J. Partial Diff. Eqs. **19**(2006), 208–217

Vol.19 No.3

SOME NONLINEAR ELLIPTIC EQUATIONS HAVE ONLY CONSTANT SOLUTIONS*

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Dedicated to K. C. Chang with high esteem and warm friendship

(Received Apr. 11, 2006)

Abstract We study some nonlinear elliptic equations on compact Riemannian manifolds. Our main concern is to find conditions which imply that such equations admit only constant solutions.

Key Words Nonlinear elliptic equations; constant solutions.
2000 MR Subject Classification 35J60.
Chinese Library Classification O175.29, O175.25.

1. Introduction

Motivated by some recent results and questions raised in [1], we study some nonlinear elliptic equations of the form

$$\begin{cases} -\Delta_g u = f(u) & \text{on } M, \\ \quad u > 0 & \text{on } M, \end{cases} \tag{1.1}$$

where (M, g) is a compact Riemannian manifold of dimension $n \geq 2$, without boundary, and $f : (0, +\infty) \to \mathbb{R}$ is a smooth function. Our main concern is to find conditions on M and f which imply that (1.1) admits only constant solutions.

We will present results in two directions:

1) The case where $M = S^n, n \geq 3$, equipped with its standard metric g_0

*The first author (H.B) is also a member of the Institut Universitaire de France and his work is partially supported by an EC Grant through the RTN Program "Front-Singularities" HPRN-CT-2002-00274. The second author (Y. L.) is partially supported by NSF Grant DMS-0401118.

In this case our first result is

Theorem 1 *Assume that* $(M,g)=(S^n,g_0), n\geq 3,$ *and*

$$h(t):=t^{-\frac{n+2}{n-2}}\left(f(t)+\frac{n(n-2)}{4}t\right) \text{ is decreasing on } (0,\infty). \tag{1.2}$$

Then any solution of (1.1) *is constant.*

A typical example is the case

$$f(t)=t^p-\lambda t,\ p>1, \lambda>0, \tag{1.3}$$

so that (1.1) becomes

$$\begin{cases} -\Delta_g u=u^p-\lambda u & \text{on } S^n, \\ u>0 & \text{on } S^n. \end{cases} \tag{1.4}$$

Corollary 1 *Assume that* $p\leq (n+2)/(n-2)$ *and* $\lambda\leq n(n-2)/4$, *and at least one of these inequalities is strict. Then the only solution of* (1.4) *is the constant* $u=\lambda^{1/(p-1)}$.

In fact, Corollary 1 is originally due to Gidas-Spruck [2]. But our argument is quite different from theirs; they rely on some remarkable identities while our method uses moving planes.

When $p=(n+2)/(n-2)$ the conclusion of Corollary 1 is sharp. Indeed if $\lambda=n(n-2)/4$ there is a well-known family of nonconstant solutions; moreover all solutions of (1.4) belong to this family. However when $p<(n+2)/(n-2)$, B. Gidas and J. Spruck established a better result which was later sharpened by M.F. Bidaut-Veron and L. Veron. Namely they proved

Theorem 2([2],[3]) *Assume that* $p<(n+2)/(n-2)$ *and* $\lambda\leq n/(p-1)$. *Then the only solution of* (1.4) *is the constant* $u=\lambda^{1/(p-1)}$.

Remark 1 The proof of Theorem 2 in [2] and [3] is based on some remarkable identities. Our proof of Theorem 1 uses the method of moving planes. It would be very interesting to find a proof of Theorem 2 based on moving planes.

On the other hand, bifurcation analysis (see [3] and Section 4 below) yields

Theorem 3 *Assume* $p<(n+2)/(n-2)$ *and* $\lambda>n/(p-1)$ *with* $|\lambda-n/(p-1)|$ *small. Then there exist nonconstant solutions of* (1.4).

