Reduced measures on the boundary

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Abstract

We study the existence of solutions of the nonlinear problem

\[
\begin{align*}
-\Delta u + g(u) &= 0 \quad \text{in } \Omega, \\
u &= \mu \quad \text{on } \partial \Omega,
\end{align*}
\]

where $\mu$ is a bounded measure and $g : \mathbb{R} \to \mathbb{R}$ is a nondecreasing continuous function with $g(t) = 0$, $\forall t \leq 0$. Problem (0.1) admits a solution for every $\mu \in L^1(\partial \Omega)$, but this need not be the case when $\mu$ is a general bounded measure. We introduce a concept of reduced measure $\mu^*$ (in the spirit of Brezis et al. (Ann. Math. Stud., to appear)); this is the “closest” measure to $\mu$ for which (0.1) admits a solution.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a smooth bounded domain. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous, nondecreasing function such that $g(0) = 0$. In this paper, we are interested in the problem

$$
\begin{cases}
-\Delta u + g(u) = 0 & \text{in } \Omega, \\
u = \mu & \text{on } \partial \Omega,
\end{cases}
$$

(1.1)

where $\mu$ is a bounded measure on $\partial \Omega$. The right concept of weak solution of (1.1) is the following:

$$
\begin{cases}
u \in L^1(\Omega), \ g(u)\rho_0 \in L^1(\Omega) \text{ and} \\
-\int_{\Omega} u\Delta \zeta + \int_{\Omega} g(u)\zeta = -\int_{\partial \Omega} \frac{\partial \zeta}{\partial n} \ d\mu \ \forall \zeta \in C^2_0(\overline{\Omega}),
\end{cases}
$$

(1.2)

where $\rho_0(x) = d(x, \partial \Omega)$, $\forall x \in \Omega$, $\frac{\partial}{\partial n}$ denotes the derivative with respect to the outward normal of $\partial \Omega$, and $C^2_0(\overline{\Omega}) = \{\zeta \in C^2(\overline{\Omega}); \zeta = 0 \text{ on } \partial \Omega\}$.

If $u$ is a solution of (1.1), then $u \in W^{2,p}_{\text{loc}}(\Omega)$, $\forall p < \infty$ (see [3, Theorem 5]).

It has been proved by Brezis (1972, unpublished; see [15]) that (1.1) admits a unique weak solution when $\mu$ is any $L^1$-function (for a general nonlinearity $g$). When $g$ is a power, the study of (1.1) for measures was initiated by Gmira–Véron [15] (in the same spirit as [1]). They proved that if $g(t) = |t|^{p-1}t$ and $1 < p < \frac{N+1}{N-1}$, then (1.1) has a solution for any measure $\mu$. They also showed that if $p \geq \frac{N+1}{N-1}$ and $\mu = \delta_a$, $a \in \partial \Omega$, then (1.1) has no solution. The set of measures $\mu$ for which (1.1) has a solution has been completely characterized when $p \geq \frac{N+1}{N-1}$. In this case, (1.1) has a solution if and only if $\mu(A) = 0$ for every Borel set $A \subset \partial \Omega$ such that $C_{2/p,p'}(A) = 0$, where $C_{2/p,p'}$ denotes the Bessel capacity on $\partial \Omega$ associated to $W^{2/p,p'}$. This result was established by Le Gall [17] (for $p = 2$) and by Dynkin–Kuznetsov [12] (for $p < 2$) using probabilistic tools and by Marcus–Véron [20] (for $p > 2$) using purely analytical methods; see also Marcus–Véron [21] for a unified approach for any $p \geq \frac{N+1}{N-1}$. We refer the reader to [18,19,22] for other related results.

Our goal in this paper is to develop for (1.1) the same program as in [4] for the problem

$$
\begin{cases}
-\Delta u + g(u) = \lambda & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

(1.3)

where $\lambda$, in this case, is a measure in $\Omega$. We shall analyze the nonexistence mechanism behind (1.1) for a general nonlinearity $g$. In [4] we have shown that the Newtonian
(H^1) capacity in \( \Omega \), \( \text{cap}_{H^1} \), plays a major role in the study of (1.3); one of the main results there asserts that (1.3) has a solution for every \( g \) if and only if \( \lambda(E) = 0 \) for every Borel set \( E \subset \Omega \) such that \( \text{cap}_{H^1}(E) = 0 \). For problem (1.1), the analogous quantity is the Hausdorff measure \( \mathcal{H}^{N-1} \) on \( \partial\Omega \) (i.e., \((N-1)\)-dimensional Lebesgue measure on \( \partial\Omega \)). In fact, many of the results in [4] remain valid provided one replaces in the statements the \( H^1 \)-capacity by the \((N-1)\)-Hausdorff measure. Some of the proofs, however, have to be substantially modified.

Concerning the function \( g \) we will assume throughout the rest of the paper that \( g : \mathbb{R} \to \mathbb{R} \) is continuous, nondecreasing, and that
\[
g(t) = 0 \quad \forall t \leq 0. \tag{1.4}
\]

The space of bounded measures on \( \partial\Omega \) is denoted by \( \mathcal{M}(\partial\Omega) \) and is equipped with the standard norm
\[
\|\mu\|_{\mathcal{M}} = \sup \left\{ \int_{\partial\Omega} \phi \, d\mu; \, \phi \in C(\partial\Omega) \text{ and } \|\phi\|_{L^\infty} \leq 1 \right\}.
\]

By a (weak) solution \( u \) of (1.1) we mean that (1.2) holds. A (weak) subsolution of (1.1) is a function \( v \) satisfying
\[
\begin{cases}
  v \in L^1(\Omega), \quad g(v)\rho_0 \in L^1(\Omega) \quad \text{and} \\
  -\int_{\Omega} v\Delta \zeta + \int_{\Omega} g(v)\zeta \leq -\int_{\partial\Omega} \frac{\partial \zeta}{\partial n} \, d\mu \quad \forall \zeta \in C_0^2(\overline{\Omega}), \ \zeta \geq 0 \quad \text{in } \Omega.
\end{cases} \tag{1.5}
\]

We will say that \( \mu \in \mathcal{M}(\partial\Omega) \) is a good measure if (1.1) admits a solution. If \( \mu \) is a good measure, then Eq. (1.1) has exactly one solution \( u \) (see [20]; although this result is stated there when \( g \) is a power, the proof remains unchanged for a general nonlinearity \( g \)). We denote by \( \mathcal{G} \) the set of good measures (relative to \( g \)); when we need to make explicit the dependence on \( g \) we shall write \( \mathcal{G}(g) \). Recall that \( L^1 \)-functions on \( \partial\Omega \) belong to \( \mathcal{G}(g) \) for every \( g \).

In the sequel we denote by \((g_k)\) a sequence of functions \( g_k : \mathbb{R} \to \mathbb{R} \) which are continuous, nondecreasing and satisfy the following conditions:
\[
0 \leq g_1(t) \leq g_2(t) \leq \cdots \leq g(t) \quad \forall t \in \mathbb{R}, \tag{1.6}
\]
\[
g_k(t) \to g(t) \quad \forall t \in \mathbb{R}. \tag{1.7}
\]

We assume in addition that each \( g_k \) has subcritical growth, i.e., that there exist \( C > 0 \) and \( p < \frac{N+1}{N-1} \) (possibly depending on \( k \)) such that
\[
g_k(t) \leq C(|t|^p + 1) \quad \forall t \in \mathbb{R}. \tag{1.8}
\]

A good example to keep in mind is \( g_k(t) = \min \{ g(t), k \}, \forall t \in \mathbb{R} \).
Since (1.8) holds, then for every $\mu \in \mathcal{M}(\partial \Omega)$ there exists a unique solution $u_k$ of
\[
\begin{aligned}
-\Delta u_k + g_k(u_k) &= 0 \quad \text{in } \Omega, \\
 u_k &= \mu \quad \text{on } \partial \Omega.
\end{aligned}
\] (1.9)

The convergence of the sequence $(u_k)$ follows from the next result, established in [4, Section 9.3]:

**Theorem 1.** As $k \uparrow \infty$, $u_k \downarrow u^*$ in $L^1(\Omega)$, with $g(u^*)\rho_0 \in L^1(\Omega)$, and $u^*$ satisfies
\[
\begin{aligned}
-\Delta u^* + g(u^*) &= 0 \quad \text{in } \Omega, \\
 u^* &= \mu^* \quad \text{on } \partial \Omega
\end{aligned}
\] (1.10)

for some $\mu^* \in \mathcal{M}(\partial \Omega)$ such that $\mu^* \leq \mu$. In addition, $u^*$ is the largest subsolution of (1.1).

