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On a Degenerate Elliptic-Parabolic Equation Occurring in the Theory of Probability*

H. BREZIS, W. ROSENKRANTZ AND B. SINGER
Courant Institute, Columbia University
with an appendix by PETER D. LAX

1. Introduction

In this paper we present results on the differentiability properties of solutions to the following singular parabolic partial differential equation:

$$U_t(x, t) = \tfrac{1}{2}U_{xx}(x, t) + \frac{\gamma}{x} U_x(x, t), \qquad \gamma > -\tfrac{1}{2}, \tag{1}$$

$$0 \leqq x < +\infty, \quad 0 \leqq t < +\infty;$$

the solution U satisfies the initial condition $U(x, 0) = f(x)$ and the boundary condition $f'(0) = 0 = U_x(0, t)$.

Such equations occur in the theory of probability. For example, when $\gamma = \frac{1}{2}(n - 1)$, equation (1) is the backward differential equation corresponding to the radial component of n-dimensional Brownian motion (see [3]). For other values of γ, equation (1) is the backward differential equation corresponding to a stochastic process which is the limit of a sequence of random walks (see [5]).

It is easily checked that the function $V(x, t) = U(\sqrt{x}, \frac{1}{2}t)$ satisfies the following degenerate elliptic-parabolic equation:

$$\begin{aligned} V_t(x, t) &= xV_{xx}(x, t) + aV_x(x, t), \\ V(x, 0) &= g(x) = f(\sqrt{x}), \end{aligned} \tag{2}$$

where $a = \gamma + \frac{1}{2} > 0$.

In [1], Feller investigated a class of degenerate elliptic-parabolic equations which includes our equation (2). However, Feller discussed only the existence and uniqueness of solutions to equation (2), he did not study the differentiability properties of their solutions and this is our main concern here.

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Before stating our results it will be necessary to introduce some notation and definitions. Let

$$R_+ = \{x\colon 0 < x < +\infty\} \quad \text{and} \quad \bar{R}_+ = \{x\colon\ 0 \leqq x < +\infty\}\,.$$

We define B as follows:

$$B = \{f(x)\colon f \text{ is a bounded continuous function on } \bar{R}_+\}\,, \tag{3}$$

$$B^k = \{f(x)\colon f^{(l)}(x) \in B,\ 0 \leqq l \leqq k\}\,, \tag{4}$$

where $f^{(l)}$ denotes the l-th derivative of the function f. We define norms $\|f\|_k$ on the spaces B^k as

$$\|f\|_k = \sum_{l=0}^{k} \|f^{(l)}\|\,, \tag{5}$$

where $\|f\| = \max_{0 \leqq x < +\infty} |f(x)|$.

It is understood that $B^0 = B$ and $\|f\|_0 = \|f\|$. With these norms the spaces B^k, as is well known, become Banach spaces. The spaces B_0 and B_0^k play an important role in our work and are defined below:

$$\begin{aligned} B_0 &= \left\{f(x)\colon f \in B \text{ and } \lim_{x\to\infty} f(x) = 0\right\}, \\ B_0^k &= \{f(x)\colon f^{(l)}(x) \in B_0\,, 0 \leqq l \leqq k\}\,. \end{aligned} \tag{6}$$

We define $C(\bar{R}_+)$, $C(R_+)$, $C^k(\bar{R}_+)$, $C^k(R_+)$ in the following way:

$$\begin{aligned} &\text{(i) } C(\bar{R}_+) = \{f(x)\colon f \text{ continuous for } x \in \bar{R}_+\}\,, \\ &\text{(ii) } C^k(\bar{R}_+) = \{f(x)\colon f^{(l)}(x) \in C(\bar{R}_+),\ 0 \leqq l \leqq k\}\,, \\ &\text{(iii) } C(R_+) = \{f(x)\colon f \text{ continuous for } x \in R_+\}\,, \\ &\text{(iv) } C^k(R_+) = \{f(x)\colon f^{(l)}(x) \in C(R_+),\ 0 \leqq l \leqq k\}\,. \end{aligned} \tag{7}$$