Remark 2 When $p>\frac{n+2}{n-2}$, there exist nonconstant solutions of (1.4) for some values of $\lambda<\frac{n(n-2)}{4}$. Indeed bifurcation theory (see Section 4 and Remark 7 there) implies the existence of a branch of nonconstant solutions emanating from the constant solutions at the value $\lambda=\frac{\nu}{p-1}$ where $\nu=n$ is the second eigenvalue of $-\Delta_{g_0}$ on S^n; note that $\frac{\nu}{p-1}<\frac{n(n-2)}{4}$ since $p>\frac{n+2}{n-2}$. These solutions exist for $\lambda<\frac{\nu}{p-1}$ and $|\lambda-\frac{\nu}{p-1}|$ sufficiently small.

Open Problem 1 When $p>\frac{n+2}{n-2}$, we do not know any result asserting that for some value of $\lambda>0,\lambda$ small, equation (1.4) admits only the constant solution

$u = \lambda^{1/(p-1)}$. In particular, it would be very interesting to decide what happens when $n = 3$, $p > 5$ and $\lambda > 0$ small.

Remark 3 Theorem 1 is reminiscent of Theorem 1.1 in [4], dealing with (1.1) on $M = \mathbb{R}^n$. One could start with (1.4) on S^n and transport it by stereographic projection to $\mathbb{R}^n$; however the resulting equation does not satisfy the assumptions from [4]. Still there are some analogies.

2) The case of a general manifold

Here our main results are the following

Theorem 4 *Assume $n = 3$. Then there exists some $\lambda^* = \lambda^*(M, g) > 0$ such that* (1.1) *with $f(u) = u^5 - \lambda u, 0 < \lambda < \lambda^*$, admits only the constant solution $u = \lambda^{1/4}$.*

Remark 4 A similar result on a three dimensional smooth convex domain with zero Neumann boundary data was established in [5].

Theorem 5([6]) *Let $n \geq 2$, and assume $1 < p < (n+2)/(n-2)$ (any finite $p > 1$ when $n = 2$). Then there exists some $\lambda^* = \lambda^*(M, g, p) > 0$ such that* (1.1) *with*

$$f(u) = u^p - \lambda u,\ 0 < \lambda < \lambda^*,$$

admits only the constant solution $u = \lambda^{1/(p-1)}$.

Remark 5 A similar result on a smooth domain in the Euclidean space with zero Neumann boundary data was established in [7].

Open Problem 2 Is the conclusion of Theorem 5 valid for $n > 3$ and $p = (n+2)/(n-2)$? If not, identify necessary and sufficient conditions on (M, g), $n \geq 4$, under which the conclusion of Theorem 5 is valid.

The issue concerning Open Problem 2 is whether or not there exist some $\bar{\lambda} > 0$ and $\bar{C} > 0$, depending on (M, g), such that $u \leq \bar{C}$ for all solutions of (1.1) with $f(u) = u^{\frac{n+2}{n-2}} - \lambda u$, $0 < \lambda < \bar{\lambda}$. This is true in dimension $n = 3$ (a consequence of results in [8]), but in dimension $n \geq 4$, we do not expect this to be true for all manifolds. To solve the open problem, efforts can be made in two directions. One is to establish the L^∞ estimates of solutions under appropriate conditions on the manifold. The other is to construct blow-up solutions $\{u_{\lambda_i}\}$ for a sequence of $\lambda_i \to 0^+$ under appropriate conditions on the manifold. Such issues for related problems have been studied, see e.g.[9-11], and the references therein.

Remark 6 A sufficient condition in Open Problem 2 is that the Ricci curvature is positive — this is a consequence of Theorem B.1 in [2]. We have been informed by S.S. Bahoura that he has recently proved that the positivity of the scalar curvature is enough.

2. Proof of Theorem 1

Let u be a solution of (1.1) on $M = S^n$. Let P be an *arbitrary* point on S^n, which we will rename the north pole N. Let S: $S^n \setminus \{N\} \to \mathbb{R}^n$ be the stereographic projection, and let

$$\xi(y) = \left(\frac{2}{1+|y|^2}\right)^{\frac{n-2}{2}}, \quad y \in \mathbb{R}^n. \tag{2.1}$$

Consider the new unknown v, defined on $\mathbb{R}^n$, by

$$v(y) = \xi(y)\, u\,(S^{-1}(y)). \tag{2.2}$$

A standard computation gives

$$-\Delta v = F(y, v), v > 0, \text{ in } \mathbb{R}^n, \tag{2.3}$$

where

$$F(y,v) = \xi(y)^{\frac{n+2}{n-2}} f\left(\frac{v}{\xi(y)}\right) + \frac{n(n-2)}{4}\xi(y)^{\frac{4}{n-2}} v. \tag{2.4}$$

Since ξ depends only on $r = |y|$, we will write $\xi(r)$ and $F(r, v)$.

