KATO’S INEQUALITY UP TO THE BOUNDARY

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We show that if $\Delta u$ is a finite measure in $\Omega$ then, under suitable assumptions on $u$ near $\partial \Omega$, $\Delta u^+$ is also a finite measure in $\Omega$. We also study properties of the normal derivatives $\frac{\partial u}{\partial n}$ and $\frac{\partial u^+}{\partial n}$ on $\partial \Omega$.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Given $u \in L^1(\Omega)$ with $\Delta u \in L^1(\Omega)$, Kato’s inequality (see [9]; see also [4]) asserts that

$$\Delta u^+ \geq \chi_{\{u \geq 0\}} \Delta u \quad \text{in } D'(\Omega). \quad (1.1)$$

In particular, (1.1) implies that $\Delta u^+$ is a locally finite measure in $\Omega$. Our goal in this paper is to address the question whether $\Delta u^+$ is a finite measure up to the boundary of $\Omega$, i.e. whether

$$\int_{\Omega} |\Delta u^+| < \infty.$$ 

In general, the answer is negative: one can even construct harmonic functions $u \in C(\overline{\Omega}) \cap H^1(\Omega)$ such that $\Delta u^+$ is not a finite measure in $\Omega$; see Proposition A.1 below. With further assumptions on $u$ (for instance if $u \in W^{2,1}(\Omega)$ or if $u$ vanishes on the boundary) we will see that the answer is positive.
The following class of functions will play a central role. We say that \( u \in X \) if
\[
\int_{\Omega} \nabla u \cdot \nabla \psi \leq C \|\psi\|_{L^\infty} \quad \forall \psi \in C^1(\overline{\Omega}),
\]
in which case we set
\[
[u]_X = \sup_{\psi \in C^1(\overline{\Omega})} \int_{\Omega} \nabla u \cdot \nabla \psi.
\]

Note that if \( u \in X \), then there exists a unique \( T \in [C(\overline{\Omega})]^* = \mathcal{M}(\overline{\Omega}) \) such that
\[
\langle T, \psi \rangle = \int_{\Omega} \nabla u \cdot \nabla \psi \quad \forall \psi \in C^1(\overline{\Omega}).
\]
On the other hand, by the Riesz Representation Theorem any \( T \in \mathcal{M}(\overline{\Omega}) \) admits a unique decomposition
\[
\langle T, \psi \rangle = \int_{\partial \Omega} \psi \, d\nu + \int_{\Omega} \psi \, d\mu \quad \forall \psi \in C(\overline{\Omega}),
\]
where \( \mu \in \mathcal{M}(\Omega) \) and \( \nu \in \mathcal{M}(\partial \Omega) \). As usual, \( \mathcal{M}(\Omega) \) and \( \mathcal{M}(\partial \Omega) \) denote the spaces of finite measures in \( \Omega \) and \( \partial \Omega \), respectively, equipped with the norm \( \| \cdot \|_{\mathcal{M}} \); measures in \( \mathcal{M}(\Omega) \) are identified with measures in \( \Omega \) which do not charge \( \partial \Omega \). When \( u \in X \), we will denote
\[
\mu = -\Delta u \quad \text{and} \quad \nu = \frac{\partial u}{\partial n}.
\]
Throughout the paper, whenever \( u \in X \) we use the notation \( \Delta u \) and \( \frac{\partial u}{\partial n} \) in the above sense. If \( u \in X \), then
\[
\int_{\Omega} \nabla u \cdot \nabla \psi = \int_{\partial \Omega} \psi \, d\nu - \int_{\Omega} \psi \, d\mu \quad \forall \psi \in C^1(\overline{\Omega}),
\]
and consequently,
\[
\int_{\partial \Omega} \frac{\partial u}{\partial n} - \int_{\Omega} \Delta u = \int_{\partial \Omega} \psi \, \frac{\partial u}{\partial n} - \int_{\Omega} \psi \, \Delta u \quad \forall \psi \in C^2(\overline{\Omega}).
\]
Also, note that if \( u \in X \), then
\[
[u]_X = \int_{\Omega} |\Delta u| + \int_{\partial \Omega} \left| \frac{\partial u}{\partial n} \right|.
\]
In particular, \( \cdot \) _\(X\) defines a seminorm in \( X \) and \( [u]_X = 0 \) if, and only if, \( u \) is constant in \( \Omega \). In order to verify this last assertion, one may use the fact that for every \( h \in C^\infty(\Omega) \) with \( \int_{\Omega} h = 0 \), there exists \( \psi \in C^\infty(\Omega) \) such that \(-\Delta \psi = h \) in \( \Omega \) with \( \frac{\partial \psi}{\partial n} = 0 \) on \( \partial \Omega \).

Clearly, any function \( u \in W^{2,1}(\Omega) \) belongs to \( X \) and our notation is consistent with the usual meaning of \( \Delta u \) and \( \frac{\partial u}{\partial n} \). Recall that, for any function \( u \in L^1(\Omega) \), \( \Delta u \) is well-defined as a distribution. When \( u \in X \), the distribution \( \Delta u \) belongs to \( \mathcal{M}(\Omega) \), but the converse is not true; see e.g. Proposition A.1 below.
We now present our main results.

**Theorem 1.1.** If \( u \in X \) then \( u^+ \in X \) and

\[
[u^+]_X \leq [u]_X. \tag{1.3}
\]

In other words,

\[
\int_\Omega |\Delta u^+| + \int_{\partial \Omega} \left| \frac{\partial u^+}{\partial n} \right| \leq \int_\Omega |\Delta u| + \int_{\partial \Omega} \left| \frac{\partial u}{\partial n} \right|. \tag{1.4}
\]

Our next result gives additional properties when \( u \) vanishes on the boundary:

**Theorem 1.2.** If \( u \in W^{1,1}_0(\Omega) \) and \( \Delta u \in M(\Omega) \) (in the sense of distributions), then \( u \in X \) (hence, \( u^+ \in X \)). Moreover,

\[
\int_\Omega |\Delta u^+| \leq \int_\Omega |\Delta u|. \tag{1.5}
\]

In addition, \( \frac{\partial u}{\partial n} \in L^1(\partial \Omega) \) with

\[
\int_{\partial \Omega} \left| \frac{\partial u}{\partial n} \right| \leq \int_\Omega |\Delta u|. \tag{1.6}
\]

Note that assertions (1.5) and (1.6) fail if \( u \) does not vanish on \( \partial \Omega \); simply take \( \Omega = B_1 \), the unit ball in \( \mathbb{R}^N \), and \( u(x) = x_1 \).

We now state our extension of Kato’s inequality up to the boundary:

**Theorem 1.3.** Let \( u \in X \) be such that \( \Delta u \in L^1(\Omega) \) and \( \frac{\partial u}{\partial n} \in L^1(\partial \Omega) \). Then,

\[
\int_{\partial \Omega} \nabla u^+ \cdot \nabla \psi \leq \int_{\partial \Omega} H \psi - \int_\Omega G \psi \quad \forall \psi \in C^1(\overline{\Omega}), \psi \geq 0 \text{ in } \Omega, \tag{1.7}
\]

where \( G \in L^1(\Omega) \) and \( H \in L^1(\partial \Omega) \) are given by

\[
G = \begin{cases} 
\Delta u & \text{on } [u > 0], \\
0 & \text{on } [u \leq 0],
\end{cases} \quad \text{and} \quad H = \begin{cases} 
\frac{\partial u}{\partial n} & \text{on } [u > 0], \\
0 & \text{on } [u < 0], \\
\min \left\{ \frac{\partial u}{\partial n}, 0 \right\} & \text{on } [u = 0].
\end{cases} \tag{1.8}
\]

Thus,

\[
\begin{cases} 
\Delta u^+ \geq G & \text{in } \Omega, \\
\frac{\partial u^+}{\partial n} \leq H & \text{on } \partial \Omega. \tag{1.9}
\end{cases}
\]
We conclude this introduction with the following problems:

**Open Problem 1.** Let \( u \in X \). Is it true that
\[
\left| \frac{\partial u^+}{\partial n} \right| \leq \left| \frac{\partial u}{\partial n} \right| \text{ on } \partial \Omega?
\] (1.10)

This problem is open even under the additional assumption that \( u \in W^{1,1}_0(\Omega) \).

**Open Problem 2.** Assume that \( u \in X \) and \( \frac{\partial u}{\partial n} \in L^1(\partial \Omega) \). Is it true that \( \frac{\partial u^+}{\partial n} \in L^1(\partial \Omega) \)? More precisely, does one have
\[
\frac{\partial u^+}{\partial n} = H,
\] (1.11)

where \( H \) is the function given by (1.8)?

The answer to both Open Problems 1 and 2 is positive if \( u \in W^{2,1}(\Omega) \); see Theorem 7.1 below.

**Addendum.** Recently, A. Ancona informed us that he gave a positive answer to Open Problems 1 and 2 in full generality. His argument strongly relies on tools from Potential Theory; see [2] and a detailed paper to appear.

