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Calculus of Variations

Boundary estimates for elliptic systems with L^1–data

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Abstract We obtain boundary estimates for the gradient of solutions to elliptic systems with Dirichlet or Neumann boundary conditions and L^1–data, under some condition on the divergence of the data. Similar boundary estimates are obtained for div–curl and Hodge systems.

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1 Introduction

Recently, new estimates for L^1–vector fields have been discovered by Bourgain and Brezis [1,2], which yield in particular improved estimates for the solutions of elliptic systems in $\mathbf{R}^N$ or in a cube $Q \subset \mathbf{R}^N$ with periodic boundary conditions. Simplified proofs of some of the results have been given by Van Schaftingen [10]. Here are two typical results:

Theorem 1.1 *Let $f \in \mathrm{L}^1(\mathbf{R}^N;\mathbf{R}^N)$, $N \geq 3$. If $\operatorname{div} f = 0$, then the system*

$$-\Delta u = f \quad \text{in} \quad \mathbf{R}^N,$$

admits a unique solution $u \in \mathrm{L}^{N/(N-2)}(\mathbf{R}^N;\mathbf{R}^N)$ with $\nabla u \in \mathrm{L}^{N/(N-1)}$.

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A similar conclusion holds for the same problem in a cube with periodic boundary conditions.

Theorem 1.2 *Let $f \in \mathrm{L}^1(\mathbf{R}^2;\mathbf{R}^2)$. If $\operatorname{div} f = 0$, then the system*

$$-\Delta u = f \quad \text{in} \quad \mathbf{R}^N,$$

admits a unique solution $u \in (\mathrm{L}^\infty \cap C)(\mathbf{R}^2;\mathbf{R}^2)$ with $\nabla u \in \mathrm{L}^2$.

Our main goal in this paper is to address similar questions in domains $\Omega \subset \mathbf{R}^N$ with Dirichlet or Neumann boundary conditions. Interior estimates can be easily derived from the results in [1,2,10]. However, the question of estimates up to the boundary requires some further work.

In Sect. 2, we study the system

$$-\Delta u = f \quad \text{in} \quad \Omega \subset \mathbf{R}^2,$$

together with the Dirichlet boundary condition $u = 0$ on $\partial\Omega$ or the Neumann boundary condition $\partial u/\partial n = 0$ on $\partial\Omega$. For the Dirichlet problem, we show that if $f \in \mathrm{L}^1(\Omega;\mathbf{R}^2)$ and $\operatorname{div} f = 0$, then $u \in C(\overline{\Omega};\mathbf{R}^2) \cap \mathrm{W}^{1,2}(\Omega;\mathbf{R}^2)$. For the Neumann problem, we obtain the same conclusion under the additional assumption that $(f \cdot n) = 0$ on $\partial\Omega$; such a condition plays an essential role, see Remark 2.3. The proofs are elementary; they involve sharp estimates for Green's functions. These are well-known to the experts and are presented in Appendix A for the convenience of the reader.

In Sect. 3, we start with the system

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

with $\Omega \subset \mathbf{R}^N$, $N \geq 2$ and $f \in \mathrm{L}^1(\mathbf{R}^N;\mathbf{R}^N)$. The heart of the matter is the inequality

$$\left| \int_\Omega f \cdot \varphi \right| \leq \|f\|_{\mathrm{L}^1} \|D\varphi\|_{\mathrm{L}^N}, \tag{1.2}$$

for every $\varphi \in (\mathrm{W}_0^{1,N} \cap \mathrm{L}^\infty)(\Omega;\mathbf{R}^N)$ and every $f \in \mathrm{L}^1(\mathbf{R}^N;\mathbf{R}^N)$ with $\operatorname{div} f = 0$, which we derive from similar estimates in [1,2,10]. Therefore, it admits an elementary proof in the spirit of [10]. Next, we combine (1.2) with standard L^p regularity theory to conclude that $u \in \mathrm{W}^{1,N/(N-1)}(\Omega;\mathbf{R}^N)$ when $f \in \mathrm{L}^1(\Omega;\mathbf{R}^N)$ and $\operatorname{div} f = 0$.

A much more delicate result asserts that if $f \in \mathrm{L}^1(\Omega;\mathbf{R}^N)$ and $\operatorname{div} f \in (\mathrm{W}_0^{2,N}(\mathbf{R}^N))^*$, one still has $u \in \mathrm{W}^{1,N/(N-1)}(\Omega;\mathbf{R}^N)$. The main ingredient is due to Bourgain and Brezis and asserts that every vector field in $\mathrm{W}_0^{1,N}$ belongs to L^∞ modulo gradients (see the precise statements in Theorem 3.2 and Lemma 3.3).

The remainder of Sect. 3 is devoted to the pure Neumann boundary condition and to various mixtures of Dirichlet and Neumann boundary conditions. We also consider the problem

$$\begin{cases} -\Delta u = 0 & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega. \end{cases}$$

In Sect. 4, we present estimates up to the boundary for the problem

$$\begin{cases} \operatorname{div} Z = 0 & \text{in} \quad \Omega, \\ \operatorname{curl} Z = Y & \text{in} \quad \Omega, \end{cases}$$

where $\Omega \subset \mathbf{R}^3$, together with the boundary conditions $Z \cdot n = 0$ or $Z \times n = 0$. Next, we present some results for first-order systems of k–forms, $2 \le k \le N-2$, such as

$$\begin{cases} d\omega = \alpha & \text{in } \Omega, \\ d^*\omega = \beta & \text{in } \Omega. \end{cases}$$

2 Elliptic systems in $\mathbf{R}^2$

Theorem 2.1 *Let $\Omega \subset \mathbf{R}^2$ be a smooth simply-connected domain and let $f \in L^1(\Omega; \mathbf{R}^2)$. If $\operatorname{div} f = 0$ in the sense of distributions, i.e.,*

$$\int_\Omega f \cdot \nabla \zeta = 0, \quad \forall \zeta \in C_0^1(\overline{\Omega}),$$

then the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique solution $u \in W^{1,2}(\Omega; \mathbf{R}^2) \cap C(\overline{\Omega}; \mathbf{R}^2)$ satisfying

$$\|u\|_{W^{1,2}} + \|u\|_{L^\infty} \le C\|f\|_{L^1}. \tag{2.1}$$

Proof By classical regularity estimates, there is a solution $u \in W^{1,q}$, for $q < 2$. Since $\operatorname{div} f = 0$, there exists $F \in W^{1,1}(\Omega)$ such that $f = (\nabla F)^\perp = (-\partial_2 F, \partial_1 F)$ and $\int_\Omega F = 0$. By Sobolev's inequality, $\|F\|_{L^2} \le C\|f\|_{L^1}$. For every $\varphi \in C_c^\infty(\Omega; \mathbf{R}^2)$, one has

$$\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega f \cdot \varphi = \int_\Omega F(\partial_2 \varphi_1 - \partial_1 \varphi_2) \le C\|f\|_{L^1}\|\nabla \varphi\|_{L^2}.$$

It is easy to deduce that $u \in W^{1,2}(\Omega; \mathbf{R}^2)$ with the required estimate.

