Communications in Contemporary Mathematics
Vol. 10, No. 6 (2008) 1217–1241

KATO'S INEQUALITY UP TO THE BOUNDARY

HAÏM BREZIS

Department of Mathematics, Rutgers University
Hill Center, Busch Campus, 110 Frelinghuysen Rd.
Piscataway, NJ 08854, USA
and
Department of Mathematics, Technion, 32000 Haifa, Israel
brezis@math.rutgers.edu

AUGUSTO C. PONCE

Département de Mathématique
Université catholique de Louvain
Chemin du Cyclotron 2, 1348 Louvain-la-Neuve, Belgium
Augusto.Ponce@uclouvain.be

Received 8 January 2008
Revised 26 November 2008

We show that if Δu is a finite measure in Ω then, under suitable assumptions on u near $\partial\Omega$, Δu^+ is also a finite measure in Ω. We also study properties of the normal derivatives $\frac{\partial u}{\partial n}$ and $\frac{\partial u^+}{\partial n}$ on $\partial\Omega$.

Keywords: Elliptic equations; boundary value problems.

Mathematics Subject Classification 2000: 31B25, 31B35, 35J15

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Given $u \in L^1(\Omega)$ with $\Delta u \in L^1(\Omega)$, Kato's inequality (see [9]; see also [4]) asserts that

$$\Delta u^+ \geq \chi_{[u\geq 0]}\Delta u \quad \text{in } \mathcal{D}'(\Omega). \tag{1.1}$$

In particular, (1.1) implies that Δu^+ is a locally finite measure in Ω. Our goal in this paper is to address the question whether Δu^+ is a *finite* measure up to the boundary of Ω, i.e. whether

$$\int_\Omega |\Delta u^+| < \infty.$$

In general, the answer is negative: one can even construct harmonic functions $u \in C(\overline{\Omega}) \cap H^1(\Omega)$ such that Δu^+ is not a finite measure in Ω; see Proposition A.1 below. With further assumptions on u (for instance if $u \in W^{2,1}(\Omega)$ or if u vanishes on the boundary) we will see that the answer is positive.

The following class of functions will play a central role. We say that $u \in \mathbb{X}$ if $u \in W^{1,1}(\Omega)$ and if there exists a constant $C > 0$ such that

$$\left| \int_\Omega \nabla u \cdot \nabla \psi \right| \leq C \|\psi\|_{L^\infty} \quad \forall \psi \in C^1(\overline{\Omega}), \tag{1.2}$$

in which case we set

$$[u]_{\mathbb{X}} = \sup_{\substack{\psi \in C^1(\overline{\Omega}) \\ \|\psi\|_{L^\infty} \leq 1}} \int_\Omega \nabla u \cdot \nabla \psi.$$

Note that if $u \in \mathbb{X}$, then there exists a unique $T \in \left[C(\overline{\Omega})\right]^* = \mathcal{M}(\overline{\Omega})$ such that

$$\langle T, \psi \rangle = \int_\Omega \nabla u \cdot \nabla \psi \quad \forall \psi \in C^1(\overline{\Omega}).$$

On the other hand, by the Riesz Representation Theorem any $T \in \mathcal{M}(\overline{\Omega})$ admits a unique decomposition

$$\langle T, \psi \rangle = \int_{\partial\Omega} \psi \, d\nu + \int_\Omega \psi \, d\mu \quad \forall \psi \in C(\overline{\Omega}),$$

where $\mu \in \mathcal{M}(\Omega)$ and $\nu \in \mathcal{M}(\partial\Omega)$. As usual, $\mathcal{M}(\Omega)$ and $\mathcal{M}(\partial\Omega)$ denote the spaces of finite measures in Ω and $\partial\Omega$, respectively, equipped with the norm $\|\cdot\|_{\mathcal{M}}$; measures in $\mathcal{M}(\Omega)$ are identified with measures in $\overline{\Omega}$ which do not charge $\partial\Omega$. When $u \in \mathbb{X}$, we will denote

$$\mu = -\Delta u \quad \text{and} \quad \nu = \frac{\partial u}{\partial n}.$$

Throughout the paper, whenever $u \in \mathbb{X}$ we use the notation Δu and $\frac{\partial u}{\partial n}$ in the above sense. If $u \in \mathbb{X}$, then

$$\int_\Omega \nabla u \cdot \nabla \psi = \int_{\partial\Omega} \psi \frac{\partial u}{\partial n} - \int_\Omega \psi \, \Delta u \quad \forall \psi \in C^1(\overline{\Omega}),$$

and consequently,

$$\int_{\partial\Omega} u \frac{\partial \psi}{\partial n} - \int_\Omega u \, \Delta \psi = \int_{\partial\Omega} \psi \frac{\partial u}{\partial n} - \int_\Omega \psi \, \Delta u \quad \forall \psi \in C^2(\overline{\Omega}).$$

Also, note that if $u \in \mathbb{X}$, then

$$[u]_{\mathbb{X}} = \int_\Omega |\Delta u| + \int_{\partial\Omega} \left| \frac{\partial u}{\partial n} \right|.$$

In particular, $[\,\cdot\,]_{\mathbb{X}}$ defines a seminorm in $\mathbb{X}$ and $[u]_{\mathbb{X}} = 0$ if, and only if, u is constant in Ω. In order to verify this last assertion, one may use the fact that for every $h \in C^\infty(\overline{\Omega})$ with $\int_\Omega h = 0$, there exists $\psi \in C^\infty(\overline{\Omega})$ such that $-\Delta\psi = h$ in Ω with $\frac{\partial \psi}{\partial n} = 0$ on $\partial\Omega$.

Clearly, any function $u \in W^{2,1}(\Omega)$ belongs to $\mathbb{X}$ and our notation is consistent with the usual meaning of Δu and $\frac{\partial u}{\partial n}$. Recall that, for any function $u \in L^1(\Omega)$, Δu is well-defined as a distribution. When $u \in \mathbb{X}$, the distribution Δu belongs to $\mathcal{M}(\Omega)$, but the converse is not true; see e.g. Proposition A.1 below.

We now present our main results.

Theorem 1.1. *If $u \in \mathbb{X}$, then $u^+ \in \mathbb{X}$ and*

$$[u^+]_{\mathbb{X}} \leq [u]_{\mathbb{X}}. \tag{1.3}$$

In other words,

$$\int_\Omega |\Delta u^+| + \int_{\partial\Omega} \left|\frac{\partial u^+}{\partial n}\right| \leq \int_\Omega |\Delta u| + \int_{\partial\Omega} \left|\frac{\partial u}{\partial n}\right|. \tag{1.4}$$

Our next result gives additional properties when u vanishes on the boundary:

Theorem 1.2. *If $u \in W_0^{1,1}(\Omega)$ and $\Delta u \in \mathcal{M}(\Omega)$ (in the sense of distributions), then $u \in \mathbb{X}$ (hence, $u^+ \in \mathbb{X}$). Moreover,*

$$\int_\Omega |\Delta u^+| \leq \int_\Omega |\Delta u|. \tag{1.5}$$

In addition, $\frac{\partial u}{\partial n} \in L^1(\partial\Omega)$ with

$$\int_{\partial\Omega} \left|\frac{\partial u}{\partial n}\right| \leq \int_\Omega |\Delta u|. \tag{1.6}$$

Note that assertions (1.5) and (1.6) fail if u does not vanish on $\partial\Omega$; simply take $\Omega = B_1$, the unit ball in $\mathbb{R}^N$, and $u(x) = x_1$.

We now state our extension of Kato's inequality up to the boundary:

Theorem 1.3. *Let $u \in \mathbb{X}$ be such that $\Delta u \in L^1(\Omega)$ and $\frac{\partial u}{\partial n} \in L^1(\partial\Omega)$. Then,*

$$\int_{\partial\Omega} \nabla u^+ \cdot \nabla\psi \leq \int_{\partial\Omega} H\psi - \int_\Omega G\psi \quad \forall \psi \in C^1(\overline{\Omega}),\ \psi \geq 0 \text{ in } \Omega, \tag{1.7}$$

where $G \in L^1(\Omega)$ and $H \in L^1(\partial\Omega)$ are given by

$$G = \begin{cases} \Delta u & \text{on } [u > 0], \\ 0 & \text{on } [u \leq 0], \end{cases} \quad \text{and} \quad H = \begin{cases} \dfrac{\partial u}{\partial n} & \text{on } [u > 0], \\ 0 & \text{on } [u < 0], \\ \min\left\{\dfrac{\partial u}{\partial n}, 0\right\} & \text{on } [u = 0]. \end{cases} \tag{1.8}$$

Thus,

$$\begin{cases} \Delta u^+ \geq G & \text{in } \Omega, \\ \dfrac{\partial u^+}{\partial n} \leq H & \text{on } \partial\Omega. \end{cases} \tag{1.9}$$

We conclude this introduction with the following problems:

Open Problem 1. *Let $u \in \mathbb{X}$. Is it true that*

$$\left|\frac{\partial u^+}{\partial n}\right| \le \left|\frac{\partial u}{\partial n}\right| \quad \text{on } \partial\Omega? \tag{1.10}$$

This problem is open even under the additional assumption that $u \in W_0^{1,1}(\Omega)$.

Open Problem 2. *Assume that $u \in \mathbb{X}$ and $\frac{\partial u}{\partial n} \in L^1(\partial\Omega)$. Is it true that $\frac{\partial u^+}{\partial n} \in L^1(\partial\Omega)$? More precisely, does one have*

$$\frac{\partial u^+}{\partial n} = H, \tag{1.11}$$

where H is the function given by (1.8)?

The answer to both Open Problems 1 and 2 is positive if $u \in W^{2,1}(\Omega)$; see Theorem 7.1 below.