## 2. Properties of Functions in \( X \)

In this section, we investigate properties satisfied by elements in \( X \). We first show that condition (1.2) required for a function to belong to \( X \) can be replaced by

\[
\left| \int_{\Omega} u \Delta \zeta \right| \leq C \| \zeta \|_{L^\infty} \quad \forall \zeta \in C_0^2(\Omega),
\] (2.1)

where

\[
C_0^2(\Omega) = \left\{ \zeta \in C^2(\Omega); \frac{\partial \zeta}{\partial n} = 0 \text{ on } \partial \Omega \right\}.
\] (2.2)

**Proposition 2.1.** Let \( u \in L^1(\Omega) \). Then, \( u \in X \) if, and only if,
\[
\sup_{\zeta \in C_0^2(\Omega), \| \zeta \|_{L^\infty}}} \left| \int_{\Omega} u \Delta \zeta \right| < \infty.
\] (2.3)

Moreover,

(i) the quantity in (2.3) equals \( [u]_X \);
(ii) \( u \in W^{1,p}(\Omega) \) for every \( 1 \leq p < \frac{N}{N-1} \); moreover, \( \| \nabla u \|_{L^p(\Omega)} \leq C [u]_X \).
In the proof of Proposition 2.1, we need the following variant of the classical De Giorgi–Stampacchia estimate (see [7, 8]) for the Neumann problem:

**Lemma 2.1.** Given \( F \in C_0^\infty(\Omega; \mathbb{R}^N) \), let \( w \) be the unique solution of

\[
\begin{cases}
-\Delta w = \text{div } F & \text{in } \Omega, \\
\frac{\partial w}{\partial n} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

such that \( \int \Omega w \, d\nu = 0 \). Then, for every \( q > N \) we have

\[
\|w\|_{L^\infty} \leq C \|F\|_{L^q}.
\]

We present a sketch of the proof of Lemma 2.1 in Appendix C.

**Proof of Proposition 2.1** Note that if \( u \in \mathbb{X} \), then

\[
\left| \int \Omega u \Delta \zeta \right| = \left| \int \Omega \nabla u \cdot \nabla \zeta \right| \leq [u] \| \zeta \|_{L^\infty} \quad \forall \zeta \in C_0^2(\overline{\Omega}).
\]

This gives the implication “\( \Rightarrow \)”. We now assume that (2.3) holds. We split the proof of the converse into two steps:

**Step 1.** \( u \in W^{1,p}(\Omega) \) for every \( 1 \leq p < \frac{N}{N-1} \) and

\[
\|\nabla u\|_{L^p(\Omega)} \leq CK,
\]

where \( K \) denotes the quantity in (2.3).

Clearly, we may assume that \( 1 < p < \frac{N}{N-1} \). Given \( F \in C_0^\infty(\Omega; \mathbb{R}^N) \), let \( w \) be the unique solution of (2.4) such that \( \int \Omega w = 0 \). By (2.3) and (2.5), we have

\[
\left| \int \Omega u \text{div } F \right| = \left| \int \Omega u \Delta w \right| \leq K \|w\|_{L^\infty} \leq KC \|F\|_{L^{p'}} \quad \forall F \in C_0^\infty(\Omega; \mathbb{R}^N).
\]

The conclusion follows by duality.

**Step 2.** \( u \in \mathbb{X} \) and \( [u] \mathbb{X} = K \).

It suffices to show that

\[
\left| \int \Omega \nabla u \cdot \nabla \psi \right| \leq K \|\psi\|_{L^\infty} \quad \forall \psi \in C^1(\overline{\Omega}).
\]

Indeed, this implies \( u \in \mathbb{X} \) and \( [u] \mathbb{X} \leq K \). Since by (2.6), \( K \leq [u] \mathbb{X} \), equality must hold. We now turn ourselves to the proof of (2.7). Given \( \psi \in C^2(\overline{\Omega}) \), we first show that there exists a sequence \( (\zeta_k) \) such that

\[
\zeta_k \in C_0^2(\overline{\Omega}), \quad \|\nabla \zeta_k\|_{L^\infty} \leq C, \quad \zeta_k \rightarrow \psi \quad \text{uniformly in } \Omega
\]

and

\[
\nabla \zeta_k \rightarrow \nabla \psi \quad \text{a.e. in } \Omega.
\]
Indeed, let $\Phi \in C^\infty_0(\mathbb{R})$ and $\eta \in C^2(\overline{\Omega})$ with $\eta = 0$ on $\partial \Omega$ be such that $\Phi(t) = t \quad \forall t \in [-1, 1]$ and $\frac{\partial \eta}{\partial n} = \frac{\partial \psi}{\partial n}$ on $\partial \Omega$.

Take $\zeta_k = \psi - \frac{1}{k} \Phi(k\eta)$ in $\Omega$.

Clearly, (2.8) holds. On the other hand,

$$\nabla \left[ \frac{1}{k} \Phi(k\eta) \right] = \Phi'(k\eta) \nabla \eta \rightarrow \chi_{[\eta = 0]} \nabla \eta \quad \text{in } \Omega.$$  

Since $\nabla \eta = 0$ a.e. on the set $[\eta = 0]$, (2.9) follows. For every $k \geq 1$, we thus have

$$\left\| \int_\Omega \nabla u \cdot \nabla \zeta_k \right\| \leq |\zeta_k|_{L^\infty}.$$  

As $k \to \infty$, we obtain (2.7) with test functions $\psi \in C^2(\overline{\Omega})$. Using a density argument, one then gets (2.7). The proof is complete. □

**Remark 2.1.** Using Proposition 2.1, one deduces that given measures $\mu \in \mathcal{M}(\Omega)$ and $\nu \in \mathcal{M}(\partial \Omega)$, the Neumann problem

$$\begin{cases} 
-\Delta u = \mu & \text{in } \Omega, \\
\frac{\partial u}{\partial n} = \nu & \text{on } \partial \Omega,
\end{cases}$$

has a solution $u \in X$ if, and only if,

$$\mu(\Omega) + \nu(\partial \Omega) = 0.$$  

The solution is unique up to an additive constant and belongs to $u \in W^{1,p}(\Omega)$ for every $1 \leq p < \frac{N}{N-1}$. In particular, if $\int_\Omega u = 0$, then

$$\|u\|_{W^{1,p}(\Omega)} \leq C[u]_X.$$  

The following result complements Proposition 2.1:

**Proposition 2.2.** Let $u \in L^1(\Omega)$ be such that

$$-\int_\Omega u \Delta \zeta \leq \int_{\partial \Omega} \zeta d\nu + \int_\Omega \zeta \mu \quad \forall \zeta \in C^2_N(\overline{\Omega}), \quad \zeta \geq 0 \text{ in } \overline{\Omega}$$

for some $\mu \in \mathcal{M}(\Omega)$ and $\nu \in \mathcal{M}(\partial \Omega)$. Then, $u \in X$,

$$[u]_X \leq 2(|\mu^+|_{\mathcal{M}(\Omega)} + |\nu^+|_{\mathcal{M}(\partial \Omega)})$$

and

$$\begin{cases} 
-\Delta u \leq \mu & \text{in } \Omega, \\
\frac{\partial u}{\partial n} \leq \nu & \text{on } \partial \Omega.
\end{cases}$$
Proof. By (2.12), we have
\[ -\int_{\Omega} u \Delta \zeta \leq \int_{\partial \Omega} \zeta \, d\nu^+ + \int_{\Omega} \zeta \, d\mu^+ \quad \forall \zeta \in C^2_N(\Omega), \quad \zeta \geq 0 \text{ in } \overline{\Omega}. \tag{2.15} \]
For every \( \zeta \in C^2_N(\Omega) \), we apply (2.15) with test functions \( \|\zeta\|_{L^\infty} \pm \zeta \) to get
\[ \left| \int_{\Omega} u \Delta \zeta \right| \leq 2(\|\mu^+\|_{M(\Omega)} + \|\nu^+\|_{M(\partial \Omega)})\|\zeta\|_{L^\infty}. \tag{2.16} \]
By Proposition 2.1, it follows that \( u \in \mathcal{X} \) and (2.13) holds. Proceeding as in Step 2 of the proof of Proposition 2.1 (more precisely, using (2.8) and (2.9)), one deduces from (2.12) that
\[ \int_{\Omega} \nabla u \cdot \nabla \psi \leq \int_{\partial \Omega} \psi \, d\nu + \int_{\Omega} \psi \, d\mu \quad \forall \psi \in C^2(\Omega), \quad \psi \geq 0 \text{ in } \overline{\Omega}. \]
This gives (2.14).