We now prove that $u \in L^\infty$. Let G_Ω denote Green's function associated to Dirichlet boundary condition for Ω. Integrating by parts, one has, for every $x \in \Omega$,

$$\begin{aligned} u(x) &= \int_\Omega G_\Omega(x,y) f(y)\, dy \\ &= -\int_\Omega (x-y)\big(\nabla_y G_\Omega(x,y) \cdot f(y)\big)\, dy - \int_\Omega G_\Omega(x,y)(x-y) \operatorname{div} f(y)\, dy \\ &= -\int_\Omega (x-y)\big(\nabla_y G_\Omega(x,y) \cdot f(y)\big)\, dy, \end{aligned} \tag{2.2}$$

since $\operatorname{div} f = 0$. By Proposition A.1, $|\nabla G(x,y)||x-y|$ is uniformly bounded for $x, y \in \Omega$. Hence u satisfies the required estimate. To prove that u is continuous, assume that $x_n \to x$. Then $u(x_n) \to u(x)$ by (2.2) and Lebesgue's dominated convergence theorem. □

Remark 2.1 In the more general case where $f \in L^1(\Omega; \mathbf{R}^2)$ and $\operatorname{div} f \in L^1(\Omega)$, one obtains the continuity and the boundedness of u from the boundedness of

$|\nabla_y G(x,y)||x-y|$ on $\Omega\times\Omega$. Using the the estimate

$$\int_\Omega |\nabla u|^2 = \int_\Omega u\cdot f \le \|u\|_{\mathrm{L}^\infty}\|f\|_{\mathrm{L}^1} \le C(\|f\|_{\mathrm{L}^1}+\|\operatorname{div} f\|_{\mathrm{L}^1})\|f\|_{\mathrm{L}^1},$$

one concludes $u\in \mathrm{W}^{1,2}(\Omega;\mathbf{R}^2)$. Theorem 3.1 in the next section gives $\mathrm{W}^{1,2}$ (but not L^∞!) estimates under the weaker condition $\operatorname{div} f\in(\mathrm{W}^{2,2}\cap \mathrm{W}^{1,2}_0)^*$.

Remark 2.2 The conclusion of Theorem 2.1 remains valid if f is a measure (more precisely, f belongs to the dual space of $C_0(\overline{\Omega})$). The proof of the bounds in $\mathrm{W}^{1,2}$ and in L^∞ are unchanged. To prove that u is continuous, assume that $x_n\to x\in\overline{\Omega}$. Then

$$(x_n-y)\big(\nabla_y G(x_n,y)\big)\to (x-y)\big(\nabla_y G(x,y)\big)$$

for every $y\in\overline{\Omega}\setminus\{x\}$. On the other hand, the measure f belongs to the dual space of $\mathrm{W}^{1,2}_0(\Omega)$ and thus it does not charge points. One can conclude as before by Lebesgue's dominated convergence theorem.

Theorem 2.2 *Let $\Omega\subset\mathbf{R}^2$ be a smooth simply-connected domain and let $f\in\mathrm{L}^1(\Omega;\mathbf{R}^2)$. If $\operatorname{div} f=0$ in Ω and $f\cdot n=0$ on $\partial\Omega$ in the sense that*

$$\int_\Omega f\cdot\nabla\zeta=0,\quad \forall\zeta\in C^1(\overline{\Omega}),$$

then $\int_\Omega f=0$ and the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega,\\ \frac{\partial u}{\partial n}=0 & \text{on } \partial\Omega,\\ \int_\Omega u=0, \end{cases}$$

has a unique solution $u\in\mathrm{W}^{1,2}(\Omega;\mathbf{R}^2)\cap C(\overline{\Omega};\mathbf{R}^2)$. Moreover (2.1) holds.

Proof The proof of Theorem 2.2 is similar to the proof of Theorem 2.1. Since $f\cdot n=0$ on $\partial\Omega$, one can construct F such that $F=0$ on $\partial\Omega$ (in place of $\int_\Omega F=0$ in the previous proof). Since $F\in\mathrm{W}^{1,1}_0(\Omega)$, one has

$$\int_\Omega f=\int_\Omega(\nabla F)^\perp=0.$$

One concludes as before that $u\in\mathrm{W}^{1,2}$ and, using Proposition A.5, that u is bounded and continuous up to the boundary. □

Remark 2.3 If one replaces the condition $f\cdot n=0$ on $\partial\Omega$ in Theorem 2.2 by $\int_\Omega f=0$, the conclusion is not necessarily true. Assume without loss of generality that $0\in\partial\Omega$ and that the outward normal to the boundary at 0 is e_1, the first coordinate axis. Choose a function $\rho\in C^\infty_c(\mathbf{R}^2)$ such that $\rho\ge 0$ and $\int_{\mathbf{R}}\rho(0,x_2)\,dx_2=1$. Define $\rho_\varepsilon(x)=\rho(x/\varepsilon)/\varepsilon$ and

$$f_\varepsilon=(\nabla\rho_\varepsilon)^\perp-\frac{1}{|\Omega|}\int_\Omega(\nabla\rho_\varepsilon)^\perp.$$

For every $\varphi \in C(\overline{\Omega}; \mathbf{R}^2)$,

$$\int_\Omega f_\varepsilon \cdot \varphi \to e_2 \cdot \Big(\varphi(0) - \frac{1}{|\Omega|}\int_\Omega \varphi\Big).$$

Let now $u_\varepsilon \in W^{1,2}(\Omega; \mathbf{R}^2)$ be the solution of

$$\begin{cases} -\Delta u_\varepsilon = f_\varepsilon & \text{in } \mathbf{R}^2, \\ \frac{\partial u_\varepsilon}{\partial n} = 0 & \text{on } \partial\Omega, \\ \int_\Omega u_\varepsilon = 0, \end{cases}$$

and assume by contradiction that $\|\nabla u_\varepsilon\|_{L^2}$ remains bounded as $\varepsilon \to 0$. One has then, for every $\varphi \in C^\infty(\overline{\Omega}; \mathbf{R}^2)$,

$$\Big|e_2 \cdot \Big(\varphi(0) - \frac{1}{|\Omega|}\int_\Omega \varphi\Big)\Big| = \lim_{\varepsilon\to 0}\Big|\int_\Omega f_\varepsilon \cdot \varphi\Big| = \lim_{\varepsilon\to 0}\Big|\int_\Omega \nabla u_\varepsilon \cdot \nabla\varphi\Big| \le C\|\nabla\varphi\|_{L^2}.$$

This is not possible, since $W^{1,2}(\mathbf{R}^2)$ is not imbedded in $L^\infty(\mathbf{R}^2)$. Note that

$$\int_\Omega |\nabla u_\varepsilon|^2 \le \|f_\varepsilon\|_{L^1}\|u_\varepsilon\|_{L^\infty},$$

and thus $\|u_\varepsilon\|_{L^\infty}$ is not bounded as $\varepsilon \to 0$.

3 Second-order systems in higher dimensions

In this section, assume $N \ge 2$.

3.1 The Laplace equation with zero Dirichlet boundary condition

Theorem 3.1 *Let $\Omega \subset \mathbf{R}^N$ be a smooth bounded domain and let $f \in L^1(\Omega; \mathbf{R}^N)$. If*

$$[f] = \sup\Big\{\int_\Omega f \cdot \nabla\zeta \ : \ \zeta \in C^2(\overline{\Omega}),\ \zeta = 0 \text{ and } \nabla\zeta = 0 \text{ on } \partial\Omega \text{ and } \|D^2\zeta\|_{L^N} \le 1\Big\} < \infty, \tag{3.1}$$

then the system

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

has a unique weak solution $u \in W^{1,N/(N-1)}(\Omega; \mathbf{R}^N)$ satisfying

$$\|u\|_{W^{1,N/(N-1)}} \le C\big(\|f\|_{L^1} + [f]\big).$$

Remark 3.1 If $\operatorname{div} f = 0$ in the sense of distributions, then $[f] = 0$. More generally,

$$[f] = \|\operatorname{div} f\|_{(W_0^{2,N})^*} = \|\operatorname{div} f\|_{W^{-2,N/(N-1)}}.$$

Remark 3.2 Note that the assumption (3.1) is also *necessary* in order to obtain the conclusion of Theorem 3.1. Indeed, if $\zeta \in \mathrm{W}^{2,N}_0(\Omega)$, we have

$$\left|\int_\Omega f \cdot \nabla\zeta\right| = \left|\int_\Omega \nabla u \cdot \nabla(\nabla\zeta)\right| \leq \|u\|_{\mathrm{W}^{1,N/(N-1)}} \|D^2\zeta\|_{\mathrm{L}^N}.$$

