SOME NONLINEAR ELLIPTIC EQUATIONS HAVE ONLY CONSTANT SOLUTIONS*

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Dedicated to K. C. Chang with high esteem and warm friendship

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Abstract We study some nonlinear elliptic equations on compact Riemannian manifolds. Our main concern is to find conditions which imply that such equations admit only constant solutions.

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1. Introduction

Motivated by some recent results and questions raised in [1], we study some nonlinear elliptic equations of the form

\[
\begin{cases}
-\Delta_g u = f(u) & \text{on } M, \\
u > 0 & \text{on } M,
\end{cases}
\]

where \((M, g)\) is a compact Riemannian manifold of dimension \(n \geq 2\), without boundary, and \(f : (0, +\infty) \rightarrow \mathbb{R}\) is a smooth function. Our main concern is to find conditions on \(M\) and \(f\) which imply that (1.1) admits only constant solutions.

We will present results in two directions:

1) The case where \(M = S^n, n \geq 3\), equipped with its standard metric \(g_0\)

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In this case our first result is

**Theorem 1** Assume that \((M, g) = (S^n, g_0), n \geq 3,\) and

\[
h(t) := t^{-\frac{n+2}{n-2}} \left( f(t) + \frac{n(n-2)}{4} t \right)
\]

is decreasing on \((0, \infty).\) (1.2)

Then any solution of (1.1) is constant.

A typical example is the case \(f(t) = t^p - \lambda t, p > 1, \lambda > 0,\) (1.3) so that (1.1) becomes

\[
\begin{aligned}
-\Delta_g u &= u^p - \lambda u & \text{on } S^n, \\
u &> 0 & \text{on } S^n.
\end{aligned}
\] (1.4)

**Corollary 1** Assume that \(p \leq \frac{n+2}{n-2}\) and \(\lambda \leq n(n-2)/4,\) and at least one of these inequalities is strict. Then the only solution of (1.4) is the constant \(u = \lambda^{1/(p-1)}.\)

In fact, Corollary 1 is originally due to Gidas-Spruck [2]. But our argument is quite different from theirs; they rely on some remarkable identities while our method uses moving planes.

When \(p = \frac{n+2}{n-2}\) the conclusion of Corollary 1 is sharp. Indeed if \(\lambda = n(n-2)/4\) there is a well-known family of nonconstant solutions; moreover all solutions of (1.4) belong to this family. However when \(p < \frac{n+2}{n-2},\) B. Gidas and J. Spruck established a better result which was later sharpened by M.F. Bidaut-Veron and L. Veron. Namely they proved

**Theorem 2** ([2],[3]) Assume that \(p < \frac{n+2}{n-2}\) and \(\lambda \leq n/(p-1).\) Then the only solution of (1.4) is the constant \(u = \lambda^{1/(p-1)}.\)

**Remark 1** The proof of Theorem 2 in [2] and [3] is based on some remarkable identities. Our proof of Theorem 1 uses the method of moving planes. It would be very interesting to find a proof of Theorem 2 based on moving planes.

On the other hand, bifurcation analysis (see [3] and Section 4 below) yields

**Theorem 3** Assume \(p < \frac{n+2}{n-2}\) and \(\lambda > n/(p-1)\) with \(|\lambda - n/(p-1)|\) small. Then there exist nonconstant solutions of (1.4).

**Remark 2** When \(p > \frac{n+2}{n-2},\) there exist nonconstant solutions of (1.4) for some values of \(\lambda < \frac{n(n-2)}{4}.\) Indeed bifurcation theory (see Section 4 and Remark 7 there) implies the existence of a branch of nonconstant solutions emanating from the constant solutions at the value \(\lambda = \nu\) where \(\nu = n\) is the second eigenvalue of \(-\Delta_{g_0}\) on \(S^n;\) note that \(\frac{\nu}{p-1} < \frac{n(n-2)}{4}\) since \(p > \frac{n+2}{n-2}.\) These solutions exist for \(\lambda < \frac{\nu}{p-1}\) and \(|\lambda - \frac{\nu}{p-1}|\) sufficiently small.

**Open Problem 1** When \(p > \frac{n+2}{n-2},\) we do not know any result asserting that for some value of \(\lambda > 0, \lambda\) small, equation (1.4) admits only the constant solution
\[ u = \lambda^{1/(p-1)}. \] In particular, it would be very interesting to decide what happens when \( n = 3, \ p > 5 \) and \( \lambda > 0 \) small.