**Remark 1.** An alternative approximation mechanism consists of keeping $g$ fixed and considering a sequence of functions $\mu_k \in L^1(\partial \Omega)$ weakly converging to $\mu$. Let $v_k$ be the solution of (1.1) associated to $\mu_k$. It would be interesting to prove that $v_k \rightarrow u^*$ in $L^1(\Omega)$ for some appropriate choices of sequences $(\mu_k)$ (for measures in $\Omega$, see [4, Theorem 11]).

An important consequence of Theorem 1 is that $u^*$—and thus $\mu^*$—does not depend on the choice of the truncating sequence $(g_k)$. We call $\mu^*$ the reduced measure associated to $\mu$. If $g$ has subcritical growth, then $\mu^* = \mu$ for every $\mu \in \mathcal{M}(\partial \Omega)$ (see Example 1 below). However, if $g$ has critical or supercritical growth, then $\mu^*$ might be different from $\mu$. In this case, $\mu^*$ depends both on the measure $\mu$ and on the nonlinearity $g$.

By definition, $\mu^*$ is a good measure $\leq \mu$ (since (1.10) has a solution $u^*$). One of the main properties satisfied by $\mu^*$ is the following:

**Theorem 2.** The reduced measure $\mu^*$ is the largest good measure $\leq \mu$.

A consequence of Theorem 2 is

**Corollary 1.** There exists a Borel set $\Sigma \subset \partial \Omega$ with $\mathcal{H}^{N-1}(\Sigma) = 0$ such that
\[
(\mu - \mu^*)(\partial \Omega \setminus \Sigma) = 0.
\] (1.11)

To see this, let $\mu_a$ and $\mu_s$ denote, respectively, the absolutely continuous and the singular parts of $\mu$ with respect to $\mathcal{H}^{N-1}$. Since $\mu_a \in L^1(\partial \Omega)$, then $\mu_a$ is good. Thus, $\mu_a - \mu_s^-$ is also a good measure (see Proposition 1 below). We then conclude from
Theorem 2 that $\mu_a - \mu_s^- \leq \mu^* \leq \mu$. Hence,

$$0 \leq \mu - \mu^* \leq \mu - \mu_a + \mu_s^- = \mu_s^+$$

and so $\mu - \mu^*$ is concentrated on a set of zero $\mathcal{H}^{N-1}$-measure.

**Remark 2.** Corollary 1 is the “best one can say” about $\mu - \mu^*$ for a general nonlinearity $g$. In fact, given any measure $\mu \geq 0$ concentrated on a set of zero $\mathcal{H}^{N-1}$-measure, there exists some $g$ such that $\mu^* = 0$ (see Theorem 7 below). In particular, $\mu - \mu^*$ can be any nonnegative measure concentrated on a set of zero $\mathcal{H}^{N-1}$-measure in $\partial \Omega$.

It is not difficult to see that if $\mu \in \mathcal{M}(\partial \Omega)$ and $\mu^+ \in L^1(\partial \Omega)$, then $\mu \in \mathcal{G}(g)$ for every $g$ (see Proposition 5 below). The converse is also true:

**Theorem 3.** Let $\mu \in \mathcal{M}(\partial \Omega)$. If $\mu \in \mathcal{G}(g)$ for every $g$, then $\mu^+ \in L^1(\partial \Omega)$.

A key ingredient in the proof of Theorem 3 is the following:

**Theorem 4.** For every compact set $K \subset \partial \Omega$, we have

$$\mathcal{H}^{N-1}(K) = \inf \left\{ \int_{\Omega} | \nabla \zeta |; \zeta \in C^2_0(\overline{\Omega}), -\frac{\partial \zeta}{\partial n} \geq 1 \text{ in some neighborhood of } K \right\}.$$  

**Remark 3.** As we have already pointed out, the measure $\mathcal{H}^{N-1}$ plays here the same role as $\text{cap}_{H^1}$ in [4]. There, for every compact set $K \subset \Omega$ we showed that

$$\text{cap}_{H^1}(K) = \frac{1}{2} \inf \left\{ \int_{\Omega} | \nabla \varphi |; \varphi \in C^\infty_c(\Omega), \varphi \geq 1 \text{ in some neighborhood of } K \right\},$$

which is the counterpart of Theorem 4.

We now address a different question. Could it happen that, for some fixed $g_0$, the only good measures $\mu$ are those satisfying $\mu^+ \in L^1(\partial \Omega)$? The answer is negative. In fact,

**Theorem 5.** For any $g$, there exists a good measure $\mu \geq 0$ such that $\mu \notin L^1(\partial \Omega)$.

A natural question is to combine the results of [4] with those in the present paper, i.e., consider the problem

$$\begin{cases}
-\Delta u + g(u) = \lambda & \text{in } \Omega, \\
u = \mu & \text{on } \partial \Omega,
\end{cases}$$  

(1.12)
where \( \lambda \in \mathcal{M}(\Omega) \) and \( \mu \in \mathcal{M}(\partial\Omega) \). We say that the pair \((\lambda, \mu)\) is good if (1.12) has a solution in the usual weak sense (with \( g(u)\rho_0 \in L^1(\Omega) \)). Surprisingly, the problem “uncouples”. More precisely,

**Theorem 6.** Let \( \lambda \in \mathcal{M}(\Omega) \) and \( \mu \in \mathcal{M}(\partial\Omega) \). The pair \((\lambda, \mu)\) is good if and only if \( \lambda \) is a good measure for (1.3) and \( \mu \) is a good measure for (1.1). Furthermore, 
\[
(\lambda, \mu)^* = (\lambda^*, \mu^*).
\]

This paper is organized as follows. In the next section we prove Theorem 2. In Section 3, we present several properties satisfied by the mapping \( \mu \mapsto \mu^* \) and by the set of good measures \( \mathcal{G} \). Theorem 4 will be established in Section 4. We show in Section 5 that for every singular measure \( \mu \geq 0 \) there exists some \( g \) such that \( \mu^* = 0 \); we then deduce Theorem 3 as a corollary. Theorem 5 will be proved in Section 6. In Section 7, we give the explicit value of \( \mu^* \) in the case where \( g(t) = t^p, t \geq 0 \), for any \( p > 1 \). In the last section we present the proof of Theorem 6.

Some of the results in this paper were announced in [4].

**2. Proof of Theorem 2**

The main ingredient in the proof of Theorem 2 is the following:

**Lemma 1.** Given \( f \in L^1(\Omega; \rho_0 \, dx) \), \( \lambda \in \mathcal{M}(\Omega) \) and \( \mu \in \mathcal{M}(\partial\Omega) \), let \( w \in L^1(\Omega) \) be the unique solution of

\[
- \int_{\Omega} w \Delta \zeta = \int_{\Omega} f \zeta + \int_{\Omega} \zeta \, d\lambda - \int_{\partial\Omega} \frac{\partial \zeta}{\partial n} \, d\mu \quad \forall \zeta \in C_0^2(\Omega).
\]

If \( w \geq 0 \) a.e. in \( \Omega \), then \( \mu \geq 0 \) on \( \partial\Omega \).

This result is fairly well-known. We present a proof for the convenience of the reader. For measures in \( \Omega \), the counterpart of Lemma 1 is the “Inverse” maximum principle of [8] (see [4]).