In order to prove Theorem 3.1, first recall

Theorem 3.2 (Bourgain and Brezis [2, Theorem 5″]) *If $\varphi \in \mathrm{W}^{1,N}_0(Q;\mathbf{R}^N)$, then there exist $\psi \in (\mathrm{W}^{1,N}_0 \cap \mathrm{L}^\infty)(Q;\mathbf{R}^N)$ and $\eta \in \mathrm{W}^{2,N}_0(Q)$ such that*

$$\varphi = \psi + \nabla\eta,$$

where Q denotes the unit cube in $\mathbf{R}^N$. Moreover,

$$\|\psi\|_{\mathrm{W}^{1,N}} + \|\psi\|_{\mathrm{L}^\infty} + \|\eta\|_{\mathrm{W}^{2,N}} \leq C\|\varphi\|_{\mathrm{W}^{1,N}}.$$

Theorem 3.2 can be extended to any smooth domain:

Lemma 3.3 *Let $\Omega \subset \mathbf{R}^N$ be a smooth domain. For every $\varphi \in \mathrm{W}^{1,N}_0(\Omega;\mathbf{R}^N)$, there exist $\psi \in (\mathrm{W}^{1,N}_0 \cap \mathrm{L}^\infty)(\Omega;\mathbf{R}^N)$ and $\eta \in \mathrm{W}^{2,N}_0(\Omega)$ such that*

$$\varphi = \psi + \nabla\eta.$$

Moreover,

$$\|\psi\|_{\mathrm{W}^{1,N}} + \|\psi\|_{\mathrm{L}^\infty} + \|\eta\|_{\mathrm{W}^{2,N}} \leq C\|\varphi\|_{\mathrm{W}^{1,N}}.$$

Proof Taking a partition of unity we may assume that φ lives in a small neighborhood U near a point $x_0 \in \partial\Omega$, φ vanishes on $U \cap \partial\Omega$ and that G is a smooth diffeomorphism of a cube Q onto $U \cap \Omega$. Set $H = G^{-1}$ and $M = \operatorname{Jac} G$ be the Jacobian matrix of M defined on Q. Set

$$\tilde{\varphi} = {}^tM\varphi \circ G \quad \text{on} \quad Q.$$

By Theorem 3.2, there exists $\tilde{\psi} \in (\mathrm{W}^{1,N}_0 \cap \mathrm{L}^\infty)(Q;\mathbf{R}^N)$ and $\tilde{\eta} \in \mathrm{W}^{2,N}_0(Q)$ such that

$$\tilde{\varphi} = \tilde{\psi} + \nabla\tilde{\eta} \quad \text{on} \quad Q.$$

Set

$$\psi = {}^tN\tilde{\psi} \circ H \quad \text{on} \quad U$$

where $N = \operatorname{Jac} H$ and

$$\eta = \tilde{\eta} \circ H \quad \text{on} \quad U.$$

Clearly

$$\varphi = \psi + \nabla\eta.$$

□

With Lemma 3.3, we can now prove that the data f in Theorem 3.1 belongs to the dual of $\mathrm{W}^{1,N}_0(\Omega)$, more precisely:

Lemma 3.4 *Let $\Omega \subset \mathbf{R}^N$ be a smooth bounded domain and let $f \in L^1(\Omega;\mathbf{R}^N)$ be such that (3.1) holds. For every $\varphi \in (W_0^{1,N} \cap L^\infty)(\Omega;\mathbf{R}^N)$,*

$$\left|\int_\Omega f\cdot\varphi\right| \le C\big(\|f\|_{L^1}+[f]\big)\|D\varphi\|_{L^N}.$$

Proof Write $\varphi \in (W_0^{1,N} \cap L^\infty)(\Omega;\mathbf{R}^N)$ as $\varphi = \psi + \nabla\eta$ according to Lemma 3.3. One has

$$\begin{aligned}\int_\Omega f\cdot\varphi &= \int_\Omega f\cdot(\psi+\nabla\eta)\\ &\le \|f\|_{L^1}\|\psi\|_{L^\infty}+[f]\|D^2\eta\|_{L^N} \le C(\|f\|_{L^1}+[f])\|D\varphi\|_{L^N}.\end{aligned}$$

□

Remark 3.3 Starting from the estimate

$$\left|\int_{\mathbf{R}^N} f\cdot\varphi\right| \le C\big(\|f\|_{L^1}\|D\varphi\|_{L^N}+\|\mathrm{div}\, f\|_{L^1}\|\varphi\|_{L^N}\big), \tag{3.3}$$

which has an elementary proof [10], we deduce that

$$\left|\int_\Omega f\cdot\varphi\right| \le C\big(\|f\|_{L^1}+\|\mathrm{div}\, f\|_{L^1}\big)\|\varphi\|_{W^{1,N}}, \tag{3.4}$$

for every $f \in L^1(\Omega;\mathbf{R}^N)$ and $\varphi \in (W_0^{1,N} \cap L^\infty)(\Omega;\mathbf{R}^N)$. This is proved by extension of f to a small neighbourhood of Ω and multiplication of this extension by a suitable cutoff function. Note that, trivially,

$$[f] \le C\|\mathrm{div}\, f\|_{L^1},$$

and thus (3.4) is an immediate consequence of Lemma 3.4. However, we call the attention of the reader to the fact that (3.3) has an elementary proof, while the proof of Theorem 3.2 is quite elaborate.

The other tool to obtain our regularity result is the following elliptic regularity result for data in $W^{-1,p}(\Omega)$ that is well-known to the experts, but difficult to find in the litterature. Much more general estimates are obtained e.g. [4].

Lemma 3.5 *Let $\Omega \subset \mathbf{R}^N$ be a smooth domain. Let $F \in L^p(\Omega;\mathbf{R}^N)$, $1 < p < \infty$. There is a unique $u \in W_0^{1,p}(\Omega)$ that solves*

$$\begin{cases} -\Delta u = -\,\mathrm{div}\, F & \text{in } \Omega\\ u = 0 & \text{on } \partial\Omega.\end{cases}$$

Moreover,

$$\|u\|_{W^{1,p}} \le C\|F\|_{L^p}.$$

The proof of Theorem 3.1 is a direct consequence of Lemmas 3.4 and 3.5.

Remark 3.4 Most of the results of this paper can be easily localized. Here is a typical localization: let $u \in \mathrm{W}^{1,1}(\Omega \cap B(x_0,R);\mathbf{R}^N)$ with $x_0 \in \partial\Omega$ and let $f \in \mathrm{L}^1(\Omega \cap B(x_0,R);\mathbf{R}^N)$. If

$$\begin{cases} -\Delta u = f & \text{in } \ \Omega \cap B(x_0,R), \\ u = 0 & \text{on } \ (\partial\Omega) \cap B(x_0,R), \end{cases}$$

and

$$\sup\Big\{ \int_\Omega f \cdot \nabla\zeta \ : \ \zeta \in C^2\big(\overline{\Omega} \cap B(x_0,R)\big), \\ \zeta = 0 \text{ and } \nabla\zeta = 0 \text{ on } \partial\big(\Omega \cap B(x_0,R)\big) \text{ and } \|D^2\zeta\|_{\mathrm{L}^N} \le 1 \Big\} < \infty$$

then $u \in \mathrm{W}^{1,N/(N-1)}\big(\Omega \cap B(x_0,R/2);\mathbf{R}^N\big)$.

3.2 The Laplace equation with Neumann boundary condition

Theorem 3.6 *Let $\Omega \subset \mathbf{R}^N$ be a smooth and bounded domain. Let $f \in \mathrm{L}^1(\Omega;\mathbf{R}^N)$ and $g \in \mathrm{L}^1(\partial\Omega;\mathbf{R}^N)$. If*

$$[f,g] = \sup\Big\{ \int_\Omega f \cdot \nabla\zeta + \int_{\partial\Omega} g \cdot \nabla\zeta \ : \ \zeta \in C^2(\overline{\Omega}) \\ \text{and } \|D^2\zeta\|_{\mathrm{L}^N} \le 1 \Big\} < \infty, \tag{3.5}$$

then the system

$$\begin{cases} -\Delta u = f & \text{in } \ \Omega, \\ \frac{\partial u}{\partial n} = g & \text{on } \ \partial\Omega, \\ \int_\Omega u = 0, \end{cases}$$

has a unique weak solution $u \in \mathrm{W}^{1,N/(N-1)}(\Omega;\mathbf{R}^N)$ and

$$\|u\|_{\mathrm{W}^{1,N/(N-1)}} \le C\big(\|f\|_{\mathrm{L}^1} + [f,g]\big).$$

Remark 3.5 It is easily seen that $[f,g] = 0$ if and only if $\operatorname{div} f = 0$ in Ω, $g_n = 0$ and $\operatorname{div}_{\partial\Omega} g_t = f_n$ on $\partial\Omega$, where g_n, resp. g_t, denotes the normal, resp. tangential, component of g and $\operatorname{div}_{\partial\Omega} g_t$ denotes the tangential divergence. As in Remark 3.2, condition (3.5) is necessary and sufficient in order to obtain the conclusion of Theorem 3.6. As explained in Remark 2.3, the conditions $g = 0$ on $\partial\Omega$, $\operatorname{div} f = 0$ in Ω and $\int_\Omega f = 0$ do not imply the conclusion of Theorem 3.6.