By (1.2) and (2.4),

$$F(r,v) = v^{\frac{n+2}{n-2}} h\left(\frac{v}{\xi(r)}\right).$$

Thus, by (1.2),

$$\text{for every fixed } v > 0, r \longmapsto F(r, v) \text{ is decreasing in } r > 0. \tag{2.5}$$

Since u is regular at N, it is easy to see from (2.1) and (2.2) that $\frac{1}{|y|^{n-2}} v\left(\frac{y}{|y|^2}\right)$ is smooth and positive near $y = 0$. From the theory of Gidas, Ni and Nirenberg, see [12], we know that any solution v of (2.3), with F satisfying (2.5), must be radially symmetric about the origin. Going back to u, this means that u is constant on every $(n-1)$- sphere $|x - N| =$ constant. Since P is arbitrary on S^n, u must be a constant.

3. Proof of Theorem 4

To prove Theorem 4, we first apply the results in [8] to establish

Lemma 1 *Assume $n = 3$. Then there exist some constants $C_1, \varepsilon_1 > 0$ such that for $0 < \lambda < \varepsilon_1$, any solution u of* (1.1)*, with $f(u) = u^5 - \lambda u$, satisfies*

$$u \le C_1.$$

Proof Suppose the contrary; then there exist $\lambda_i \to 0^+, u_i$ satisfies (1.1) with $f(u) = u^5 - \lambda_i u$, such that

$$\max_M u_i \to \infty.$$

By the results in [8] (see in particular Theorem 0.2, Proposition 5.2, Proposition 4.1 and Proposition 3.1), there exist distinct points $p_1, \ldots, p_m$ on M, $m \geq 1$, and $p_\ell^{(i)} \to p_\ell$ as $i \to \infty$, and $\ell = 1, \ldots, m$, such that

$$u_i(p_1^{(i)})u_i \to \eta \text{ in } C^2_{\text{loc}}(M \setminus \{p_1, \ldots p_m\}), \quad \text{as } i \to \infty,$$

where η satisfies

$$\eta > 0 \text{ in } M \setminus \{p_1, \ldots, p_m\},$$

$$\Delta_g \eta = 0 \text{ in } M \setminus \{p_1, \ldots, p_m\},$$

$$\lim_{p \to p_\ell} \eta(p) = \infty \ , \ \ell = 1, 2, \ldots, m.$$

But this violates the maximum principle, since η clearly has an interior minimum point in $M \setminus \{p_1, \ldots, p_m\}$.

Proof of Theorem 4 Integrating equation (1.1) on M leads to, using Hölder inequality,

$$\|u\|_{L^5(M)} \leq C\lambda^{1/4}. \tag{3.1}$$

Here and in the following, C denotes some positive constant depending only on (M, g).

By Lemma 1 and the equation satisfied by u,

$$|\Delta_g u| \leq Cu.$$

By elliptic estimates, in view of (3.1),

$$\|u\|_{L^\infty(M)} \leq C\lambda^{1/4}. \tag{3.2}$$

Next, we use an argument due to J.R. Licois and L. Veron [6]. From (1.4) we have

$$\int_M \nabla u \nabla(u - \bar{u}) + \lambda \int_M u(u - \bar{u}) = \int_M u^5(u - \bar{u}) \tag{3.3}$$

where $\bar{u} = ⨍_M u$. Clearly

$$\int_M \bar{u}(u - \bar{u}) = \int_M \bar{u}^5(u - \bar{u}) = 0. \tag{3.4}$$

By (3.3) and (3.4) we have

$$\int_M |\nabla(u - \bar{u})|^2 + \lambda \int_M |u - \bar{u}|^2 = \int_M (u^5 - \bar{u}^5)(u - \bar{u}). \tag{3.5}$$

Let ν_1 be the first positive eigenvalue of $-\Delta_g$. From (3.5) we deduce that

$$(\nu_1 + \lambda)\|u - \bar{u}\|^2_{L^2} \leq 5\|u\|^4_{L^\infty}\|u - \bar{u}\|^2_{L^2}. \tag{3.6}$$

Combining (3.2) and (3.6) yields $u = \bar{u} = \lambda^{1/4}$ when λ is sufficiently small.