Our results for equation (2) are obtained by first studying properties of the solution ϕ to the stationary equation

$$x\phi''(x) + a\phi'(x) - \lambda\phi(x) = f(x)\,, \qquad \lambda > 0,\ f \in B\,. \tag{8}$$

The results obtained concerning equation (8) are derived so as to enable us to apply the methods of the Hille-Yosida theorem to equation (2). In particular we obtain the following result (Theorem 6):

We denote by A the second order linear differential operator

$$Ag(x) = xg''(x) + ag'(x)\,. \tag{9}$$

We then show that A is the infinitesimal generator of a contraction semigroup $T(t)\colon B_0^k \to B_0^k$ with respect to the norm $\|f\|_k$, i.e., $\|T(t)f\|_k \leqq \|f\|_k$. From these estimates we deduce the differentiability properties of solutions to equation (2).

2. The Stationary Equation

THEOREM 1. *Let* $a > 0$, $\lambda > 0$, $f \in B$. *Then there exists a unique function* ϕ *satisfying*

$$\phi \in C^2(R_+) \cap C^1(\bar{R}_+) \cap B\,, \tag{10}$$

$$x\phi''(x) + a\phi'(x) - \lambda\phi(x) = (A - \lambda)\phi(x) = f(x)\,, \qquad 0 \leqq x < +\infty\,, \tag{11}$$

$$\lim_{x\to 0} x\phi''(x) = 0\,. \tag{12}$$

In addition, we have

$$\|\phi\| \leqq \frac{\|f\|}{\lambda}\,, \tag{13}$$

$$\phi \in B^1 \quad \textit{and} \quad \|\phi'\| \leqq \frac{2}{a}\|f\|\,. \tag{14}$$

The proof of Theorem 1 is divided into four lemmas.

LEMMA 1. *Let us assume that there exists a* ϕ *satisfying conditions* (10), (11), (12) *of Theorem* 1. *Then we have*

$$\|\phi\| \leqq \frac{\|f\|}{\lambda}\,.$$

Proof: We introduce the auxiliary function $\psi(x) = \phi(x)/(1 + \varepsilon x)$, $\varepsilon > 0$. One easily checks that $\lim_{x\to 0} x\psi''(x) = 0$ and that $\lim_{x\to\infty} \psi(x) = 0$. A routine calculation shows that ψ satisfies the differential equation

$$x(1 + \varepsilon x)\psi''(x) + (2\varepsilon x + a(1 + \varepsilon x))\psi'(x) + (\varepsilon a - \lambda - \varepsilon\lambda x)\psi(x) = f(x)\,. \tag{15}$$

Clearly, if $\max_{0\leqq x<+\infty} \psi(x) = \lim_{x\to\infty} \psi(x)$, then

$$\max_{0\leqq x<+\infty} \psi(x) \leqq \lim_{x\to\infty} \psi(x) = 0$$

and hence

$$\psi(x) \leqq 0 \leqq \frac{\|f\|}{\lambda}\,.$$

We can assume, therefore, that ψ achieves its maximum at a point x_0, $0 \leqq x_0 < +\infty$.

Case 1: $0 < x_0 < +\infty$. Then $\psi'(x_0) = 0$, $\psi''(x_0) \leqq 0$, and hence

$$(\varepsilon a - \lambda - \varepsilon\lambda x_0)\psi(x_0) \geqq f(x_0) .$$

For $\varepsilon < \lambda/a$ this yields the inequality

$$\psi(x_0) \leqq \frac{-f(x_0)}{\lambda - a\varepsilon + \varepsilon\lambda x_0} \leqq \frac{\|f\|}{\lambda - a\varepsilon} .$$

Case 2: $x_0 = 0$. Letting $x \to 0$ in (15) we get $a\psi'(0) + (\varepsilon a - \lambda)\psi(0) = f(0)$. Note that if the maximum of ψ occurs at 0, then $\psi'(0) \leqq 0$ and we can conclude, as in case 1, that

$$\psi(0) \leqq \frac{-f(0)}{\lambda - a\varepsilon} \leqq \frac{\|f\|}{\lambda - a\varepsilon}, \qquad \varepsilon < \frac{\lambda}{a} .$$

Since $\psi(x) \leqq \psi(0)$, we have $\psi(x) \leqq \|f\|/(\lambda - a\varepsilon)$, $\varepsilon < \lambda/a$. Thus

$$\phi(x) \leqq (1 + \varepsilon x)\frac{\|f\|}{\lambda - a\varepsilon} \qquad \text{for} \qquad 0 \leqq x < +\infty , \qquad \varepsilon < \frac{\lambda}{a} .$$

For fixed x, we let $\varepsilon \to 0$ and see that $\phi(x) \leqq \|f\|/\lambda$.