First, we observe that the necessary condition for the existence of a solution, $\int_\Omega f + \int_{\partial\Omega} g = 0$, is satisfied:

Lemma 3.7 *Let $\Omega \subset \mathbf{R}^N$ be a smooth and bounded domain, then $f \in \mathrm{L}^1(\Omega;\mathbf{R}^N)$ and $g \in \mathrm{L}^1(\partial\Omega;\mathbf{R}^N)$ satisfy the assumptions of Theorem 3.6 if and only if $\int_\Omega f + \int_{\partial\Omega} g = 0$ and*

$$[f,g]' = \sup\Big\{ \int_\Omega f \cdot \nabla\zeta + \int_{\partial\Omega} g \cdot \nabla\zeta \ : \ \zeta \in C^2(\overline{\Omega}) \\ \text{and } \|\zeta\|_{\mathrm{W}^{2,N}} \le 1 \Big\} < \infty. \tag{3.6}$$

Proof Suppose first that f satisfies the assumptions of Theorem 3.6. It is then clear that $[f,g]' \le [f,g] < \infty$. Moreover, for every $1 \le i \le N$, taking $\zeta_i(x) = x_i$, one has $D^2\zeta_i = 0$, so that since $[f,g] < \infty$, $\int_\Omega f_i + \int_{\partial\Omega} g_i = 0$.

On the other hand, assume $\int_\Omega f + \int_{\partial\Omega} g = 0$ and (3.6) holds. For every $\zeta \in C^2(\bar\Omega)$, there is ζ' such that $D^2\zeta' = D^2\zeta$, $\int_\Omega \zeta' = 0$ and $\int_\Omega \nabla\zeta' = 0$, so that $\nabla\zeta - \nabla\zeta'$ is constant and by Poincaré's inequality, $\|\zeta'\|_{W^{2,N}} \le C\|D^2\zeta'\|_{L^N}$. Therefore,

$$\int_\Omega f\cdot\nabla\zeta + \int_{\partial\Omega} g\cdot\nabla\zeta = \int_\Omega f\cdot\nabla\zeta' + \int_{\partial\Omega} g\cdot\nabla\zeta'$$
$$\le [f,g]'\|\zeta'\|_{W^{2,N}} \le C[f,g]'\|D^2\zeta\|_{W^{2,N}}.$$

□

In order to prove Theorem 3.6, also recall

Theorem 3.8 (Bourgain and Brezis [2, Theorem 5′]) *If $\varphi \in W^{1,N}(Q;\mathbf{R}^N)$, then there exist $\psi \in (W^{1,N}\cap L^\infty)(Q;\mathbf{R}^N)$ and $\eta \in W^{2,N}(Q)$ such that*

$$\varphi = \psi + \nabla\eta.$$

Moreover,

$$\|\psi\|_{W^{1,N}} + \|\psi\|_{L^\infty} + \|\eta\|_{W^{2,N}} \le C\|\varphi\|_{W^{1,N}}.$$

Theorem 3.8 can be extended to any smooth domain and slightly refined:

Lemma 3.9 *Let $\Omega \subset \mathbf{R}^N$ be a smooth domain. For every $\varphi \in W^{1,N}(\Omega;\mathbf{R}^N)$, there exist $\psi \in (W^{1,N}\cap L^\infty)(\Omega;\mathbf{R}^N)$ and $\eta \in W^{2,N}(\Omega)$ such that*

$$\varphi = \psi + \nabla\eta \quad \text{in} \quad \Omega,$$

and

$$\psi_n = 0 \quad \text{on} \quad \partial\Omega.$$

Moreover,

$$\|\psi\|_{W^{1,N}} + \|\psi\|_{L^\infty} + \|\eta\|_{W^{2,N}} \le C\|\varphi\|_{W^{1,N}}.$$

Proof Since Ω is bounded, up to translation and scaling, $\Omega \subset Q$. Let $\tilde\varphi \in W^{1,N}(Q;\mathbf{R}^N)$ be an extension of φ. Applying Theorem 3.8, one obtains $\tilde\psi \in (W^{1,N}\cap L^\infty)(Q;\mathbf{R}^N)$ and $\tilde\eta \in W^{2,N}(Q)$ such that $\tilde\varphi = \tilde\psi + \nabla\tilde\eta$ in Q, and therefore in Ω. By the next lemma with $\gamma = \tilde\psi_n$, there exists α such that $\nabla\alpha \in L^\infty \cap W^{1,N}$ and $(\nabla\alpha)_n = \tilde\psi_n$. One has thus the conclusion with $\psi = \tilde\psi - \nabla\alpha$ and $\eta = \tilde\eta + \alpha$. □

Lemma 3.10 *If Ω is a smooth bounded domain and $1 < p < \infty$, for every $\gamma \in (L^\infty \cap W^{1-1/p,p})(\partial\Omega)$, there exists $\alpha \in (W_0^{1,\infty}\cap W^{2,p})(\Omega)$ such that*

$$\frac{\partial\alpha}{\partial n} = \gamma \quad \text{on} \quad \partial\Omega,$$

satisfying

$$\|\alpha\|_{W^{1,\infty}} \le C\|\gamma\|_{L^\infty(\partial\Omega)} \quad \text{and} \quad \|\alpha\|_{W^{2,p}} \le C\|\gamma\|_{W^{1-1/p,p}(\partial\Omega)}.$$

Proof By local charts, it is sufficient to consider the case where Ω is a half-space. Let $\rho \in C_c^\infty(\mathbf{R}^{N-1})$ be such that $\int_{\mathbf{R}^{N-1}} \rho = 1$. One checks that

$$\alpha(y,t) = \frac{1}{t^{N-2}} \int\limits_{\mathbf{R}^{N-1}} \rho\Big(\frac{y-z}{t}\Big)\gamma(z)\,dz$$

has the required properties. Since

$$|\alpha(y,t)| \leq t\|\rho\|_{\mathrm{L}^1}\|\gamma\|_{L^\infty},$$

so that $\alpha(\cdot,t) = 0$. One can also write

$$\partial_t\alpha(y,t) = \frac{1}{t^{N-1}} \int\limits_{\mathbf{R}^{N-1}} \tilde{\rho}\Big(\frac{y-z}{t}\Big)\gamma(z)\,dz,$$

where $\tilde{\rho} \in C_c^\infty(\mathbf{R}^{N-1})$ is defined as

$$\tilde{\rho}(z) = -z\cdot\nabla\rho(z) - (N-2)\rho(z).$$

and $\tilde{\rho}_t(z) = \tilde{\rho}_t(z/t)/t^{N-1}$. Hence,

$$\partial_t\alpha(y,t) = (\tilde{\rho}_t * \gamma)(y),$$

so that $\partial_t\alpha(y,t) \to \gamma(y)$ as $t \to 0$. Moreover,

$$|\partial_t\alpha(y,t)| \leq \|\tilde{\rho}\|_{\mathrm{L}^1}\|\gamma\|_{\mathrm{L}^\infty}.$$

The proof for the tangential components of the derivative is similar. One also has

$$\partial_t^2\alpha(y,t) = \frac{1}{t^N} \int\limits_{\mathbf{R}^{N-1}} \hat{\rho}\Big(\frac{y-z}{t}\Big)\gamma(z)\,dz,$$

where $\hat{\rho} \in C_c^\infty(\mathbf{R}^{N-1})$ is defined as

$$\hat{\rho}(z) = z\cdot D^2\rho(z)\cdot z + (2N-2)z\cdot\nabla\rho(z) + (N-2)(N-1)\rho(z).$$