3. Proof of Theorem 1.1
We begin by establishing the following lemma:

Lemma 3.1. If \( u \in C^2(\overline{\Omega}) \), then
\[ \int_{\Omega} \nabla u^+ \cdot \nabla \psi \leq \int_{\partial \Omega} \psi \frac{\partial u}{\partial n} \bigg|_{[u \geq 0]} - \int_{\Omega} \psi \Delta u \quad \forall \psi \in C^1(\overline{\Omega}), \quad \psi \geq 0 \text{ in } \overline{\Omega}. \tag{3.1} \]

Proof. We first prove the

Claim. If \( u \in C^2(\overline{\Omega}) \) and \( \Phi \in C^2(\mathbb{R}) \) is convex, then
\[ \int_{\Omega} \nabla \Phi(u) \cdot \nabla \psi \leq \int_{\partial \Omega} \psi \Phi'(u) \frac{\partial u}{\partial n} - \int_{\Omega} \psi \Phi'(u) \Delta u \quad \forall \psi \in C^1(\overline{\Omega}), \quad \psi \geq 0 \text{ in } \overline{\Omega}. \tag{3.2} \]

Note that
\[ \frac{\partial \Phi(u)}{\partial n} = \Phi'(u) \frac{\partial u}{\partial n} \quad \text{on } \partial \Omega \]
and, by the convexity of \( \Phi \),
\[ \Delta \Phi(u) \geq \Phi'(u) \Delta u \quad \text{in } \Omega. \]
Thus, for every \( \psi \in C^1(\overline{\Omega}), \psi \geq 0 \text{ in } \overline{\Omega}, \)
\[ \int_{\Omega} \nabla \Phi(u) \cdot \nabla \psi = \int_{\partial \Omega} \psi \frac{\partial \Phi(u)}{\partial n} - \int_{\Omega} \psi \Delta \Phi(u) \leq \int_{\partial \Omega} \psi \Phi'(u) \frac{\partial u}{\partial n} - \int_{\Omega} \psi \Phi'(u) \Delta u. \]
This establishes the claim.
We now apply (3.2) with $\Phi = \Phi_k$, where $(\Phi_k)$ is a sequence of smooth convex functions such that $\Phi_k(0) = 0$, $\|\Phi'_k\|_{L^\infty} \leq 1$ and satisfying

$$\Phi'_k(t) \rightarrow \begin{cases} 1 & \text{if } t \geq 0, \\ 0 & \text{if } t < 0. \end{cases}$$

As $k \to \infty$, we obtain (3.1). \hfill \square

We now prove a special case of Theorem 1.1 for functions in $C^2(\Omega)$:

**Lemma 3.2.** Let $u \in C^2(\Omega)$. Then, $u^+ \in X$ and

$$|u^+|_X \leq [u]_X.$$  

**Proof.** Note that $u^+ \in W^{1,1}(\Omega)$. In order to establish the lemma, it thus suffices to show that

$$\left| \int_{\Omega} \nabla u^+ \cdot \nabla \psi \right| \leq |u|_X \|\psi\|_{L^\infty} \quad \forall \psi \in C^1(\Omega).$$  

(3.4)

For this purpose, given $\tilde{\psi} \in C^1(\Omega)$ we apply (3.1) with $\psi = \|\tilde{\psi}\|_{L^\infty} + \tilde{\psi}$. We then get

$$\int_{\Omega} \nabla u^+ \cdot \nabla \tilde{\psi} \leq \left( \int_{\partial \Omega} \frac{\partial u}{\partial n} \left[ u^{\geq 0} \right]- \int_{\Omega} \Delta u \right) \|\tilde{\psi}\|_{L^\infty} + \int_{\partial \Omega} \tilde{\psi} \frac{\partial u}{\partial n} - \int_{\Omega} \tilde{\psi} \Delta u.$$  

(3.5)

Since

$$\int_{\partial \Omega} \frac{\partial u}{\partial n} \left[ u^{\geq 0} \right]- \int_{\Omega} \Delta u = - \int_{\partial \Omega} \frac{\partial u}{\partial n} \left[ u^{< 0} \right]+ \int_{\Omega} \Delta u,$$

estimate (3.5) becomes

$$\int_{\Omega} \nabla u^+ \cdot \nabla \tilde{\psi} \leq \left( \int_{\partial \Omega} \frac{\partial u}{\partial n} \left[ u^{< 0} \right]- \int_{\Omega} \Delta u \right) \|\tilde{\psi}\|_{L^\infty} + \int_{\partial \Omega} \tilde{\psi} \frac{\partial u}{\partial n} - \int_{\Omega} \tilde{\psi} \Delta u$$

$$\leq \left( \int_{\partial \Omega} \frac{\partial u}{\partial n} \left[ u^{\geq 0} \right]+ \int_{\Omega} |\Delta u| \right) \|\tilde{\psi}\|_{L^\infty} = [u]_X \|\tilde{\psi}\|_{L^\infty}.$$

This relation holds for every $\tilde{\psi} \in C^1(\Omega)$. Replacing $\tilde{\psi}$ by $-\tilde{\psi}$, we obtain (3.4). This establishes the lemma. \hfill \square

**Proof of Theorem 1.1.** Since $u \in X$,

$$\int_{\Omega} \nabla u \cdot \nabla \psi = \int_{\partial \Omega} \psi \frac{\partial u}{\partial n} - \int_{\Omega} \psi \Delta u \quad \forall \psi \in C^1(\Omega).$$

Taking $\psi = 1$ as a test function, we get

$$\int_{\partial \Omega} \frac{\partial u}{\partial n} = \int_{\Omega} \Delta u.$$  

(3.6)
Let \((\mu_k) \subset C^\infty(\Omega)\) and \((\nu_k) \subset C^\infty(\partial \Omega)\) be two sequences such that
\[
\mu_k \rightharpoonup^* -\Delta u \text{ weak* in } M(\Omega) \quad \text{and} \quad \|\mu_k\|_{L^1(\Omega)} \to \|\Delta u\|_{M(\Omega)},
\]
\[
\nu_k \rightharpoonup^* \frac{\partial u}{\partial n} \text{ weak* in } M(\partial \Omega) \quad \text{and} \quad \|\nu_k\|_{L^1(\partial \Omega)} \to \left\| \frac{\partial u}{\partial n} \right\|_{M(\partial \Omega)}.
\]

In view of (3.6), we may also assume that
\[
\int_{\partial \Omega} \nu_k = -\int_{\Omega} \mu_k \quad \forall k \geq 1.
\]

For each \(k \geq 1\), let \(u_k \in C^2(\Omega)\) be the unique function such that
\[
\begin{cases}
-\Delta u_k = \mu_k & \text{in } \Omega, \\
\frac{\partial u_k}{\partial n} = \nu_k & \text{on } \partial \Omega,
\end{cases}
\]
and
\[
\int_{\Omega} u_k = \int_{\Omega} u.
\]

Then, by Remark 2.1 applied to \(u_k - \int_{\Omega} u\), the sequence \((u_k)\) is bounded in \(W^{1,p}(\Omega)\) for every \(1 \leq p < \frac{N}{N-1}\). Since \(u_k \to u\) a.e., one deduces that
\[
\nabla u_k^+ \rightharpoonup \nabla u^+ \text{ weakly in } L^1(\Omega).
\]

On the other hand, applying Lemma 3.2 to \(u_k\), we get
\[
\left| \int_{\Omega} \nabla u_k^+ \cdot \nabla \psi \right| \leq [u_k^+]_X \|\psi\|_{L^\infty} \leq [u_k]_X \|\psi\|_{L^\infty} \quad \forall \psi \in C^1(\overline{\Omega}).
\]

As \(k \to \infty\), we obtain
\[
\left| \int_{\Omega} \nabla u^+ \cdot \nabla \psi \right| \leq [u]_X \|\psi\|_{L^\infty} \quad \forall \psi \in C^1(\overline{\Omega}),
\]
from which the conclusion follows. \(\square\)

4. Properties of \(\frac{\partial u}{\partial n}\)

We start with a result which seems intuitively true, but still requires a proof:

**Proposition 4.1.** Let \(u \in W^{1,\infty}(\Omega)\). Then, \(u \in X\ if, and only if, \Delta u \in M(\Omega)\) (in the sense of distributions). In this case, \(\frac{\partial u}{\partial n} \in L^\infty(\partial \Omega)\) and
\[
\left\| \frac{\partial u}{\partial n} \right\|_{L^\infty(\partial \Omega)} \leq \|\nabla u\|_{L^\infty(\Omega)}.
\] (4.1)

If \(u \in C^1(\overline{\Omega}) \cap X\), then \(\frac{\partial u}{\partial n}\) coincides with the standard normal derivative on \(\partial \Omega\).
Proof. We first assume that \( u \in W^{1,\infty}(\Omega) \) and \( \Delta u \in M(\Omega) \). Given a sequence of mollifiers \((\rho_k)\) such that \( \text{supp} \rho_k \subset B_{1/k}, \) let 
\[
  u_k(x) = \int_{\Omega} \rho_k(x-y)u(y) \, dy \quad \forall x \in \Omega.
\]
Note that if \( d(x, \partial \Omega) > 1/k \), then 
\[
  \nabla u_k(x) = \int_{\Omega} \rho_k(x-y)\nabla u(y) \, dy \quad \text{and} \quad \Delta u_k(x) = \int_{\Omega} \rho_k(x-y)\Delta u(y) \, dy.
\]
Denote 
\[
  \Omega_\delta = \{ x \in \Omega; \ d(x, \partial \Omega) > \delta \}; \quad (4.2)
\]
for \( \delta_0 > 0 \) small enough, \( \Omega_\delta \) is smooth for every \( \delta \in (0, \delta_0) \).