**Proof of Lemma 1.** Given \( \phi \in C^\infty(\partial\Omega) \), \( \phi \geq 0 \) on \( \partial\Omega \), let \( \zeta \in C_0^2(\overline{\Omega}) \), \( \zeta > 0 \) in \( \Omega \), be such that \( -\frac{\partial \zeta}{\partial n} = \phi \) on \( \partial\Omega \). Let \( \delta_j \downarrow 0 \) be a sequence of regular values of \( \zeta \). For each \( j \geq 1 \), set \( \zeta_j = \zeta - \delta_j \) and \( \omega_j = [\zeta > \delta_j] \). In particular, \( \zeta_j \in C_0^2(\overline{\omega_j}) \), \( \zeta_j \geq 0 \) in \( \omega_j \), and \( -\frac{\partial \zeta_j}{\partial n} \geq 0 \) on \( \partial\omega_j \). By standard elliptic estimates (see [25]), we know that \( w \in W^{1,p}_{\text{loc}}(\Omega) \), \( \forall p < \frac{N}{N-1} \); thus, \( w \) has a nonnegative \( L^1 \)-trace on \( \partial\omega_j \). Therefore,

\[
- \int_{\omega_j} w \Delta \zeta_j = \int_{\omega_j} f \zeta_j + \int_{\omega_j} \zeta_j \, d\lambda - \int_{\partial\omega_j} \frac{\partial \zeta_j}{\partial n} \, w \geq \int_{\omega_j} f \zeta_j + \int_{\omega_j} \zeta_j \, d\lambda.
\]
As $j \to \infty$, we conclude that

$$\int_{\Omega} w \Delta \zeta + \int_{\Omega} f \zeta + \int_{\Omega} \zeta \, d\lambda \leq 0.$$ 

Thus,

$$\int_{\partial \Omega} \phi \, d\mu = -\int_{\partial \Omega} \frac{\partial \zeta}{\partial n} \, d\mu = -\left( \int_{\Omega} w \Delta \zeta + \int_{\Omega} f \zeta + \int_{\Omega} \zeta \, d\lambda \right) \geq 0.$$ 

Since $\phi \geq 0$ was arbitrary, we conclude that $\mu \geq 0$. □

We can now establish Theorem 2:

**Proof of Theorem 2.** Assume $\nu$ is a good measure $\leq \mu$. Let $\nu$ denote the solution of

$$\begin{cases}
-\Delta \nu + g(\nu) = 0 & \text{in } \Omega, \\
\nu = \nu & \text{on } \partial \Omega.
\end{cases}$$

Since $\nu \leq \mu$, it follows that $\nu$ is a subsolution of (1.1). Thus, by Theorem 1, $\nu \leq u^*$ a.e. Applying Lemma 1 to the function $w = u^* - \nu$, we then conclude that $u^* - \nu \geq 0$. □

3. Some properties of $G$ and $\mu^*$

Here is a list of properties which can be established exactly as in [4]. For this reason, we shall omit their proofs.

**Proposition 1.** Suppose $\mu_1$ is a good measure. Then, any measure $\mu_2 \leq \mu_1$ is also a good measure.

**Proposition 2.** If $\mu_1, \mu_2$ are good measures, then so is $\text{sup} \{\mu_1, \mu_2\}$.

**Proposition 3.** The set $G$ of good measures is convex.

**Proposition 4.** We have

$$G + L^1(\partial \Omega) \subset G.$$ 

**Proposition 5.** Let $\mu \in \mathcal{M}(\partial \Omega)$. Then, $\mu \in G$ if and only if $\mu^+ \in G$.

**Proposition 6.** Let $\mu \in \mathcal{M}(\partial \Omega)$. Then, $\mu \in G$ if and only if $\mu_s \in G$, where $\mu_s$ denotes the singular part of $\mu$ with respect to $\mathcal{H}^{N-1}$. 
Proposition 7. Let \( \mu \in \mathcal{M}(\partial \Omega) \). Then, \( \mu \in \mathcal{G} \) if and only if there exist functions \( f_0 \in L^1(\Omega; \rho_0 \, dx) \) and \( v_0 \in L^1(\Omega) \) such that \( g(v_0) \in L^1(\Omega; \rho_0 \, dx) \) and
\[
\int_{\partial \Omega} \frac{\partial \zeta}{\partial n} \, d\mu = \int_{\Omega} f_0 \zeta + \int_{\Omega} v_0 \Delta \zeta \quad \forall \zeta \in C^2_0(\overline{\Omega}).
\] (3.1)

Proposition 7 is the analog of a result of Gallouët–Morel [14]; see also [4, Theorem 6].

Proposition 8. For every measure \( \mu \), we have
\[
0 \leq \mu - \mu^* \leq \mu^+. \tag{3.2}
\]

Proposition 9. For every measure \( \mu \), we have
\[
(\mu^*)^+ = (\mu^+)^* \quad \text{and} \quad (\mu^*)^- = \mu^- \tag{3.3}
\]

Proposition 10. Let \( \mu \in \mathcal{M}(\partial \Omega) \). Then,
\[
\|\mu - \mu^*\|_{\mathcal{M}} = \min_{\nu \in \mathcal{G}} \|\mu - \nu\|_{\mathcal{M}}. \tag{3.4}
\]

Moreover, \( \mu^* \) is the unique good measure which achieves the minimum in (3.4).

Proposition 11. Let \( \mu \in \mathcal{M}(\partial \Omega) \) and \( h \in L^1(\Omega; \rho_0 \, dx) \). The problem
\[
\begin{cases}
-\Delta v + g(v) = h & \text{in } \Omega, \\
v = \mu & \text{on } \partial \Omega,
\end{cases}
\tag{3.5}
\]
has a solution if and only if \( \mu \in \mathcal{G}(g) \).

By a solution \( v \) of (3.5) we mean that \( v \in L^1(\Omega) \) satisfies \( g(v) \in L^1(\Omega; \rho_0 \, dx) \) and
\[
-\int_{\Omega} v \Delta \zeta + \int_{\Omega} g(v) \zeta = \int_{\Omega} h \zeta - \int_{\partial \Omega} \frac{\partial \zeta}{\partial n} \, dv \quad \forall \zeta \in C^2_0(\overline{\Omega}). \tag{3.6}
\]

In view of Lemma 2 below such a solution, whenever it exists, is unique.

The proofs of Propositions 7 and 11 require an extra argument. We shall present a proof based on Lemmas 2–6 below.

Given \( h \in L^1(\Omega; \rho_0 \, dx) \), let \( \mathcal{A}_h(h) \) denote the set of measures \( \mu \) for which (3.5) has a solution. By Lemma 2 below, \( \mathcal{A}_h(h) \) is closed with respect to the strong topology in
Our goal is to show that $A_g(h)$ is independent of $h$ and $A_g(h) = G(g)$, $\forall h$. In the sequel, we shall denote by $\zeta_0$ the solution of

$$\begin{cases}
-\Delta \zeta_0 = 1 & \text{in } \Omega, \\
\zeta_0 = 0 & \text{on } \partial \Omega.
\end{cases}$$

We start with the following:

**Lemma 2.** Let $h_i \in L^1(\Omega; \rho_0 \,dx)$, $i = 1, 2$. Given $\mu_i \in A_g(h_i)$, let $v_i$ denote the solution of (3.5) corresponding to $h_i, \mu_i$. Then,

$$\int_{\Omega} |v_1 - v_2| + \int_{\Omega} |g(v_1) - g(v_2)| \zeta_0 \leq \int_{\partial \Omega} |h_1 - h_2| \zeta_0 + C \int_{\partial \Omega} |\mu_1 - \mu_2|.$$  (3.7)

**Proof.** Apply Lemma 1.5 in [20].

**Lemma 3.** Assume $g$ satisfies

$$g(t) \leq C(|t|^p + 1) \quad \forall t \in \mathbb{R},$$

for some $p < \frac{N+1}{N-1}$. Then, for every $h \in L^1(\Omega; \rho_0 \,dx)$, we have $A_g(h) = \mathcal{M}(\partial \Omega)$.

**Proof.** This result is established in [15] for $h = 0$. The same proof there also applies for $h \in L^\infty(\Omega)$. The general case when $h \in L^1(\Omega; \rho_0 \,dx)$ then follows by density using Lemma 2 above.

Given $\mu \in \mathcal{M}(\partial \Omega)$, let $v_k$ be the solution of

$$\begin{cases}
-\Delta v_k + g_k(v_k) = h & \text{in } \Omega, \\
v_k = \mu & \text{on } \partial \Omega,
\end{cases}$$

where $(g_k)$ is a sequence of functions satisfying (1.6)–(1.8).