Addendum. *Recently, A. Ancona informed us that he gave a positive answer to Open Problems* 1 *and* 2 *in full generality. His argument strongly relies on tools from Potential Theory; see* [2] *and a detailed paper to appear.*

2. Properties of Functions in $\mathbb{X}$

In this section, we investigate properties satisfied by elements in $\mathbb{X}$. We first show that condition (1.2) required for a function to belong to $\mathbb{X}$ can be replaced by

$$\left|\int_\Omega u\,\Delta\zeta\right| \le C\|\zeta\|_{L^\infty} \quad \forall \zeta \in C^2_{\mathrm{N}}(\overline{\Omega}), \tag{2.1}$$

where

$$C^2_{\mathrm{N}}(\overline{\Omega}) = \left\{\zeta \in C^2(\overline{\Omega});\ \frac{\partial\zeta}{\partial n} = 0 \text{ on } \partial\Omega\right\}. \tag{2.2}$$

Proposition 2.1. *Let $u \in L^1(\Omega)$. Then, $u \in \mathbb{X}$ if, and only if,*

$$\sup_{\substack{\zeta \in C^2_{\mathrm{N}}(\overline{\Omega}) \\ \|\zeta\|_{L^\infty} \le 1}} \left|\int_\Omega u\,\Delta\zeta\right| < \infty. \tag{2.3}$$

Moreover,

(i) *the quantity in* (2.3) *equals* $[u]_{\mathbb{X}}$;
(ii) $u \in W^{1,p}(\Omega)$ *for every* $1 \le p < \frac{N}{N-1}$; *moreover,* $\|\nabla u\|_{L^p(\Omega)} \le C[u]_{\mathbb{X}}$.

In the proof of Proposition 2.1, we need the following variant of the classical De Giorgi–Stampacchia estimate (see [7, 8]) for the Neumann problem:

Lemma 2.1. *Given $F \in C_0^\infty(\Omega;\mathbb{R}^N)$, let w be the unique solution of*

$$\begin{cases} -\Delta w = \operatorname{div} F & \text{in } \Omega, \\ \dfrac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

such that $\int_\Omega w = 0$. Then, for every $q > N$ we have

$$\|w\|_{L^\infty} \le C\|F\|_{L^q}. \tag{2.5}$$

We present a sketch of the proof of Lemma 2.1 in Appendix C.

Proof of Proposition 2.1 Note that if $u \in \mathbb{X}$, then

$$\left|\int_\Omega u\,\Delta\zeta\right| = \left|\int_\Omega \nabla u\cdot\nabla\zeta\right| \le [u]_{\mathbb{X}}\,\|\zeta\|_{L^\infty} \quad \forall \zeta \in C^2_{\mathrm{N}}(\overline{\Omega}). \tag{2.6}$$

This gives the implication "$\Rightarrow$". We now assume that (2.3) holds. We split the proof of the converse into two steps:

Step 1. $u \in W^{1,p}(\Omega)$ for every $1 \le p < \frac{N}{N-1}$ and

$$\|\nabla u\|_{L^p(\Omega)} \le CK,$$

where K denotes the quantity in (2.3).

Clearly, we may assume that $1 < p < \frac{N}{N-1}$. Given $F \in C_0^\infty(\Omega;\mathbb{R}^N)$, let w be the unique solution of (2.4) such that $\int_\Omega w = 0$. By (2.3) and (2.5), we have

$$\left|\int_\Omega u \operatorname{div} F\right| = \left|\int_\Omega u\,\Delta w\right| \le K\|w\|_{L^\infty} \le KC\|F\|_{L^{p'}} \quad \forall F \in C_0^\infty(\Omega;\mathbb{R}^N).$$

The conclusion follows by duality.

Step 2. $u \in \mathbb{X}$ and $[u]_{\mathbb{X}} = K$.

It suffices to show that

$$\left|\int_\Omega \nabla u\cdot\nabla\psi\right| \le K\|\psi\|_{L^\infty} \quad \forall \psi \in C^1(\overline{\Omega}). \tag{2.7}$$

Indeed, this implies $u \in \mathbb{X}$ and $[u]_{\mathbb{X}} \le K$. Since by (2.6), $K \le [u]_{\mathbb{X}}$, equality must hold. We now turn ourselves to the proof of (2.7). Given $\psi \in C^2(\overline{\Omega})$, we first show that there exists a sequence (ζ_k) such that

$$\zeta_k \in C^2_{\mathrm{N}}(\overline{\Omega}), \quad \|\nabla\zeta_k\|_{L^\infty} \le C, \quad \zeta_k \to \psi \quad \text{uniformly in } \Omega \tag{2.8}$$

and

$$\nabla\zeta_k \to \nabla\psi \quad \text{a.e. in } \Omega. \tag{2.9}$$

Indeed, let $\Phi \in C_0^\infty(\mathbb{R})$ and $\eta \in C^2(\overline{\Omega})$ with $\eta = 0$ on $\partial\Omega$ be such that

$$\Phi(t) = t \quad \forall t \in [-1,1] \quad \text{and} \quad \frac{\partial \eta}{\partial n} = \frac{\partial \psi}{\partial n} \quad \text{on } \partial\Omega.$$

Take

$$\zeta_k = \psi - \frac{1}{k}\Phi(k\eta) \quad \text{in } \overline{\Omega}.$$

Clearly, (2.8) holds. On the other hand,

$$\nabla\left[\frac{1}{k}\Phi(k\eta)\right] = \Phi'(k\eta)\nabla\eta \to \chi_{[\eta=0]}\nabla\eta \quad \text{in } \Omega.$$

Since $\nabla\eta = 0$ a.e. on the set $[\eta = 0]$, (2.9) follows. For every $k \geq 1$, we thus have

$$\left|\int_\Omega \nabla u \cdot \nabla\zeta_k\right| = \left|\int_\Omega u\,\Delta\zeta_k\right| \leq K\|\zeta_k\|_{L^\infty}.$$

As $k \to \infty$, we obtain (2.7) with test functions $\psi \in C^2(\overline{\Omega})$. Using a density argument, one then gets (2.7). The proof is complete. □

Remark 2.1. Using Proposition 2.1, one deduces that given measures $\mu \in \mathcal{M}(\Omega)$ and $\nu \in \mathcal{M}(\partial\Omega)$, the Neumann problem

$$\begin{cases} -\Delta u = \mu & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = \nu & \text{on } \partial\Omega, \end{cases} \tag{2.10}$$

has a solution $u \in \mathbb{X}$ if, and only if,

$$\mu(\Omega) + \nu(\partial\Omega) = 0. \tag{2.11}$$

The solution is unique up to an additive constant and belongs to $u \in W^{1,p}(\Omega)$ for every $1 \leq p < \frac{N}{N-1}$. In particular, if $\int_\Omega u = 0$, then

$$\|u\|_{W^{1,p}(\Omega)} \leq C[u]_{\mathbb{X}}.$$

The following result complements Proposition 2.1:

Proposition 2.2. *Let $u \in L^1(\Omega)$ be such that*

$$-\int_\Omega u\,\Delta\zeta \leq \int_{\partial\Omega} \zeta\,d\nu + \int_\Omega \zeta\,d\mu \quad \forall \zeta \in C^2_{\mathrm{N}}(\overline{\Omega}), \quad \zeta \geq 0 \text{ in } \overline{\Omega} \tag{2.12}$$

for some $\mu \in \mathcal{M}(\Omega)$ and $\nu \in \mathcal{M}(\partial\Omega)$. Then, $u \in \mathbb{X}$,

$$[u]_{\mathbb{X}} \leq 2(\|\mu^+\|_{\mathcal{M}(\Omega)} + \|\nu^+\|_{\mathcal{M}(\partial\Omega)}) \tag{2.13}$$

and

$$\begin{cases} -\Delta u \leq \mu & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} \leq \nu & \text{on } \partial\Omega. \end{cases} \tag{2.14}$$

Proof. By (2.12), we have

$$-\int_\Omega u\,\Delta\zeta \le \int_{\partial\Omega} \zeta\,d\nu^+ + \int_\Omega \zeta\,d\mu^+ \quad \forall \zeta \in C^2_{\mathrm{N}}(\overline{\Omega}), \quad \zeta \ge 0 \text{ in } \overline{\Omega}. \tag{2.15}$$

For every $\zeta \in C^2_{\mathrm{N}}(\overline{\Omega})$, we apply (2.15) with test functions $\|\zeta\|_{L^\infty} \pm \zeta$ to get

$$\left|\int_\Omega u\,\Delta\zeta\right| \le 2(\|\mu^+\|_{\mathcal{M}(\Omega)} + \|\nu^+\|_{\mathcal{M}(\partial\Omega)})\|\zeta\|_{L^\infty}. \tag{2.16}$$

By Proposition 2.1, it follows that $u \in \mathbb{X}$ and (2.13) holds. Proceeding as in Step 2 of the proof of Proposition 2.1 (more precisely, using (2.8) and (2.9)), one deduces from (2.12) that

$$\int_\Omega \nabla u \cdot \nabla\psi \le \int_{\partial\Omega} \psi\,d\nu + \int_\Omega \psi\,d\mu \quad \forall \psi \in C^2(\overline{\Omega}), \quad \psi \ge 0 \text{ in } \overline{\Omega}.$$

Therefore,

$$\int_{\partial\Omega} \psi\,\frac{\partial u}{\partial n} - \int_\Omega \psi\,\Delta u \le \int_{\partial\Omega} \psi\,d\nu + \int_\Omega \psi\,d\mu \quad \forall \psi \in C^2(\overline{\Omega}), \quad \psi \ge 0 \text{ in } \overline{\Omega}.$$

This gives (2.14). □

3. Proof of Theorem 1.1

We begin by establishing the following lemma:

Lemma 3.1. *If $u \in C^2(\overline{\Omega})$, then*

$$\int_\Omega \nabla u^+ \cdot \nabla\psi \le \int_{\substack{\partial\Omega \\ [u\ge 0]}} \psi\,\frac{\partial u}{\partial n} - \int_{\substack{\Omega \\ [u\ge 0]}} \psi\,\Delta u \quad \forall \psi \in C^1(\overline{\Omega}), \quad \psi \ge 0 \text{ in } \overline{\Omega}. \tag{3.1}$$

Proof. We first prove the

Claim. *If $u \in C^2(\overline{\Omega})$ and $\Phi \in C^2(\mathbb{R})$ is convex, then*

$$\int_\Omega \nabla\Phi(u) \cdot \nabla\psi \le \int_{\partial\Omega} \psi\,\Phi'(u)\frac{\partial u}{\partial n} - \int_\Omega \psi\,\Phi'(u)\Delta u \quad \forall \psi \in C^1(\overline{\Omega}), \quad \psi \ge 0 \text{ in } \overline{\Omega}. \tag{3.2}$$

Note that

$$\frac{\partial\Phi(u)}{\partial n} = \Phi'(u)\frac{\partial u}{\partial n} \quad \text{on } \partial\Omega$$

and, by the convexity of Φ,

$$\Delta\Phi(u) \ge \Phi'(u)\Delta u \quad \text{in } \Omega.$$

Thus, for every $\psi \in C^1(\overline{\Omega})$, $\psi \ge 0$ in $\overline{\Omega}$,

$$\int_\Omega \nabla\Phi(u) \cdot \nabla\psi = \int_{\partial\Omega} \psi\,\frac{\partial\Phi(u)}{\partial n} - \int_\Omega \psi\,\Delta\Phi(u) \le \int_{\partial\Omega} \psi\,\Phi'(u)\frac{\partial u}{\partial n} - \int_\Omega \psi\,\Phi'(u)\Delta u.$$

This establishes the claim.

We now apply (3.2) with $\Phi = \Phi_k$, where (Φ_k) is a sequence of smooth convex functions such that $\Phi_k(0) = 0$, $\|\Phi_k'\|_{L^\infty} \le 1$ and satisfying

$$\Phi_k'(t) \to \begin{cases} 1 & \text{if } t \ge 0, \\ 0 & \text{if } t < 0. \end{cases}$$

As $k \to \infty$, we obtain (3.1). □

We now prove a special case of Theorem 1.1 for functions in $C^2(\overline{\Omega})$:

Lemma 3.2. *Let $u \in C^2(\overline{\Omega})$. Then, $u^+ \in \mathbb{X}$ and*

$$[u^+]_{\mathbb{X}} \le [u]_{\mathbb{X}}. \tag{3.3}$$

Proof. Note that $u^+ \in W^{1,1}(\Omega)$. In order to establish the lemma, it thus suffices to show that

$$\left| \int_\Omega \nabla u^+ \cdot \nabla \psi \right| \le [u]_{\mathbb{X}} \, \|\psi\|_{L^\infty} \quad \forall \psi \in C^1(\overline{\Omega}). \tag{3.4}$$

For this purpose, given $\tilde{\psi} \in C^1(\overline{\Omega})$ we apply (3.1) with $\psi = \|\tilde{\psi}\|_{L^\infty} + \tilde{\psi}$. We then get

$$\int_\Omega \nabla u^+ \cdot \nabla \tilde{\psi} \le \Big(\int_{\substack{\partial\Omega \\ [u \ge 0]}} \frac{\partial u}{\partial n} - \int_{\substack{\Omega \\ [u \ge 0]}} \Delta u \Big) \|\tilde{\psi}\|_{L^\infty} + \int_{\substack{\partial\Omega \\ [u \ge 0]}} \tilde{\psi} \, \frac{\partial u}{\partial n} - \int_{\substack{\Omega \\ [u \ge 0]}} \tilde{\psi} \, \Delta u. \tag{3.5}$$

Since

$$\int_{\substack{\partial\Omega \\ [u \ge 0]}} \frac{\partial u}{\partial n} - \int_{\substack{\Omega \\ [u \ge 0]}} \Delta u = - \int_{\substack{\partial\Omega \\ [u < 0]}} \frac{\partial u}{\partial n} + \int_{\substack{\Omega \\ [u < 0]}} \Delta u,$$

estimate (3.5) becomes

$$\begin{aligned} \int_\Omega \nabla u^+ \cdot \nabla \tilde{\psi} &\le - \Big(\int_{\substack{\partial\Omega \\ [u < 0]}} \frac{\partial u}{\partial n} - \int_{\substack{\Omega \\ [u < 0]}} \Delta u \Big) \|\tilde{\psi}\|_{L^\infty} + \int_{\substack{\partial\Omega \\ [u \ge 0]}} \tilde{\psi} \, \frac{\partial u}{\partial n} - \int_{\substack{\Omega \\ [u \ge 0]}} \tilde{\psi} \, \Delta u \\ &\le \Big(\int_{\partial\Omega} \Big| \frac{\partial u}{\partial n} \Big| + \int_\Omega |\Delta u| \Big) \|\tilde{\psi}\|_{L^\infty} = [u]_{\mathbb{X}} \, \|\tilde{\psi}\|_{L^\infty}. \end{aligned}$$

This relation holds for every $\tilde{\psi} \in C^1(\overline{\Omega})$. Replacing $\tilde{\psi}$ by $-\tilde{\psi}$, we obtain (3.4). This establishes the lemma. □

Proof of Theorem 1.1. Since $u \in \mathbb{X}$,

$$\int_\Omega \nabla u \cdot \nabla \psi = \int_{\partial\Omega} \psi \, \frac{\partial u}{\partial n} - \int_\Omega \psi \, \Delta u \quad \forall \psi \in C^1(\overline{\Omega}).$$

Taking $\psi = 1$ as a test function, we get

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} = \int_\Omega \Delta u. \tag{3.6}$$

Let $(\mu_k) \subset C^\infty(\overline{\Omega})$ and $(\nu_k) \subset C^\infty(\partial\Omega)$ be two sequences such that

$$\begin{aligned} &\mu_k \overset{*}{\rightharpoonup} -\Delta u \quad \text{weak}^* \text{ in } \mathcal{M}(\overline{\Omega}) \quad \text{and} \quad \|\mu_k\|_{L^1(\Omega)} \to \|\Delta u\|_{\mathcal{M}(\Omega)}, \\ &\nu_k \overset{*}{\rightharpoonup} \frac{\partial u}{\partial n} \quad \text{weak}^* \text{ in } \mathcal{M}(\partial\Omega) \quad \text{and} \quad \|\nu_k\|_{L^1(\partial\Omega)} \to \left\| \frac{\partial u}{\partial n} \right\|_{\mathcal{M}(\partial\Omega)}. \end{aligned}$$

In view of (3.6), we may also assume that

$$\int_{\partial\Omega} \nu_k = -\int_\Omega \mu_k \quad \forall k \geq 1.$$

For each $k \geq 1$, let $u_k \in C^2(\overline{\Omega})$ be the unique function such that

$$\begin{cases} -\Delta u_k = \mu_k & \text{in } \Omega, \\ \dfrac{\partial u_k}{\partial n} = \nu_k & \text{on } \partial\Omega, \end{cases}$$

and

$$\int_\Omega u_k = \int_\Omega u.$$

Then, by Remark 2.1 applied to $u_k - \int_\Omega u$, the sequence (u_k) is bounded in $W^{1,p}(\Omega)$ for every $1 \leq p < \frac{N}{N-1}$. Since $u_k \to u$ a.e., one deduces that

$$\nabla u_k^+ \rightharpoonup \nabla u^+ \quad \text{weakly in } L^1(\Omega).$$

On the other hand, applying Lemma 3.2 to u_k, we get

$$\left| \int_\Omega \nabla u_k^+ \cdot \nabla\psi \right| \leq [u_k^+]_{\mathbb{X}} \|\psi\|_{L^\infty} \leq [u_k]_{\mathbb{X}} \|\psi\|_{L^\infty} \quad \forall \psi \in C^1(\overline{\Omega}).$$

As $k \to \infty$, we obtain

$$\left| \int_\Omega \nabla u^+ \cdot \nabla\psi \right| \leq [u]_{\mathbb{X}} \|\psi\|_{L^\infty} \quad \forall \psi \in C^1(\overline{\Omega}),$$

from which the conclusion follows. □

4. Properties of $\frac{\partial u}{\partial n}$

We start with a result which seems intuitively true, but still requires a proof:

Proposition 4.1. *Let $u \in W^{1,\infty}(\Omega)$. Then, $u \in \mathbb{X}$ if, and only if, $\Delta u \in \mathcal{M}(\Omega)$ (in the sense of distributions). In this case, $\frac{\partial u}{\partial n} \in L^\infty(\partial\Omega)$ and*

$$\left\| \frac{\partial u}{\partial n} \right\|_{L^\infty(\partial\Omega)} \leq \|\nabla u\|_{L^\infty(\Omega)}. \tag{4.1}$$

If $u \in C^1(\overline{\Omega}) \cap \mathbb{X}$, then $\frac{\partial u}{\partial n}$ coincides with the standard normal derivative on $\partial\Omega$.

Proof. We first assume that $u \in W^{1,\infty}(\Omega)$ and $\Delta u \in \mathcal{M}(\Omega)$. Given a sequence of mollifiers (ρ_k) such that $\operatorname{supp} \rho_k \subset B_{1/k}$, let

$$u_k(x) = \int_\Omega \rho_k(x-y)u(y)\,dy \quad \forall x \in \Omega.$$

Note that if $d(x, \partial\Omega) > 1/k$, then

$$\nabla u_k(x) = \int_\Omega \rho_k(x-y)\nabla u(y)\,dy \quad \text{and} \quad \Delta u_k(x) = \int_\Omega \rho_k(x-y)\Delta u(y)\,dy.$$

Denote

$$\Omega_\delta = \big\{x \in \Omega;\ d(x, \partial\Omega) > \delta\big\}; \tag{4.2}$$

for $\delta_0 > 0$ small enough, Ω_δ is smooth for every $\delta \in (0, \delta_0)$.