**Remark 3** Theorem 1 is reminiscent of Theorem 1.1 in [4], dealing with (1.1) on \( M = \mathbb{R}^n \). One could start with (1.4) on \( S^n \) and transport it by stereographic projection to \( \mathbb{R}^n \); however the resulting equation does not satisfy the assumptions from [4]. Still there are some analogies.

**2) The case of a general manifold**

Here our main results are the following

**Theorem 4** Assume \( n = 3 \). Then there exists some \( \lambda^* = \lambda^*(M, g) > 0 \) such that (1.1) with \( f(u) = u^5 - \lambda u, 0 < \lambda < \lambda^* \), admits only the constant solution \( u = \lambda^{1/4} \).

**Remark 4** A similar result on a three dimensional smooth convex domain with zero Neumann boundary data was established in [5].

**Theorem 5**([6]) Let \( n \geq 2 \), and assume \( 1 < p < (n+2)/(n-2) \) (any finite \( p > 1 \) when \( n = 2 \)). Then there exists some \( \lambda^* = \lambda^*(M, g, p) > 0 \) such that (1.1) with

\[ f(u) = u^p - \lambda u, \ 0 < \lambda < \lambda^*, \]

admits only the constant solution \( u = \lambda^{1/(p-1)} \).

**Remark 5** A similar result on a smooth domain in the Euclidean space with zero Neumann boundary data was established in [7].

**Open Problem 2** Is the conclusion of Theorem 5 valid for \( n > 3 \) and \( p = (n+2)/(n-2) \)? If not, identify necessary and sufficient conditions on \( (M, g), n \geq 4 \), under which the conclusion of Theorem 5 is valid.

The issue concerning Open Problem 2 is whether or not there exist some \( \overline{\lambda} > 0 \) and \( \overline{C} > 0 \), depending on \( (M, g) \), such that \( u \leq \overline{C} \) for all solutions of (1.1) with \( f(u) = u^{n+2} - \lambda u, 0 < \lambda < \overline{\lambda} \). This is true in dimension \( n = 3 \) (a consequence of results in [8]), but in dimension \( n \geq 4 \), we do not expect this to be true for all manifolds. To solve the open problem, efforts can be made in two directions. One is to establish the \( L^\infty \) estimates of solutions under appropriate conditions on the manifold. The other is to construct blow-up solutions \( \{u_{\lambda_i}\} \) for a sequence of \( \lambda_i \to 0^+ \) under appropriate conditions on the manifold. Such issues for related problems have been studied, see e.g.[9-11], and the references therein.

**Remark 6** A sufficient condition in Open Problem 2 is that the Ricci curvature is positive — this is a consequence of Theorem B.1 in [2]. We have been informed by S.S. Bahoura that he has recently proved that the positivity of the scalar curvature is enough.
2. Proof of Theorem 1

Let \( u \) be a solution of (1.1) on \( M = S^n \). Let \( P \) be an arbitrary point on \( S^n \), which we will rename the north pole \( N \). Let \( S : S^n \setminus \{N\} \to \mathbb{R}^n \) be the stereographic projection, and let

\[
\xi(y) = \left( \frac{2}{1 + |y|^2} \right)^{\frac{n-2}{2}}, \quad y \in \mathbb{R}^n.
\]  
(2.1)

Consider the new unknown \( v \), defined on \( \mathbb{R}^n \), by

\[
v(y) = \xi(y) u(S^{-1}(y)).
\]  
(2.2)

A standard computation gives

\[-\Delta v = F(y, v), \quad v > 0, \quad \text{in} \quad \mathbb{R}^n,
\]  
(2.3)

where

\[
F(y, v) = \xi(y)^{\frac{n+2}{n-2}} f \left( \frac{v}{\xi(y)} \right) + \frac{n(n-2)}{4} \xi(y)^{\frac{4}{n-2}} v.
\]  
(2.4)

Since \( \xi \) depends only on \( r = |y| \), we will write \( \xi(r) \) and \( F(r, v) \).

By (1.2) and (2.4),

\[
F(r, v) = v^{\frac{n+2}{n-2}} h \left( \frac{v}{\xi(r)} \right).
\]

Thus, by (1.2),

\[
\text{for every fixed } v > 0, r \mapsto F(r, v) \text{ is decreasing in } r > 0.
\]  
(2.5)

Since \( u \) is regular at \( N \), it is easy to see from (2.1) and (2.2) that \( \frac{1}{|y|^{n-2}} v \left( \frac{y}{|y|^2} \right) \) is smooth and positive near \( y = 0 \). From the theory of Gidas, Ni and Nirenberg, see [12], we know that any solution \( v \) of (2.3), with \( F \) satisfying (2.5), must be radially symmetric about the origin. Going back to \( u \), this means that \( u \) is constant on every \((n-1)\)-sphere \(|x-N| = \text{constant}\). Since \( P \) is arbitrary on \( S^n \), \( u \) must be a constant.