For every \( k \geq 1 \) and \( \delta \in (0, \delta_0) \) such that \( 1/k < \delta \), we then have 
\[
  \left\| \frac{\partial u_k}{\partial n} \right\|_{L^\infty(\partial \Omega_\delta)} \leq \left\| \nabla u_k \right\|_{L^\infty(\Omega_\delta)} \leq \left\| \nabla u \right\|_{L^\infty(\Omega)} \cdot (4.3)
\]
Thus, for every \( \psi \in C^1(\overline{\Omega}) \), 
\[
  \left| \int_{\Omega_\delta} \psi \Delta u_k + \int_{\Omega_\delta} \nabla \psi \cdot \nabla u_k \right| \leq \left\| \nabla u \right\|_{L^\infty(\Omega)} \left\| \psi \right\|_{L^1(\partial \Omega_\delta)} . \quad (4.4)
\]
Note that for a.e. \( \delta \in (0, \delta_0) \) 
\[
  \int_{\partial \Omega_\delta} |\Delta u| = 0; \quad (4.5)
\]
hence, for any such \( \delta > 0 \), 
\[
  \int_{\Omega_\delta} \psi \Delta u_k \to \int_{\Omega_\delta} \psi \Delta u \quad \text{as} \ k \to \infty.
\]
Indeed, this is a general fact (see, e.g., [5, Theorem 1, p. 54]): if \( \mu \in M(\Omega) \) and \( |\mu|(\partial \Omega_\delta) = 0 \), then 
\[
  \int_{\Omega_\delta} \psi(\rho_k * \mu) \to \int_{\Omega_\delta} \psi \, d\mu \quad \forall \psi \in C^0(\overline{\Omega_\delta}).
\]
For any \( \delta \in (0, \delta_0) \) verifying (4.5), as \( k \to \infty \) in (4.4), we get 
\[
  \left| \int_{\Omega_\delta} \psi \Delta u + \int_{\Omega_\delta} \nabla \psi \cdot \nabla u \right| \leq \left\| \nabla u \right\|_{L^\infty(\Omega)} \left\| \psi \right\|_{L^1(\partial \Omega_\delta)} \quad \forall \psi \in C^1(\overline{\Omega}). \quad (4.6)
\]
From this estimate, one deduces that for every \( \psi \in C^1(\overline{\Omega}) \), 
\[
  \left| \int_{\Omega_\delta} \nabla \psi \cdot \nabla u \right| \leq \left\| \Delta u \right\|_{M(\Omega)} \left\| \psi \right\|_{L^\infty(\Omega_\delta)} + \left\| \nabla u \right\|_{L^\infty(\Omega)} \left\| \psi \right\|_{L^1(\partial \Omega_\delta)} \leq (\left\| \Delta u \right\|_{M(\Omega)} + \left\| \nabla u \right\|_{L^\infty(\Omega)} |\partial \Omega_\delta|) \left\| \psi \right\|_{L^\infty(\Omega)}.
\]
As \( \delta \to 0 \), we conclude that \( u \in \mathbb{X} \).
In order to prove that $\frac{\partial u}{\partial n} \in L^\infty(\partial\Omega)$, we return to estimate (4.6). Given $\phi \in C^1(\partial\Omega)$, we fix an extension $\psi \in C^1(\overline{\Omega})$ of $\phi$; note that

$$\|\psi\|_{L^1(\partial\Omega)} \leq \|\phi\|_{L^1(\partial\Omega)} + C\delta \quad \forall \delta \in (0, \delta_0),$$

for some constant $C > 0$. Insert this test function $\psi$ in (4.6). As $\delta \to 0$ we obtain, by dominated convergence,

$$\left| \int_{\Omega} \psi \Delta u + \int_{\Omega} \nabla \psi \cdot \nabla u \right| \leq \|\nabla u\|_{L^\infty(\Omega)} \|\phi\|_{L^1(\partial\Omega)}.$$ 

Hence,

$$\left| \int_{\partial\Omega} \phi \frac{\partial u}{\partial n} \right| \leq \|\nabla u\|_{L^\infty(\Omega)} \|\phi\|_{L^1(\partial\Omega)} \quad \forall \phi \in C^1(\partial\Omega).$$

Therefore, by duality $\frac{\partial u}{\partial n} \in L^\infty(\partial\Omega)$ and (4.1) holds.

We now assume that $u \in C^1(\overline{\Omega}) \cap X$ and we denote by $h$ the normal derivative of $u$ in the standard sense. By Lemma B.1 and Remark B.1, there exists a sequence $(u_k) \subset C^\infty(\Omega)$ satisfying (B.2) and (B.3) and such that

$$u_k \to u \quad \text{in} \quad C^1(\overline{\Omega}).$$

In particular,

$$\frac{\partial u_k}{\partial n} \to h \quad \text{uniformly on} \quad \partial\Omega.$$

Thus,

$$\int_{\Omega} \nabla u \cdot \nabla \psi + \int_{\Omega} \psi \Delta u = \int_{\partial\Omega} h \psi \quad \forall \psi \in C^1(\overline{\Omega}). \quad (4.7)$$

Hence, the normal derivative $\frac{\partial u}{\partial n}$ in the sense of the space $X$ coincides with $h$. □

When $u \in X$ the measure $\frac{\partial u}{\partial n}$ need not be an $L^1$-function. Surprisingly, this is always true if $u$ vanishes on $\partial\Omega$:

**Proposition 4.2.** Let $u \in W_{0}^{1,1}(\Omega)$. Then, $u \in X$ if, and only if, $\Delta u \in \mathcal{M}(\Omega)$ (in the sense of distributions). Moreover, $\frac{\partial u}{\partial n} \in L^1(\partial\Omega)$ and

$$\left| \frac{\partial u}{\partial n} \right|_{L^1(\partial\Omega)} \leq \|\Delta u\|_{\mathcal{M}(\Omega)}. \quad (4.8)$$

**Proof.** We split the proof into two steps:

**Step 1.** Proof of (4.8) if $u$ is smooth in a neighborhood of $\partial\Omega$.

Under this assumption, $\frac{\partial u}{\partial n}$ is a smooth function on $\partial\Omega$. Denote by $v_1$ and $v_2$ the solutions of

$$\begin{cases}
-\Delta v_1 = \mu^+ \quad \text{in} \; \Omega, \\
v_1 = 0 \quad \text{on} \; \partial\Omega,
\end{cases} \quad \begin{cases}
-\Delta v_2 = \mu^- \quad \text{in} \; \Omega, \\
v_2 = 0 \quad \text{on} \; \partial\Omega,
\end{cases}$$

where

$$\mu^+ = \text{the positive part of } \frac{\partial u}{\partial n}, \quad \mu^- = \text{the negative part of } \frac{\partial u}{\partial n}.$$
where $\mu = -\Delta u$. In particular,

$$u = v_1 - v_2 \quad \text{in } \Omega.$$  

Since $\mu$ is smooth in a neighborhood of $\partial \Omega$, $\mu^+$ and $\mu^-$ are Lipschitz continuous near $\partial \Omega$. Hence, $v_1$ and $v_2$ are of class $C^2$ near $\partial \Omega$. Moreover, $v_1 \geq 0$ in $\Omega$ and $v_1 = 0$ on $\partial \Omega$; thus,

$$\frac{\partial v_1}{\partial n} \leq 0 \quad \text{on } \partial \Omega.$$ 

It follows that

$$\int_{\partial \Omega} \left| \frac{\partial v_1}{\partial n} \right| = - \int_{\partial \Omega} \frac{\partial v_1}{\partial n} = \int_{\Omega} \mu^+. $$

Similarly,

$$\int_{\partial \Omega} \left| \frac{\partial v_2}{\partial n} \right| = \int_{\Omega} \mu^-.$$ 

Therefore,

$$\int_{\partial \Omega} \left| \frac{\partial u}{\partial n} \right| \leq \int_{\partial \Omega} \left| \frac{\partial v_1}{\partial n} \right| + \int_{\partial \Omega} \left| \frac{\partial v_2}{\partial n} \right| = \int_{\Omega} (\mu^+ + \mu^-) = \int_{\Omega} |\Delta u|.$$ 

**Step 2.** Proof of the proposition completed.

Let $(\varphi_k) \subset C_0^\infty (\Omega)$ be a sequence of test functions such that

$$0 \leq \varphi_k \leq 1 \quad \text{in } \bar{\Omega} \quad \text{and} \quad \varphi_k(x) = 1 \quad \text{if } d(x, \partial \Omega) \geq \frac{1}{k}.$$ 

Take $\mu_k = -\varphi_k \Delta u$, $\forall k \geq 1$. Then, $(\mu_k) \subset M(\Omega)$ is a sequence of measures such that $\text{supp } \mu_k \subset \Omega$ and, by dominated convergence,

$$\mu_k \rightarrow -\Delta u \quad \text{strongly in } M(\Omega). \quad (4.9)$$

For each $k \geq 1$, let $u_k$ be the unique solution of

$$\begin{cases}
-\Delta u_k = \mu_k & \text{in } \Omega, \\
u_k = 0 & \text{on } \partial \Omega.
\end{cases}$$

Note that $u_k$ is harmonic in a neighborhood of $\partial \Omega$. We claim that

$$\int_{\partial \Omega} \phi \frac{\partial u_k}{\partial n} \rightarrow \int_{\partial \Omega} \phi \frac{\partial u}{\partial n} \quad \forall \phi \in C^1(\partial \Omega). \quad (4.10)$$

Indeed, since $u_k \rightarrow u$ in $L^1(\Omega)$ and $(\nabla u_k)$ is bounded in $W_0^{1,p}(\Omega)$ for every $1 \leq p < \frac{N}{N-1}$ (see [10]), we have

$$\int_{\Omega} \nabla \psi \cdot \nabla u_k \rightarrow \int_{\Omega} \nabla \psi \cdot \nabla u \quad \forall \psi \in C^1(\bar{\Omega}). \quad (4.11)$$

Assertion (4.10) then follows from (4.9) and (4.11).