For every $k \geq 1$ and $\delta \in (0, \delta_0)$ such that $1/k < \delta$, we then have

$$\left\| \frac{\partial u_k}{\partial n} \right\|_{L^\infty(\partial\Omega_\delta)} \leq \|\nabla u_k\|_{L^\infty(\Omega_\delta)} \leq \|\nabla u\|_{L^\infty(\Omega)}\,. \tag{4.3}$$

Thus, for every $\psi \in C^1(\overline{\Omega})$,

$$\left| \int_{\Omega_\delta} \psi\,\Delta u_k + \int_{\Omega_\delta} \nabla\psi \cdot \nabla u_k \right| \leq \|\nabla u\|_{L^\infty(\Omega)} \|\psi\|_{L^1(\partial\Omega_\delta)}. \tag{4.4}$$

Note that for a.e. $\delta \in (0, \delta_0)$

$$\int_{\partial\Omega_\delta} |\Delta u| = 0; \tag{4.5}$$

hence, for any such $\delta > 0$,

$$\int_{\Omega_\delta} \psi\,\Delta u_k \to \int_{\Omega_\delta} \psi\,\Delta u \quad \text{as } k \to \infty.$$

Indeed, this is a general fact (see, e.g., [5, Theorem 1, p. 54]): if $\mu \in \mathcal{M}(\Omega)$ and $|\mu|(\partial\Omega_\delta) = 0$, then

$$\int_{\Omega_\delta} \psi(\rho_k * \mu) \to \int_{\Omega_\delta} \psi\,d\mu \quad \forall \psi \in C^0(\overline{\Omega}_\delta).$$

For any $\delta \in (0, \delta_0)$ verifying (4.5), as $k \to \infty$ in (4.4), we get

$$\left| \int_{\Omega_\delta} \psi\,\Delta u + \int_{\Omega_\delta} \nabla\psi \cdot \nabla u \right| \leq \|\nabla u\|_{L^\infty(\Omega)} \|\psi\|_{L^1(\partial\Omega_\delta)} \quad \forall \psi \in C^1(\overline{\Omega}). \tag{4.6}$$

From this estimate, one deduces that for every $\psi \in C^1(\overline{\Omega})$,

$$\begin{aligned} \left| \int_{\Omega_\delta} \nabla\psi \cdot \nabla u \right| &\leq \|\Delta u\|_{\mathcal{M}(\Omega)} \|\psi\|_{L^\infty(\Omega_\delta)} + \|\nabla u\|_{L^\infty(\Omega)} \|\psi\|_{L^1(\partial\Omega_\delta)} \\ &\leq \big(\|\Delta u\|_{\mathcal{M}(\Omega)} + \|\nabla u\|_{L^\infty(\Omega)} |\partial\Omega_\delta|\big) \|\psi\|_{L^\infty(\Omega)}. \end{aligned}$$

As $\delta \to 0$, we conclude that $u \in \mathbb{X}$.

In order to prove that $\frac{\partial u}{\partial n} \in L^\infty(\partial\Omega)$, we return to estimate (4.6). Given $\phi \in C^1(\partial\Omega)$, we fix an extension $\psi \in C^1(\overline{\Omega})$ of ϕ; note that

$$\|\psi\|_{L^1(\partial\Omega_\delta)} \leq \|\phi\|_{L^1(\partial\Omega)} + C\delta \quad \forall \delta \in (0, \delta_0),$$

for some constant $C > 0$. Insert this test function ψ in (4.6). As $\delta \to 0$ we obtain, by dominated convergence,

$$\left| \int_\Omega \psi\, \Delta u + \int_\Omega \nabla\psi \cdot \nabla u \right| \leq \|\nabla u\|_{L^\infty(\Omega)} \|\phi\|_{L^1(\partial\Omega)}.$$

Hence,

$$\left| \int_{\partial\Omega} \phi\, \frac{\partial u}{\partial n} \right| \leq \|\nabla u\|_{L^\infty(\Omega)} \|\phi\|_{L^1(\partial\Omega)} \quad \forall \phi \in C^1(\partial\Omega).$$

Therefore, by duality $\frac{\partial u}{\partial n} \in L^\infty(\partial\Omega)$ and (4.1) holds.

We now assume that $u \in C^1(\overline{\Omega}) \cap \mathbb{X}$ and we denote by h the normal derivative of u in the standard sense. By Lemma B.1 and Remark B.1, there exists a sequence $(u_k) \subset C^\infty(\overline{\Omega})$ satisfying (B.2) and (B.3) and such that

$$u_k \to u \quad \text{in } C^1(\overline{\Omega}).$$

In particular,

$$\frac{\partial u_k}{\partial n} \to h \quad \text{uniformly on } \partial\Omega.$$

Thus,

$$\int_\Omega \nabla u \cdot \nabla\psi + \int_\Omega \psi\, \Delta u = \int_{\partial\Omega} h\, \psi \quad \forall \psi \in C^1(\overline{\Omega}). \tag{4.7}$$

Hence, the normal derivative $\frac{\partial u}{\partial n}$ in the sense of the space $\mathbb{X}$ coincides with h. □

When $u \in \mathbb{X}$ the measure $\frac{\partial u}{\partial n}$ need not be an L^1-function. Surprisingly, this is always true if u vanishes on $\partial\Omega$:

Proposition 4.2. *Let $u \in W_0^{1,1}(\Omega)$. Then, $u \in \mathbb{X}$ if, and only if, $\Delta u \in \mathcal{M}(\Omega)$ (in the sense of distributions). Moreover, $\frac{\partial u}{\partial n} \in L^1(\partial\Omega)$ and*

$$\left\| \frac{\partial u}{\partial n} \right\|_{L^1(\partial\Omega)} \leq \|\Delta u\|_{\mathcal{M}(\Omega)}. \tag{4.8}$$

Proof. We split the proof into two steps:

Step 1. Proof of (4.8) if u is smooth in a neighborhood of $\partial\Omega$.

Under this assumption, $\frac{\partial u}{\partial n}$ is a smooth function on $\partial\Omega$. Denote by v_1 and v_2 the solutions of

$$\begin{cases} -\Delta v_1 = \mu^+ & \text{in } \Omega, \\ v_1 = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta v_2 = \mu^- & \text{in } \Omega, \\ v_2 = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mu = -\Delta u$. In particular,

$$u = v_1 - v_2 \quad \text{in } \Omega.$$

Since μ is smooth in a neighborhood of $\partial\Omega$, μ^+ and μ^- are Lipschitz continuous near $\partial\Omega$. Hence, v_1 and v_2 are of class C^2 near $\partial\Omega$. Moreover, $v_1 \geq 0$ in Ω and $v_1 = 0$ on $\partial\Omega$; thus,

$$\frac{\partial v_1}{\partial n} \leq 0 \quad \text{on } \partial\Omega.$$

It follows that

$$\int_{\partial\Omega} \left|\frac{\partial v_1}{\partial n}\right| = -\int_{\partial\Omega} \frac{\partial v_1}{\partial n} = \int_\Omega \mu^+.$$

Similarly,

$$\int_{\partial\Omega} \left|\frac{\partial v_2}{\partial n}\right| = \int_\Omega \mu^-.$$

Therefore,

$$\int_{\partial\Omega} \left|\frac{\partial u}{\partial n}\right| \leq \int_{\partial\Omega} \left|\frac{\partial v_1}{\partial n}\right| + \int_{\partial\Omega} \left|\frac{\partial v_2}{\partial n}\right| = \int_\Omega (\mu^+ + \mu^-) = \int_\Omega |\Delta u|.$$

Step 2. Proof of the proposition completed.

Let $(\varphi_k) \subset C_0^\infty(\Omega)$ be a sequence of test functions such that

$$0 \leq \varphi_k \leq 1 \quad \text{in } \overline{\Omega} \quad \text{and} \quad \varphi_k(x) = 1 \quad \text{if } d(x, \partial\Omega) \geq \frac{1}{k}.$$

Take $\mu_k = -\varphi_k \, \Delta u$, $\forall k \geq 1$. Then, $(\mu_k) \subset \mathcal{M}(\Omega)$ is a sequence of measures such that $\operatorname{supp} \mu_k \subset \Omega$ and, by dominated convergence,

$$\mu_k \to -\Delta u \quad \text{strongly in } \mathcal{M}(\Omega). \tag{4.9}$$

For each $k \geq 1$, let u_k be the unique solution of

$$\begin{cases} -\Delta u_k = \mu_k & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that u_k is harmonic in a neighborhood of $\partial\Omega$. We claim that

$$\int_{\partial\Omega} \phi \, \frac{\partial u_k}{\partial n} \to \int_{\partial\Omega} \phi \, \frac{\partial u}{\partial n} \quad \forall \phi \in C^1(\partial\Omega). \tag{4.10}$$

Indeed, since $u_k \to u$ in $L^1(\Omega)$ and (∇u_k) is bounded in $W_0^{1,p}(\Omega)$ for every $1 \leq p < \frac{N}{N-1}$ (see [10]), we have

$$\int_\Omega \nabla\psi \cdot \nabla u_k \to \int_\Omega \nabla\psi \cdot \nabla u \quad \forall \psi \in C^1(\overline{\Omega}). \tag{4.11}$$

Assertion (4.10) then follows from (4.9) and (4.11).