4. Bifurcation analysis. Proof of Theorem 3

We now return to equation (1.1) with f given by (1.3), i.e.,

$$\begin{cases} -\Delta_g u = u^p - \lambda u & \text{on } M, \\ u > 0 & \text{on } M, \end{cases} \tag{4.1}$$

where $1 < p < \infty$ and $\lambda > 0$.

Writing the solution u as

$$u = \lambda^{1/(p-1)} v,$$

equation (4.1) becomes

$$\begin{cases} -\Delta_g v = \lambda(v^p - v) & \text{on } M, \\ v > 0 & \text{on } M. \end{cases}$$

Next we set

$$w = v - 1$$

and we are led to

$$\begin{cases} -\Delta_g w = \lambda F(w) & \text{on } M, \\ w > -1 & \text{on } M, \end{cases} \tag{4.2}$$

where

$$F(w) = (w+1)^p - w - 1.$$

Clearly,

$$F(0) = 0, \quad F'(0) = p-1, \quad F''(0) = p(p-1), \quad F'''(0) = p(p-1)(p-2).$$

Bifurcation theory asserts that, under some assumptions, a branch of solutions of (4.2), parametrized as $(\lambda(t), w(t))$, bifurcates from the 0-solution with

$$\lambda(0)F'(0) = \lambda(0)(p-1) = \nu \tag{4.3}$$

and ν is an eigenvalue of $-\Delta_g$. In particular, if ν is a simple eigenvalue the result of Crandall-Rabinowitz [13, Theorem 1.7] applies and yields the existence of a smooth branch of solutions of (4.2) of the form $(\lambda(t), w(t)), t \in (-a, +a)$ satisfying (4.3) and

$$w(t) = t\varphi + \psi(t)$$

where

$$-\Delta_g \varphi = \nu\varphi, \varphi \neq 0$$

$$\psi(0) = 0, \quad \psi'(0) = 0,$$

$$\int_M \varphi\psi(t) = 0 \quad \forall t \in (-a, +a).$$

We now differentiate (4.2) with respect to t and obtain

$$-\Delta_g w' = \lambda F'(w)w' + \lambda' F(w),$$

$$-\Delta_g w'' = \lambda[F''(w)(w')^2 + F'(w)w''] + 2\lambda' F'(w)w' + \lambda'' F(w). \tag{4.4}$$

Taking $t = 0$ in (4.4) yields

$$-\Delta_g \psi''(0) - \nu\psi''(0) = \nu p\varphi^2 + 2\lambda'(0)(p-1)\varphi$$

and thus

Lemma 2 *We have*

$$\lambda'(0) = -\frac{\nu p \int \varphi^3}{2(p-1)\int \varphi^2}.$$

When $\int \varphi^3 \neq 0$ we may be satisfied with the information $\lambda'(0) \neq 0$ which gives the existence of nonconstant solutions of (4.1), close to the constant solution $u = \lambda^{1/(p-1)}$, for all values of λ with $|\lambda - \nu/(p-1)|$ sufficiently small.

