{\frac12} = |\Sigma|^{1/2}\Big(\int_{\mathbf{R}^2}|\nabla\vec\varphi|^2\Big)^{\frac12}\ ,\end{aligned}\tag{2.6}$$

where $\nabla \wedge \vec{\varphi} = \partial_1 \varphi_2 - \partial_2 \varphi_1$. This proves that $A_\Gamma \le |\Sigma|^{\frac{1}{2}}$.

In order to obtain the equality, choose $\vec{\varphi} \in \dot{\mathrm{H}}^1(\mathbf{R}^2; \mathbf{R}^2)$ to be the solution of

$$\begin{cases} \nabla \wedge \vec{\varphi} = 1 & \text{in } \Sigma\ , \\ \nabla \wedge \vec{\varphi} = 0 & \text{in } \mathbf{R}^2 \setminus \Sigma\ , \\ \operatorname{div} \vec{\varphi} = 0 & \text{in } \mathbf{R}^2\ . \end{cases}$$

By construction, $\vec{\varphi}$ realizes the equality in (2.6); hence, $A_\Gamma = |\Sigma|^{\frac{1}{2}}$. □

The main result is

PROPOSITION 2.3. *One has*

$$A_\Gamma \le \frac{|\Gamma|}{2\sqrt{\pi}}\ . \tag{2.7}$$

Moreover, equality holds in (2.7) *if, and only if,* Γ *is a circle.*

PROOF. Writing $\Gamma = \partial \Sigma$, we combine Proposition 2.2 together with the classical isoperimetric inequality, in order to obtain

$$A_\Gamma = |\Sigma|^{\frac{1}{2}} \le \frac{|\Gamma|}{2\sqrt{\pi}}.$$

Moreover, one has equality if and only if $\Gamma = \partial\Sigma$ is a circle. □

2.2. A relaxed problem. Let $\vec{\mu}$ be a bounded $\mathbf{R}^2$–valued vector measure such that $\operatorname{div} \vec{\mu} = 0$ in the sense of distributions, and set

$$A_{\vec{\mu}} = \sup\Big\{ \int_{\mathbf{R}^2} \vec{\varphi} \cdot \vec{\mu}\ :\ \vec{\varphi} \in C_c^\infty(\mathbf{R}^2; \mathbf{R}^2) \text{ and } \int_{\mathbf{R}^2} |\nabla \vec{\varphi}|^2 \le 1 \Big\}\ .$$

A particular case is when Γ is as in Section 2.1 and the measure is $\vec{\mu} = \vec{t}\,\mathcal{H}^1 \llcorner \Gamma$; one has then $A_{\vec{\mu}} = A_\Gamma$.

By Theorem 2, the integral $\int_{\mathbf{R}^2} \vec{\varphi} \cdot \vec{\mu}$ is well-defined when $\vec{\varphi} \in \dot{\mathrm{H}}^1(\mathbf{R}^2; \mathbf{R}^2)$ as an extension of a linear functional[1]. One can thus relax the supremum to

$$A_{\vec{\mu}} = \sup\Big\{ \int_{\mathbf{R}^2} \vec{\varphi} \cdot \vec{\mu}\ :\ \vec{\varphi} \in \dot{\mathrm{H}}^1(\mathbf{R}^2; \mathbf{R}^2) \text{ and } \int_{\mathbf{R}^2} |\nabla \vec{\varphi}|^2 \le 1 \Big\}\ .$$

As previously, one can show that if $\vec{\mu} \ne 0$, $A_{\vec{\mu}}$ is attained by a vector field $\vec{\varphi} \in (\dot{\mathrm{H}}^1 \cap \mathrm{L}^\infty)(\mathbf{R}^2; \mathbf{R}^2)$ that satisfies

$$-\Delta \vec{\varphi} = \frac{1}{A_{\vec{\mu}}} \vec{\mu}\ , \tag{2.8}$$

so that in particular $\operatorname{div} \vec{\varphi} = 0$. By the result of Bourgain and Brezis mentioned above, $\vec{\varphi}$ is a bounded continuous vector field.

One can also compute explicitely the value $A_{\vec{\mu}}$. Recall that by the Sobolev–Nirenberg embedding, if $\vec{\mu}$ is divergence-free, it can be written as $\vec{\mu} = \nabla^\perp \zeta$, for some unique $\zeta \in \mathrm{L}^2(\mathbf{R}^2)$, where $\nabla^\perp \zeta = (\partial_2 \zeta, -\partial_1 \zeta)$.

PROPOSITION 2.4. *Let* $\vec{\mu}$ *and* ζ *be as above, then*

$$A_{\vec{\mu}} = \Big(\int_{\mathbf{R}^2} |\zeta|^2 \Big)^{\frac{1}{2}}\ .$$

[1] In fact, one can even show using Theorem 2 that null sets for the H^1 capacity in $\mathbf{R}^2$ are null sets for the variation $|\vec{\mu}|$ of $\vec{\mu}$, so that every H^1– quasicontinuous function is $\vec{\mu}$–measurable. In particular, the integral makes sense for every $\vec{\varphi} \in (\dot{\mathrm{H}}^1 \cap \mathrm{L}^\infty)(\mathbf{R}^2; \mathbf{R}^2)$.