**Lemma 4.** Given $\mu \in A_g(h)$, let $v$ denote the solution of (3.5). Assume $v_k$ satisfies (3.9). Then,

$$v_k \to v \quad \text{in } L^1(\Omega) \quad \text{and} \quad g_k(v_k) \to g(v) \quad \text{in } L^1(\Omega; \rho_0 \,dx).$$  (3.10)

**Proof.** The lemma follows by mimicking the proof of Proposition 3 in [4] and using Lemma 2 above.

**Lemma 5.** Let $h_1, h_2 \in L^1(\Omega; \rho_0 \,dx)$. If $h_1 \leq h_2$ a.e., then $A_g(h_1) \supset A_g(h_2)$. 
Proof. Let \( \mu \in \mathcal{A}_g(h_2) \) and let \( (g_k) \) be a sequence satisfying (1.6)–(1.8). Denote by \( v_{i,k}, \ i = 1, 2, \) the solution of
\[
\begin{cases}
-\Delta v_{i,k} + g_k(v_{i,k}) = h_i & \text{in } \Omega, \\
v_{i,k} = \mu & \text{on } \partial \Omega.
\end{cases}
\]
Let \( v_i \) be such that \( v_{i,k} \downarrow v_i \) in \( L^1(\Omega) \) as \( k \uparrow \infty \). By Lemma 4 above, we have
\[ g_k(v_{2,k}) \to g(v_2) \text{ in } L^1(\Omega; \rho_0 \, dx). \]
By [4, Corollary B.2], \( h_1 \leq h_2 \) a.e. implies \( v_{1,k} \leq v_{2,k} \) a.e.; thus, \( g_k(v_{1,k}) \leq g_k(v_{2,k}) \) a.e. It then follows by dominated convergence that
\[ g_k(v_{1,k}) \to g(v_1) \text{ in } L^1(\Omega; \rho_0 \, dx). \]
Therefore, \( \mu \in \mathcal{A}_g(h_1) \). This concludes the proof of the lemma. \( \square \)

Lemma 6. Assume \( \mu \) satisfies (3.1) for some \( f_0 \in L^1(\Omega; \rho_0 \, dx) \) and \( v_0 \in L^1(\Omega) \), with \( g(v_0) \in L^1(\Omega; \rho_0 \, dx) \). Then, problem (3.5) has a solution for every \( h \in L^1(\Omega; \rho_0 \, dx) \).

Proof. Fix \( \varepsilon < 1 \). Given \( m \geq 1 \), let \( M_m = \frac{m\|\xi_0\|_\infty}{1-\varepsilon} \). Since
\[ \varepsilon v_0 + m\xi_0 \leq v_0 \text{ a.e. on the set } [v_0 \geq M_m], \]
we have \( g(\varepsilon v_0 + m\xi_0) \in L^1(\Omega; \rho_0 \, dx) \); moreover,
\[ -\int_\Omega (\varepsilon v_0 + m\xi_0) \Delta \zeta = \int_\Omega (\varepsilon f_0 + m)\zeta - \varepsilon \int_{\Omega \cap \partial} \frac{\partial \zeta}{\partial n} \, d\mu \quad \forall \zeta \in C^2_0(\overline{\Omega}). \]
Thus, \( \varepsilon \mu \in \mathcal{A}_g(\tilde{h}_m) \), where
\[ \tilde{h}_m = \varepsilon f_0 + m + g(\varepsilon v_0 + m\xi_0). \]
Given \( h \in L^1(\Omega; \rho_0 \, dx) \), let
\[ h_m = \min \{h, \tilde{h}_m\}. \]
Since \( h_m \leq \tilde{h}_m \) a.e., it follows from Lemma 5 that \( \varepsilon \mu \in \mathcal{A}_g(h_m) \), \( \forall m \geq 1 \). Note that \( h_m \to h \) in \( L^1(\Omega; \rho_0 \, dx) \) as \( m \to \infty \); thus, by Lemma 2 we get \( \varepsilon \mu \in \mathcal{A}_g(h) \). Since this holds true for every \( \varepsilon < 1 \), we must have \( \mu \in \mathcal{A}_g(h) \). \( \square \)
Proof of Proposition 7. Clearly, if $\mu$ is a good measure, then (3.1) holds. Conversely, assume $\mu$ satisfies (3.1) for some $v_0, f_0$. It then follows from the previous lemma that (3.5) has a solution for $h = 0$. In other words, $\mu$ is good. □

Proof of Proposition 11. If $\mu$ is good, then (3.1) holds. Thus, by Lemma 6 above we conclude that problem (3.5) has a solution for every $h \in L^1(\Omega; \rho_0 \, dx)$. Conversely, if (3.5) has a solution for some $h \in L^1(\Omega; \rho_0 \, dx)$, then (3.1) holds. Applying Proposition 7, we deduce that $\mu$ is good. □

4. Proof of Theorem 4

Given a compact set $K \subset \partial \Omega$, we define the capacity

$$ c_{\partial \Omega}(K) = \inf \left\{ \int_{\Omega} |\Delta \zeta|; \zeta \in C^2_0(\Omega), -\frac{\partial \zeta}{\partial n} \geq 1 \text{ in some neighborhood of } K \right\}. $$

In order to establish Theorem 4 we will need a few preliminary results. We start with

Lemma 7. Let $K \subset \partial \Omega$ be a compact set. Given $\varepsilon > 0$, there exists $\psi \in C^2_0(\Omega)$ such that $\psi \geq 0$ in $\Omega$, $-\frac{\partial \psi}{\partial n} \geq 1$ in some neighborhood of $K$ and

$$ \int_{\Omega} |\Delta \psi| \leq c_{\partial \Omega}(K) + \varepsilon. $$

(4.1)

Proof. Given $\varepsilon > 0$, let $\zeta \in C^2_0(\Omega)$ be such that $-\frac{\partial \zeta}{\partial n} \geq 1$ in some neighborhood of $K$ and

$$ \int_{\Omega} |\Delta \zeta| \leq c_{\partial \Omega}(K) + \frac{\varepsilon}{2}. $$

(4.2)

We now extend $\zeta$ as a $C^2$-function in the whole space $\mathbb{R}^N$. We then let

$$ f_k(x) = \int_{\mathbb{R}^N} \rho_k(x - y) |\Delta \zeta(y)| \, dy \quad \forall x \in \Omega, $$

where $(\rho_k)$ is any sequence of nonnegative modifiers such that $\text{supp} \rho_k \subset B_{1/k}$, $\forall k \geq 1$. As $k \to \infty$, we have

$$ f_k \to |\Delta \zeta| \quad \text{uniformly in } \Omega. $$

(4.3)
Let \( v_k \in C^2_0(\Omega) \) be the solution of
\[
\begin{cases}
-\Delta v_k = f_k & \text{in } \Omega, \\
v_k = 0 & \text{on } \partial \Omega.
\end{cases}
\]
Since \( f_k \geq 0 \), we have \( v_k \geq 0 \) in \( \Omega \). Moreover, (4.3) implies
\[
\frac{\partial v_k}{\partial n} \to \frac{\partial v}{\partial n} \text{ uniformly on } \partial \Omega,
\]
where \( v \) is the solution of
\[
\begin{cases}
-\Delta v = \vert \Delta \zeta \vert & \text{in } \Omega, \\
v = 0 & \text{on } \partial \Omega.
\end{cases}
\]
By the maximum principle, \( \zeta \leq v \) in \( \Omega \). Since \( \zeta = v = 0 \) on \( \partial \Omega \), we have
\[
-\frac{\partial \zeta}{\partial n} \leq -\frac{\partial v}{\partial n} \text{ on } \partial \Omega,
\]
which implies that \( -\frac{\partial v}{\partial n} \geq 1 \) in some neighborhood of \( K \). In view of (4.4), we can fix \( k_0 \geq 1 \) sufficiently large so that \( \frac{\partial v_k}{\partial n} \geq \alpha \) in some neighborhood of \( K \), where \( \alpha < 1 \). We may also assume that
\[
\int_{A_{k_0}} \vert \Delta \zeta \vert < \frac{\varepsilon}{4},
\]
where \( A_{k_0} = \mathcal{N}_{k_0}(\Omega) \setminus \overline{\Omega} \).
Set
\[
\psi = \frac{1}{\alpha} v_{k_0},
\]
so that \( \psi \geq 0 \) in \( \Omega \) and \( -\frac{\partial \psi}{\partial n} \geq 1 \) in some neighborhood of \( K \). Moreover,
\[
\int_{\Omega} \vert \Delta \psi \vert = \frac{1}{\alpha} \int_{\Omega} \vert \Delta v_{k_0} \vert \leq \frac{1}{\alpha} \left( \int_{\Omega} \vert \Delta \zeta \vert + \frac{\varepsilon}{4} \right) \leq \frac{1}{\alpha} \left( c_{\partial \Omega}(K) + \frac{3\varepsilon}{4} \right).
\]
Therefore, by taking
\[
\alpha = \frac{c_{\partial \Omega}(K) + \frac{3\varepsilon}{4}}{c_{\partial \Omega}(K) + \varepsilon} < 1,
\]
we conclude that \( \psi \) satisfies (4.1). \( \Box \)
We next prove