Applying Step 1 to the function $u_i - u_j$, we have

$$\left\| \frac{\partial u_i}{\partial n} - \frac{\partial u_j}{\partial n} \right\|_{L^1(\partial \Omega)} \leq \|\mu_i - \mu_j\|_{M(\Omega)} \quad \forall i, j \geq 1.$$
In view of the strong convergence of \((\mu_k)\) in \(M(\Omega)\), \((\frac{\partial u_k}{\partial n})\) is a Cauchy sequence in \(L^1(\partial \Omega)\). Hence, this sequence converges in \(L^1(\partial \Omega)\) to some function \(h\). By (4.10), 
\[
\frac{\partial u_k}{\partial n} \rightarrow \frac{\partial u}{\partial n} \quad \text{in} \quad L^1(\partial \Omega).
\]
Moreover, since (4.8) holds for every \(u_k\), it also holds for \(u\). The proof is complete.

We now show that if \(u \in W^{1,1}(\Omega)\) and \(\nabla u \in BV(\Omega)\) then the normal derivative \(\frac{\partial u}{\partial n}\) in the sense of the space \(X\) coincides with the function \(n \cdot \nabla u\) on \(\partial \Omega\) defined in the sense of traces:

**Proposition 4.3.** Assume that \(u \in W^{1,1}(\Omega)\) and \(\nabla u \in BV(\Omega)\); hence,
\[
\Delta u = \text{div}(\nabla u) \in M(\Omega).
\]
Then, \(u \in X\) and \(\frac{\partial u}{\partial n}\) coincides with \(n \cdot \nabla u|_{\partial \Omega}\) on \(\partial \Omega\), where \(\nabla u|_{\partial \Omega}\) is understood in the sense of traces. In particular, \(\frac{\partial u}{\partial n} \in L^1(\partial \Omega)\) and
\[
\left\| \frac{\partial u}{\partial n} \right\|_{L^1(\partial \Omega)} \leq C \|\nabla u\|_{BV(\Omega)}.
\]

In the proof of Proposition 4.3 we use the notion of strict convergence in \(BV(A)\), where \(A \subset \mathbb{R}^N\) is a Lipschitz domain. We recall that a sequence \((f_k) \subset BV(A)\) converges strictly to \(f \in BV(A)\) if
\[
f_k \rightarrow f \quad \text{strongly in} \quad L^1(A) \quad \text{and} \quad \int_A |Df_k| \rightarrow \int_A |Df|.
\]
By [1, Theorem 3.88], the trace operator
\[
f \in BV(A) \mapsto f|_{\partial A} \in L^1(\partial A)
\]
is continuous from \(BV(A)\) (under strict convergence) into \(L^1(\partial A)\) (under strong convergence).

**Proof of Proposition 4.3.** By Lemma B.1 and Remark B.1, there exists a sequence \((u_k) \subset C^{\infty}(\overline{\Omega})\) satisfying (B.1)–(B.3) and (B.12). Since \((\nabla u_k)\) converges strictly to \(\nabla u\) in \(BV(\Omega)\), we have
\[
\nabla u_k|_{\partial \Omega} \rightarrow \nabla u|_{\partial \Omega} \quad \text{in} \quad L^1(\partial \Omega).
\]
Hence,
\[
\int_\Omega \nabla u \cdot \nabla \psi + \int_\Omega \psi \Delta u = \int_{\partial \Omega} (n \cdot \nabla u|_{\partial \Omega}) \psi \quad \forall \psi \in C^1(\partial \Omega).
\]
This implies that \(\frac{\partial u}{\partial n}\) \(L^1(\partial \Omega)\) and equals \(n \cdot \nabla u|_{\partial \Omega}\). By the \(BV\)-trace theory, (4.12) holds. \(\square\)
5. Proof of Theorem 1.2

We first establish Theorem 1.2 for functions in \( C^2 \mathcal{D} \Omega \), where
\[
C^2 \mathcal{D} \Omega = \{ \zeta \in C^2 \Omega; \zeta = 0 \text{ on } \partial \Omega \}. \tag{5.1}
\]

**Lemma 5.1.** Let \( u \in C^2 \mathcal{D} \Omega \). Then, \( \Delta u^+ \in \mathcal{M} \Omega \) and
\[
\| \Delta u^+ \|_\mathcal{M} \leq \| \Delta u \|_{L^1}. \tag{5.2}
\]

**Proof.** Apply (3.3) with \( u + a \), where \( a > 0 \). We deduce that
\[
[(u + a)^+] \leq [u + a] = [u]. \tag{5.3}
\]

Since \((u + a)^+ = u + a\) in a neighborhood of \( \partial \Omega \),
\[
\frac{\partial}{\partial n}(u + a)^+ = \frac{\partial u}{\partial n} \text{ on } \partial \Omega. \tag{5.4}
\]

Note that
\[
[(u + a)^+] = \| \Delta (u + a)^+ \|_{\mathcal{M} \Omega} + \left\| \frac{\partial}{\partial n}(u + a)^+ \right\|_{L^1(\partial \Omega)},
\]
\[
[u] = \| \Delta u \|_{L^1(\Omega)} + \left\| \frac{\partial u}{\partial n} \right\|_{L^1(\partial \Omega)}.
\]

By (5.3)–(5.4), we then have
\[
\| \Delta (u + a)^+ \|_{\mathcal{M}} \leq \| \Delta u \|_{L^1} \quad \forall a > 0.
\]

The result follows from the lower semicontinuity of the norm \( \| \cdot \|_{\mathcal{M}} \) with respect to the weak* convergence as \( a \to 0 \).

**Proof of Theorem 1.2.** Since \( u \in \mathcal{X} \), \( \Delta u \in \mathcal{M} \Omega \). Take a sequence \((\mu_k) \subset C^\infty(\overline{\Omega})\) such that
\[
\mu_k \rightharpoonup -\Delta u \text{ weak* in } \mathcal{M} \Omega \quad \text{and} \quad \| \mu_k \|_{L^1} \to \| \Delta u \|_\mathcal{M}.
\]

For each \( k \geq 1 \), let \( u_k \in C^2 \mathcal{D} \Omega \) be the solution of
\[-\Delta u_k = \mu_k \text{ in } \Omega.
\]

Then, by standard elliptic estimates,
\[
u_k \to u \text{ in } L^1(\Omega).
\]

On the other hand, it follows from Lemma 5.1 that \( \Delta u_k^+ \in \mathcal{M} \Omega \) and
\[
\| \Delta u_k^+ \|_\mathcal{M} \leq \| \Delta u_k \|_{L^1}.
\]

Thus,
\[
\int \Omega u_k^+ \Delta \zeta \leq \| \Delta u_k \|_{L^1} \| \zeta \|_{L^\infty} = \| \mu_k \|_{L^1} \| \zeta \|_{L^\infty} \quad \forall \zeta \in C^2 \mathcal{D} \Omega.
\]
As \( k \to \infty \) we obtain
\[
\left| \int_{\Omega} u^+ \Delta \zeta \right| \leq \| \Delta u \|_{M} \| \zeta \|_{L^\infty} \quad \forall \zeta \in C^2_0(\overline{\Omega}).
\]
This gives (1.5). From Proposition 4.2, we know that \( \frac{\partial u}{\partial n}, \frac{\partial u^+}{\partial n} \in L^1(\partial \Omega) \) and (1.6) holds.

\[
6. \text{ Kato’s Inequality up to the Boundary}
\]
Before proving Theorem 1.3, we first present some variants of Kato’s inequality when \( \Delta u \) and \( \frac{\partial u}{\partial n} \) are not necessarily \( L^1 \)-functions but only finite measures. We prove for instance the following companion to [3, Proposition 4.5]:

**Proposition 6.1.** Let \( u \in L^1(\Omega) \) be such that
\[
-\int_{\Omega} u \Delta \zeta \leq \int_{\partial \Omega} h \zeta + \int_{\Omega} g \zeta \quad \forall \zeta \in C^2(\overline{\Omega}), \quad \zeta \geq 0 \text{ in } \Omega \quad (6.1)
\]
for some \( g \in L^1(\Omega) \) and \( h \in L^1(\partial \Omega) \). Then, \( u \in W^{1,1}(\Omega) \) and
\[
\int_{\Omega} \nabla u^+ \cdot \nabla \psi \leq \int_{\partial \Omega} h \psi + \int_{\Omega} g \psi \quad \forall \psi \in C^1(\overline{\Omega}), \quad \psi \geq 0 \text{ in } \Omega. \quad (6.2)
\]

**Proof.** By Proposition 2.2, \( u \in X \). Moreover,
\[
\begin{cases}
-\Delta u \leq g & \text{in } \Omega, \\
\frac{\partial u}{\partial n} \leq h & \text{on } \partial \Omega.
\end{cases} \quad (6.3)
\]
We now split the proof into two steps:

**Step 1.** Let \( \Phi \in C^2(\mathbb{R}) \) be a nondecreasing convex function such that \( \Phi' \in L^\infty(\mathbb{R}) \). Then,
\[
\int_{\Omega} \nabla \Phi(u) \cdot \nabla \psi \leq \int_{\partial \Omega} \psi \Phi'(u) h + \int_{\Omega} \psi \Phi'(u) g \quad (6.4)
\]
for every \( \psi \in C^1(\overline{\Omega}) \) such that \( \psi \geq 0 \) in \( \overline{\Omega} \).