However when

$$\int \varphi^3 = 0 \tag{4.5}$$

we have $\lambda'(0) = 0$ and we must study $\lambda''(0)$. First observe that if (4.5) holds then $\psi''(0)$ is uniquely determined by the relations

$$-\Delta_g \psi''(0) - \nu\psi''(0) = \nu p\varphi^2 \tag{4.6}$$

$$\int \varphi\psi''(0) = 0. \tag{4.7}$$

Differentiating (4.4) with respect to t once more gives

$$\begin{aligned} -\Delta_g w''' = {} & \lambda[F'''(w)(w')^3 + 3F''(w)w'w'' + F'(w)w''']+ \\ & + 3\lambda'[F''(w)(w')^2 + F'(w)w''] + 3\lambda'' F'(w)w' + \lambda''' F(w). \end{aligned} \tag{4.8}$$

Evaluating (4.8) at $t = 0$ yields

$$-\Delta_g \psi'''(0) - \nu\psi'''(0) = \nu[p(p-2)\varphi^3 + 3p\varphi\psi''(0)] + 3\lambda''(0)(p-1)\varphi$$

and thus

Lemma 3 *We have*

$$\lambda''(0) = -\frac{\nu p[(p-2)\int \varphi^4 + 3\int \varphi^2\psi''(0)]}{3(p-1)\int \varphi^2}. \tag{4.9}$$

We are now more specific and take $M = S^n$ equipped with its standard metric g_0. The first positive eigenvalue of $-\Delta_{g_0}$ is $\nu_1 = n$. Its multiplicity is $(n+1)$ and the corresponding eigenvalues are the functions $\{x_1, x_2, \dots, x_n, x_{n+1}\}$ restricted to S^n. We are going to look for solutions of (1.4) which are radial about a point N on S^n, say $N = (0, 0, \dots, 1)$. Restricted to the class of radial functions the eigenvalue $\nu_1 = n$ becomes simple and the corresponding eigenfunction is

$$\varphi = x_{n+1}.$$

It is convenient to work with the variable $\theta = d_{S^n}(x, N) =$ geodesic distance between x and N on S^n. In the θ-variable we have

$$\varphi(\theta) = \cos\theta$$

so that

$$\int_{S^n} \varphi^3 = C_n \int_0^\pi \cos^3\theta d\theta = 0,$$

and thus $\lambda'(0) = 0$ by Lemma 2. We now proceed to compute $\lambda''(0)$ using Lemma 3.

Lemma 4 *We have*

$$\lambda''(0) = K_{p,m}\left[-p + \frac{(n+2)}{(n-2)}\right] \tag{4.10}$$

where $K_{p,m}$ is a positive constant depending only on p and n.

Proof For simplicity we write Δ instead of Δ_{g_0}. We first determine $\psi''(0)$ using (4.6) - (4.7). Note that

$$\Delta\varphi^2 = 2\varphi\Delta\varphi + 2|\nabla\varphi|^2 = -2n\varphi^2 + 2|\nabla\varphi|^2. \tag{4.11}$$

On the other hand

$$|\nabla\varphi| = |\varphi_\theta| = \sin\theta$$

and therefore

$$|\nabla\varphi|^2 = 1 - \varphi^2 \tag{4.12}$$

Inserting this into (4.11) yields

$$\Delta\varphi^2 = -2(n+1)\varphi^2 + 2.$$

Thus the solution $\psi''(0)$ of (4.6)-(4.7) is given by

$$\psi''(0) = a\varphi^2 + b$$

with

$$a = \frac{np}{n+2} \tag{4.13}$$

$$b = \frac{-2p}{n+2}. \tag{4.14}$$

Going back to (4.9) we find

$$\lambda''(0) = -np\frac{[(p-2)+3a]}{3(p-1)}\frac{\displaystyle\int \varphi^4}{\displaystyle\int \varphi^2} - \frac{npb}{(p-1)}. \tag{4.15}$$

It remains to compute $\int \varphi^4 / \int \varphi^2$. For this purpose we write

$$\begin{aligned} \Delta\varphi^4 &= 4\varphi^3\Delta\varphi + 12\varphi^2|\nabla\varphi|^2 \\ &= -4n\varphi^4 + 12\varphi^2(1-\varphi^2) \text{ by (4.12)}. \end{aligned} \tag{4.16}$$

Integrating (4.16) gives

$$\frac{\displaystyle\int \varphi^4}{\displaystyle\int \varphi^2} = \frac{3}{n+3} \tag{4.17}$$