3. Proof of Theorem 4

To prove Theorem 4, we first apply the results in [8] to establish

**Lemma 1** Assume \( n = 3 \). Then there exist some constants \( C_1, \varepsilon_1 > 0 \) such that for \( 0 < \lambda < \varepsilon_1 \), any solution \( u \) of (1.1), with \( f(u) = u^5 - \lambda u \), satisfies

\[ u \leq C_1. \]

**Proof** Suppose the contrary; then there exist \( \lambda_i \to 0^+ \), \( u_i \) satisfies (1.1) with \( f(u) = u^5 - \lambda_i u \), such that

\[ \max_M u_i \to \infty. \]
By the results in [8] (see in particular Theorem 0.2, Proposition 5.2, Proposition 4.1 and Proposition 3.1), there exist distinct points $p_1, \ldots, p_m$ on $M$, $m \geq 1$, and $p_\ell \rightarrow p_\ell$ as $i \rightarrow \infty$, and $\ell = 1, \ldots, m$, such that

$$u_i(p_\ell) \rightarrow u_\ell^2 \text{ in } C_{\text{loc}}(M \setminus \{p_1, \ldots, p_m\}), \quad \text{as } i \rightarrow \infty,$$

where $u_\ell$ satisfies

$$\eta > 0 \text{ in } M \setminus \{p_1, \ldots, p_m\},$$

$$\Delta g \eta = 0 \text{ in } M \setminus \{p_1, \ldots, p_m\},$$

$$\lim_{p \rightarrow p_\ell} \eta(p) = \infty, \quad \ell = 1, 2, \ldots, m.$$

But this violates the maximum principle, since $\eta$ clearly has an interior minimum point in $M \setminus \{p_1, \ldots, p_m\}$.

**Proof of Theorem 4** Integrating equation (1.1) on $M$ leads to, using Hölder inequality,

$$\|u\|_{L^5(M)} \leq C\lambda^{1/4}. \tag{3.1}$$

Here and in the following, $C$ denotes some positive constant depending only on $(M, g)$.

By Lemma 1 and the equation satisfied by $u$,

$$|\Delta g u| \leq Cu.$$

By elliptic estimates, in view of (3.1),

$$\|u\|_{L^\infty(M)} \leq C\lambda^{1/4}. \tag{3.2}$$

Next, we use an argument due to J.R. Licois and L. Veron [6]. From (1.4) we have

$$\int_M \nabla u \nabla (u - \bar{u}) + \lambda \int_M u(u - \bar{u}) = \int_M u^5(u - \bar{u}) \tag{3.3}$$

where $\bar{u} = \int_M u$. Clearly

$$\int_M \bar{u}(u - \bar{u}) = \int_M \bar{u}^5(u - \bar{u}) = 0. \tag{3.4}$$

By (3.3) and (3.4) we have

$$\int_M |\nabla (u - \bar{u})|^2 + \lambda \int_M |u - \bar{u}|^2 = \int_M (u^5 - \bar{u}^5)(u - \bar{u}). \tag{3.5}$$

Let $\nu_1$ be the first positive eigenvalue of $-\Delta g$. From (3.5) we deduce that

$$(\nu_1 + \lambda)||u - \bar{u}||_{L^2}^2 \leq 5||u||_{L^\infty}^4||u - \bar{u}||_{L^2}^2. \tag{3.6}$$

Combining (3.2) and (3.6) yields $u = \bar{u} = \lambda^{1/4}$ when $\lambda$ is sufficiently small.
4. Bifurcation analysis. Proof of Theorem 3

We now return to equation (1.1) with \( f \) given by (1.3), i.e.,

\[
\begin{aligned}
-\Delta_g u &= u^p - \lambda u \quad \text{on } M, \\
u &= 0 \quad \text{on } M, \tag{4.1}
\end{aligned}
\]

where \( 1 < p < \infty \) and \( \lambda > 0 \).