**Lemma 8.** Let $K \subset \partial \Omega$ be a compact set. Given $\varepsilon > 0$, there exists $\psi \in C^2_0(\overline{\Omega})$ such that $0 \leq \psi \leq \varepsilon$ in $\Omega$, $-\frac{\partial \psi}{\partial n} \geq 1$ in some neighborhood of $K$,

$$\int_{\Omega} |\Delta \psi| \leq \mathcal{H}^{N-1}(K) + \varepsilon \quad \text{and} \quad \left\| \frac{\psi}{\rho_0} \right\|_{L^\infty} \leq 1 + \varepsilon. \quad (4.5)$$

**Proof.** Let $\delta > 0$ be such that

$$\mathcal{H}^{N-1}(N_\delta(K) \cap \partial \Omega) \leq \mathcal{H}^{N-1}(K) + \varepsilon.$$

We now fix $\zeta \in C^2_0(\overline{\Omega})$ such that $\zeta > 0$ in $\Omega$, $-\frac{\partial \zeta}{\partial n} = 1$ in $N_\frac{\delta}{2}(K) \cap \partial \Omega$, $\frac{\partial \zeta}{\rho_0} = 0$ in $\partial \Omega \setminus N_\delta(K)$, $0 \leq -\frac{\partial \zeta}{\partial n} \leq 1$ on $\partial \Omega$, and $\left\| \frac{\zeta}{\rho_0} \right\|_{L^\infty} \leq 1 + \varepsilon$. Let $a \in (0, \varepsilon)$ be sufficiently small so that

$$\int_{[\zeta < a]} |\Delta \zeta| < \varepsilon.$$

Let

$$u = a - (a - \zeta)^+ \quad \text{in} \quad \overline{\Omega}.$$

In particular, $0 \leq u < \varepsilon$ in $\Omega$. It is easy to see that $\Delta u \in \mathcal{M}(\Omega)$ and $\Delta u = \Delta \zeta$ in $[\zeta < a]$. Since $u$ is bounded and achieves its maximum everywhere on the set $[\zeta \geq a]$, we can apply Corollary 1.3 in [5] to deduce that

$$-\Delta u \geq 0 \quad \text{in} \quad [\zeta \geq a],$$

in the sense of measures. Thus,

$$\|\Delta u\|_{\mathcal{M}} = -\int_{[\zeta \geq a]} \Delta u + \int_{[\zeta < a]} |\Delta \zeta| \leq -\int_{\Omega} \Delta u + 2 \int_{[\zeta < a]} |\Delta \zeta| \leq -\int_{\Omega} \Delta u + 2\varepsilon. \quad (4.6)$$

On the other hand, proceeding as in the proof of Lemma 7, one can find $\psi \in C^2_0(\overline{\Omega})$ such that $0 \leq \psi \leq \varepsilon$ in $\Omega$, $-\frac{\partial \psi}{\partial n} \geq 1$ on $\partial \Omega$,

$$\left\| \frac{\psi}{\rho_0} \right\|_{L^\infty} \leq \left\| \frac{u}{\rho_0} \right\|_{L^\infty} + \varepsilon \leq 1 + 2\varepsilon \quad (4.7)$$
By (4.6) and (4.8), we have
\begin{equation}
\int_\Omega |\Delta \psi| \leq - \int_\Omega \Delta u + 3\varepsilon.
\end{equation}

Since \( u = \zeta \) in a neighborhood of \( \partial \Omega \),
\begin{equation*}
\int_\Omega \Delta u = \int_{\partial \Omega} \frac{\partial u}{\partial n} = \int_{\partial \Omega} \frac{\partial \zeta}{\partial n}.
\end{equation*}

Thus,
\begin{equation*}
\int_\Omega |\Delta \psi| \leq - \int_{\partial \Omega} \frac{\partial \zeta}{\partial n} + 3\varepsilon \leq H^{-1}(N_\delta(K) \cap \partial \Omega) + 3\varepsilon \leq H^{-1}(K) + 4\varepsilon.
\end{equation*}

This concludes the proof of the lemma. \( \square \)

**Proof of Theorem 4.** Given \( \varepsilon > 0 \), let \( \psi \in C^2_0(\overline{\Omega}) \) be the function given by Lemma 7. Since \( \psi \geq 0 \) in \( \Omega \), we have \( -\frac{\partial \psi}{\partial n} \geq 0 \) on \( \partial \Omega \). Thus, integrating by parts and using (4.1) we get
\begin{equation*}
H^{-1}(K) \leq - \int_{\partial \Omega} \frac{\partial \psi}{\partial n} = - \int_{\partial \Omega} \Delta \psi \leq \int_{\partial \Omega} |\Delta \psi| \leq c_{\partial \Omega}(K) + \varepsilon.
\end{equation*}

Since \( \varepsilon > 0 \) was arbitrary, we deduce that
\begin{equation*}
H^{-1}(K) \leq c_{\partial \Omega}(K).
\end{equation*}

The reverse inequality immediately follows from Lemma 8. \( \square \)

**5. Nonnegative measures which are good for every \( g \) must belong to \( L^1(\partial \Omega) \)**

We start with

**Theorem 7.** Given a Borel set \( \Sigma \subset \partial \Omega \) of zero \( \mathcal{H}^N \)-measure, there exists \( g \) such that
\begin{equation*}
\mu^+ = -\mu^- \quad \text{for every measure} \ \mu \ \text{concentrated on} \ \Sigma.
\end{equation*}
In particular, for every nonnegative $\mu \in \mathcal{M}(\partial \Omega)$ concentrated on a set of zero $\mathcal{H}^{N-1}$-measure, there exists some $g$ such that $\mu^* = 0$.

**Proof.** Let $\Sigma \subset \partial \Omega$ be a Borel set such that $\mathcal{H}^{N-1}(\Sigma) = 0$. Let $(K_k)$ be an increasing sequence of compact subsets of $\Sigma$ such that

$$\mu^+(\Sigma \setminus \bigcup_k K_k) = 0. \quad (5.1)$$

For each $k \geq 1$, $K_k$ has zero $\mathcal{H}^{N-1}$-measure. By Lemma 8, one can find $\psi_k \in C^2_0(\overline{\Omega})$ such that $0 \leq \psi_k \leq \min \{ \frac{1}{k}, 2\rho_0 \}$ in $\Omega$, $-\frac{\partial \psi_k}{\partial n} \geq 1$ in some neighborhood of $K_k$, and

$$\int_{\Omega} |\nabla \psi_k| \leq \frac{1}{k} \quad \forall k \geq 1.$$

In particular,

$$\frac{\Delta \psi_k}{\rho_0} \to 0 \quad \text{in} \quad L^1(\Omega; \rho_0 \, dx).$$

Passing to a subsequence if necessary, we may assume that

$$\frac{\Delta \psi_k}{\rho_0} \to 0 \quad \text{a.e. and} \quad \frac{|\nabla \psi_k|}{\rho_0} \leq G \in L^1(\Omega; \rho_0 \, dx) \quad \forall k \geq 1.$$

According to a theorem of De La Vallée-Poussin (see [6, Remarque 23] or [7, Théorème II.22]), there exists a convex function $h : [0, \infty) \to [0, \infty)$ such that $h(0) = 0$, $h(s) > 0$ for $s > 0$,

$$\lim_{t \to \infty} \frac{h(t)}{t} = +\infty, \quad \text{and} \quad h(G) \in L^1(\Omega; \rho_0 \, dx).$$

Set $h(s) = +\infty$ for $s < 0$. Let $g = h^*$ be the convex conjugate of $h$. Note that $h^*$ is finite in view of the coercivity of $h$, and we have $h^*(t) = 0$ if $t \leq 0$.