Since $\int_{\mathbf{R}^{N-1}} \hat{\rho} = 0$, one has

$$\partial_t^2\alpha(y,t) = \frac{1}{t^N} \int\limits_{\mathbf{R}^{N-1}} \hat{\rho}\Big(\frac{y-z}{t}\Big)\big(\gamma(z)-\gamma(y)\big)\,dz.$$

By Hölder's inequality, one has then

$$\begin{aligned}
&\int\limits_{\mathbf{R}^{N-1}}\int\limits_{\mathbf{R}^+} |\partial_t^2\alpha(y,t)|^p\,dt\,dy \\
&\quad\leq \int\limits_{\mathbf{R}^{N-1}}\int\limits_{\mathbf{R}^+} \frac{1}{t^p}\Big(\int\limits_{\mathbf{R}^{N-1}} \frac{1}{t^{N-1}}\Big|\hat{\rho}\Big(\frac{y-z}{t}\Big)\Big|\big|\gamma(z)-\gamma(y)\big|\,dz\Big)^p\,dt\,dy \\
&\quad\leq \|\hat{\rho}\|_{\mathrm{L}^1}^{p-1} \int\limits_{\mathbf{R}^{N-1}\times\mathbf{R}^{N-1}}\int\limits_{\mathbf{R}^+} \frac{1}{t^{p+N-1}}\Big|\hat{\rho}\Big(\frac{y-z}{t}\Big)\Big|\,dt\,\big|\gamma(z)-\gamma(y)\big|^p\,dz\,dy.
\end{aligned}$$

Noting that

$$\int_{\mathbf{R}^+} \frac{1}{t^{p+N-1}} \left|\hat{\rho}\left(\frac{y-z}{t}\right)\right| dt \le \frac{C}{|y-z|^{N+p-2}},$$

one obtains the required estimate

$$\int_{\mathbf{R}^{N-1}} \int_{\mathbf{R}^+} |\partial_t^2 \alpha(y,t)|^p \, dt \, dy \le C \int_{\mathbf{R}^{N-1}\times\mathbf{R}^{N-1}} \frac{|\gamma(z)-\gamma(y)|^p}{|z-y|^{N+p-2}} \, dz \, dy.$$

The other second-order derivatives of α are treated in a similar fashion. □

With Lemma 3.9, we can now prove

Lemma 3.11 *Let $\Omega \subset \mathbf{R}^N$ be a smooth and bounded domain. Let $f \in L^1(\Omega;\mathbf{R}^N)$ and $g \in L^1(\partial\Omega;\mathbf{R}^N)$. If (3.5) holds, then for every $\varphi \in (W^{1,N} \cap L^\infty)(\Omega;\mathbf{R}^N)$,*

$$\left|\int_\Omega f\cdot\varphi + \int_{\partial\Omega} g\cdot\varphi\right| \le C\big(\|f\|_{L^1(\Omega)} + \|g_t\|_{L^1(\partial\Omega)} + [f,g]\big)\|D\varphi\|_{L^N}.$$

Proof Write $\varphi \in (W^{1,N} \cap L^\infty)(\Omega;\mathbf{R}^N)$ as $\varphi = \psi + \nabla\eta$ according to Lemma 3.9. One has

$$\begin{aligned}\int_\Omega f\cdot\varphi + \int_{\partial\Omega} g\cdot\varphi &= \int_\Omega f\cdot(\psi+\nabla\eta) + \int_{\partial\Omega} g\cdot(\psi+\nabla\eta)\\ &\le \big(\|f\|_{L^1(\Omega)} + \|g_t\|_{L^1(\partial\Omega)}\big)\|\psi\|_{L^\infty} + [f,g]\|D^2\eta\|_{L^N}\\ &\le C\big(\|f\|_{L^1(\Omega)} + \|g_t\|_{L^1(\partial\Omega)} + [f,g]\big)\|\varphi\|_{W^{1,N}}.\end{aligned}$$

Applying this to $\varphi - \frac{1}{|\Omega|}\int_\Omega \varphi$ and using Lemma 3.7 yields the desired inequality. □

The last tool to obtain our regularity result is the counterpart of Lemma 3.5 for Neumann boundary condition.

Lemma 3.12 *Let $\Omega \subset \mathbf{R}^N$ be a bounded smooth domain and let $1 < p < \infty$. If $F \in L^p(\Omega;\mathbf{R}^N)$, there is a unique $u \in W^{1,p}(\Omega)$ such that $\int_\Omega u = 0$ that solves*

$$\int_\Omega \nabla u\cdot\nabla\varphi = \int_\Omega F\cdot\nabla\varphi, \quad \forall\varphi \in C^1(\overline{\Omega}).$$

Moreover,

$$\|u\|_{W^{1,p}} \le C\|F\|_{L^p}.$$

3.3 The Laplace equation with nonzero Dirichlet boundary condition

Theorem 3.13 *Let $\Omega \subset \mathbf{R}^N$ be a smooth and bounded open set. Let $g \in L^1(\partial\Omega;\mathbf{R}^N)$. If*

$$[g] = \sup\left\{\int_{\partial\Omega} g\cdot\nabla\zeta \ : \ \zeta \in C^2(\overline{\Omega}) \text{ and } \|\zeta\|_{W^{2,N}} \le 1\right\} < \infty,$$

then the system

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

has a unique weak solution $u \in \mathrm{L}^{N/(N-1)}(\Omega; \mathbf{R}^N)$ *and*

$$\|u\|_{\mathrm{L}^{N/(N-1)}} \leq C\big(\|g_t\|_{\mathrm{L}^1(\partial\Omega)} + [g]\big).$$

Remark 3.6 One has $[g] = 0$ if and only if $g_n = 0$ and $\operatorname{div}_{\partial\Omega} g = 0$ on $\partial\Omega$.

Proof Assume without loss of generality that $\int_{\partial\Omega} g = 0$. By classical regularity estimates, there exists a solution $u \in \mathrm{L}^q$ for $q < N/(N-1)$. Let now $\varphi \in C^\infty(\overline{\Omega}; \mathbf{R}^N)$ and let Φ be the solution of

$$\begin{cases} -\Delta \Phi = \varphi & \text{in } \Omega, \\ \Phi = 0 & \text{on } \partial\Omega. \end{cases}$$

By classical regularity estimates, $\Phi \in \mathrm{W}^{2,N}(\mathbf{R}^N; \mathbf{R}^N)$ and

$$\|\Phi\|_{W^{2,N}(\mathbf{R}^N;\mathbf{R}^N)} \leq \|\varphi\|_{\mathrm{L}^N(\mathbf{R}^N;\mathbf{R}^N)}.$$

Combining the definitions of Φ and u, one has

$$\int_\Omega u \cdot \varphi = -\int_{\partial\Omega} g \cdot \frac{\partial \Phi}{\partial n}. \tag{3.7}$$

Since Ω is smooth, the smooth vector field n defined on $\partial\Omega$ can be extended smoothly to $\overline{\Omega}$; we denote it $\tilde{n}$. The vector field g satisfies the assumptions of Lemma 3.11 with $f = 0$ and $\varphi = \tilde{n} \cdot D\Phi$, one has therefore

$$\begin{aligned} \int_{\partial\Omega} g \cdot \frac{\partial \Phi}{\partial n} &\leq C\big(\|g_t\|_{\mathrm{L}^1} + [g]\big)\|\tilde{n} \cdot D\Phi\|_{\mathrm{W}^{1,N}} \\ &\leq C'\big(\|g_t\|_{\mathrm{L}^1} + [g]\big)\|\Phi\|_{W^{2,N}} \leq C''\big(\|g_t\|_{\mathrm{L}^1} + [g]\big)\|\varphi\|_{\mathrm{L}^N}. \end{aligned}$$

Combining this with (3.7), one deduces, since φ is arbitrary, that $u \in \mathrm{L}^{N/(N-1)}$ with the desired estimate. □

3.4 The Laplace equation with other boundary conditions

Since we are dealing with systems, we can prescribe Dirichlet on some components and Neumann on the others. The assumptions on f on the boundary have to be made accordingly.