PROOF. One has, by the Cauchy-Schwarz inequality, for every $\vec{\varphi} \in C_c^\infty(\mathbf{R}^2;\mathbf{R}^2)$,

$$\begin{aligned}\int_{\mathbf{R}^2} \vec{\varphi}\cdot\vec{\mu} = \int_{\mathbf{R}^2} \nabla^\perp\zeta\cdot\vec{\varphi} &= \int_{\mathbf{R}^2} \zeta(\nabla\wedge\vec{\varphi})\\ &\le \Big(\int_{\mathbf{R}^2}|\zeta|^2\Big)^{\frac12}\Big(\int_{\mathbf{R}^2}|\nabla\wedge\vec{\varphi}|^2\Big)^{\frac12}\\ &\le \Big(\int_{\mathbf{R}^2}|\zeta|^2\Big)^{\frac12}\Big(\int_{\mathbf{R}^2}|\nabla\wedge\vec{\varphi}|^2+|\operatorname{div}\vec{\varphi}|^2\Big)^{\frac12}\\ &= \Big(\int_{\mathbf{R}^2}|\zeta|^2\Big)^{\frac12}\Big(\int_{\mathbf{R}^2}|\nabla\vec{\varphi}|^2\Big)^{\frac12}\ ,\end{aligned}$$

whence one obtains an upper bound

$$A_{\vec{\mu}} \le \Big(\int_{\mathbf{R}^2}|\zeta|^2\Big)^{\frac12}\ .$$

For the equality, observe that the solution $\vec{\varphi}\in \dot{\mathrm{H}}^1(\mathbf{R}^2;\mathbf{R}^2)$ to

$$\begin{cases}\nabla\wedge\vec{\varphi} = \zeta\ ,\\ \operatorname{div}\vec{\varphi} = 0\ ,\end{cases}$$

achieves equality in the above inequality. □

PROPOSITION 2.5. *One has*

$$A_{\vec{\mu}} \le \frac{\|\vec{\mu}\|}{2\sqrt{\pi}}\ , \tag{2.9}$$

where

$$\|\vec{\mu}\| = \sup\Big\{\int_{\mathbf{R}^2}\vec{\varphi}\cdot\vec{\mu}\ :\ \forall x\in\mathbf{R}^2,\ |\vec{\varphi}(x)|\le 1\Big\}\ .$$

Moreover, equality holds in (2.9) *if, and only if,*

$$\vec{\mu} = \lambda\nabla^\perp\chi_{B(x,r)}$$

for some $\lambda\in\mathbf{R}$, $x\in\mathbf{R}^2$ and $r>0$.

PROOF. By the optimal Sobolev–Nirenberg inequality, one has

$$\|\zeta\|_{\mathrm{L}^2(\mathbf{R}^2)} \le \frac{1}{2\sqrt{\pi}}\|\nabla\zeta\|_{\mathrm{L}^1(\mathbf{R}^2;\mathbf{R}^2)} = \frac{1}{2\sqrt{\pi}}\|\vec{\mu}\|\ ,$$

with equality if and only if ζ is a multiple of a characteristic function of a ball; this result is originaly due to H. Federer and W. H. Fleming [**8**], and independently to V. G. Maz'ya [**13**], see e.g. [**14**, Lemma 3.2.3/1] and also [**6, 9**]. □

REMARK 2.6. In view of the result of Smirnov [**16**] representing any divergence-free measure $\vec{\mu}$ with $\|\vec{\mu}\| = 1$ as a "convex combination" of measures of the form

$$\frac{\vec{t}\,\mathcal{H}^1\llcorner\Gamma}{|\Gamma|}\ ,$$

it is clear that (2.7) implies (2.9). The main interest of Proposition 2.5 is that the relaxation to Lipschitz curves (having possibly self-intersections), or to divergence-free measures, does not introduce new maximizers.

There is a slight improvement of Proposition 2.5

Corollary 2.7. *For every* $\vec\varphi \in \dot{\mathrm{H}}^1(\mathbf{R}^2;\mathbf{R}^2)$,

$$\int_{\mathbf{R}^2} \vec\varphi\cdot\vec\mu \le \frac{\|\vec\mu\|}{2\sqrt{\pi}}\Bigl(\int_{\mathbf{R}^2}|\nabla\wedge\vec\varphi|^2\Bigr)^{\frac12}\ .$$

Moreover there is equality in the nontrivial case if, and only if,

$$\vec\mu = \lambda\nabla^\perp\chi_{B(x,r)} \tag{2.10}$$

and

$$\nabla\wedge\vec\varphi = \nu\lambda\chi_{B(x,r)}\ , \tag{2.11}$$

for some $\lambda\in\mathbf{R}$ *and* $\nu>0$, $x\in\mathbf{R}^2$ *and* $r>0$.

Proof. Given $\vec\varphi \in \dot{\mathrm{H}}^1(\mathbf{R}^2;\mathbf{R}^2)$, let $\vec\psi$ solve

$$\begin{cases} \nabla\wedge\vec\psi = \nabla\wedge\vec\varphi\ ,\\ \operatorname{div}\vec\psi = 0\ . \end{cases}$$

One then has

$$\begin{aligned}\int_{\mathbf{R}^2}\vec\varphi\cdot\vec\mu = \int_{\mathbf{R}^2}\vec\psi\cdot\vec\mu &\le \frac{\|\vec\mu\|}{2\sqrt{\pi}}\Bigl(\int_{\mathbf{R}^2}|\nabla\vec\psi|^2\Bigr)^{\frac12}\\ &= \frac{\|\vec\mu\|}{2\sqrt{\pi}}\Bigl(\int_{\mathbf{R}^2}|\nabla\wedge\vec\psi|^2\Bigr)^{\frac12} = \frac{\|\vec\mu\|}{2\sqrt{\pi}}\Bigl(\int_{\mathbf{R}^2}|\nabla\wedge\vec\varphi|^2\Bigr)^{\frac12}\ .\end{aligned}$$