We claim that $g$ satisfies all the required properties. In fact, let $\mu$ be any measure concentrated on $\Sigma$ and set $\nu = (\mu^*)^+$, where the reduced measure $\mu^*$ is computed with respect to $g$. By Proposition 5, $\nu$ is a good measure. Let $u \in L^1(\Omega)$, $u \geq 0$ a.e., be such that $g(u) \rho_0 \in L^1(\Omega)$ and

$$-\int_{\Omega} u \Delta \zeta + \int_{\Omega} g(u) \zeta = -\int_{\partial \Omega} \frac{\partial \zeta}{\partial n} \, d\nu \quad \forall \zeta \in C^2_0(\overline{\Omega}). \quad (5.2)$$
Recall that $\psi_k \geq 0$ in $\Omega$ and $\psi_k = 0$ on $\partial \Omega$; thus, $-\frac{\partial \psi_k}{\partial n} \geq 0$ on $\partial \Omega$. Using $\psi_k$ as a test function in (5.2), we get

$$v(K_k) \leq -\int_{\partial \Omega} \frac{\partial \psi_k}{\partial n} \, d\nu \leq -\int_{\Omega} |u\Delta \psi_k + g(u)\psi_k|. \quad (5.3)$$

Note that

$$|u\Delta \psi_k + g(u)\psi_k| \to 0 \quad \text{a.e.}$$

and

$$|u\Delta \psi_k + g(u)\psi_k| \leq u \frac{|\Delta \psi_k|}{\rho_0} + g(u)\frac{\psi_k}{\rho_0} \rho_0$$

$$\leq g(u)\rho_0 + h \left( \frac{|\Delta \psi_k|}{\rho_0} \right) \rho_0 + 2g(u)\rho_0$$

$$\leq 3g(u)\rho_0 + G \rho_0 \in L^1(\Omega).$$

By dominated convergence, we conclude that the right-hand side of (5.3) converges to 0 as $k \to \infty$. Thus,

$$(\mu^*)^+(K_k) = v(K_k) = 0 \quad \forall k \geq 1,$$

so that, by (5.1) and Proposition 8, $(\mu^*)^+(\Sigma) = 0$. Since $\mu$ is concentrated on $\Sigma$, we have $(\mu^*)^+ = 0$; thus, by Proposition 9,

$$\mu^* = (\mu^*)^+ - (\mu^*)^- = -\mu^-,$$

which is the desired result. □

We now present the

**Proof of Theorem 3.** Assume $\mu \in \mathcal{M}(\partial \Omega)$ is good for every $g$. Given a Borel set $\Sigma \subset \partial \Omega$ of zero $\mathcal{H}^{N-1}$-measure, let $v = \mu^+ |_\Sigma$. By Theorem 7, there exists some $g_0$ such that $v^* = 0$. On the other hand, by Propositions 1 and 5, $v$ is good for $g_0$. Thus, $v = v^* = 0$. In other words,

$$\mu^+(\Sigma) = 0 \quad \text{for every Borel set } \Sigma \subset \partial \Omega \text{ such that } \mathcal{H}^{N-1}(\Sigma) = 0.$$ We conclude that $\mu^+ \in L^1(\partial \Omega)$. □
6. How to construct good measures which are not in $L^1(\partial \Omega)$

In this section, we establish Theorem 5. We shall closely follow the strategy used in [24] to construct good measures for problem (1.3) which are not diffuse.

Let $(\ell_k)$ be a decreasing sequence of positive numbers such that

$$\ell_1 < \frac{1}{2} \quad \text{and} \quad \ell_{k+1} < \frac{1}{2} \ell_k \quad \forall k \geq 1. \quad (6.1)$$

We start by briefly recalling the construction of the Cantor set $F \subset [-\frac{1}{2}, \frac{1}{2}]^{N-1}$ associated to the subsequence $(\ell_{kj})$. We refer the reader to [24, Section 2] for details.

We proceed by induction as follows. Let $F_0 = [-\frac{1}{2}, \frac{1}{2}]^{N-1}$, $\ell_0 = 1$ and $k_0 = 0$. Let $F_j$ be the set obtained after the $j$th step; $F_j$ is the union of $2^{(N-1)kj}$ cubes $Q_i$ of side $\ell_{kj}$. Inside each $Q_i$, select $2^{(N-1)(kj+1-k_j)}$ cubes $Q_{i,n}$ of side $\ell_{kj+1}$ uniformly distributed in $Q_i$; the distance between the centers of any two cubes $Q_{i,n}$ is $\gtrsim \ell_{kj}^{2(k_j+1-k_j)}$. Let

$$F_{j+1} = \bigcup_{i,n} Q_{i,n}.$$ 

The set $F$ is given by

$$F = \bigcap_{j=0}^{\infty} F_j.$$ 

We now fix a diffeomorphism

$$\Phi: (-1, 1)^{N-1} \to \Phi((-1, 1)^{N-1}) \subset \partial \Omega$$

and define $\hat{F} = \Phi(F)$. From now on, we shall identify $\hat{F}$ with $F$, and simply denote $\hat{F}$ by $F$. For each $j \geq 1$, let

$$\mu_j = \frac{1}{\mathcal{H}^{N-1}(F_{j+1})} \mathcal{H}_{F_{j+1}};$$

in particular, $\mu_j \in L^1(\partial \Omega)$. The uniform measure concentrated on $F$, $\mu_F$, is the weak* limit of $(\mu_j)$ in $\mathcal{M}(\partial \Omega)$ as $j \to \infty$. In particular, $\mu_F \geq 0$ and $\mu_F(\partial \Omega) = 1$. An important property satisfied by $\mu_F$ is given by the next
Lemma 9. For every \( x \in \partial \Omega \), we have

\[
\mu_F(B_r(x) \cap \partial \Omega) \lesssim \begin{cases} 
\frac{1}{2} \frac{(N-1)k_{j+1}}{(N-1)k_j} & \text{if } \ell_{k_{j+1}} \lesssim r \lesssim 2^{(k_{j+1} - k_j)} \\
\frac{1}{2} \left( \frac{r}{\ell_{k_j}} \right)^{N-1} & \text{if } \frac{\ell_{k_j}}{2^{(k_{j+1} - k_j)}} \lesssim r \lesssim \ell_{k_j}.
\end{cases}
\tag{6.2}
\]

We say that \( a \lesssim b \) if there exists \( C > 0 \), depending only on \( N \), such that \( a \leq Cb \). By \( a \sim b \), we mean that \( a \lesssim b \) and \( b \lesssim a \). We refer the reader to [24] for a proof of Lemma 9; although a slightly stronger assumption than (6.1) is made there, the proof of (6.2) remains unchanged.

Let \( v \in L^1(\Omega) \) be the unique solution of

\[
\begin{cases}
-\Delta v = 0 & \text{in } \Omega, \\
v = \mu_F & \text{on } \partial \Omega.
\end{cases}
\tag{6.3}
\]

Our next step is to establish the following:

Proposition 12. Let \( F \subset \partial \Omega \) be the Cantor set associated to the subsequence \( (\ell_{k_j}) \) and let \( v \) be the solution of (6.3). Assume that

\[
\frac{2^{k_{j+1}} \ell_{k_{j+1}}}{2^{k_j} \ell_{k_j}} \sim 1 \quad \forall \ j \geq 1.
\tag{6.4}
\]

Then, there exists \( C > 0 \) such that

\[
v(x) \leq C \left\{ \frac{1}{\ell_{k_1}^{N-1}} + \sum_{i=1}^{j} \frac{1}{2^{(N-1)k_i} \ell_{k_i}^{N-1}} \left( \frac{\ell_{k_j}}{\ell_{k_i}} \right) + \sum_{i=j+1}^{\infty} \frac{1}{2^{(N-1)k_i} \ell_{k_i}^{N-1}} \left( \frac{\ell_{k_i}}{\ell_{k_{j+1}}} \right)^{N+1} \right\}
\tag{6.5}
\]

for every \( x \in \Omega \) such that \( \ell_{k_{j+1}} < d(x, \partial \Omega) \leq \ell_{k_j}, \ j \geq 1 \).