Let \( (g_k) \subset C^\infty(\overline{\Omega}) \) and \( (h_k) \subset C^\infty(\partial \Omega) \) be such that
\[
g_k \to g \text{ in } L^1(\Omega) \text{ and } a.e. \quad \text{and} \quad h_k \to h \text{ in } L^1(\partial \Omega) \text{ and } a.e.
\]
Next, take \( (\mu_k) \subset C^\infty(\overline{\Omega}) \) and \( (\nu_k) \subset C^\infty(\partial \Omega) \) such that
\[
\mu_k \rightharpoonup -\Delta u \text{ weak* in } M(\overline{\Omega}) \quad \text{and} \quad \nu_k \rightharpoonup \frac{\partial u}{\partial n} \text{ weak* in } M(\partial \Omega).
\]
In view of (6.3) and
\[
\int_{\partial \Omega} \frac{\partial u}{\partial n} = \int_{\Omega} \Delta u,
\]
we may assume that
\[ \mu_k \leq g_k \text{ in } \Omega, \quad \nu_k \leq h_k \text{ on } \partial \Omega \quad \text{and} \quad \int_{\partial \Omega} \nu_k = - \int_{\Omega} \mu_k \quad \forall k \geq 1. \]

Let \( u_k \in C^\infty(\overline{\Omega}) \) be the unique solution of
\[
\begin{aligned}
-\Delta u_k &= \mu_k \quad \text{in } \Omega, \\
\frac{\partial u_k}{\partial n} &= \nu_k \quad \text{on } \partial \Omega,
\end{aligned}
\]
such that \( \int_{\Omega} u_k = \int_{\Omega} u \). By Remark 2.1, the sequence \((u_k)\) is bounded in \( W^{1,p}(\Omega) \) for every \( 1 \leq p < \frac{N}{N-1} \). Passing to a subsequence if necessary, we have
\[ \nabla \Phi(u_k) \rightharpoonup \nabla \Phi(u) \text{ weakly in } L^1(\Omega). \]

Let \( \psi \in C^1(\overline{\Omega}), \psi \geq 0 \text{ in } \Omega \). As in Lemma 3.1, for every \( k \geq 1 \) we have
\[
\int_{\Omega} \nabla \Phi(u_k) \cdot \nabla \psi \leq \int_{\partial \Omega} \psi \Phi'(u_k) \frac{\partial u_k}{\partial n} - \int_{\Omega} \psi \Phi'(u_k) \Delta u_k
\]
\[ \leq \int_{\partial \Omega} \psi \Phi'(u_k) h_k + \int_{\Omega} \psi \Phi'(u_k) g_k. \]

By dominated convergence we obtain (6.4) as \( k \to \infty \).

**Step 2.** Proof of the proposition completed.

Apply (6.4) with \( \Phi = \Phi_k \), where \((\Phi_k)\) is a sequence of smooth convex functions such that \( \Phi_k(0) = 0, 0 \leq \Phi_k' \leq 1 \) and
\[ \Phi_k'(t) \to \begin{cases} 1 & \text{if } t \geq 0, \\ 0 & \text{if } t < 0. \end{cases} \]

The result follows as we let \( k \to \infty \).

The following variant of Proposition 6.1 will be needed below:

**Proposition 6.2.** Let \( u \in L^1(\Omega) \) be such that
\[ -\int_{\Omega} u \Delta \zeta \leq \int_{\partial \Omega} h \zeta + \int_{\Omega} \zeta \, d\mu \quad \forall \zeta \in C^2_0(\Omega), \quad \zeta \geq 0 \text{ in } \overline{\Omega} \quad (6.5) \]
for some \( \mu \in \mathcal{M}(\Omega), \mu \geq 0 \), and \( h \in L^1(\partial \Omega) \). Then, \( u \in W^{1,1}(\Omega) \) and
\[ \int_{\Omega} \nabla u^+ \cdot \nabla \psi \leq \int_{\partial \Omega} h \psi + \int_{\Omega} \psi \, d\mu \quad \forall \psi \in C^1(\Omega), \quad \psi \geq 0 \text{ in } \overline{\Omega}. \quad (6.6) \]

**Proof.** One can proceed as in the proof of Proposition 6.1. In Step 1, one should replace (6.4) by
\[ \int_{\Omega} \nabla \Phi(u) \cdot \nabla \psi \leq \int_{\partial \Omega} \psi \Phi'(u) h + \|\Phi'\|_{L^\infty} \int_{\Omega} \psi \, d\mu. \quad (6.4') \]
Inequality (6.4’) is easily obtained by approximation, where the sequence \((g_k) \subset C^\infty(\Omega)\) is chosen so that

\[ g_k \rightharpoonup^\ast \mu \text{ weak}^\ast \text{ in } M(\overline{\Omega}). \]

The rest of the argument remains unchanged.

We now prove the

**Proposition 6.3.** Let \(u \in X\). If \(\frac{\partial u}{\partial n} \in L^1(\partial \Omega)\), then

\[
\frac{\partial u^+}{\partial n} \leq \begin{cases} \\
\frac{\partial u}{\partial n} & \text{on } [u > 0], \\
0 & \text{on } [u < 0], \\
\min\left\{\frac{\partial u}{\partial n}, 0\right\} & \text{on } [u = 0].
\end{cases}
\] (6.7)

**Proof.** Denoting by \(\mu = (\Delta u)^+ + h\) and \(h = \frac{\partial u}{\partial n}\), we have

\[-\int_\Omega u \Delta \zeta \leq \int_{\partial \Omega} h \zeta + \int_\Omega \zeta \, d\mu \quad \forall \zeta \in C^2(\overline{\Omega}), \quad \zeta \geq 0 \text{ in } \Omega.\]

Therefore, by Proposition 6.2, \(u^+\) satisfies

\[
\int_\Omega \nabla u^+ \cdot \nabla \psi \leq \int_{\partial \Omega} h \psi + \int_\Omega \psi \, d\mu \quad \forall \psi \in C^1(\overline{\Omega}), \quad \psi \geq 0 \text{ in } \overline{\Omega}. \] (6.8)

By Theorem 1.1, we know that \(u^+ \in X\). It thus follows that

\[
\frac{\partial u^+}{\partial n} \leq \chi_{[u \geq 0]} h = \chi_{[u \geq 0]} \frac{\partial u}{\partial n} \quad \text{on } \partial \Omega. \] (6.9)

Given \(a > 0\), we now apply (6.8) with \(u\) replaced by \(u - a\). As \(a \to 0\), we obtain

\[
\int_{\partial \Omega} u^+ \frac{\partial \psi}{\partial n} - \int_\Omega u^+ \Delta \psi \leq \int_{\partial \Omega} h \psi + \int_\Omega \psi \, d\mu \quad \forall \psi \in C^1(\overline{\Omega}), \quad \psi \geq 0 \text{ in } \Omega. \] (6.10)

Hence,

\[
\frac{\partial u^+}{\partial n} \leq \chi_{[u > 0]} h = \chi_{[u > 0]} \frac{\partial u}{\partial n} \quad \text{on } \partial \Omega. \]

In particular,

\[
\frac{\partial u^+}{\partial n} \leq 0 \quad \text{on } [u = 0]. \] (6.11)

Assertion (6.7) follows by combining (6.9) and (6.11). □
We state the following consequence of Proposition 6.3:

**Corollary 6.1.** Let \( u \in X \cap W^{1,1}_0(\Omega) \). If \( u \geq 0 \) in \( \Omega \), then
\[
\frac{\partial u}{\partial n} \leq 0 \quad \text{on } \partial \Omega.
\]

**Proof.** Since \( u = u^+ \) in \( \Omega \) and \( u = 0 \) on \( \partial \Omega \), applying Proposition 6.3 above we get
\[
\frac{\partial u}{\partial n} = \frac{\partial u^+}{\partial n} \leq \min\left\{ \frac{\partial u}{\partial n}, 0 \right\} \leq 0 \quad \text{on } \partial \Omega.
\]

We now present the

**Proof of Theorem 1.3.** By Theorem 1.1, \( u^+ \in X \). Applying Kato's inequality to \( u - a \), we have
\[
\Delta(u - a)^+ \geq \chi_{[u \geq a]} \Delta u \quad \text{in } \Omega \quad (6.12)
\]
for every \( a \in \mathbb{R} \). As \( a \downarrow 0 \) in (6.12) we get
\[
\Delta u^+ \geq \chi_{[u > 0]} \Delta u = G \quad \text{in } \Omega.
\]
By this estimate and (6.7), for every \( \psi \in C^1(\Omega) \) with \( \psi \geq 0 \) in \( \Omega \),
\[
\int_{\Omega} \nabla u^+ \cdot \nabla \psi = \int_{\partial \Omega} \psi \frac{\partial u^+}{\partial n} - \int_{\partial \Omega} \psi \Delta u^+ \leq \int_{\partial \Omega} H \psi - \int_{\Omega} G \psi.
\]
The proof is complete. \( \square \)

**7. Computing \( \frac{\partial u^+}{\partial n} \) for \( W^{2,1} \)-Functions**

Our goal in this section is to give a positive answer to Open Problems 1 and 2 under the additional assumption that \( u \in W^{2,1}(\Omega) \):

**Theorem 7.1.** If \( u \in W^{2,1}(\Omega) \), then \( \nabla u^+ \in BV(\Omega) \) (so that, \( u^+ \in X \) by Proposition 4.3) and
\[
\frac{\partial u^+}{\partial n} = \begin{cases} 
\frac{\partial u}{\partial n} & \text{on } [u > 0], \\
0 & \text{on } [u < 0], \\
\min\left\{ \frac{\partial u}{\partial n}, 0 \right\} & \text{on } [u = 0].
\end{cases}
\]