If equality holds, one has by Proposition 2.5,

$$\vec\mu = \lambda\nabla^\perp\chi_{B(x,r)}\ ,$$

so that, by (2.8),

$$-\Delta\vec\psi = \nu\vec\mu = \nu\lambda\nabla^\perp\chi_{B(x,r)}\ .$$

On the other hand,

$$\nabla^\perp(\nabla\wedge\vec\varphi) = \nabla^\perp(\nabla\wedge\vec\psi) = -\Delta\vec\psi$$

since $\operatorname{div}\vec\psi = 0$. Therefore,

$$\nabla^\perp(\nabla\wedge\vec\varphi) = \nu\lambda\nabla^\perp\chi_{B(x,r)}$$

and consequently $\nabla\wedge\vec\varphi = \nu\lambda\chi_{B(x,r)}$. □

2.3. Confinement to domains. Let $\Omega\subset\mathbf{R}^2$ be a smooth, bounded, simply connected domain with normal vector field $\vec n$. We will work with the class of measures on $\bar\Omega$ satisfying $\operatorname{div}\vec\mu = 0$ in Ω and $\vec\mu\cdot\vec n = 0$ on $\partial\Omega$. This class is defined as

$$\mathcal{M}_0^\#(\bar\Omega,\mathbf{R}^2) = \Bigl\{\vec\mu\in C(\bar\Omega;\mathbf{R}^2)^*\ :\ \forall\zeta\in C^1(\bar\Omega),\ \int_{\bar\Omega}\nabla\zeta\cdot\vec\mu = 0\Bigr\}\ .$$

Note that $\int_{\bar\Omega}\vec\mu = 0$ for every $\vec\mu\in\mathcal{M}_0^\#(\bar\Omega,\mathbf{R}^2)$ (just take $\zeta(x) = x_i$, $i = 1,2$ in the definition).

Proposition 2.8. *For every* $\vec\varphi\in\mathrm{H}^1(\Omega)\cap C(\bar\Omega)$ *and* $\vec\mu\in\mathcal{M}_0^\#(\bar\Omega,\mathbf{R}^2)$

$$\int_\Omega\vec\varphi\cdot\vec\mu \le \frac{\|\vec\mu\|}{2\sqrt{\pi}}\|\nabla\wedge\vec\varphi\|_{\mathrm{L}^2(\Omega)}\ . \tag{2.12}$$

Equality holds in the nontrivial cases if $\vec\mu = \lambda\vec t\mathcal{H}^1\llcorner\partial B(x,r)$ *with* $\partial B(x,r)\subset\bar\Omega$ *and* $\vec\varphi$ *satisfies* (2.11) *for* $\nu>0$.

PROOF. Let $\vec{\psi} \in \dot{H}^1(\mathbf{R}^2; \mathbf{R}^2)$ satisfy

$$\begin{cases} \nabla \wedge \vec{\psi} = \nabla \wedge \vec{\varphi} & \text{in } \Omega \ , \\ \nabla \wedge \vec{\psi} = 0 & \text{in } \mathbf{R}^2 \setminus \Omega \ , \\ \operatorname{div} \vec{\psi} = 0 & \text{in } \mathbf{R}^2 \ , \end{cases}$$

and define the measure $\vec{\eta} \in \mathcal{M}(\mathbf{R}^2)$ by

$$\int_{\mathbf{R}^2} \vec{\vartheta} \cdot \vec{\eta} = \int_{\bar{\Omega}} \vec{\vartheta} \cdot \vec{\mu} \ ,$$

for every $\vec{\vartheta} \in C_c(\mathbf{R}^2; \mathbf{R}^2)$. One has, by Corollary 2.7,

$$\int_{\bar{\Omega}} \vec{\varphi} \cdot \vec{\mu} = \int_{\mathbf{R}^2} \vec{\psi} \cdot \vec{\eta} \leq \frac{\|\vec{\eta}\|}{2\sqrt{\pi}} \Big(\int_{\mathbf{R}^2} |\nabla \wedge \vec{\psi}|^2 \Big)^{\frac{1}{2}} = \frac{\|\vec{\mu}\|}{2\sqrt{\pi}} \Big(\int_{\Omega} |\nabla \wedge \vec{\varphi}|^2 \Big)^{\frac{1}{2}} \ .$$

The equality cases follow again from the conclusion of Corollary 2.7. □

REMARK 2.9. If Ω is not simply connected, the inequality

$$\int_{\Omega} \vec{\varphi} \cdot \vec{\mu} \leq C \|\vec{\mu}\| \, \|\nabla \wedge \vec{\varphi}\|_{\mathrm{L}^2(\Omega)}$$

cannot be true. Indeed, assume that 0 belongs to a bounded connected component of $\mathbf{R}^2 \setminus \Omega$. Take $\Gamma \subset \Omega$ to be any closed curve of index 1 with respect to 0, and set $\vec{\mu} = \vec{t} \mathcal{H}^1 \llcorner \Gamma$ and $\vec{\varphi}(x) = \frac{x^\perp}{|x|^2}$. One then has

$$\int_{\Omega} \vec{\varphi} \cdot \vec{\mu} = 2\pi \ ,$$

while

$$\nabla \wedge \vec{\varphi} = 0$$

in Ω. □

From (2.12), we also have

$$\int_{\Omega} \vec{\varphi} \cdot \vec{\mu} \leq \frac{\|\vec{\mu}\|}{\sqrt{2\pi}} \|\nabla \vec{\varphi}\|_{\mathrm{L}^2(\Omega)} \tag{2.13}$$

since

$$\|\nabla \wedge \vec{\varphi}\|_{\mathrm{L}^2(\Omega)} \leq \sqrt{2} \|\nabla \vec{\varphi}\|_{\mathrm{L}^2(\Omega)}.$$