Theorem 3.14 *Let* $\Omega \subset \mathbf{R}^N$ *be a bounded smooth domain and let* $f \in \mathrm{L}^1(\Omega; \mathbf{R}^N)$. *If*

$$[f]_1 = \sup\Big\{ \int_\Omega f \cdot \nabla\zeta \ : \ \zeta \in C^2(\overline{\Omega}),\ \zeta = 0 \text{ on } \partial\Omega \text{ and } \|D^2\zeta\|_{\mathrm{L}^N} \leq 1 \Big\} < \infty, \tag{3.8}$$

holds, then the system

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u_t = 0 & \text{on } \partial\Omega, \\ (\frac{\partial u}{\partial n})_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.9}$$

has a unique weak solution $u \in W^{1,N/(N-1)}(\Omega;\mathbf{R}^N)$ *and*

$$\|u\|_{W^{1,N/(N-1)}} \le C\big(\|f\|_{L^1} + [f]\big).$$

Theorem 3.15 *Let* $\Omega \subset \mathbf{R}^N$ *be a bounded smooth domain. Let* $f \in L^1(\Omega;\mathbf{R}^N)$. *If*

$$[f]_2 = \sup\Big\{ \int_\Omega f \cdot \nabla\zeta \ : \ \zeta \in C^2(\overline{\Omega}),\ \frac{\partial\zeta}{\partial n} = 0 \text{ on } \partial\Omega \text{ and } \|D^2\zeta\|_{L^N} \le 1 \Big\} < \infty, \tag{3.10}$$

holds, then the system

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \\ (\frac{\partial u}{\partial n})_t = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique weak solution $u \in W^{1,N/(N-1)}(\Omega;\mathbf{R}^N)$ *and*

$$\|u\|_{W^{1,N/(N-1)}} \le C\big(\|f\|_{L^1} + [f,0]\big).$$

Problem (3.9) is weakly formulated as: $u_t = 0$ on $\partial\Omega$ and

$$\int_\Omega Du \cdot D\varphi = \int_\Omega f \cdot \varphi,$$

for every $\varphi \in C^\infty(\overline{\Omega};\mathbf{R}^N)$ such that $\varphi_t = 0$ on $\partial\Omega$. In order prove Theorem 3.14 with the strategy of the proof of Theorem 3.1, we need to refine Lemma 3.4 into

Lemma 3.16 *Let* $\Omega \subset \mathbf{R}^N$ *be a smooth bounded domain and let* $f \in L^1(\Omega;\mathbf{R}^N)$ *be such that* (3.8) *holds. For every* $\varphi \in (W^{1,N} \cap L^\infty)(\Omega;\mathbf{R}^N)$ *such that* $\varphi_t = 0$ *on* $\partial\Omega$, *we have*

$$\Big| \int_\Omega f \cdot \varphi \Big| \le C\big(\|f\|_{L^1} + [f]\big)\|D\varphi\|_{L^N}.$$

Lemma 3.16 is proved as Lemma 3.4, using, instead of Lemma 3.3, the following

Lemma 3.17 *Let* $\Omega \subset \mathbf{R}^N$ *be a smooth domain. For every* $\varphi \in W^{1,N}(\Omega;\mathbf{R}^N)$ *such that* $\varphi_t = 0$ *on* $\partial\Omega$, *there exist* $\psi \in (W_0^{1,N} \cap L^\infty)(\Omega;\mathbf{R}^N)$ *and* $\eta \in W^{2,N}(\Omega)$, *such that* $\eta = 0$ *on* $\partial\Omega$ *and*

$$\varphi = \psi + \nabla\eta.$$

Moreover,

$$\|\psi\|_{W^{1,N}} + \|\psi\|_{L^\infty} + \|\eta\|_{W^{2,N}} \le C\|\varphi\|_{W^{1,N}}.$$

Proof By Lions and Magenes [8], there exists $\theta \in \mathrm{W}^{2,N}(\Omega)$ such that

$$\begin{cases} \theta = 0 & \text{on } \partial\Omega, \\ \frac{\partial\theta}{\partial n} = \varphi_n & \text{on } \partial\Omega, \end{cases}$$

and $\|\theta\|_{\mathrm{W}^{2,N}} \leq \|\varphi\|_{\mathrm{W}^{1,N}}$. Then, $\tilde{\varphi} = \varphi - \nabla\theta$ belongs to $\mathrm{W}^{1,N}_0(\Omega)$. By applying Lemma 3.3 to $\tilde{\varphi}$, one obtains the conclusion. □

The proof of Theorem 3.15 is similar; it uses Lemma 3.9 in place of Lemma 3.17. (Note that, in Lemma 3.9, if $\varphi_n = 0$ on $\partial\Omega$, one has $\partial\eta/\partial n = 0$ on $\partial\Omega$.)

4 First-order elliptic systems

4.1 div–curl systems in 3-D

Here we consider the system

$$\begin{cases} \operatorname{div} Z = 0, \\ \operatorname{curl} Z = Y, \end{cases}$$

together with boundary conditions either on the normal or on the tangential part of Z.

Theorem 4.1 *Let $\Omega \subset \mathbf{R}^3$ be a contractible smooth domain, and let $Y \in \mathrm{L}^1(\Omega;\mathbf{R}^3)$. If* $\operatorname{div} Y = 0$*, then there exists a unique $Z \in \mathrm{L}^{3/2}(\Omega;\mathbf{R}^3)$ such that*

$$\begin{cases} \operatorname{div} Z = 0 & \textit{in } \Omega, \\ \operatorname{curl} Z = Y & \textit{in } \Omega, \\ Z_n = 0 & \textit{on } \partial\Omega. \end{cases}$$

Proof The classical construction yields a solution $Z \in \mathrm{L}^q(\Omega;\mathbf{R}^3)$, for $q < 3/2$. Let $X \in C^\infty(\overline{\Omega};\mathbf{R}^N)$. By Hodge decomposition, there are $\varphi \in C^\infty(\overline{\Omega})$ and $A \in C^\infty(\overline{\Omega};\mathbf{R}^N)$ such that

$$X = \nabla\varphi + \operatorname{curl} A,$$

$A_t = 0$ on $\partial\Omega$, and $\|\varphi\|_{\mathrm{W}^{1,3}} + \|A\|_{\mathrm{W}^{1,3}} \leq C\|X\|_{\mathrm{L}^3}$. Integrating by parts, and applying Lemma 3.16, one obtains,

$$\begin{aligned} \int_\Omega Z \cdot X = \int_\Omega Z \cdot (\nabla\varphi + \operatorname{curl} A) = \int_\Omega Y \cdot A \\ \leq C\|Y\|_{\mathrm{L}^1}\|A\|_{\mathrm{W}^{1,3}} \leq C'\|Y\|_{\mathrm{L}^1}\|X\|_{\mathrm{L}^3}. \end{aligned}$$

Since X is arbitrary, $Z \in \mathrm{L}^{3/2}$ with the required estimate. □

Remark 4.1 Since these estimates can be localized (see Remark 3.4), the result extends to smooth domains which are not contractible provided the suitable conditions of orthogonality with harmonic fields are met by the data.

One can also prescribe the tangential part of Z on $\partial\Omega$:

Theorem 4.2 *Let $\Omega \subset \mathbf{R}^3$ be a contractible smooth domain, let $Y \in L^1(\Omega; \mathbf{R}^3)$ and $V \in L^1(\partial\Omega; T\partial\Omega)$. If $\operatorname{div} Y = 0$ in Ω and $Y_n = \operatorname{div}_{\partial\Omega} V$ on $\partial\Omega$ in the sense that*

$$\forall \varphi \in C^\infty(\overline{\Omega}), \quad \int_\Omega Y \cdot \nabla\varphi = -\int_{\partial\Omega} V \cdot \nabla\varphi, \tag{4.1}$$

then there exists $Z \in L^{3/2}(\Omega; \mathbf{R}^3)$ such that

$$\begin{cases} \operatorname{div} Z = 0 & \text{in } \Omega, \\ \operatorname{curl} Z = Y & \text{in } \Omega, \\ Z_t = V & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Remark 4.2 Clearly (4.1) is a necessary condition for the solvability of (4.2).