Combining (4.15) with (4.13), (4.14) and (4.17) we are led to

$$\begin{aligned} \lambda''(0) &= \frac{-3np}{(n+3)}\left[\frac{(p-2)}{3(p-1)} + \frac{np}{(p-1)(n+2)}\right] + \frac{2np^2}{(p-1)(n+2)} \\ &= \frac{np}{(p-1)(n+2)(n+3)}\left[-(p-2)(n+2) - 3np + 2p(n+3)\right] \\ &= \frac{2np(n-2)}{(p-1)(n+2)(n+3)}\left[-p + \frac{(n+2)}{(n-2)}\right]. \end{aligned}$$

Proof of Theorem 3 When $p < (n+2)/(n-2)$ we obtain from Lemmas 3 and 4 that $\lambda'(0) = 0$ and $\lambda''(0) > 0$. Hence the branch of solutions of (4.2) (and thus (1.4)) emanating from $(\lambda(0), w(0)) = \left(\frac{n}{p-1}, 0\right)$ bends to the right of $\lambda(0)$. This was already observed in [3] based on Theorem 2.

Remark 7 When $p > (n+2)/(n-2)$ we have $\lambda'(0) = 0$ and $\lambda''(0) < 0$. In this case the branch of solutions of (4.3) emanating from $\left(\frac{n}{p-1}, 0\right)$ bends to the left of $\lambda(0)$.

Remark 8 When $p = (n+2)/(n-2)$ we have $\lambda'(0) = 0$ and $\lambda''(0) = 0$. In fact the branch of solutions of (4.2) emanating from $\left(\frac{n}{p-1}, 0\right)$ satisfies $\lambda(t) \equiv \lambda(0) = \frac{n(n-2)}{4}$, i.e., the branch is vertical and it corresponds to the standard solutions of (1.4) with $\lambda = n(n-2)/4$.

Acknowledgment We are grateful to L. A Peletier and A. Ponce for very useful discussions.

References

[1] Brezis H, Peletier L A. Elliptic equations with critical exponent on spherical caps of S^3. *Journal d'Analyse Mathematique*, to appear.

[2] Gidas B, Spruck J. Global and local behavior of positive solutions of nonlinear elliptic equations. *Comm. Pure Appl. Math.*, 1981, **34**: 525-598.

[3] Bidaut-Veron M F, Veron L. Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations. *Inventiones Math.*, 1991, **106**: 489-539.

[4] Li Y Y, Zhang L. Liouville type theorem and Harnack type inequalities for semilinear elliptic inequalities. *J. d'Analyse Math.*, 2003, **90**: 27-87.

[5] Zhu M. Uniqueness results through a priori estimates. I. A three dimensional Neumann problem. *J. Differential Equations*, 1999, **154**: 284-317.

[6] Licois J R, Veron L. A class of nonlinear conservative elliptic equations in cylinders. *Ann. Sc. Norm. Sup. Pisa, Ser. 4*, 1998, **26**: 249-283.

[7] Lin C S, Ni W M, Takagi I. Large amplitude stationary solutions to a chemotaxis system. *J. Differential Equations*, 1988, **72**: 1-27.

[8] Li Y Y, Zhu M. Yamabe type equations on three dimensional Riemannian manifolds. *Comm. in Contemp. Math.*, 1999, **1**: 1-50.

[9] Giacomoni J, Prajapat J, Ramaswamy M. Positive solution branch for elliptic problems with critical indefinite nonlinearity. *Differential Integral Equations*, 2005, **18**: 721-764.

[10] Berti M, Malchiodi A. Non-compactness and multiplicity results for the Yamabe problem on S^n. *J. Funct. Anal*, 2001, **180**: 210-241.

[11] Ge Y, Jing R, Pacard F. Bubble towers for supercritical semilinear elliptic equations. *J. Funct. Anal*, 2005, **221**: 251-302.

[12] Gidas B, Ni W M, Nirenberg L. Symmetry and related properties via the maximum principle. *Comm. Math. Phys.*, 1979, **68**: 209-243.

[13] Crandall M G, Rabinowitz P H. Bifurcation from simple eigenvalues. *J. Funct. Anal*, 1971, **8**: 321-340.