For every $\vec{\mu} \in \mathcal{M}_0^{\#}(\bar{\Omega}, \mathbf{R}^2)$, let

$$A_{\vec{\mu},\Omega} = \sup \Big\{ \int_{\bar{\Omega}} \vec{\varphi} \cdot \vec{\mu} \ : \ \vec{\varphi} \in \mathrm{H}^1(\mathbf{R}^2; \mathbf{R}^2) \text{ and } \int_{\Omega} |\nabla \vec{\varphi}|^2 \leq 1 \Big\} \ ; \tag{2.14}$$

of course the supremum in (2.14) is achieved by some unique $\vec{\varphi} \in \mathrm{H}^1(\Omega; \mathbf{R}^2)$ modulo constants; moreover, $\vec{\varphi} \in (\mathrm{L}^\infty \cap C)(\bar{\Omega}; \mathbf{R}^2)$ by Theorem 2 in [**5**] (when $\vec{\mu}$ is an L^1 function — the case of measures is similar).

Thus we have for every $\vec{\mu} \in \mathcal{M}_0^{\#}(\bar{\Omega}, \mathbf{R}^2)$

$$A_{\vec{\mu},\Omega} \leq \frac{\|\vec{\mu}\|}{\sqrt{2\pi}} \ .$$

Set

$$A_{\Omega} = \sup \big\{ A_{\vec{\mu},\Omega} \ : \ \vec{\mu} \in \mathcal{M}_0^{\#}(\bar{\Omega}, \mathbf{R}^2) \text{ and } \|\vec{\mu}\| \leq 1 \big\} \ . \tag{2.15}$$

PROPOSITION 2.10. *One has*

$$\frac{1}{2\sqrt{\pi}} < A_\Omega \leq \frac{1}{\sqrt{2\pi}} \; .$$

Moreover, if Ω *is a disc, then* $A_\Omega = \frac{1}{\sqrt{2\pi}}$ *and supremum in* (2.15) *is achieved.*

PROOF. Let $\vec{\mu} = \lambda \vec{t} \mathcal{H}^1 \llcorner \partial B(x_0, r)$ with $\partial B(x_0, r) \subset \Omega$ and

$$\vec{\varphi}(x) = \begin{cases} (x - x_0)^\perp & \text{if } |x - x_0| \leq r \; , \\ (x - x_0)^\perp \frac{r^2}{|x - x_0|^2} & \text{if } |x - x_0| > r \; . \end{cases}$$

One then has

$$\int_\Omega \vec{\varphi} \cdot \vec{\mu} = \int_{\mathbf{R}^2} \vec{\varphi} \cdot \vec{\mu} = \frac{\|\vec{\mu}\|}{2\sqrt{\pi}} \Bigl(\int_{\mathbf{R}^2} |\nabla \vec{\varphi}|^2 \Bigr)^{\frac{1}{2}} > \frac{\|\vec{\mu}\|}{2\sqrt{\pi}} \Bigl(\int_\Omega |\nabla \vec{\varphi}|^2 \Bigr)^{\frac{1}{2}} \; ,$$

so that $A_\Omega > \frac{1}{2\sqrt{\pi}}$. The inequality $A_\Omega \leq \frac{1}{\sqrt{2\pi}}$ follows immediately from (2.13). Finally, assume Ω is a disc; without loss of generality, $\Omega = B(0, 1)$. One sets then $\Gamma = \partial B(0, 1)$, $\vec{\mu} = \vec{t} \mathcal{H}^1 \llcorner \Gamma$ and $\vec{\varphi}(x) = x^\perp$; immediate computations give

$$\int_\Omega \vec{\varphi} \cdot \vec{\mu} = 2\pi \; ,$$

$$\|\vec{\mu}\| = 2\pi \; ,$$

$$\int_\Omega |\nabla \vec{\varphi}|^2 = 2\pi \; . \qquad \square$$

We have no clue about the dependence of A_Ω on Ω and whether the supremum in (2.15) is achieved. The only information we have is

PROPOSITION 2.11. *Assume that* $A_\Omega = \frac{1}{\sqrt{2\pi}}$ *and* A_Ω *is achieved. Then* Ω *is a disc.*

There are two extreme scenarios:

SCENARIO 1. $A_\Omega = \frac{1}{\sqrt{2\pi}}$ only when Ω is a disc.

SCENARIO 2. $A_\Omega = \frac{1}{\sqrt{2\pi}}$ for every domain $\Omega \subset \mathbf{R}^2$.

PROBLEM 1. Decide between Scenario 1, Scenario 2 and intermediate scenarios.

PROBLEM 2. Is it true that for every domain Ω, A_Ω is achieved?

By Proposition 2.11, a positive answer to Problem 2 would lead to Scenario 1. This scenario would be reminiscent of the situation of the balls who have the worst best Sobolev inequalities [**12**].

There is a variant of Proposition 2.8 where the boundary condition $\vec{\mu} \cdot \vec{n} = 0$ is replaced by the condition that $\vec{\varphi}$ should vanish on $\partial\Omega$. Set

$$\mathcal{M}^\#(\Omega, \mathbf{R}^2) = \Bigl\{ \vec{\mu} \in C(\Omega; \mathbf{R}^2)^* \; : \; \forall \zeta \in C^1_c(\Omega), \; \int_\Omega \nabla \zeta \cdot \vec{\mu} = 0 \Bigr\}.$$