4.2 Systems of differential forms

If $\omega : \Omega \to \Lambda^k \mathbf{R}^N$ is a differential form, $d\omega$, resp. $d^*\omega$, denotes its exterior differential, resp. codifferential, and ω_t, resp. ω_n, its tangential, resp. normal, component on the boundary. One possible extension to higher dimensions and to k–forms is

Theorem 4.3 *Let $N \geq 4$, $\Omega \subset \mathbf{R}^N$ be a contractible smooth domain, $2 \leq k \leq N-2$, $\alpha \in L^1(\Omega; \Lambda^{k+1})$, $\beta \in L^1(\Omega; \Lambda^{k-1}\mathbf{R}^N)$ and $\gamma \in L^1(\partial\Omega; \Lambda^k T\partial\Omega)$. If $d\alpha = 0$, $d^*\beta = 0$ and $\alpha_t = d_{\partial\Omega}\gamma$ in the sense that*

$$\int_\Omega \alpha \wedge d\varphi = \int_{\partial\Omega} \gamma \wedge (d\varphi)_t \quad \forall \varphi \in C^1(\overline{\Omega}; \Lambda^{N-k-2}\mathbf{R}^N),$$

$$\int_\Omega \beta \wedge d^*\varphi = 0 \quad \forall \varphi \in C^1(\overline{\Omega}; \Lambda^{N-k+2}\mathbf{R}^N) \text{ such that } \varphi_n = 0,$$

then there exists a unique $\omega \in L^{N/(N-1)}(\Omega; \Lambda^k \mathbf{R}^N)$ such that

$$\begin{cases} d\omega = \alpha & \text{in } \Omega, \\ d^*\omega = \beta & \text{in } \Omega, \\ \omega_t = \gamma & \text{on } \partial\Omega. \end{cases}$$

Remark 4.3 The counterpart in the whole space or on the torus can be found in [2,7].

The analogue of Lemma 3.3 for k-forms is

Lemma 4.4 *Let $\Omega \subset \mathbf{R}^N$ be a smooth domain and let $1 \leq k \leq N-1$. For every $\varphi \in W^{1,N}(\Omega; \Lambda^k \mathbf{R}^N)$, there exist $\psi \in (W^{1,N} \cap L^\infty)(\Omega; \Lambda^k \mathbf{R}^N)$ and $\eta \in W^{2,N}(\Omega; \Lambda^{k-1}\mathbf{R}^N)$ such that*

$$\varphi = \psi + d\eta,$$

satisfying

$$\|\psi\|_{W^{1,N}} + \|\psi\|_{L^\infty} + \|\eta\|_{W^{2,N}} \leq C\|\varphi\|_{W^{1,N}}.$$

If moreover $\varphi_t = 0$ on $\partial\Omega$, then one can choose ψ and η such that $\psi = 0$ on $\partial\Omega$ and $\eta = 0$ on $\partial\Omega$.
If, in addition, $\varphi = 0$ on $\partial\Omega$, then one can take $\eta \in W_0^{2,N}(\Omega; \Lambda^{k-1}\mathbf{R}^N)$.

Proof The k-form φ can be written as

$$\varphi = \sum_{i_1<\cdots<i_k} \varphi_{i_1\ldots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

By Lemma 3.9, there exist $\psi_{i_1\ldots i_k} \in (\mathrm{W}^{1,N} \cap \mathrm{L}^\infty)(\Omega; \Lambda^1 \mathbf{R}^N)$ and $\eta \in \mathrm{W}^{2,N}(\Omega; \Lambda^0 \mathbf{R}^N)$ such that

$$\varphi_{i_1\ldots i_k} dx_{i_1} = \psi_{i_1\ldots i_k} + d\eta_{i_1\ldots i_k}.$$

The conclusion now holds with

$$\psi = \sum_{i_1<\cdots<i_k} \psi_{i_1\ldots i_k} dx_{i_2} \wedge \cdots \wedge dx_{i_k},$$
$$\eta = \sum_{i_1<\cdots<i_k} \eta_{i_1\ldots i_k} dx_{i_2} \wedge \cdots \wedge dx_{i_k}.$$

In the case where $\varphi = 0$ on $\partial\Omega$, if one uses Lemma 3.3 in place of Lemma 3.9, one constructs $\psi \in (\mathrm{W}^{1,N}_0 \cap \mathrm{L}^\infty)(\Omega; \Lambda^k \mathbf{R}^N)$ and $\eta \in \mathrm{W}^{2,N}_0(\Omega; \Lambda^{k-1}\mathbf{R}^N)$. In the general case, as in the proof of Lemma 3.17, when $\varphi \in \mathrm{W}^{1,N}(\Omega; \Lambda^k \mathbf{R}^N)$ and $\varphi_t = 0$, there are $\tilde{\varphi} \in \mathrm{W}^{1,N}_0(\Omega; \Lambda^k \mathbf{R}^N)$ and $\tilde{\eta}$ such that $\varphi = \tilde{\varphi} + d\tilde{\eta}$ and $\tilde{\eta} = 0$ on $\partial\Omega$. We are thus reduced to the previous case with $\tilde{\varphi}$ replacing φ. □

As Lemmas 3.11 and 3.16 followed from Lemmas 3.9 and 3.17, one deduces from Lemma 4.4 the

Lemma 4.5 *Let $\Omega \subset \mathbf{R}^N$ be a smooth bounded domain and let $1 \le k \le N-1$. Let $\alpha \in \mathrm{L}^1(\Omega; \Lambda^k \mathbf{R}^N)$. If $d\alpha = 0$ in Ω, then for every $\varphi \in (\mathrm{W}^{1,N} \cap \mathrm{L}^\infty)(\Omega; \Lambda^{N-k}\mathbf{R}^N)$ such that $\varphi_t = 0$,*

$$\left| \int_\Omega \alpha \wedge \varphi \right| \le C \|\alpha\|_{\mathrm{L}^1} \|D\varphi\|_{\mathrm{L}^N}.$$

Let $\gamma \in \mathrm{L}^1(\Omega; \Lambda^{k-1} T\partial\Omega)$. If $d\alpha = 0$ in Ω and $d\gamma = \alpha_t$ on $\partial\Omega$ in the sense of (4.3), then for every $\varphi \in (\mathrm{W}^{1,N} \cap \mathrm{L}^\infty)(\Omega; \Lambda^{N-k}\mathbf{R}^N)$

$$\left| \int_\Omega \alpha \wedge \varphi + \int_{\partial\Omega} \gamma \wedge \varphi \right| \le C\big(\|\alpha\|_{\mathrm{L}^1} + \|\gamma\|_{\mathrm{L}^1}\big) \|D\varphi\|_{\mathrm{L}^N}.$$

Proof of Theorem 4.3 Let $\zeta \in C^\infty(\Omega; \Lambda^{N-k}\mathbf{R}^N)$. By the classical Hodge decomposition for k-forms on domains [5], there are $\varphi \in C^\infty(\overline{\Omega}; \Lambda^{N-k-1}\mathbf{R}^N)$ and $\psi \in C^\infty(\overline{\Omega}; \Lambda^{N-k+1}\mathbf{R}^N)$ such that $\zeta = d\varphi + d^*\psi$, $\psi_n = 0$ and

$$\|\varphi\|_{\mathrm{W}^{1,N}} + \|\psi\|_{\mathrm{W}^{1,N}} \le C \|\zeta\|_{\mathrm{L}^N}.$$

One has thus

$$\int_\Omega \omega \wedge \zeta = \int_\Omega \omega \wedge (d\varphi + d^*\psi).$$

One estimates then, by Lemma 4.5

$$\int_\Omega \omega \wedge d\varphi = \int_{\partial\Omega} \gamma \wedge \varphi + \int_\Omega \alpha \wedge \varphi$$
$$\le C\big(\|\alpha\|_{\mathrm{L}^1} + \|\gamma\|_{\mathrm{L}^1}\big) \|D\varphi\|_{\mathrm{L}^N} \le C'\big(\|\alpha\|_{\mathrm{L}^1} + \|\gamma\|_{\mathrm{L}^1}\big) \|\zeta\|_{\mathrm{L}^N}$$

Similarly, since $d(*\beta) = 0$ and $(*\psi)_t = 0$, by Lemma 4.5 again

$$\int_\Omega \omega \wedge d^*\psi = \int_\Omega *\beta \wedge *\psi \leq C\|\beta\|_{L^1}\|D\varphi\|_{L^N} \leq C\|\beta\|_{L^1}\|\zeta\|_{L^N}.$$

□

Appendix A: Estimates for Green's functions in 2-D

Throughout this appendix Ω is a smooth bounded domain in $\mathbf{R}^2$.