Proof. We shall suppose for simplicity that \( \Omega = \mathbb{R}_+^N \) is the upper-half space. In this case, the solution \( v \) of (6.3) can be explicitly written as (see Lemma 10 below)

\[
v(z, t) = Nc_N \int_0^\infty \frac{st}{(s^2 + t^2)^{N+1}} \mu_F(B_s(z) \cap \partial \mathbb{R}_+^N) \, ds \quad \forall z \in \mathbb{R}_+^{N-1} \quad \forall t > 0,
\]
where \( c_N = \frac{\Gamma(N/2)}{\pi^{N/2}} \). Applying Lemma 9, we have

\[
v(z, t) \lesssim \sum_{i=1}^{\infty} (A_i + B_i) + C_0,
\]

where

\[
A_i = \frac{1}{2(N-1)k_i} \int_{\frac{\ell_{k_i}}{2(N-1)k_i}}^{\ell_{k_i+1}} \frac{s t}{(s^2 + t^2)^{N/2} + 1} ds,
\]

\[
B_i = \frac{t}{2(N-1)k_i} \int_{\frac{\ell_{k_i}}{2(N-1)k_i}}^{\ell_{k_i+1}} \frac{s^N}{(s^2 + t^2)^{N/2} + 1} ds,
\]

\[
C_0 = \int_{\ell_{k_1}}^{\infty} \frac{s t}{(s^2 + t^2)^{N/2} + 1} ds.
\]

An elementary (but tedious) computation using (6.4) shows that

\[
A_i \lesssim \begin{cases} 
\frac{1}{2(N-1)k_i} \left( \frac{\ell_{k_i+1}}{t} \right)^{N+1} & \text{if } t > \ell_{k_i+1}, \\
\frac{1}{2(N-1)k_i} \left( \frac{t}{\ell_{k_i+1}} \right) & \text{if } t \leq \ell_{k_i+1},
\end{cases}
\]

(6.7)

\[
B_i \lesssim \begin{cases} 
\frac{1}{2(N-1)k_i} \left( \frac{\ell_{k_i}}{t} \right)^{N+1} & \text{if } \ell_{k_i+1} < t \leq \ell_{k_i}, \\
\frac{1}{2(N-1)k_i} & \text{if } \ell_{k_i+1} < t \leq \ell_{k_i},
\end{cases}
\]

(6.8)

\[
C_0 \lesssim \begin{cases} 
\frac{1}{t^{N-1}} & \text{if } t > \ell_{k_1}, \\
\frac{t}{\ell_{k_1}^N} & \text{if } t \leq \ell_{k_1},
\end{cases}
\]

(6.9)
We now assume that $\ell_{k+1} < t \leq \ell_k$. Inserting (6.7)–(6.9) into (6.6), we obtain (6.5).

In order to conclude the proof of Proposition 12, we establish the following:

**Lemma 10.** Given $v \in \mathcal{M}(\mathbb{R}^{N-1})$, let $w$ be the solution of

\[
\begin{cases}
-\Delta w = 0 & \text{in } \mathbb{R}^{N}_+, \\
w = v & \text{on } \partial \mathbb{R}^{N}_+.
\end{cases}
\]  

(6.10)

Then,

\[
w(z, t) = N c_N \int_0^\infty \frac{st}{(s^2 + t^2)^{N/2 + 1}} v(\tilde{B}_s(z)) \, ds \quad \forall z \in \mathbb{R}^{N-1} \quad \forall t > 0,
\]  

(6.11)

where $\tilde{B}_s(z)$ denotes the ball in $\partial \mathbb{R}^{N}_+$ of radius $s$ centered at $z$.

**Proof.** Assume $\mu = f \in C_\infty(\mathbb{R}^{N-1})$. Then, $w$ is given as the Poisson integral of $f$:

\[
w(z, t) = c_N \int_{\mathbb{R}^{N-1}} \frac{t}{(|x - z|^2 + t^2)^{N/2}} f(x) \, dx \quad \forall z \in \mathbb{R}^{N-1} \quad \forall t > 0.
\]

Thus,

\[
w(z, t) = c_N \int_0^\infty \frac{t}{(s^2 + t^2)^{N/2}} \left( \int_{\tilde{B}_s(z)} f \right) \, ds
\]

\[= c_N \int_0^\infty \frac{t}{(s^2 + t^2)^{N/2}} \frac{d}{ds} \left( \int_{\tilde{B}_s(z)} f \right) \, ds.
\]

Integrating by parts with respect to $s$, we obtain (6.11) for $\mu = f$. This establishes (6.11) when $\mu$ is a smooth function. The general case easily follows using a density argument (see, e.g., [20, Lemma 1.4]). \(\square\)

We may now turn to the

**Proof of Theorem 5.** Let $(k_j)$ be an increasing sequence of positive integers such that

\[g(2^{Nj}) \leq 2^{2k_j} \quad \forall j \geq 1.
\]  

(6.12)

Let $(\ell_k)$ be any sequence satisfying (6.1) and such that

\[\ell_{k_j} = \frac{1}{2^{j+k_j}} \quad \forall j \geq 1.
\]
Let $F$ be the Cantor set associated to $(\ell_{kj})$. Since

$$2^{(N-1)kj} \ell_{kj}^{N-1} = \frac{1}{2^{(N-1)j}} \to 0 \text{ as } j \to \infty,$$

we have $|F| = 0$; thus, $\mu_F \notin L^1(\partial \Omega)$. We claim that $\mu_F$ is a good measure. In fact, let $v$ be the solution of (6.3). A simple computation shows that

$$\sum_{i=1}^j \frac{1}{2^{(N-1)k_i} \ell_{k_i}^{N-1}} \left( \frac{\ell_{kj}}{\ell_{k_i}} \right) + \sum_{i=j+1}^{\infty} \frac{1}{2^{(N-1)k_i} \ell_{k_i}^{N-1}} \left( \frac{\ell_{k_i}}{\ell_{kj}^{N-1}} \right)^{N+1} \leq C 2^{(N-1)j}$$

for some constant $C > 0$ sufficiently large. It follows from Proposition 12 that

$$v(x) \leq \tilde{C} 2^{(N-1)j} \text{ if } \ell_{kj+1} < d(x, \partial \Omega) \leq \ell_{kj} \; \forall \; j \geq 1.$$

Denoting $\Omega_j = \{ x \in \Omega; d(x, \partial \Omega) > \ell_{kj} \}$, we then have

$$\int_{\Omega} g(v) \rho_0 = \sum_{j=1}^{\infty} \int_{\Omega \setminus \Omega_j} g(v) \rho_0 + \int_{\Omega_j} g(v) \rho_0 \leq C \sum_{j=1}^{\infty} g(\tilde{C} 2^{(N-1)j}) \ell_{kj} |\Omega_{j+1} \setminus \Omega_j| + O(1).$$

Since $|\Omega_{j+1} \setminus \Omega_j| \leq C \ell_{kj}$, we get

$$\int_{\Omega} g(v) \rho_0 \leq C \sum_{j=1}^{\infty} \frac{g(\tilde{C} 2^{(N-1)j})}{2^{2(j+1)k_j}} + O(1). \quad (6.13)$$

Note that, for $j \geq 1$ sufficiently large, we have $\tilde{C} 2^{(N-1)j} \leq 2^{Nj}$. We deduce from (6.12) and (6.13) that $g(v) \in L^1(\Omega; \rho_0 dx)$. By Proposition 7, we conclude that $\mu_F$ is a good measure. □

7. The case where $g(t) = t^p$

We describe here some examples where the measure $\mu^*$ can be explicitly identified.

**Example 1.** $g(t) = t^p$, $t \geq 0$, with $1 < p < \frac{N+1}{N-1}$.

In this case, every measure is good (see [15]); thus, $\mu^* = \mu$, $\forall \mu \in \mathcal{M}(\partial \Omega)$. 
Example 2. \( g(t) = t^p, \ t \geq 0, \) with \( p \geq \frac{N+1}{N-1}. \)

By [21], a nonnegative measure \( v \) is good if and only if \( v(A) = 0 \) for every Borel set \( A \subseteq \partial \Omega \) such that \( C_{2/p,p'}(A) = 0. \) Recall (see [13]) that any measure \( \mu \) can be uniquely decomposed as

\[
\mu = \mu_1 + \mu_2,
\]

where \( \mu_1(A) = 0 \) for every Borel set \( A \subseteq \partial \Omega \) such that \( C_{2/p,p'}(A) = 0, \) and \( \mu_2 \) is concentrated on a set of zero \( C_{2/p,p'} \)-capacity. Using the same argument as in [4, Section 8], one then shows that for every \( \mu \in \mathcal{M}(\partial \Omega) \) we have

\[
\mu^* = \mu - \mu_2^+.
\]

Here is an interesting

Open Problem 1. Let \( N = 2 \) and \( g(t) = e^t - 1, \ t \geq 0. \) Is there a simple characterization of the set of good measures relative to \( g? \) Is there an explicit formula of \( \mu^* \) in terms of \( \mu? \)

There are some partial results in this direction; see [16] and also [23].