PROPOSITION 2.12. *For every* $\vec{\mu} \in \mathcal{M}^\#(\Omega; \mathbf{R}^2)$ *and for every* $\vec{\varphi} \in \mathrm{H}^1_0(\Omega; \mathbf{R}^2) \cap C(\bar{\Omega}; \mathbf{R}^2)$*, one has*

$$\int_\Omega \vec{\varphi} \cdot \vec{\mu} \leq S_\Omega \|\vec{\mu}\| \Bigl(\int_\Omega |\nabla \wedge \vec{\varphi}|^2 \Bigr)^{\frac{1}{2}} \; , \tag{2.16}$$

for every $\vec{\varphi} \in \mathrm{H}^1_0(\Omega)$ *where*

$$S_\Omega = \sup\Big\{ \|u\|_{\mathrm{L}^2(\Omega)} \ :\ u \in \mathrm{BV}(\Omega),\ \|\nabla u\| \le 1 \ \textit{and} \ \int_\Omega u = 0 \Big\}\ , \tag{2.17}$$

and $\|\nabla u\|$ *denotes the total mass of the measure* ∇u*. Moreover the constant* S_Ω *in* (2.16) *cannot be improved.*

Proof. Inequality (2.16) is established as above, see also Theorem 2.1 in [**5**]. For the last statement, assume that for every $\vec{\mu} \in \mathcal{M}^{\#}(\Omega; \mathbf{R}^2)$ and for every $\vec{\varphi} \in \mathrm{H}^1_0(\Omega; \mathbf{R}^2) \cap C(\bar{\Omega}; \mathbf{R}^2)$, one has

$$\int_\Omega \vec{\varphi} \cdot \vec{\mu} \le A \|\vec{\mu}\| \Big(\int_\Omega |\nabla \wedge \vec{\varphi}|^2 \Big)^{\frac{1}{2}}\ ,$$

for some constant A. We claim that for every $u \in \mathrm{BV}(\Omega)$ with $\int_\Omega u = 0$, we have

$$\|u\|_{\mathrm{L}^2} \le A \|\nabla u\|\ .$$

Indeed, set $\vec{\mu} = \nabla^\perp u$, and choose any function $\vec{\varphi} \in \mathrm{H}^1_0(\Omega) \cap C(\bar{\Omega})$ such that $\nabla \wedge \vec{\varphi} = u$ in Ω [**1**, Theorem 3]. □

Problem 3. Is the supremum in (2.17) achieved by some $u \in \mathrm{BV}(\Omega)$? Or equivalently, does equality hold in (2.16) in the nontrivial cases?

The problem has been treated on the sphere [**19**] and on the unit ball [**11**]. For a general domain $\Omega \subset \mathbf{R}^n$, with $n \ge 3$ and when $\mathrm{BV}(\Omega)$ and $\mathrm{L}^2(\Omega)$ are replaced by the spaces $\mathrm{H}^1(\Omega)$ and $\mathrm{L}^{\frac{2n}{n-2}}(\Omega)$, an affirmative answer has been given [**10**, Proposition 1.2].

Remark 2.13. As is well known, there is no universal bound on S_Ω, even when replacing the constraint $\|\nabla u\| \le 1$ by the constraint $\|\nabla u\|_{\mathrm{L}^2} \le 1$. This is related to the eigenvalue problem for the Laplacian with Neumann boundary condition. In the similar inequality

$$\inf_{c \in \mathbf{R}} \|u - c\|_{\mathrm{L}^{\frac{n}{n-1}}} \le S'_\Omega \|\nabla u\|_{\mathrm{L}^1}\ ,$$

the best constant S'_Ω is proportional to a relative isoperimetric constant of Ω [**14**, Theorem 3.2.3 and § 6.1.7].

A consequence of Proposition 2.12 is the inequality

$$\int_\Omega \vec{\varphi} \cdot \vec{\mu} \le S_\Omega \|\vec{\mu}\| \Big(\int_\Omega |\nabla \vec{\varphi}|^2 \Big)^{\frac{1}{2}}\ ,$$

for every $\vec{\mu} \in \mathcal{M}^{\#}(\Omega; \mathbf{R}^2)$ and for every $\vec{\varphi} \in \mathrm{H}^1_0(\Omega; \mathbf{R}^2) \cap C(\bar{\Omega}; \mathbf{R}^2)$, since

$$\int_\Omega |\nabla \wedge \vec{\varphi}|^2 \le \int_\Omega |\nabla \vec{\varphi}|^2\ .$$

By analogy with the above, for $\vec{\mu} \in \mathcal{M}^{\#}(\Omega, \mathbf{R}^2)$, set

$$A'_{\vec{\mu},\Omega} = \sup\Big\{ \int_{\bar{\Omega}} \vec{\varphi} \cdot \vec{\mu} \ :\ \vec{\varphi} \in \mathrm{H}^1_0(\mathbf{R}^2; \mathbf{R}^2) \text{ and } \int_\Omega |\nabla \vec{\varphi}|^2 \le 1 \Big\}\ ; \tag{2.18}$$

and

$$A'_\Omega = \sup\big\{ A'_{\vec{\mu},\Omega} \ :\ \vec{\mu} \in \mathcal{M}^{\#}(\Omega, \mathbf{R}^2) \text{ and } \|\vec{\mu}\| \le 1 \big\}\ , \tag{2.19}$$

so that

$$A'_\Omega \le S_\Omega\ .$$

The supremum in (2.18) is uniquely achieved since $\mathcal{M}^{\#}(\Omega, \mathbf{R}^2) \subset \mathrm{H}^{-1}(\Omega; \mathbf{R}^2)$, and the maximizer is bounded and continuous [**5**].

In general, we do not expect having $A'_\Omega = S_\Omega$. Indeed, the maximizing vector fields $\vec{\varphi}$ in Proposition 2.12 need not be divergence-free.