Applying Step 1 to the function $u_i - u_j$, we have

$$\left\| \frac{\partial u_i}{\partial n} - \frac{\partial u_j}{\partial n} \right\|_{L^1(\partial\Omega)} \leq \|\mu_i - \mu_j\|_{\mathcal{M}(\Omega)} \quad \forall i, j \geq 1.$$

In view of the strong convergence of (μ_k) in $\mathcal{M}(\Omega)$, $\left(\frac{\partial u_k}{\partial n}\right)$ is a Cauchy sequence in $L^1(\partial\Omega)$. Hence, this sequence converges in $L^1(\partial\Omega)$ to some function h. By (4.10), $h = \frac{\partial u}{\partial n}$; hence,

$$\frac{\partial u_k}{\partial n} \to \frac{\partial u}{\partial n} \quad \text{in } L^1(\partial\Omega).$$

Moreover, since (4.8) holds for every u_k, it also holds for u. The proof is complete. □

We now show that if $u \in W^{1,1}(\Omega)$ and $\nabla u \in BV(\Omega)$ then the normal derivative $\frac{\partial u}{\partial n}$ in the sense of the space $\mathbb{X}$ coincides with the function $n \cdot \nabla u$ on $\partial\Omega$ defined in the sense of traces:

Proposition 4.3. *Assume that $u \in W^{1,1}(\Omega)$ and $\nabla u \in BV(\Omega)$; hence,*

$$\Delta u = \operatorname{div}(\nabla u) \in \mathcal{M}(\Omega).$$

Then, $u \in \mathbb{X}$ and $\frac{\partial u}{\partial n}$ coincides with $n \cdot \nabla u|_{\partial\Omega}$ on $\partial\Omega$, where $\nabla u|_{\partial\Omega}$ is understood in the sense of traces. In particular, $\frac{\partial u}{\partial n} \in L^1(\partial\Omega)$ and

$$\left\| \frac{\partial u}{\partial n} \right\|_{L^1(\partial\Omega)} \le C \|\nabla u\|_{BV(\Omega)}. \tag{4.12}$$

In the proof of Proposition 4.3 we use the notion of strict convergence in $BV(A)$, where $A \subset \mathbb{R}^N$ is a Lipschitz domain. We recall that a sequence $(f_k) \subset BV(A)$ converges strictly to $f \in BV(A)$ if

$$f_k \to f \quad \text{strongly in } L^1(A) \quad \text{and} \quad \int_A |Df_k| \to \int_A |Df|.$$

By [1, Theorem 3.88], the trace operator

$$f \in BV(A) \mapsto f|_{\partial A} \in L^1(\partial A)$$

is continuous from $BV(A)$ (under strict convergence) into $L^1(\partial A)$ (under strong convergence).

Proof of Proposition 4.3. By Lemma B.1 and Remark B.1, there exists a sequence $(u_k) \subset C^\infty(\overline{\Omega})$ satisfying (B.1)–(B.3) and (B.12). Since (∇u_k) converges strictly to ∇u in $BV(\Omega)$, we have

$$\nabla u_k|_{\partial\Omega} \to \nabla u|_{\partial\Omega} \quad \text{in } L^1(\partial\Omega). \tag{4.13}$$

Hence,

$$\int_\Omega \nabla u \cdot \nabla\psi + \int_\Omega \psi\, \Delta u = \int_{\partial\Omega} \left(n \cdot \nabla u|_{\partial\Omega}\right) \psi \quad \forall \psi \in C^1(\partial\Omega).$$

This implies that $\frac{\partial u}{\partial n} \in L^1(\partial\Omega)$ and equals $n \cdot \nabla u|_{\partial\Omega}$. By the BV-trace theory, (4.12) holds. □

5. Proof of Theorem 1.2

We first establish Theorem 1.2 for functions in $C^2_{\mathrm{D}}(\overline{\Omega})$, where

$$C^2_{\mathrm{D}}(\overline{\Omega}) = \{\zeta \in C^2(\overline{\Omega});\ \zeta = 0 \text{ on } \partial\Omega\}. \tag{5.1}$$

Lemma 5.1. *Let $u \in C^2_{\mathrm{D}}(\overline{\Omega})$. Then, $\Delta u^+ \in \mathcal{M}(\Omega)$ and*

$$\|\Delta u^+\|_{\mathcal{M}} \le \|\Delta u\|_{L^1}. \tag{5.2}$$

Proof. Apply (3.3) with $u + a$, where $a > 0$. We deduce that

$$\big[(u+a)^+\big]_{\mathbb{X}} \le [u+a]_{\mathbb{X}} = [u]_{\mathbb{X}}. \tag{5.3}$$

Since $(u+a)^+ = u + a$ in a neighborhood of $\partial\Omega$,

$$\frac{\partial}{\partial n}(u+a)^+ = \frac{\partial u}{\partial n} \quad \text{on } \partial\Omega. \tag{5.4}$$

Note that

$$\big[(u+a)^+\big]_{\mathbb{X}} = \big\|\Delta(u+a)^+\big\|_{\mathcal{M}(\Omega)} + \left\|\frac{\partial}{\partial n}(u+a)^+\right\|_{L^1(\partial\Omega)},$$

$$[u]_{\mathbb{X}} = \|\Delta u\|_{L^1(\Omega)} + \left\|\frac{\partial u}{\partial n}\right\|_{L^1(\partial\Omega)}.$$

By (5.3)–(5.4), we then have

$$\big\|\Delta(u+a)^+\big\|_{\mathcal{M}} \le \|\Delta u\|_{L^1} \quad \forall a > 0.$$

The result follows from the lower semicontinuity of the norm $\|\cdot\|_{\mathcal{M}}$ with respect to the weak* convergence as $a \to 0$. □

Proof of Theorem 1.2. Since $u \in \mathbb{X}$, $\Delta u \in \mathcal{M}(\Omega)$. Take a sequence $(\mu_k) \subset C^\infty(\overline{\Omega})$ such that

$$\mu_k \overset{*}{\rightharpoonup} -\Delta u \quad \text{weak}^* \text{ in } \mathcal{M}(\Omega) \quad \text{and} \quad \|\mu_k\|_{L^1} \to \|\Delta u\|_{\mathcal{M}}.$$

For each $k \ge 1$, let $u_k \in C^2_{\mathrm{D}}(\overline{\Omega})$ be the solution of

$$-\Delta u_k = \mu_k \quad \text{in } \Omega.$$

Then, by standard elliptic estimates,

$$u_k \to u \quad \text{in } L^1(\Omega).$$

On the other hand, it follows from Lemma 5.1 that $\Delta u_k^+ \in \mathcal{M}(\Omega)$ and

$$\|\Delta u_k^+\|_{\mathcal{M}} \le \|\Delta u_k\|_{L^1}.$$

Thus,

$$\left|\int_\Omega u_k^+\,\Delta\zeta\right| \le \|\Delta u_k\|_{L^1}\|\zeta\|_{L^\infty} = \|\mu_k\|_{L^1}\|\zeta\|_{L^\infty} \quad \forall \zeta \in C^2_{\mathrm{D}}(\overline{\Omega}).$$

As $k \to \infty$ we obtain

$$\left| \int_\Omega u^+ \Delta\zeta \right| \le \|\Delta u\|_{\mathcal{M}} \|\zeta\|_{L^\infty} \quad \forall \zeta \in C^2_{\mathrm{D}}(\overline{\Omega}).$$

This gives (1.5). From Proposition 4.2, we know that $\frac{\partial u}{\partial n}, \frac{\partial u^+}{\partial n} \in L^1(\partial\Omega)$ and (1.6) holds. □

6. Kato's Inequality up to the Boundary

Before proving Theorem 1.3, we first present some variants of Kato's inequality when Δu and $\frac{\partial u}{\partial n}$ are not necessarily L^1-functions but only finite measures. We prove for instance the following companion to [3, Proposition 4.B.5]:

Proposition 6.1. *Let $u \in L^1(\Omega)$ be such that*

$$-\int_\Omega u\,\Delta\zeta \le \int_{\partial\Omega} h\zeta + \int_\Omega g\zeta \quad \forall \zeta \in C^2_{\mathrm{N}}(\overline{\Omega}), \quad \zeta \ge 0 \text{ in } \overline{\Omega} \tag{6.1}$$

for some $g \in L^1(\Omega)$ and $h \in L^1(\partial\Omega)$. Then, $u \in W^{1,1}(\Omega)$ and

$$\int_\Omega \nabla u^+ \cdot \nabla\psi \le \int_{\substack{\partial\Omega \\ [u\ge 0]}} h\psi + \int_{\substack{\Omega \\ [u\ge 0]}} g\psi \quad \forall \psi \in C^1(\overline{\Omega}), \quad \psi \ge 0 \text{ in } \Omega. \tag{6.2}$$

Proof. By Proposition 2.2, $u \in \mathbb{X}$. Moreover,

$$\begin{cases} -\Delta u \le g & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} \le h & \text{on } \partial\Omega. \end{cases} \tag{6.3}$$

We now split the proof into two steps:

Step 1. Let $\Phi \in C^2(\mathbb{R})$ be a nondecreasing convex function such that $\Phi' \in L^\infty(\mathbb{R})$. Then,

$$\int_\Omega \nabla\Phi(u) \cdot \nabla\psi \le \int_{\partial\Omega} \psi\,\Phi'(u)h + \int_\Omega \psi\,\Phi'(u)g \tag{6.4}$$

for every $\psi \in C^1(\overline{\Omega})$ such that $\psi \ge 0$ in $\overline{\Omega}$.