8. Proof of Theorem 6

We start with the following:

Lemma 11. Let \( \lambda \in \mathcal{M}(\Omega) \) and \( \mu \in \mathcal{M}(\partial \Omega). \) Assume that there exists \( w \in L^1(\Omega) \) such that \( g(w) \in L^1(\Omega; \rho_0 \, dx) \) and

\[
- \int_{\Omega} w \Delta \zeta + \int_{\Omega} g(w) \zeta \geq \int_{\Omega} \zeta \, d\lambda - \int_{\partial \Omega} \frac{\partial \zeta}{\partial n} \, d\mu \quad \forall \zeta \in C_0^2(\Omega), \ \zeta \geq 0 \text{ in } \Omega. \tag{8.1}
\]

Then, the pair \((\lambda, \mu)\) is good.

Proof. Since (8.1) holds, there exist \( \mu_0 \in \mathcal{M}(\partial \Omega) \) and a locally bounded measure \( \lambda_0 \) in \( \Omega, \) with \( \int_{\Omega} \rho_0 \, d|\lambda_0| < \infty, \) such that \( \mu_0 \geq \mu \) on \( \partial \Omega, \lambda_0 \geq \lambda \) in \( \Omega, \) and

\[
- \int_{\Omega} w \Delta \zeta + \int_{\Omega} g(w) \zeta = \int_{\Omega} \zeta \, d\lambda_0 - \int_{\partial \Omega} \frac{\partial \zeta}{\partial n} \, d\mu_0 \quad \forall \zeta \in C_0^2(\Omega).
\]

(The existence of \( \lambda_0 \) and \( \mu_0 \) is sketched in [4, Remark B.1]).
Let \((g_k)\) be a sequence of bounded functions satisfying (1.6)–(1.7). Let \(u_k, w_k\) be the solutions associated to \((\lambda, \mu), (\lambda_0, \mu_0)\), resp. Then, as in the proof of Lemma 5 above, we have
\[
g_k(u_k) \leq g_k(w_k) \to g(w) \quad \text{in } L^1(\Omega; \rho_0 \, dx).
\]
On the other hand, \(u_k \downarrow u\) in \(L^1(\Omega)\). Thus, by dominated convergence,
\[
g_k(u_k) \to g(u) \quad \text{in } L^1(\Omega; \rho_0 \, dx).
\]
We conclude that \(u\) satisfies (1.12). Therefore, \((\lambda, \mu)\) is good. □

**Proof of Theorem 6.**

**Step 1:** Proof of
\[
(\lambda, \mu)^* = (\lambda^*, \mu^*). \tag{8.2}
\]
Let \(u_k\) be such that
\[
\begin{cases}
-\Delta u_k + g_k(u_k) = \lambda & \text{in } \Omega, \\
u_k = \mu & \text{on } \partial \Omega.
\end{cases}
\]
Then, \(u_k \downarrow \hat{u}\) in \(L^1(\Omega)\). By Fatou, we deduce that \(g(\hat{u}) \in L^1(\Omega; \rho_0 \, dx)\) and
\[
-\int_{\Omega} \hat{u} \Delta \zeta + \int_{\Omega} g(\hat{u}) \zeta \leq \int_{\Omega} \zeta d\lambda - \int_{\partial \Omega} \frac{\partial \zeta}{\partial n} d\mu \quad \forall \zeta \in C_0^2(\overline{\Omega}), \quad \zeta \geq 0 \quad \text{in } \Omega.
\]
By [4, Remark B.1], there exist \(\hat{\mu} \in \mathcal{M}(\hat{\partial} \Omega)\) and a locally bounded measure \(\hat{\lambda}\) in \(\Omega\), with \(\int_{\Omega} \rho_0 \, d|\hat{\lambda}| < \infty\), such that
\[
-\int_{\Omega} \hat{\mu} \Delta \zeta + \int_{\Omega} g(\hat{u}) \zeta = \int_{\hat{\Omega}} \zeta d\hat{\lambda} - \int_{\partial \Omega} \frac{\partial \zeta}{\partial n} d\hat{\mu} \quad \forall \zeta \in C_0^2(\overline{\hat{\Omega}}).
\]
Note that \(\hat{\lambda} \leq \lambda\) in \(\Omega\) and \(\hat{\mu} \leq \mu\) on \(\hat{\partial} \Omega\). We claim that
(a) \((\hat{\lambda})_d = \lambda_d = (\lambda^*)_d\);
(b) \((\hat{\lambda})_c = (\lambda^*)_c\);
(c) \(\hat{\mu} = \mu^*\).
The subscripts “d” and “c” denote the diffuse and the concentrated parts of the measure with respect to \(\text{cap}_{H^1}\) (see [13]). We then deduce from (a) and (b) that \(\hat{\lambda} = \lambda^*\); in particular, \(\hat{\lambda} \in \mathcal{M}(\Omega)\).
Proof of (a): The second equality in (a) is established in [4]. Proceeding exactly as in the proof of Lemma 1 there, one shows that
\[ \hat{\lambda} \geq \lambda_d - \lambda_c^- . \]
Thus, \((\hat{\lambda})_d \geq \lambda_d\). Since \(\hat{\lambda} \leq \lambda\), we conclude that \((\hat{\lambda})_d = \lambda_d\).

Proof of (b): Since the pair \((\lambda^*, 0)\) is good, it follows from Lemma 11 above that \((\hat{\lambda}^*, -\mu^-)\) is also good. Let \(v_1\) be the solution of (1.12) corresponding to \((\lambda^*, -\mu^-)\).

By [4, Corollary B.2], we have \(v_1 \leq u_k\) a.e., \(\forall k \geq 1\). Thus,
\[ v_1 \leq \hat{u}\ a.e. \]

By the "Inverse" maximum principle (see [8]), we obtain
\[ (\hat{\lambda}^*)_c = (-\Delta v_1)_c \leq (-\Delta \hat{u})_c = (\hat{\lambda})_c . \] (8.3)

We conclude from (a) and (8.3) that
\[ \lambda^* \leq \hat{\lambda} \leq \lambda. \]

In particular, \(\hat{\lambda} \in M(\Omega)\). Since \((\hat{\lambda}, \hat{\mu})\) is good, we can apply Lemma 11 to deduce that \((\hat{\lambda}, - (\hat{\mu})^-)\) is also good. Let \(v_2\) denote the corresponding solution. Clearly, \(v_2\) is a subsolution of (1.3). Thus,
\[ v_2 \leq v^* \ a.e., \]
where \(v^*\) is the largest subsolution of (1.3), i.e., \(v^*\) is the solution of (1.3) with data \(\lambda^*\). Applying the "Inverse" maximum principle, we conclude that
\[ (\hat{\lambda})_c = (-\Delta v_2)_c \leq (-\Delta v^*)_c = (\lambda^*)_c . \] (8.4)

We deduce from (8.3) and (8.4) that \((\hat{\lambda})_c = (\lambda^*)_c\).

Proof of (c): The argument in this case is the same as in the proof of (b) and is omitted (one should use Lemma 1 in Section 2 above, instead of the "Inverse" maximum principle).

It now follows from (a)-(c) that \(\hat{\lambda} = \lambda^*\) and \(\hat{\mu} = \mu^*\). This concludes the proof of Step 1.

Step 2: Proof of the theorem completed.

Assume \((\lambda, \mu)\) is good. Thus, \((\lambda, \mu)^* = (\lambda, \mu)\). We deduce from the previous step that \(\lambda^* = \lambda\) and \(\mu^* = \mu\). In other words, \(\lambda\) is a good measure for (1.3) and \(\mu\) is good for (1.1). Similarly, the converse follows. The proof of Theorem 6 is complete. \(\square\)
Open Direction 1. In all the problems above, the equation in $\Omega$ is nonlinear but the boundary condition is the usual Dirichlet condition. It might be interesting to investigate problems involving nonlinear boundary conditions. Here is a typical example:

$$
\begin{aligned}
-\Delta u + u &= 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n} + g(u) &= \mu \quad \text{on } \partial \Omega,
\end{aligned}
$$

(8.5)

where $g$ and $\mu$ are as in the Introduction. This type of problems arises in Physics for various choices of $g$, possibly graphs; see, e.g. [9]. They have been studied in [2] when $\mu \in L^2(\partial \Omega)$.

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