One has

PROPOSITION 2.14. *There exists $\alpha > 0$ such that for every domain $\Omega \subset \mathbf{R}^2$,*

$$A'_\Omega \geq \alpha \,.$$

PROOF. Simply take some compactly supported divergence-free measure $\vec{\mu} \in \mathcal{M}(\Omega; \mathbf{R}^2)$, and some compactly supported vector field $\vec{\psi} \in \mathcal{C}^\infty_c(\mathbf{R}^2; \mathbf{R}^2)$ such that $\int_{\mathbf{R}^2} \vec{\psi} \cdot \vec{\mu} \neq 0$. By translation and dilation, one has that

$$A'_\Omega \geq \frac{\int_{\mathbf{R}^2} \vec{\psi} \cdot \vec{\mu}}{\|\vec{\mu}\| \|\nabla \vec{\psi}\|_{\mathrm{L}^2}} \,. \qquad \square$$

This raises the question

PROBLEM 4. Compute $\inf_\Omega A'_\Omega$ and $\inf_\Omega S_\Omega$. Are they achieved?

In [**19**, Question 4.1], the question was asked whether $\inf_\Omega S_\Omega = S_{B(0,1)}$.

Remember that A_Ω does not have an upper bound independent of the geometry. If we allow Ω to be multiply connected, A'_Ω has no upper bound. On the other hand, we do not know whether A'_Ω has an upper bound independently of the geometry for simply connected domains.

PROBLEM 5. Does one have

$$\sup\{A'_\Omega \,:\, \Omega \subset \mathbf{R}^2 \text{ is a simply connected domain}\} < \infty?$$

3. Higher dimensions

3.1. Inequalities for curves. Throughout this section $\Gamma \subset \mathbf{R}^n$ is a simple, closed, rectifiable curve. The optimal constant in Theorem 2 is

$$A_\Gamma = \sup\Big\{ \int_\Gamma \vec{\varphi} \cdot \vec{t} \,:\, \vec{\varphi} \in C^\infty_c(\mathbf{R}^n; \mathbf{R}^n) \text{ and } \int_{\mathbf{R}^n} |\nabla \vec{\varphi}|^n \leq 1 \Big\} \,.$$

As in 2d, we obtain

PROPOSITION 3.1. *There exists a unique $\vec{\varphi} \in \dot{\mathrm{W}}^{1,n}(\mathbf{R}^n; \mathbf{R}^n)$ modulo constants such that*

$$\int_\Gamma \vec{\varphi} \cdot \vec{t} = A_\Gamma \,,$$
$$\|\nabla \vec{\varphi}\|_{\mathrm{L}^n} = 1 \,.$$

The vector field $\vec{\varphi}$ satisfies

$$-\sum_{i=1}^n \partial_i (|\nabla \vec{\varphi}|^{n-2} \partial_i \vec{\varphi}) = \frac{1}{A_\Gamma} \vec{t} \mathcal{H}^1 \llcorner \Gamma \,.$$

In contrast with Proposition 2.1, we do not in general expect to have $\operatorname{div} \vec{\varphi} = 0$ (find a counterexample) and we do not know whether $\vec{\varphi}$ is a bounded continuous vector field. This is an interesting open problem:

PROBLEM 6. Does one have $\vec{\varphi} \in (\mathrm{L}^\infty \cap C)(\mathbf{R}^n;\mathbf{R}^n)$. More generally, let $\vec{\mu} \in \mathcal{M}(\mathbf{R}^n;\mathbf{R}^n)$ be a bounded measure such that $\operatorname{div}\vec{\mu} = 0$. From [**2, 3**], we know that $\vec{\mu} \in \big(\dot{\mathrm{W}}^{1,n}(\mathbf{R}^n;\mathbf{R}^n)\big)^*$ and hence there exists a unique $\vec{\varphi} \in \dot{\mathrm{W}}^{1,n}(\mathbf{R}^n;\mathbf{R}^n)$ modulo constants that solves

$$-\sum_{i=1}^{n} \partial_i(|\nabla\vec{\varphi}|^{n-2}\partial_i\vec{\varphi}) = \vec{\mu}\ .$$

Does one have $\vec{\varphi} \in (\mathrm{L}^\infty \cap C)(\mathbf{R}^n;\mathbf{R}^n)$?

In two dimensions, Proposition 2.2 gives the exact value of A_Γ in terms of $|\Sigma|$, the area of the surface spanned by Γ. This is not anymore the case in higher dimensions.

Let us examine what happens when $\Gamma \subset \mathbf{R}^n$ is planar, i.e., if $\Gamma \subset \Pi$ where $\Pi \subset \mathbf{R}^n$ is a (two-dimensional) plane. Recall that the trace on Π of a function $u \in \dot{\mathrm{W}}^{1,n}(\mathbf{R}^n)$ belongs to $\dot{\mathrm{W}}^{\frac{2}{n},n}(\Pi)$ with

$$\|u|_\Pi\|_{\dot{\mathrm{W}}^{\frac{2}{n},n}(\Pi)} \le C\|u\|_{\dot{\mathrm{W}}^{1,n}(\mathbf{R}^n)}.$$

Conversely, given any $g \in \dot{\mathrm{W}}^{\frac{2}{n},n}(\Pi)$, there is an extension u to $\mathbf{R}^n$ such that

$$\|u\|_{\dot{\mathrm{W}}^{1,n}(\mathbf{R}^n)} \le C\|g\|_{\dot{\mathrm{W}}^{\frac{2}{n},n}(\Pi)}$$

(one proceeds for example by successive harmonic extensions). As a consequence, we have, when $\Gamma \subset \Pi \subset \mathbf{R}^n$,

$$A_\Gamma \simeq \sup\Big\{\int_\Gamma \vec{\varphi}\cdot\vec{t}\ :\ \vec{\varphi} \in C_c^\infty(\Pi;\mathbf{R}^2) \text{ and } \|\vec{\varphi}\|_{\dot{\mathrm{W}}^{\frac{2}{n},n}(\Pi)} \le 1\Big\}\ , \tag{3.1}$$