Let $(g_k) \subset C^\infty(\overline{\Omega})$ and $(h_k) \subset C^\infty(\partial\Omega)$ be such that

$$g_k \to g \text{ in } L^1(\Omega) \text{ and a.e.} \quad \text{and} \quad h_k \to h \text{ in } L^1(\partial\Omega) \text{ and a.e.}$$

Next, take $(\mu_k) \subset C^\infty(\overline{\Omega})$ and $(\nu_k) \subset C^\infty(\partial\Omega)$ such that

$$\mu_k \overset{*}{\rightharpoonup} -\Delta u \text{ weak}^* \text{ in } \mathcal{M}(\overline{\Omega}) \quad \text{and} \quad \nu_k \overset{*}{\rightharpoonup} \frac{\partial u}{\partial n} \text{ weak}^* \text{ in } \mathcal{M}(\partial\Omega).$$

In view of (6.3) and

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} = \int_\Omega \Delta u,$$

we may assume that

$$\mu_k \le g_k \text{ in } \Omega, \quad \nu_k \le h_k \text{ on } \partial\Omega \quad \text{and} \quad \int_{\partial\Omega} \nu_k = -\int_\Omega \mu_k \quad \forall k \ge 1.$$

Let $u_k \in C^\infty(\overline{\Omega})$ be the unique solution of

$$\begin{cases} -\Delta u_k = \mu_k & \text{in } \Omega, \\ \dfrac{\partial u_k}{\partial n} = \nu_k & \text{on } \partial\Omega, \end{cases}$$

such that $\int_\Omega u_k = \int_\Omega u$. By Remark 2.1, the sequence (u_k) is bounded in $W^{1,p}(\Omega)$ for every $1 \le p < \frac{N}{N-1}$. Passing to a subsequence if necessary, we have

$$\nabla\Phi(u_k) \rightharpoonup \nabla\Phi(u) \quad \text{weakly in } L^1(\Omega).$$

Let $\psi \in C^1(\overline{\Omega})$, $\psi \ge 0$ in $\overline{\Omega}$. As in Lemma 3.1, for every $k \ge 1$ we have

$$\begin{aligned} \int_\Omega \nabla\Phi(u_k)\cdot\nabla\psi &\le \int_{\partial\Omega} \psi\,\Phi'(u_k)\frac{\partial u_k}{\partial n} - \int_\Omega \psi\,\Phi'(u_k)\Delta u_k \\ &\le \int_{\partial\Omega} \psi\,\Phi'(u_k)h_k + \int_\Omega \psi\,\Phi'(u_k)g_k. \end{aligned}$$

By dominated convergence we obtain (6.4) as $k \to \infty$.

Step 2. Proof of the proposition completed.

Apply (6.4) with $\Phi = \Phi_k$, where (Φ_k) is a sequence of smooth convex functions such that $\Phi_k(0) = 0$, $0 \le \Phi_k' \le 1$ and

$$\Phi_k'(t) \to \begin{cases} 1 & \text{if } t \ge 0, \\ 0 & \text{if } t < 0. \end{cases}$$

The result follows as we let $k \to \infty$. □

The following variant of Proposition 6.1 will be needed below:

Proposition 6.2. *Let $u \in L^1(\Omega)$ be such that*

$$-\int_\Omega u\,\Delta\zeta \le \int_{\partial\Omega} h\zeta + \int_\Omega \zeta\,d\mu \quad \forall\zeta \in C^2_{\mathrm{N}}(\overline{\Omega}), \quad \zeta \ge 0 \text{ in } \overline{\Omega} \tag{6.5}$$

for some $\mu \in \mathcal{M}(\Omega)$, $\mu \ge 0$, and $h \in L^1(\partial\Omega)$. Then, $u \in W^{1,1}(\Omega)$ and

$$\int_\Omega \nabla u^+ \cdot \nabla\psi \le \int_{\substack{\partial\Omega \\ [u\ge 0]}} h\psi + \int_\Omega \psi\,d\mu \quad \forall\psi \in C^1(\overline{\Omega}), \quad \psi \ge 0 \text{ in } \overline{\Omega}. \tag{6.6}$$

Proof. One can proceed as in the proof of Proposition 6.1. In Step 1, one should replace (6.4) by

$$\int_\Omega \nabla\Phi(u)\cdot\nabla\psi \le \int_{\partial\Omega} \psi\,\Phi'(u)h + \|\Phi'\|_{L^\infty}\int_\Omega \psi\,d\mu. \tag{6.4$'$}$$

Inequality (6.4′) is easily obtained by approximation, where the sequence $(g_k) \subset C^\infty(\overline{\Omega})$ is chosen so that

$$g_k \overset{*}{\rightharpoonup} \mu \quad \text{weak}^* \text{ in } \mathcal{M}(\overline{\Omega}).$$

The rest of the argument remains unchanged. □

We now prove the

Proposition 6.3. *Let $u \in \mathbb{X}$. If $\frac{\partial u}{\partial n} \in L^1(\partial\Omega)$, then*

$$\frac{\partial u^+}{\partial n} \le \begin{cases} \dfrac{\partial u}{\partial n} & \text{on } [u > 0], \\ 0 & \text{on } [u < 0], \\ \min\left\{\dfrac{\partial u}{\partial n}, 0\right\} & \text{on } [u = 0]. \end{cases} \tag{6.7}$$

Proof. Denoting by $\mu = (-\Delta u)^+$ and $h = \frac{\partial u}{\partial n}$, we have

$$-\int_\Omega u\,\Delta\zeta \le \int_{\partial\Omega} h\zeta + \int_\Omega \zeta\,d\mu \quad \forall \zeta \in C^2_{\mathrm{N}}(\overline{\Omega}), \quad \zeta \ge 0 \text{ in } \overline{\Omega}.$$

Therefore, by Proposition 6.2, u^+ satisfies

$$\int_\Omega \nabla u^+ \cdot \nabla\psi \le \int_{\substack{\partial\Omega \\ [u\ge 0]}} h\psi + \int_\Omega \psi\,d\mu \quad \forall \psi \in C^1(\overline{\Omega}), \quad \psi \ge 0 \text{ in } \overline{\Omega}. \tag{6.8}$$

By Theorem 1.1, we know that $u^+ \in \mathbb{X}$. It thus follows that

$$\frac{\partial u^+}{\partial n} \le \chi_{[u\ge 0]} h = \chi_{[u\ge 0]} \frac{\partial u}{\partial n} \quad \text{on } \partial\Omega. \tag{6.9}$$

Given $a > 0$, we now apply (6.8) with u replaced by $u - a$. As $a \to 0$, we obtain

$$\int_{\partial\Omega} u^+ \frac{\partial\psi}{\partial n} - \int_\Omega u^+\,\Delta\psi \le \int_{\substack{\partial\Omega \\ [u>0]}} h\psi + \int_\Omega \psi\,d\mu \quad \forall \psi \in C^1(\overline{\Omega}), \quad \psi \ge 0 \text{ in } \Omega. \tag{6.10}$$

Hence,

$$\frac{\partial u^+}{\partial n} \le \chi_{[u>0]} h = \chi_{[u>0]} \frac{\partial u}{\partial n} \quad \text{on } \partial\Omega.$$

In particular,

$$\frac{\partial u^+}{\partial n} \le 0 \quad \text{on } [u = 0]. \tag{6.11}$$

Assertion (6.7) follows by combining (6.9) and (6.11). □

We state the following consequence of Proposition 6.3:

Corollary 6.1. *Let $u \in \mathbb{X} \cap W_0^{1,1}(\Omega)$. If $u \geq 0$ in Ω, then*

$$\frac{\partial u}{\partial n} \leq 0 \quad \text{on } \partial\Omega.$$

Proof. Since $u = u^+$ in Ω and $u = 0$ on $\partial\Omega$, applying Proposition 6.3 above we get

$$\frac{\partial u}{\partial n} = \frac{\partial u^+}{\partial n} \leq \min\left\{\frac{\partial u}{\partial n}, 0\right\} \leq 0 \quad \text{on } \partial\Omega. \qquad \square$$

We now present the

Proof of Theorem 1.3. By Theorem 1.1, $u^+ \in \mathbb{X}$. Applying Kato's inequality to $u - a$, we have

$$\Delta(u-a)^+ \geq \chi_{[u \geq a]} \Delta u \quad \text{in } \Omega \tag{6.12}$$

for every $a \in \mathbb{R}$. As $a \downarrow 0$ in (6.12) we get

$$\Delta u^+ \geq \chi_{[u>0]} \Delta u = G \quad \text{in } \Omega.$$

By this estimate and (6.7), for every $\psi \in C^1(\overline{\Omega})$ with $\psi \geq 0$ in Ω,

$$\int_\Omega \nabla u^+ \cdot \nabla \psi = \int_{\partial\Omega} \psi \frac{\partial u^+}{\partial n} - \int_\Omega \psi \, \Delta u^+ \leq \int_{\partial\Omega} H\psi - \int_\Omega G\psi.$$

The proof is complete. $\square$

7. Computing $\frac{\partial u^+}{\partial n}$ for $W^{2,1}$-Functions

Our goal in this section is to give a positive answer to Open Problems 1 and 2 under the additional assumption that $u \in W^{2,1}(\Omega)$:

Theorem 7.1. *If $u \in W^{2,1}(\Omega)$, then $\nabla u^+ \in BV(\Omega)$ (so that, $u^+ \in \mathbb{X}$ by Proposition 4.3) and*

$$\frac{\partial u^+}{\partial n} = \begin{cases} \dfrac{\partial u}{\partial n} & \text{on } [u > 0], \\ 0 & \text{on } [u < 0], \\ \min\left\{\dfrac{\partial u}{\partial n}, 0\right\} & \text{on } [u = 0]. \end{cases} \tag{7.1}$$

We first prove the

Lemma 7.1. *If $v \in W^{1,1}(\Omega)$ and $\nabla v \in BV(\Omega)$, then*

$$\frac{\partial v}{\partial n}(x) = \lim_{t \downarrow 0} \frac{v(x) - v(x - tn(x))}{t} \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega. \tag{7.2}$$

In (7.2), we identify v with its precise representative, which is well-defined outside a set of zero $\mathcal{H}^{N-1}$-Hausdorff measure; see [5, Sec. 4.8, Theorem 1 and Sec. 5.6, Theorem 3].