We first prove the

**Lemma 7.1.** If \( v \in W^{1,1}(\Omega) \) and \( \nabla v \in BV(\Omega) \), then
\[
\frac{\partial v}{\partial n}(x) = \lim_{t \downarrow 0} \frac{v(x) - v(x - tn(x))}{t} \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial \Omega.
\]

In (7.2), we identify \( v \) with its precise representative, which is well-defined outside a set of zero \( \mathcal{H}^{N-1} \)-Hausdorff measure; see [5, Sec. 4.8, Theorem 1 and Sec. 5.6, Theorem 3].
Proof. Since $v \in W^{1,1}(\Omega)$, for $\mathcal{H}^{N-1}$-a.e. $x \in \partial \Omega$ the function
\[ t \in (0, \delta) \mapsto v(x - tn(x)) \]
is well-defined for some $\delta > 0$ and belongs to $W^{1,1}(0, \delta)$. Thus,
\[ \frac{v(x - tn(x)) - v(x)}{t} = -n(x) \cdot \int_0^1 \nabla v(x - stn(x)) \, ds. \quad (7.3) \]
Moreover, since $\nabla v \in BV(\Omega)$, for $\mathcal{H}^{N-1}$-a.e. $x \in \partial \Omega$ the function
\[ r \in (0, \delta) \mapsto \nabla v(x - rn(x)) \]
belongs to $BV(0, \delta) \subset L^\infty(0, \delta)$ and (see [1, Theorem 3.108])
\[ \lim_{r \downarrow 0} \nabla v(x - rn(x)) = \nabla v|_{\partial \Omega}(x). \quad (7.4) \]
We deduce from (7.3)--(7.4) that
\[ \lim_{t \downarrow 0} \frac{v(x - tn(x)) - v(x)}{t} = -n(x) \cdot \nabla v|_{\partial \Omega}(x). \]
By Proposition 4.3 above, $\frac{\partial u}{\partial n} = n \cdot \nabla v|_{\partial \Omega}$ and the conclusion follows. $\square$

We also need the following elementary lemma whose proof is left to the reader:

Lemma 7.2. Let $v : [0, \delta] \to \mathbb{R}$ be such that
\[ \lim_{t \downarrow 0} \frac{v(0) - v(t)}{t} = \alpha \in \mathbb{R}. \quad (7.5) \]
Then,
\[ \lim_{t \downarrow 0} \frac{v^+(0) - v^+(t)}{t} = \begin{cases} 
\alpha & \text{if } v(0) > 0, \\
0 & \text{if } v(0) < 0, \\
\min\{\alpha, 0\} & \text{if } v(0) = 0.
\end{cases} \quad (7.6) \]

We now present the proof of Theorem 7.1. We split the proof into three steps:

Step 1. Proof of the assertion: $\nabla u^+ \in BV(\Omega)$.

Extending $u$ to $\mathbb{R}^N$, we may assume that $u \in W^{2,1}(\mathbb{R}^N)$. We claim that
\[ \frac{\partial^2 u^+}{\partial e^2} \geq \chi_{[u \geq 0]} \frac{\partial^2 u}{\partial e^2} \quad \text{in } D'(\mathbb{R}^N) \quad (7.7) \]
for every $e \in \mathbb{R}^N \setminus \{0\}$. Indeed, let $(\Phi_k)$ be a sequence of smooth convex functions such that $\Phi_k(0) = 0$, $\|\Phi'_k\|_{L^\infty} \leq 1$ and
\[ \Phi_k(t) = \begin{cases} 
1 & \text{if } t \geq 0, \\
0 & \text{if } t < 0.
\end{cases} \quad (7.8) \]
Then,
\[
\frac{\partial^2 \Phi_k(u)}{\partial e^2} = \Phi'_k(u) \frac{\partial^2 u}{\partial e^2} + \Phi''_k(u) \left( \frac{\partial u}{\partial e} \right)^2 \geq \Phi'_k(u) \frac{\partial^2 u}{\partial e^2} \quad \text{in } \mathbb{R}^N.
\]

As \( k \to \infty \), we obtain (7.7).

It follows from (7.7) that
\[
\frac{\partial^2 u}{\partial e^2} \in M(\Omega) \quad \text{for every } e \in \mathbb{R}^N \setminus \{0\}.
\]

Applying the conclusion with \( e = e_i, e_j, e_i + e_j \) for every \( i, j \in \{1, \ldots, N\} \) we deduce that \( \nabla u^+ \) is a finite measure in \( \Omega \). Thus, \( \nabla u^+ \in BV(\Omega) \).

**Step 2.** Proof of (7.1).

By Lemma 7.1, for \( \mathcal{H}^{N-1} \)-a.e. \( x \in \partial \Omega \), \( u \) satisfies
\[
\lim_{t \downarrow 0} \frac{u(x) - u(x - tn(x))}{t} = \frac{\partial u}{\partial n}(x) 
\]
(7.9)

Hence, by (7.2) applied to \( u^+ \) and by (7.6) applied to \( v(t) = u(x - tn(x)) \),
\[
\frac{\partial u^+}{\partial n}(x) = \lim_{t \downarrow 0} \frac{u^+(x) - u^+(x - tn(x))}{t} = \begin{cases} 
\frac{\partial u}{\partial n}(x) & \text{if } u(x) > 0, \\
0 & \text{if } u(x) < 0, \\
\min\left\{\frac{\partial u}{\partial n}(x), 0\right\} & \text{if } u(x) = 0,
\end{cases}
\]
for every \( x \in \partial \Omega \) for which (7.9) holds. Since this is true \( \mathcal{H}^{N-1} \)-a.e. on \( \partial \Omega \), (7.1) follows. The proof of Theorem 7.1 is complete. \( \square \)

**Appendix A. The Measure \( \Delta u^+ \) Need not be Finite**

In this appendix, we construct a harmonic function in dimension 2 such that
\[
\int_\Omega |\Delta u^+| = \infty:
\]

**Proposition A.1.** Let
\[
\Omega = \{ (x, y) \in \mathbb{R}^2; \ x^2 + y^2 < 1 \text{ and } x > 0 \}.
\]

There exists a harmonic function \( u \in C(\Omega) \cap H^1(\Omega) \) with \( u|_{\partial \Omega} \in W^{1,1}(\partial \Omega) \) such that

(i) \( u \notin \mathcal{X} \) and \( u^+ \notin \mathcal{X} \);

(ii) \( \Delta u^+ \geq 0 \) in the sense of distributions;

(iii) \( \Delta u^+ \) is not a finite measure in \( \Omega \).
Proof. Let $u$ be the function in $\Omega$ given in polar coordinates by

$$u(r, \theta) = \sum_{k=1}^{\infty} r^{a_k} \sin(a_k \theta)$$

(A.1)

where $(a_k) \subset (0, 1)$ is a sequence such that

$$\sum_{k=1}^{\infty} k a_k < \infty.$$

Since

$$\sum_{k=1}^{\infty} |r^{a_k} \sin(a_k \theta)| \leq \frac{\pi}{2} \sum_{k=1}^{\infty} a_k,$$

it follows that $u \in C(\Omega)$ and $u$ is harmonic in $\Omega$ ($u$ is a series of harmonic functions).

Note that

$$|\nabla u|^2 = \sum_{j,k=1}^{\infty} a_j a_k r^{a_j + a_k - 2} \cos((a_j - a_k)\theta).$$

Thus,

$$\int_{\Omega} |\nabla u|^2 \leq \pi \sum_{j,k=1}^{\infty} a_j a_k \leq 2\pi \sum_{j=1}^{\infty} \sum_{k \leq j} a_j a_k \leq 2\pi \sum_{k=1}^{\infty} k a_k < \infty;$$

in other words, $u \in H^1(\Omega)$. Denoting by $\tau$ the tangential unit vector of $\partial \Omega$, we have

$$\int_{\partial \Omega} |\frac{\partial u}{\partial \tau}| = 4 \sum_{k=1}^{\infty} \sin\left(a_k \frac{\pi}{2}\right) \leq 2\pi \sum_{k=1}^{\infty} a_k < \infty;$$

hence, $u \in W^{1,1}(\partial \Omega)$.

Since $u$ is harmonic in $\Omega$, $u^+$ is subharmonic. Thus, $\Delta u^+ \geq 0$ in $\Omega$. We show that $\Delta u^+$ is not a finite measure in $\Omega$. Note that $u$ vanishes only on the $x$-axis. Denoting by $dx$ ($= dr$) the 1-dimensional Lebesgue measure on the segment $(0, 1) \times \{0\}$, we then have

$$\Delta u^+ = \frac{\partial u}{\partial y}(x,0) dx = \frac{1}{r} \frac{\partial u}{\partial \theta}(r,0) dr = \sum_{k=1}^{\infty} a_k r^{a_k - 1} dr.$$

Therefore,

$$\int_{\Omega} |\Delta u^+| = \sum_{k=1}^{\infty} \int_{0}^{1} a_k r^{a_k - 1} dr = \sum_{k=1}^{\infty} 1 = \infty.$$ 

Hence, $u^+ \notin \mathcal{X}$ and, by Theorem 1.1, this means that $u \notin \mathcal{X}$.  $\Box$
Remark A.1. This example also shows that given $\varphi \in W^{1,1}(\partial \Omega)$, it is in general not possible to construct a function $v \in W^{2,1}(\Omega)$ such that $v|_{\partial \Omega} = \varphi$. This is in contrast with the well-known result of Gagliardo [6] which asserts that the map

$$w \in W^{1,1}(\Omega) \mapsto w|_{\partial \Omega} \in L^1(\partial \Omega)$$

is surjective.