Since $\dot{\mathrm{W}}^{1,2}(\mathbf{R}^2;\mathbf{R}^2) \subset \dot{\mathrm{W}}^{\frac{2}{n},n}(\mathbf{R}^2;\mathbf{R}^2)$ [**15**, Theorem 2.2.3], we see, by Proposition 2.2, that

$$|\Sigma|^{\frac{1}{2}} \le CA_\Gamma\ , \tag{3.2}$$

for every planar curve $\Gamma \subset \mathbf{R}^n$, where $\Sigma \subset \Pi$ is the surface spanned by Γ. This leads to the problem

PROBLEM 7. Let $\Gamma \subset \mathbf{R}^n$ be a simple, closed, rectifiable curve, and let $|\Sigma|$ be the area of an area-minizing surface spanned by Γ. Does (3.2) hold?

On the other hand, one cannot find an upper bound on A_Γ in terms of $|\Sigma|$:

PROPOSITION 3.2. *There exists a sequence of planar surfaces Σ_k and planar curves $\Gamma_k = \partial\Sigma_k$ such that, as $k \to \infty$,*

$$A_{\Gamma_k} \to \infty$$

while

$$|\Sigma_k| \le C\ .$$

In view of (3.1), the conclusion follows from

LEMMA 3.3. *If $p > 2$ and $s = 2/p$, then there exists a sequence of planar surfaces Σ_k and planar curves $\Gamma_k = \partial\Sigma_k$, and a sequence of vector fields $\vec{\varphi}_k \in C_c^\infty(\mathbf{R}^2;\mathbf{R}^2)$ such that, as $k \to \infty$,*

$$\int_{\Gamma_k} \vec{\varphi}_k\cdot\vec{t} \to \infty$$

while

$$|\Sigma_k| \le C$$

and

$$\|\vec{\varphi}_k\|_{\mathrm{W}^{s,p}} \le C\ .$$

PROOF. Consider $\psi \in C_c^\infty(\mathbf{R}^2)$ and $\Gamma = \partial\Sigma \subset \mathbf{R}^2$ such that

$$\int_\Gamma \psi\, t_2 \neq 0\ ,$$

where $t_2 = \vec{t} \cdot \vec{e}_2$. Set

$$T_k(x_1, x_2) = (\frac{1}{k}x_1, kx_2)\ .$$

and define

$$\begin{aligned}\Gamma_k &= T_k(\Gamma)\ ,\\ \Sigma_k &= T_k(\Sigma)\ ,\\ \vec{\psi}_k &= k^{-s}\vec{e}_2(\psi \circ T_k^{-1})\ .\end{aligned}$$

One has

$$\int_{\Gamma_k} \vec{\psi}_k \cdot \vec{t}_k = k^{1-s} \int_\Gamma \psi\, t_2 \to \infty\ .$$

On the other hand, one has

$$|\Sigma_k| = |\Sigma|\ ,$$

and

$$\|\vec{\psi}_\varepsilon\|_{\dot{\mathrm{W}}^{s,p}} \le C\|\psi\|_{\dot{\mathrm{W}}^{s,p}}\ . \qquad \square$$

Let $\Gamma \subset \mathbf{R}^n$ be a simple closed rectifiable curve. Recall that (see [**2, 18**])

$$\int_\Gamma \vec{\varphi} \cdot \vec{t} \le C_{s,p}|\Gamma|\, \|\vec{\varphi}\|_{\dot{\mathrm{W}}^{s,p}}$$

for every $0 < s < n$ and $sp = n$. Recall also that (see [**7**, 4.2.10])

$$|\Gamma| \ge C|\Sigma|^{\frac{1}{2}}\ ,$$

where $|\Sigma|$ is the area of an area-minimizing surface spanned by Γ. A natural question is whether for every $\vec{\varphi} \in \dot{\mathrm{W}}^{s,p}(\mathbf{R}^n; \mathbf{R}^n)$,

$$\int_\Gamma \vec{\varphi} \cdot \vec{t} \le C_{s,p}|\Sigma|^{\frac{1}{2}}\, \|\vec{\varphi}\|_{\dot{\mathrm{W}}^{s,p}}\ , \tag{3.3}$$

with $0 < s < n$ and $sp = n$. From Lemma 3.3, we deduce that (3.3) fails when $p > 2$. We ask the question whether (3.3) holds when $p = 2$ and $s = \frac{n}{2}$:

PROBLEM 8. Does (3.3) holds when $p = 2$ and $s = \frac{n}{2}$?

The final question concerning the inequality for curves is finding the optimal constant among all curves:

PROBLEM 9. When $n \ge 3$, is

$$A = \sup\bigl\{A_\Gamma\ :\ \Gamma \subset \mathbf{R}^n \text{ is a closed rectifiable curve}\bigr\}$$

attained? By which curve?

The answer to Problem 9 is open even when Γ is a planar curve of $\mathbf{R}^n$, $n \geq 3$. There are numerous variants of Problem 9. In particular, one can define

$$\tilde{A}_\Gamma = \sup\Big\{ \int_\Gamma \vec{\varphi} \cdot \vec{t} \, : \, \vec{\varphi} \in C_c^\infty(\mathbf{R}^n; \mathbf{R}^n) \text{ and } \int_{\mathbf{R}^n} |\nabla \wedge \vec{\varphi}|^n \leq 1 \Big\} \,,$$

or one could work with different norms on $\dot{\mathrm{W}}^{1,n}(\mathbf{R}^n; \mathbf{R}^n)$, or even on $\dot{\mathrm{W}}^{s,p}(\mathbf{R}^n; \mathbf{R}^n)$, with $0 < s < n$ and $sp = n$.