Writing the solution \( u \) as

\[
u = \lambda^{1/(p-1)}v,
\]
equation (4.1) becomes

\[
\begin{aligned}
-\Delta_g v &= \lambda(v^p - v) \quad \text{on } M, \\
v &= 0 \quad \text{on } M.
\end{aligned}
\]

Next we set

\[
w = v - 1
\]

and we are led to

\[
\begin{aligned}
-\Delta_g w &= \lambda F(w) \quad \text{on } M, \\
w &= -1 \quad \text{on } M, \tag{4.2}
\end{aligned}
\]

where

\[
F(w) = (w + 1)^p - w - 1.
\]

Clearly,

\[
F(0) = 0, \quad F'(0) = p - 1, \quad F''(0) = p(p - 1), \quad F'''(0) = p(p - 1)(p - 2).
\]

Bifurcation theory asserts that, under some assumptions, a branch of solutions of (4.2), parametrized as \((\lambda(t), w(t))\), bifurcates from the 0-solution with

\[
\lambda(0)F'(0) = \lambda(0)(p - 1) = \nu \tag{4.3}
\]

and \( \nu \) is an eigenvalue of \(-\Delta_g\). In particular, if \( \nu \) is a simple eigenvalue the result of Crandall-Rabinowitz [13, Theorem 1.7] applies and yields the existence of a smooth branch of solutions of (4.2) of the form \((\lambda(t), w(t))\), \( t \in (-a, a) \) satisfying (4.3) and

\[
w(t) = t\varphi + \psi(t)
\]

where

\[
-\Delta_g \varphi = \nu \varphi, \varphi \neq 0
\]

\[
\psi(0) = 0, \quad \psi'(0) = 0,
\]
\[ \int_M \varphi \psi(t) = 0 \quad \forall t \in (-a, +a). \]

We now differentiate (4.2) with respect to \( t \) and obtain
\[
-\Delta_g w' = \lambda F'(w)w' + \lambda' F(w),
\]
\[
-\Delta_g w'' = \lambda [F''(w)(w')^2 + F'(w)w''] + 2\lambda' F'(w)w' + \lambda'' F(w). \tag{4.4}
\]

Taking \( t = 0 \) in (4.4) yields
\[
-\Delta_g \psi''(0) - \nu \psi''(0) = \nu \varphi^2 + 2\lambda'(0)(p - 1)\varphi
\]

and thus

**Lemma 2** We have
\[
\lambda'(0) = -\frac{\nu p \int \varphi^3}{2(p - 1) \int \varphi^2}.
\]

When \( \int \varphi^3 \neq 0 \) we may be satisfied with the information \( \lambda'(0) \neq 0 \) which gives the existence of nonconstant solutions of (4.1), close to the constant solution \( u = \lambda^{1/(p-1)} \), for all values of \( \lambda \) with \( |\lambda - \nu/(p - 1)| \) sufficiently small.

However when
\[
\int \varphi^3 = 0 \tag{4.5}
\]
we have \( \lambda'(0) = 0 \) and we must study \( \lambda''(0) \). First observe that if (4.5) holds then \( \psi''(0) \) is uniquely determined by the relations
\[
-\Delta_g \psi''(0) - \nu \psi''(0) = \nu \varphi^2 \tag{4.6}
\]
\[
\int \varphi \psi''(0) = 0. \tag{4.7}
\]

Differentiating (4.4) with respect to \( t \) once more gives
\[
-\Delta_g w''' = \lambda[F'''(w)(w')^3 + 3F''(w)w' + F'(w)w''] +
+ 3\lambda' F'(w)(w')^2 + F'(w)w'' + 3\lambda'' F'(w)w' + \lambda''' F(w). \tag{4.8}
\]

Evaluating (4.8) at \( t = 0 \) yields
\[
-\Delta_g \psi'''(0) - \nu \psi'''(0) = \nu [p(p - 2)\varphi^3 + 3p \varphi \psi''(0)] + 3\lambda''(0)(p - 1)\varphi
\]

and thus

**Lemma 3** We have
\[
\lambda''(0) = -\frac{\nu p [(p - 2) \int \varphi^4 + 3 \int \varphi^2 \psi''(0)]}{3(p - 1) \int \varphi^2}. \tag{4.9}
\]
We are now more specific and take $M = S^n$ equipped with its standard metric $g_0$. The first positive eigenvalue of $-\Delta_{g_0}$ is $\nu_1 = n$. Its multiplicity is $(n + 1)$ and the corresponding eigenvalues are the functions $\{x_1, x_2, \ldots, x_n, x_{n+1}\}$ restricted to $S^n$. We are going to look for solutions of (1.4) which are radial about a point $N$ on $S^n$, say $N = (0, 0, \ldots, 1)$. Restricted to the class of radial functions the eigenvalue $\nu_1 = n$ becomes simple and the corresponding eigenfunction is

$$\varphi = x_{n+1}.$$ 

It is convenient to work with the variable $\theta = d_{S^n}(x, N) =$ geodesic distance between $x$ and $N$ on $S^n$. In the $\theta$-variable we have

$$\varphi(\theta) = \cos \theta$$

so that

$$\int_{S^n} \varphi^3 = C_n \int_0^\pi \cos^3 \theta d\theta = 0,$$

and thus $\lambda'(0) = 0$ by Lemma 2. We now proceed to compute $\lambda''(0)$ using Lemma 3.