Proof. Since $v \in W^{1,1}(\Omega)$, for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$ the function

$$t \in (0,\delta) \mapsto v\big(x - tn(x)\big)$$

is well-defined for some $\delta > 0$ and belongs to $W^{1,1}(0,\delta)$. Thus,

$$\frac{v\big(x - tn(x)\big) - v(x)}{t} = -n(x) \cdot \int_0^1 \nabla v\big(x - stn(x)\big)\, ds. \tag{7.3}$$

Moreover, since $\nabla v \in BV(\Omega)$, for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$ the function

$$r \in (0,\delta) \mapsto \nabla v\big(x - rn(x)\big)$$

belongs to $BV(0,\delta) \subset L^\infty(0,\delta)$ and (see [1, Theorem 3.108])

$$\lim_{r\downarrow 0} \nabla v\big(x - rn(x)\big) = \nabla v|_{\partial\Omega}(x). \tag{7.4}$$

We deduce from (7.3)–(7.4) that

$$\lim_{t\downarrow 0} \frac{v\big(x - tn(x)\big) - v(x)}{t} = -n(x) \cdot \nabla v|_{\partial\Omega}(x).$$

By Proposition 4.3 above, $\frac{\partial v}{\partial n} = n \cdot \nabla v|_{\partial\Omega}$ and the conclusion follows. □

We also need the following elementary lemma whose proof is left to the reader:

Lemma 7.2. *Let $v : [0,\delta] \to \mathbb{R}$ be such that*

$$\lim_{t\downarrow 0} \frac{v(0) - v(t)}{t} = \alpha \in \mathbb{R}. \tag{7.5}$$

Then,

$$\lim_{t\downarrow 0} \frac{v^+(0) - v^+(t)}{t} = \begin{cases} \alpha & \text{if } v(0) > 0, \\ 0 & \text{if } v(0) < 0, \\ \min\{\alpha, 0\} & \text{if } v(0) = 0. \end{cases} \tag{7.6}$$

We now present the

Proof of Theorem 7.1. We split the proof into three steps:

Step 1. Proof of the assertion: $\nabla u^+ \in BV(\Omega)$.

Extending u to $\mathbb{R}^N$, we may assume that $u \in W^{2,1}(\mathbb{R}^N)$. We claim that

$$\frac{\partial^2 u^+}{\partial \mathrm{e}^2} \geq \chi_{[u\geq 0]} \frac{\partial^2 u}{\partial \mathrm{e}^2} \quad \text{in } \mathcal{D}'(\mathbb{R}^N) \tag{7.7}$$

for every $\mathrm{e} \in \mathbb{R}^N \backslash \{0\}$. Indeed, let (Φ_k) be a sequence of smooth convex functions such that $\Phi_k(0) = 0$, $\|\Phi_k'\|_{L^\infty} \leq 1$ and

$$\Phi_k'(t) \to \begin{cases} 1 & \text{if } t \geq 0, \\ 0 & \text{if } t < 0. \end{cases} \tag{7.8}$$

Then,

$$\frac{\partial^2 \Phi_k(u)}{\partial \mathrm{e}^2} = \Phi_k'(u)\frac{\partial^2 u}{\partial \mathrm{e}^2} + \Phi_k''(u)\left(\frac{\partial u}{\partial \mathrm{e}}\right)^2 \geq \Phi_k'(u)\frac{\partial^2 u}{\partial \mathrm{e}^2} \quad \text{in } \mathbb{R}^N.$$

As $k \to \infty$, we obtain (7.7).

It follows from (7.7) that $\frac{\partial^2 u^+}{\partial \mathrm{e}^2} \in \mathcal{M}(\Omega)$ for every $\mathrm{e} \in \mathbb{R}^N \backslash \{0\}$. Applying the conclusion with $\mathrm{e} = \mathrm{e}_i, \mathrm{e}_j, \mathrm{e}_i + \mathrm{e}_j$ for every $i, j \in \{1, \ldots, N\}$ we deduce that $D^2 u^+$ is a finite measure in Ω. Thus, $\nabla u^+ \in BV(\Omega)$.

Step 2. Proof of (7.1).

By Lemma 7.1, for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$, u satisfies

$$\lim_{t \downarrow 0} \frac{u(x) - u\big(x - tn(x)\big)}{t} = \frac{\partial u}{\partial n}(x). \tag{7.9}$$

Hence, by (7.2) applied to u^+ and by (7.6) applied to $v(t) = u(x - tn(x))$,

$$\frac{\partial u^+}{\partial n}(x) = \lim_{t \downarrow 0} \frac{u^+(x) - u^+\big(x - tn(x)\big)}{t} = \begin{cases} \dfrac{\partial u}{\partial n}(x) & \text{if } u(x) > 0, \\ 0 & \text{if } u(x) < 0, \\ \min\left\{\dfrac{\partial u}{\partial n}(x), 0\right\} & \text{if } u(x) = 0, \end{cases}$$

for every $x \in \partial\Omega$ for which (7.9) holds. Since this is true $\mathcal{H}^{N-1}$-a.e. on $\partial\Omega$, (7.1) follows. The proof of Theorem 7.1 is complete. □

Appendix A. The Measure Δu^+ Need not be Finite

In this appendix, we construct a harmonic function in dimension 2 such that $\int_\Omega |\Delta u^+| = \infty$:

Proposition A.1. *Let*

$$\Omega = \big\{(x, y) \in \mathbb{R}^2;\ x^2 + y^2 < 1 \text{ and } x > 0\big\}.$$

There exists a harmonic function $u \in C(\overline{\Omega}) \cap H^1(\Omega)$ with $u|_{\partial\Omega} \in W^{1,1}(\partial\Omega)$ such that

(i) *$u \notin \mathbb{X}$ and $u^+ \notin \mathbb{X}$;*
(ii) *$\Delta u^+ \geq 0$ in the sense of distributions;*
(iii) *Δu^+ is **not** a finite measure in Ω.*

Proof. Let u be the function in $\overline{\Omega}$ given in polar coordinates by

$$u(r,\theta) = \sum_{k=1}^{\infty} r^{a_k} \sin(a_k\theta) \tag{A.1}$$

where $(a_k) \subset (0,1)$ is a sequence such that

$$\sum_{k=1}^{\infty} k a_k < \infty.$$

Since

$$\sum_{k=1}^{\infty} |r^{a_k} \sin(a_k\theta)| \le \frac{\pi}{2} \sum_{k=1}^{\infty} a_k,$$

it follows that $u \in C(\overline{\Omega})$ and u is harmonic in Ω (u is a series of harmonic functions). Note that

$$|\nabla u|^2 = \sum_{j,k=1}^{\infty} a_j a_k r^{a_j+a_k-2} \cos((a_j - a_k)\theta).$$

Thus,

$$\int_\Omega |\nabla u|^2 \le \pi \sum_{j,k=1}^{\infty} \frac{a_j a_k}{a_j + a_k} \le 2\pi \sum_{\substack{j,k=1\\ j\le k}}^{\infty} \frac{a_j a_k}{a_j + a_k} \le 2\pi \sum_{k=1}^{\infty} k a_k < \infty;$$

in other words, $u \in H^1(\Omega)$. Denoting by τ the tangential unit vector of $\partial\Omega$, we have

$$\int_{\partial\Omega} \left|\frac{\partial u}{\partial \tau}\right| = 4 \sum_{k=1}^{\infty} \sin\Big(a_k \frac{\pi}{2}\Big) \le 2\pi \sum_{k=1}^{\infty} a_k < \infty;$$

hence, $u \in W^{1,1}(\partial\Omega)$.

Since u is harmonic in Ω, u^+ is subharmonic. Thus, $\Delta u^+ \ge 0$ in Ω. We show that Δu^+ is not a finite measure in Ω. Note that u vanishes only on the x-axis. Denoting by dx $(= dr)$ the 1-dimensional Lebesgue measure on the segment $(0,1) \times \{0\}$, we then have

$$\Delta u^+ = \frac{\partial u}{\partial y}(x,0)\,dx = \frac{1}{r}\frac{\partial u}{\partial \theta}(r,0)\,dr = \sum_{k=1}^{\infty} a_k r^{a_k-1}\,dr.$$

Therefore,

$$\int_\Omega |\Delta u^+| = \sum_{k=1}^{\infty} \int_0^1 a_k r^{a_k-1}\,dr = \sum_{k=1}^{\infty} 1 = \infty.$$

Hence, $u^+ \notin \mathbb{X}$ and, by Theorem 1.1, this means that $u \notin \mathbb{X}$. $\square$

Remark A.1. This example also shows that given $\varphi \in W^{1,1}(\partial\Omega)$, it is in general not possible to construct a function $v \in W^{2,1}(\Omega)$ such that $v|_{\partial\Omega} = \varphi$. This is in contrast with the well-known result of Gagliardo [6] which asserts that the map

$$w \in W^{1,1}(\Omega) \mapsto w|_{\partial\Omega} \in L^1(\partial\Omega)$$

is surjective.

Indeed, take $\varphi = u|_{\partial\Omega}$, where u is given by (A.1). Suppose by contradiction that there exists some $v \in W^{2,1}(\Omega)$ such that $v|_{\partial\Omega} = \varphi$. Applying Proposition 4.2 to $u - v \in W_0^{1,1}(\Omega)$, we would deduce that $\frac{\partial}{\partial n}(u - v) \in L^1(\partial\Omega)$. But $v \in W^{2,1}(\Omega)$ implies $\frac{\partial v}{\partial n} \in L^1(\partial\Omega)$ and therefore

$$\frac{\partial u}{\partial n} = \frac{\partial}{\partial n}(u - v) + \frac{\partial v}{\partial n} \in L^1(\partial\Omega),$$

a contradiction.