3.2. Inequalities for measures. As in two dimensions, we can also consider the relaxed problem with measures. When $\vec{\mu}$ is a finite vector measure such that $\operatorname{div} \vec{\mu} = 0$, define

$$A_{\vec{\mu}} = \sup\Big\{ \int_{\mathbf{R}^2} \vec{\varphi} \cdot \vec{\mu} \, : \, \vec{\varphi} \in C_c^\infty(\mathbf{R}^n; \mathbf{R}^n) \text{ and } \|\nabla \vec{\varphi}\|_{\mathrm{L}^n} \leq 1 \Big\} \,.$$

As explained in [**2**] and in Remark 2.6, the optimal constants in Theorems 1 and 2 are the same, i.e.

PROPOSITION 3.4. *One has*

$$A = \sup\Big\{ A_{\vec{\mu}} \, : \, \vec{\mu} \textit{ is a measure}, \operatorname{div} \vec{\mu} = 0 \textit{ and } \|\vec{\mu}\| \leq 1 \Big\}.$$

In view of Proposition 3.4, Problem 9 can be relaxed to

PROBLEM 10. Is the supremum in Proposition 3.4 attained? By what measure?

The advantage of the formulation of Problem 10 versus Problem 9, is that while Γ was taken among closed curves, $\vec{\mu}$ is taken in the vector space of divergence-free measures. One could then hope that some kind of concentration-compactness could provide the existence of an optimizer. The divergence-free condition however is quite rigid for this kind of approach.

In two dimensions, the maximizing measures are integrals along circles. In higher dimensions, we ask

PROBLEM 11. Let $\vec{\mu}$ be a maximizing measure in Proposition 3.4 (assuming that the supremum is achived). Is $\vec{\mu}$ an integral along a curve?

A partial answer is given by

PROPOSITION 3.5. *If $\vec{\mu}$ achieves the supremum of Proposition 3.4, then $\vec{\mu}$ is an extremal point of the unit ball in $\mathcal{M}^\#(\mathbf{R}^n; \mathbf{R}^n)$.*

Proposition 3.5 means that maximizing measures are atomic, i.e., they do not have any nontrivial decomposition preserving the mass into divergence-free measures. One might be tempted to claim that atomic divergence-free measures are circulation integrals. However, as explained by Smirnov [**16**], there are divergence-free atomic measure that are not circulation integrals: Consider for $k \geq 2$ a k–dimensional torus $\mathbf{T}^k$ and a constant vector field $\vec{v}$ on $\mathbf{T}^k$ such that the equation $\dot{x} = \vec{v}$ does not have periodic solutions. If $\Phi : \mathbf{T}^k \to \mathbf{R}^n$ maps $\mathbf{T}^k$ on Θ and $\vec{v}$ on $\vec{\vartheta}$, one has that $\vec{\mu}$ defined by

$$\int_{\mathbf{R}^n} \vec{\varphi} \cdot \vec{\mu} = \int_\Theta \vec{\varphi} \cdot \vec{\vartheta}$$

is atomic but is clearly not a circulation integral.

Proof of Proposition 3.5. Since one has clearly $\|\vec{\mu}\| = 1$, assume by contradiction that $\vec{\mu} = \lambda\vec{\mu}_1 + (1-\lambda)\vec{\mu}_2$, where $\lambda \in (0,1)$, $\vec{\mu}_1, \vec{\mu}_2 \in \mathcal{M}^{\#}(\mathbf{R}^n; \mathbf{R}^n)$, $\|\vec{\mu}_i\| = 1$ and $\vec{\mu}_i \neq \vec{\mu}$. Let $\vec{\varphi} \in \dot{\mathrm{W}}^{1,n}(\mathbf{R}^n; \mathbf{R}^n)$ such that $\|\nabla\vec{\varphi}\|_{\mathrm{L}^n} = 1$ and

$$\int_{\mathbf{R}^n} \vec{\varphi} \cdot \vec{\mu} = A_{\vec{\mu}} \ .$$

Because $\vec{\varphi}$ is a maximizer for $A_{\vec{\mu}}$,

$$-\sum_{i=1}^{n} \partial_i(|\nabla\vec{\varphi}|^{n-2}\partial_i\vec{\varphi}) = \frac{\vec{\mu}}{A_{\vec{\mu}}} \neq \frac{\vec{\mu}_i}{A_{\vec{\mu}_i}} \ .$$

Therefore, $\vec{\varphi}$ cannot be a maximizer for $A_{\vec{\mu}_i}$, and

$$\int_{\mathbf{R}^n} \vec{\varphi} \cdot \vec{\mu}_i < A_{\vec{\mu}_1} \ ,$$

whence

$$\begin{aligned} A = A_{\vec{\mu}} = \int_{\mathbf{R}^n} \vec{\varphi}\cdot\vec{\mu} = \lambda \int_{\mathbf{R}^n} \vec{\varphi}\cdot\vec{\mu}_1 + (1-\lambda)\int_{\mathbf{R}^n} \vec{\varphi}\cdot\vec{\mu}_2 \\ < \lambda A_{\vec{\mu}_1} + (1-\lambda)A_{\vec{\mu}_1} \leq A \ , \end{aligned}$$

which is a contradiction. □

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Added in proof. We have been informed by N. Fusco that he has obtained jointly with A. Pratelli partial answers to our Problems 3 and 4. More precisely, $\inf_\Omega S_\Omega = S_{B(0,1)}$ and if $\partial\Omega$ is Lipschitz, the supremum in (2.17) is achieved by the characteristic function of a subdomain (paper in preparation).

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