**Lemma 4** We have

$$\lambda''(0) = K_{p,m} \left[ -p + \frac{(n + 2)}{(n - 2)} \right]$$

(4.10)

where $K_{p,m}$ is a positive constant depending only on $p$ and $n$.

**Proof** For simplicity we write $\Delta$ instead of $\Delta_{g_0}$. We first determine $\psi''(0)$ using (4.6) - (4.7). Note that

$$\Delta \varphi^2 = 2\varphi \Delta \varphi + 2|\nabla \varphi|^2 = -2n\varphi^2 + 2|\nabla \varphi|^2.$$  

(4.11)

On the other hand

$$|\nabla \varphi| = |\varphi_\theta| = \sin \theta$$

and therefore

$$|\nabla \varphi|^2 = 1 - \varphi^2$$  

(4.12)

Inserting this into (4.11) yields

$$\Delta \varphi^2 = -2(n + 1)\varphi^2 + 2.$$ 

Thus the solution $\psi''(0)$ of (4.6)-(4.7) is given by

$$\psi''(0) = a\varphi^2 + b$$

with

$$a = \frac{np}{n + 2}.$$  

(4.13)
\[ b = \frac{-2p}{n+2}. \]  
(4.14)

Going back to (4.9) we find

\[ \lambda''(0) = -np \left[ (p-2) + 3a \right] \int \frac{\varphi^4}{\varphi^2} - npb \int \frac{\varphi^2}{(p-1)}. \]  
(4.15)

It remains to compute \( \int \varphi^4/ \int \varphi^2 \). For this purpose we write

\[ \Delta \varphi^4 = 4\varphi^3 \Delta \varphi + 12\varphi^2 |\nabla \varphi|^2 \]
\[ = -4n\varphi^4 + 12\varphi^2 (1 - \varphi^2) \] by (4.12).  
(4.16)

Integrating (4.16) gives

\[ \int \frac{\varphi^4}{\varphi^2} = \frac{3}{n+3} \]  
(4.17)

Combining (4.15) with (4.13), (4.14) and (4.17) we are led to

\[ \lambda''(0) = \frac{-3np}{n+3} \left[ \frac{(p-2)}{3(p-1)} + \frac{np}{(p-1)(n+2)} \right] + \frac{2np^2}{(p-1)(n+2)} \]
\[ = \frac{np}{(p-1)(n+2)(n+3)} \left[ -(p-2)(n+2) - 3np + 2p(n+3) \right] \]
\[ = \frac{2np(n-2)}{(p-1)(n+2)(n+3)} \left[ -p + \frac{(n+2)}{(n-2)} \right]. \]

**Proof of Theorem 3**  When \( p < (n+2)/(n-2) \) we obtain from Lemmas 3 and 4 that \( \lambda'(0) = 0 \) and \( \lambda''(0) > 0 \). Hence the branch of solutions of (4.2) (and thus (1.4)) emanating from \( (\lambda(0), w(0)) = \left( \frac{n}{p-1}, 0 \right) \) bends to the right of \( \lambda(0) \). This was already observed in [3] based on Theorem 2.

**Remark 7**  When \( p > (n+2)/(n-2) \) we have \( \lambda'(0) = 0 \) and \( \lambda''(0) < 0 \). In this case the branch of solutions of (4.3) emanating from \( \left( \frac{n}{p-1}, 0 \right) \) bends to the left of \( \lambda(0) \).

**Remark 8**  When \( p = (n+2)/(n-2) \) we have \( \lambda'(0) = 0 \) and \( \lambda''(0) = 0 \). In fact the branch of solutions of (4.2) emanating from \( \left( \frac{n}{p-1}, 0 \right) \) satisfies \( \lambda(t) \equiv \lambda(0) = \frac{n(n-2)}{4} \), i.e., the branch is vertical and it corresponds to the standard solutions of (1.4) with \( \lambda = n(n-2)/4 \).

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References