Definition A.1 Green's function G_Ω of Ω with Dirichlet boundary condition is defined by $G_\Omega(x,y) = 0$ if $x \in \partial\Omega$ and $G_\Omega(\cdot,y) - \frac{1}{2\pi}\log\frac{1}{\cdot - y}$ is harmonic in Ω for every $y \in \Omega$.

Definition A.2 Green's function N_Ω of Ω with Neumann boundary condition (also called *Neumann's function*) is defined by

$$\nabla_x N_\Omega(x,y)\cdot n = 0$$

for $(x,y) \in \partial\Omega \times \Omega$, where n denotes the normal to the boundary at x, and

$$-\Delta\Big(N_\Omega(\cdot,y) - \frac{1}{2\pi}\log\frac{1}{|\cdot - y|}\Big) = -\frac{1}{|\Omega|},$$

in Ω, where $|\Omega|$ denotes the area of Ω.

Green's function can be computed explicitly for the unit disk $D \subset \mathbf{R}^2 \equiv \mathbf{C}$: One has, for $x, y \in D$,

$$G_D(x,y) = \frac{1}{2\pi}\log\frac{|1-\bar{y}x|}{|x-y|},$$

where $\bar{y}$ is the complex conjugate of y and $\bar{y}x$ is the product of $\bar{y}$ and x in $\mathbf{C}$. For the Neumann boundary condition one has

$$N_D(x,y) = \frac{1}{2\pi}\Big(\log\frac{1}{|x-y||1-\bar{y}x|} - \frac{|x|^2+|y|^2}{2} - \frac{1}{4}\Big).$$

One has the following estimate on the derivative of Green's function with Dirichlet boundary condition.

Proposition A.1 (Bramble and Payne [3]) *If Ω is a smooth bounded domain, then there exists a constant C such that for every $x, y \in \Omega$,*

$$|\nabla_y G_\Omega(x,y)| \leq \frac{C}{|x-y|}. \tag{A.1}$$

This is proved by the maximum principle for Green's function with Dirichlet boundary conditions, see [3].

Another method to obtain it is to express Green's function by conformal mapping:

Proposition A.2 *Let Ω be a simply-connected domain. If $\psi : \Omega \to D$ is conformal, then*

$$G_\Omega(x,y) = G_D(\psi(x),\psi(y))$$

Proposition A.2 is a well-known consequence of the invariance of the Dirichlet integral under conformal mapping.

Proof of Theorem A.1 One has

$$|\nabla_y G_\Omega(x,y)| \le |(\nabla_y G_D)(\psi(x),\psi(y))|\,|\psi'(y)| \le \frac{C}{|\psi(x)-\psi(y)|}|\psi'(y)|.$$

where the last inequality comes from the explicit form of G_D. One concludes using the bi-Lipschitzian character of ψ by Theorem A.3. □

The regularity of the conformal mapping and of the boundary of Ω are connected by.

Theorem A.3 [6,11] *Let $\Omega \subset \mathbf{R}^2$ be a bounded simply-connected domain and let $\psi : \Omega \to D$ be a conformal mapping. If $\partial\Omega$ is of class $C^{k,\alpha}$, then $\psi \in C^{k,\alpha}(\overline{\Omega})$ and $\psi^{-1} \in C^{k,\alpha}(\overline{\Omega})$.*

Proof A modern version of this can be found [9]. Theorem 3.6 in [9] gives the Hölder continuity of the derivatives of ψ^{-1}, and Theorem 3.5 in [9] states that $(\psi^{-1})' \neq 0$, and therefore, by the implicit function theorem $\psi \in C^{k,\alpha}(\overline{\Omega})$. □

Turning to the Neumann condition we have

Proposition A.4 *Let Ω be a bounded simply-connected domain. If $\psi \in C^{k,\alpha}(\overline{\Omega}; D)$ is conformal, then there exists $w \in C^{k+1,\alpha}(\overline{\Omega})$ such that*

$$N_\Omega(x,y) = N_D(\psi(x),\psi(y)) + w(x) + w(y). \tag{A.2}$$

Proof Set

$$\tilde{N}(x,y) = N_D(\psi(x),\psi(y)).$$

For every $\varphi \in C^1(\overline{\Omega})$ and $y \in \Omega$, since ψ is conformal,

$$\begin{aligned}\int_\Omega \nabla_x \tilde{N}(x,y)\cdot\nabla u(x)\,dx &= \int_D \nabla_\xi N_D(\xi,\psi(y))\cdot\nabla u(\psi^{-1}(\xi))\,d\xi \\ &= u(y) - \frac{1}{2\pi}\int_D u(\psi^{-1}(\xi))\,d\xi = u(y) - \frac{1}{2\pi}\int_\Omega |\psi'(\xi)|^2 u(\xi)\,d\xi,\end{aligned}$$

i.e., for every $y \in \Omega$, $\tilde{N}$ satisfies the equation:

$$\begin{cases} -\Delta\tilde{N}(x,y) = \delta_y - |\psi'(x)|^2/(2\pi) & \text{in } \Omega, \\ \partial\tilde{N}/\partial n(x,y) = 0 & \text{on } \partial\Omega. \end{cases}$$

By the classical Schauder estimates, there is $\tilde{w} \in C^{k+1\alpha}(\overline{\Omega})$ that solves the problem

$$\begin{cases} -\Delta\tilde{w} = |\psi'(x)|^2/(2\pi) - 1/|\Omega| & \text{in } \Omega, \\ \partial\tilde{w}/\partial n = 0 & \text{on } \partial\Omega, \\ \int_\Omega w\,dx = 0. \end{cases}$$

Moreover, for every $y \in \Omega$,

$$\begin{aligned}\frac{1}{|\Omega|}\int_\Omega \tilde{N}(x,y)\,dx &= \int_\Omega \tilde{N}(x,y)|\psi'(x)|^2\,dx + \int_\Omega \tilde{N}(x,y)\Big(\frac{1}{|\Omega|} - \frac{|\psi'(x)|^2}{2\pi}\Big)\,dx\\ &= \frac{1}{2\pi}\int_\Omega N_D(\xi,\psi(y))\,d\xi + \int_\Omega \tilde{N}(x,y)\Delta\tilde{w}(x)\,dx\\ &= -\tilde{w}(y) + \int_\Omega \tilde{w}(x)\frac{|\psi'(x)|^2}{2\pi}\,dx = \tilde{w}(y) - \int_\Omega \tilde{w}(x)\Delta\tilde{w}(x)\,dx\\ &= \tilde{w}(y) + \int_\Omega |\nabla\tilde{w}(x)|^2\,dx.\end{aligned}$$

Therefore one has

$$N_\Omega(x,y) = N_D(\psi(x),\psi(y)) + \tilde{w}(x) + \tilde{w}(y) + \int_\Omega |\nabla\tilde{w}|^2\,dx.$$

which yields the desired conclusion (A.2). □

One can thus conclude

Proposition A.5 *If Ω is a bounded simply-connected domain with $C^{1,\alpha}$ boundary, then*

$$\sup_{x,y\in\partial\Omega}\Big|\nabla_x N_\Omega(x,y)\Big| \le \frac{C}{|x-y|}.$$

Proof The proof is similar to the proof of Theorem A.1 above. □

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