Appendix B. Approximation by Smooth Functions in $\overline{\Omega}$

In this appendix, we establish the following

Lemma B.1. *Given $u \in \mathbb{X}$, there exists a sequence $(u_k) \subset C^\infty(\overline{\Omega})$ such that*

$$u_k \to u \quad \text{in } W^{1,1}\big(\Omega\big), \tag{B.1}$$

$$\int_\Omega \psi\, \Delta u_k \to \int_\Omega \psi\, \Delta u \quad \forall \psi \in C^1(\overline{\Omega}) \tag{B.2}$$

and

$$\int_{\partial\Omega} \psi\, \frac{\partial u_k}{\partial n} \to \int_{\partial\Omega} \psi\, \frac{\partial u}{\partial n} \quad \forall \psi \in C^1(\overline{\Omega}). \tag{B.3}$$

Proof. We split the proof into two steps:

Step 1. Given $x_0 \in \partial\Omega$, there exist $\delta > 0$ and a sequence $(v_k) \subset C^\infty(\overline{\Omega})$ such that

$$v_k \to u \quad \text{in } W^{1,1}\big(B_\delta(x_0) \cap \Omega\big), \tag{B.4}$$

$$\int_\Omega \psi\, \Delta v_k \to \int_\Omega \psi\, \Delta u \quad \forall \psi \in C^1(\overline{\Omega}) \quad \text{with} \quad \operatorname{supp} \psi \subset B_\delta(x_0). \tag{B.5}$$

Since $\partial\Omega$ is smooth, there exist $\delta_1 > 0$ and an open cone $T \subset \mathbb{R}^N$ (with vertex at $0 \in \mathbb{R}^N$) such that

$$(x + T) \cap B_{\delta_1}(x) \subset \Omega \quad \forall x \in B_{\delta_1}(x_0) \cap \overline{\Omega}. \tag{B.6}$$

Let $\delta = \delta_1/2$ and $\rho \in C_0^\infty(B_\delta)$, $\rho \geq 0$, be such that $\int_{B_\delta} \rho = 1$ and

$$\operatorname{supp} \rho \subset -T. \tag{B.7}$$

Set

$$\rho_k(x) = k^N \rho(kx) \quad \forall x \in \mathbb{R}^N.$$

We show that the sequence $(v_k) \subset C^\infty(\overline{\Omega})$ given by

$$v_k(x) = \int_\Omega \rho_k(x-y)u(y)\,dy \quad \forall x \in \overline{\Omega} \tag{B.8}$$

satisfies (B.4) and (B.5).

Note that given any $x \in B_\delta(x_0) \cap \Omega$, by (B.7) $v_k(x)$ depends only on the values of u on a compact subset of $(x+T) \cap B_{\delta_1}(x)$. In fact, from (B.6) and (B.7) and a change of variable, we can rewrite (B.8) as

$$v_k(x) = \int_{T \cap B_{\delta_1}(0)} \rho_k(-z)u(x+z)\,dz \quad \forall x \in B_\delta(x_0) \cap \Omega. \tag{B.9}$$

Therefore,

$$\nabla v_k = \rho_k * (\nabla u) \quad \text{and} \quad \Delta v_k = \rho_k * (\Delta u) \quad \text{in } B_\delta(x_0) \cap \Omega. \tag{B.10}$$

In particular, (B.4) and (B.5) hold.

Step 2. Proof of the proposition completed.

By compactness of $\partial\Omega$, we can cover this set with finitely many balls $B_\delta(x_1), \ldots, B_\delta(x_t)$ such that (B.4) and (B.5) hold on each ball $B_\delta(x_i)$ for some sequence $(v_k^i) \subset C^\infty(\overline{\Omega})$. We now take $(v_k^0) \subset C^\infty(\overline{\Omega})$ and $\omega \Subset \Omega$ such that $\Omega \setminus \bigcup_{i=1}^t B_\delta(x_i) \subset \omega$,

$$v_k^0 \to u \quad \text{in } W^{1,1}(\omega) \quad \text{and} \quad \Delta v_k^0 \overset{*}{\rightharpoonup} \Delta u \quad \text{weak}^* \text{ in } \mathcal{M}(\omega)$$

(such sequence can be obtained via convolution of u).

Let (φ_i) be a partition of unity subordinated to the covering $\omega, B_\delta(x_1), \ldots, B_\delta(x_t)$ of $\overline{\Omega}$. One verifies that (B.1) and (B.2) hold for the sequence (u_k) given by

$$u_k = \sum_{i=0}^t \varphi_i v_k^i.$$

Assertion (B.3) immediately follows from (B.1) and (B.2). □

Remark B.1. An inspection of the proof of Lemma B.1 shows that

(i) if $u \in C^1(\overline{\Omega})$, then

$$u_k \to u \quad \text{in } C^1(\overline{\Omega}); \tag{B.11}$$

(ii) if $\nabla u \in BV(\Omega)$, then

$$\|D^2 u_k\|_{L^1(\Omega)} \to \|D^2 u\|_{\mathcal{M}(\Omega)}. \tag{B.12}$$

Appendix C. Proof of Lemma 2.1

The proof of Lemma 2.1 we present below follows the lines of [8, Lemma 7.3] (see also [7, Theorem 8.15]) with some minor modifications. We first need the following variant of the Gagliardo–Nirenberg inequality:

Proposition C.1. *Let*

$$\mathcal{A} = \left\{ v \in W^{1,1}(\Omega);\ |[v = 0]| \geq \frac{|\Omega|}{3} \right\}. \tag{C.1}$$

Then,

$$\|v\|_{L^{\frac{N}{N-1}}} \leq C\|\nabla v\|_{L^1} \quad \forall v \in \mathcal{A}. \tag{C.2}$$

We denote by $|E|$ the Lebesgue measure of a set $E \subset \mathbb{R}^N$.

Proof. By a variant of the Poincaré inequality (easily proved by contradiction), we have

$$\|v\|_{L^1} \leq C\|\nabla v\|_{L^1} \quad \forall v \in \mathcal{A}. \tag{C.3}$$

On the other hand, by the standard Gagliardo–Nirenberg inequality and an extension argument,

$$\|v\|_{L^{\frac{N}{N-1}}} \leq C(\|\nabla v\|_{L^1} + \|v\|_{L^1}) \quad \forall v \in W^{1,1}(\Omega). \tag{C.4}$$

Combining (C.3) and (C.4), we obtain (C.2). □

Proof of Lemma 2.1. Replacing w by $w - a$ for some suitable constant $a \in \mathbb{R}$ if necessary, we may assume that

$$|[w \leq 0]| \geq \frac{|\Omega|}{3} \quad \text{and} \quad |[w \geq 0]| \geq \frac{|\Omega|}{3}. \tag{C.5}$$

Given $t > 0$, let

$$v_t(x) = [w(x) - t]^+ \quad \forall x \in \Omega. \tag{C.6}$$

Using v_t as a test function in (2.4), one shows that

$$\|\nabla v_t\|_{L^2} \leq \|F\|_{L^q} |[w > t]|^{\frac{1}{2} - \frac{1}{q}}.$$

On the other hand, by Hölder's inequality and Proposition C.1,

$$\|v_t\|_{L^1} \leq C\|\nabla v_t\|_{L^2} |[w > t]|^{\frac{1}{2} + \frac{1}{N}}.$$

Thus,

$$\|v_t\|_{L^1} \leq C\|F\|_{L^q} |[w > t]|^{\alpha} \quad \forall t > 0, \tag{C.7}$$

where $\alpha = 1 + \frac{1}{N} - \frac{1}{q}$. Recall that

$$\|v_t\|_{L^1} = \int_0^\infty |[v_t > r]|\, dr = \int_t^M |[w > s]|\, ds, \tag{C.8}$$

where $M = \|w^+\|_{L^\infty}$. Since $\alpha > 1$, one deduces using (C.7) and (C.8) that

$$\|w^+\|_{L^\infty} \leq C\|F\|_{L^q}^{\frac{1}{\alpha}}\|w^+\|_{L^1}^{1-\frac{1}{\alpha}}. \tag{C.9}$$

From (C.9) and $\|w^+\|_{L^1} \leq |\Omega|\|w^+\|_{L^\infty}$, we then have

$$\|w^+\|_{L^\infty} \leq C\|F\|_{L^q}.$$

Replacing w by $-w$, one obtains a similar estimate for w^-. Thus,

$$\|w\|_{L^\infty} \leq \|w^+\|_{L^\infty} + \|w^-\|_{L^\infty} \leq 2C\|F\|_{L^q}.$$

□

Acknowledgment

The work of the first author (H. B.) is partially supported by NSF Grant DMS-0802958.

References

[1] L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Mathematical Monographs (Oxford University Press, New York, 2000).

[2] A. Ancona, Inégalité de Kato et inégalité de Kato jusqu'au bord, *C. R. Math. Acad. Sci. Paris Ser. I* **346** (2008) 939–944.

[3] H. Brezis, M. Marcus and A. C. Ponce, Nonlinear elliptic equations with measures revisited, in *Mathematical Aspects of Nonlinear Dispersive Equations*, eds. J. Bourgain, C. Kenig and S. Klainerman, Annals of Mathematics Studies, Vol. 163 (Princeton University Press, Princeton, NJ, 2007), pp. 55–110.

[4] H. Brezis and A. C. Ponce, Kato's inequality when Δu is a measure, *C. R. Math. Acad. Sci. Paris Ser. I* **338** (2004) 599–604.

[5] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, Studies in Advanced Mathematics (CRC Press, Boca Raton, FL, 1992).

[6] E. Gagliardo, Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili, *Rend. Sem. Mat. Univ. Padova* **27** (1957) 284–305.

[7] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Grundlehren der Mathematischen Wissenschaften, Vol. 224 (Springer-Verlag, Berlin, 1983).

[8] P. Hartman and G. Stampacchia, On some non-linear elliptic differential-functional equations, *Acta Math.* **115** (1966) 271–310.

[9] T. Kato, Schrödinger operators with singular potentials, *Israel J. Math.* **13** (1972) 135–148.

[10] G. Stampacchia, *Équations Elliptiques du Second Ordre à Coefficients Discontinus*, Séminaire de Mathématiques Supérieures, no. 16 (été, 1965) (Les Presses de l'Université de Montréal, Montréal, Québec, Canada, 1966).