Indeed, take $\varphi = u|_{\partial \Omega}$, where $u$ is given by (A.1). Suppose by contradiction that there exists some $v \in W^{2,1}(\Omega)$ such that $v|_{\partial \Omega} = \varphi$. Applying Proposition 4.2 to $u - v \in W^{1,1}(\Omega)$, we would deduce that $\frac{\partial v}{\partial n}(u - v) \in L^1(\partial \Omega)$. But $v \in W^{2,1}(\Omega)$ implies $\frac{\partial v}{\partial n} \in L^1(\partial \Omega)$ and therefore

$$\frac{\partial u}{\partial n} = \frac{\partial}{\partial n}(u - v) + \frac{\partial v}{\partial n} \in L^1(\partial \Omega),$$

a contradiction.

Appendix B. Approximation by Smooth Functions in $\Omega$

In this appendix, we establish the following

Lemma B.1. Given $u \in X$, there exists a sequence $(u_k) \subset C^\infty(\Omega)$ such that

$$u_k \to u \text{ in } W^{1,1}(\Omega),$$

(B.1)

$$\int_\Omega \psi \Delta u_k \to \int_\Omega \psi \Delta u \quad \forall \psi \in C^1(\overline{\Omega})$$

(B.2)

and

$$\int_{\partial \Omega} \psi \frac{\partial u_k}{\partial n} \to \int_{\partial \Omega} \psi \frac{\partial u}{\partial n} \quad \forall \psi \in C^1(\overline{\Omega}).$$

(B.3)

Proof. We split the proof into two steps:

Step 1. Given $x_0 \in \partial \Omega$, there exist $\delta > 0$ and a sequence $(v_k) \subset C^\infty(\overline{\Omega})$ such that

$$v_k \to u \text{ in } W^{1,1}(B_\delta(x_0) \cap \Omega),$$

(B.4)

$$\int_\Omega \psi \Delta v_k \to \int_\Omega \psi \Delta u \quad \forall \psi \in C^1(\overline{\Omega}) \text{ with supp } \psi \subset B_\delta(x_0).$$

(B.5)

Since $\partial \Omega$ is smooth, there exist $\delta_1 > 0$ and an open cone $T \subset \mathbb{R}^N$ (with vertex at $0 \in \mathbb{R}^N$) such that

$$(x + T) \cap B_{\delta_1}(x) \subset \Omega \quad \forall x \in B_{\delta_1}(x_0) \cap \overline{\Omega}.$$  (B.6)

Let $\delta = \delta_1/2$ and $\rho \in C^\infty_0(B_\delta)$, $\rho \geq 0$, be such that $\int_{B_\delta} \rho = 1$ and

$$\text{supp } \rho \subset -T.$$  (B.7)

Set

$$\rho_k(x) = k^N \rho(kx) \quad \forall x \in \mathbb{R}^N.$$
We show that the sequence \((v_k) \subset C^\infty(\Omega)\) given by
\[
v_k(x) = \int_\Omega \rho_k(x-y)u(y) \, dy \quad \forall x \in \overline{\Omega}
\] (B.8)
satisfies (B.4) and (B.5).

Note that given any \(x \in B_\delta(x_0) \cap \Omega\), by (B.7) \(v_k(x)\) depends only on the values of \(u\) on a compact subset of \((x+T) \cap B_\delta(x)\). In fact, from (B.6) and (B.7) and a change of variable, we can rewrite (B.8) as
\[
v_k(x) = \int_{T \cap B_\delta(0)} \rho_k(-z)u(x+z) \, dz \quad \forall x \in B_\delta(x_0) \cap \Omega.
\] (B.9)

Therefore,
\[
\nabla v_k = \rho_k \ast (\nabla u) \quad \text{and} \quad \Delta v_k = \rho_k \ast (\Delta u) \quad \text{in} \quad B_\delta(x_0) \cap \Omega.
\] (B.10)

In particular, (B.4) and (B.5) hold.

**Step 2.** Proof of the proposition completed.

By compactness of \(\partial \Omega\), we can cover this set with finitely many balls \(B_\delta(x_1), \ldots, B_\delta(x_t)\) such that (B.4) and (B.5) hold on each ball \(B_\delta(x_i)\) for some sequence \((v^i_k) \subset C^\infty(\Omega)\). We now take \((v^0_k) \subset C^\infty(\Omega)\) and \(\omega \subset \Omega\) such that \(\Omega \setminus \bigcup_{i=1}^t B_\delta(x_i) \subset \omega\),
\[
v^0_k \to u \quad \text{in} \quad W^{1,1}(\omega) \quad \text{and} \quad \Delta v^0_k \rightharpoonup \Delta u \quad \text{weak* in} \quad M(\omega)
\] (such sequence can be obtained via convolution of \(u\)).

Let \((\varphi_i)\) be a partition of unity subordinated to the covering \(\omega, B_\delta(x_1), \ldots, B_\delta(x_t)\) of \(\overline{\Omega}\). One verifies that (B.1) and (B.2) hold for the sequence \((u_k)\) given by
\[
u_k = \sum_{i=0}^t \varphi_i v^i_k.
\]
Assertion (B.3) immediately follows from (B.1) and (B.2).

**Remark B.1.** An inspection of the proof of Lemma B.1 shows that

(i) if \(u \in C^1(\overline{\Omega})\), then
\[
u_k \to u \quad \text{in} \quad C^1(\overline{\Omega});
\] (B.11)

(ii) if \(\nabla u \in BV(\Omega)\), then
\[
\|D^2u_k\|_{L^1(\Omega)} \to \|D^2u\|_{M(\Omega)}.
\] (B.12)
Appendix C. Proof of Lemma 2.1

The proof of Lemma 2.1 we present below follows the lines of [8, Lemma 7.3] (see also [7, Theorem 8.15]) with some minor modifications. We first need the following variant of the Gagliardo–Nirenberg inequality:

**Proposition C.1.** Let

\[ A = \left\{ v \in W^{1,1}(\Omega); \ |v = 0| \geq \frac{|\Omega|}{3} \right\}. \]  

Then,

\[ \|v\|_{L^{N/(N-1)}} \leq C\|\nabla v\|_{L^1} \quad \forall v \in A. \]  

We denote by \( |E| \) the Lebesgue measure of a set \( E \subset \mathbb{R}^N \).

**Proof.** By a variant of the Poincaré inequality (easily proved by contradiction), we have

\[ \|v\|_{L^1} \leq C\|\nabla v\|_{L^1} \quad \forall v \in A. \]  

On the other hand, by the standard Gagliardo–Nirenberg inequality and an extension argument,

\[ \|v\|_{L^{N/(N-1)}} \leq C(\|\nabla v\|_{L^1} + \|v\|_{L^1}) \quad \forall v \in W^{1,1}(\Omega). \]  

Combining (C.3) and (C.4), we obtain (C.2).

**Proof of Lemma 2.1.** Replacing \( w \) by \( w - a \) for some suitable constant \( a \in \mathbb{R} \) if necessary, we may assume that

\[ |[w \leq 0]| \geq \frac{|\Omega|}{3} \ \text{and} \ |[w \geq 0]| \geq \frac{|\Omega|}{3}. \]  

Given \( t > 0 \), let

\[ v_t(x) = |w(x) - t|^+ \quad \forall x \in \Omega. \]  

Using \( v_t \) as a test function in (2.4), one shows that

\[ \|\nabla v_t\|_{L^2} \leq \|F\|_{L^q}|[w > t]|^{\frac{1}{q} - \frac{1}{N}}. \]  

On the other hand, by Hölder’s inequality and Proposition C.1,

\[ \|v_t\|_{L^1} \leq C\|\nabla v_t\|_{L^2}|[w > t]|^{1 + \frac{1}{N}}. \]  

Thus,

\[ \|v_t\|_{L^1} \leq C\|F\|_{L^q}|[w > t]|^{\alpha} \quad \forall t > 0, \]  

where \( \alpha = 1 + \frac{1}{N} - \frac{1}{q} \). Recall that

\[ \|v_t\|_{L^1} = \int_0^\infty |[v_t > r]| \, dr = \int_t^M |[w > s]| \, ds, \]  

where
where \( M = \| w^+ \|_{L^\infty} \). Since \( \alpha > 1 \), one deduces using (C.7) and (C.8) that
\[
\| w^+ \|_{L^\infty} \leq C \| F \|_{L^{q}} \| w^+ \|_{L^1}^{1-\frac{1}{\alpha}}. \tag{C.9}
\]
From (C.9) and \( \| w^+ \|_{L^1} \leq |\Omega| \| w^+ \|_{L^\infty} \), we then have
\[
\| w^+ \|_{L^\infty} \leq C \| F \|_{L^q}.
\]
Replacing \( w \) by \( -w \), one obtains a similar estimate for \( w^- \). Thus,
\[
\| w \|_{L^\infty} \leq \| w^+ \|_{L^\infty} + \| w^- \|_{L^\infty} \leq 2C \| F \|_{L^q}.
\]
\[\square\]

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References