The same argument applied to $-\phi$ and $-f$ yields the lower bound

$$\phi(x) \geqq -\|f\|/\lambda ,$$

and this completes the proof of Lemma 1.

We now study solutions to the inhomogeneous equation

$$(A - \lambda)\phi = f, \tag{16}$$

where $f \in B$.

Let $y_0(x; \lambda)$ denote the regular power series solution to the homogeneous equation

$$(A - \lambda)y_0(x; \lambda) = 0 . \tag{17}$$

One easily checks that

$$y_0(x; \lambda) = 1 + \sum_{i=1}^{\infty} c_i x^i , \tag{18}$$

where

$$c_i = \frac{\lambda^i}{a(a+1)\cdots(a+i-1)i!}, \qquad i = 1, 2, 3, \cdots .$$

We note for future reference that $y_0(x; \lambda) \in C^\infty(\bar{R}_+)$ and that $y_0(x; \lambda) \geqq 1$ for $0 \leqq x < +\infty$.

Since λ will be held constant throughout the ensuing discussion, we shall write $y_0(x)$ for $y_0(x; \lambda)$ in order to simplify our notation.

DEFINITION. Let $\phi(x; T)$ denote the function

$$\phi(x; T) = -y_0(x) \int_x^T z_0(s)\, ds \,. \tag{19}$$

where

$$z_0(s) = s^{-a} y_0^{-2}(s) \int_0^s \tau^{a-1} y_0(\tau) f(\tau)\, d\tau \,, \tag{20}$$

and T lies in the interval $0 < T < +\infty$.

LEMMA 2. *The function $\phi(x; T)$ satisfies the equation*

$$(A - \lambda)\phi(x; T) = f(x) \,, \qquad 0 \leqq x \leqq T \,, \tag{21}$$

together with the boundary conditions:

(i) $\lim_{x\to 0} x\phi''(x; T) = 0$

and

(ii) $\phi(T; T) = 0$.

We also have the estimate

$$\max_{0 \leqq x \leqq T} |\phi(x; T)| \leqq \frac{\|f\|}{\lambda} \,.$$

Proof: We observe that the explicit form of the solution (19) is obtained by the "reduction of order method", specifically we assume that $\phi(x; T) = \eta(x) y_0(x)$, where η is an unknown function. This leads to a linear first order equation for η' which may be explicitly solved by quadratures.

The reader may verify by direct calculation that $\phi(x; T)$ satisfies (21). Postponing for the moment the verification of the fact that ϕ satisfies the boundary condition (i) ((ii) is trivial to check) we note that the method of Lemma 1 applied to ϕ yields $\max_{0 \leqq x \leqq T} |\phi(x; T)| \leqq \|f\|/\lambda$. Condition (i) is a consequence of the following more general result:

LEMMA 3. *Let $r \in C^k(\bar{R}_+)$ and set*

$$h(x) = x^{-a} \int_0^x s^{a-1} r(s)\, ds \,. \tag{22}$$

Then

(i) $h \in C^k(\bar{R}_+) \cap C^{k+1}(R_+)$

and

(ii) $\lim_{x \to 0} x h^{(k+1)}(x) = 0$.

Proof: The change of variable $u = s/x$ transforms (22) into

$$h(x) = \int_0^1 u^{a-1} r(ux) \, du \,. \tag{23}$$

Since $a > 0$, differentiating under the integral sign of (23) yields

$$h^{(l)}(x) = \int_0^1 u^{a+l-1} r^{(l)}(xu) \, du \,, \qquad 0 \leqq l \leqq k \,. \tag{24}$$

This shows that $h \in C^k(\bar{R}_+)$ and that

$$\lim_{x \to 0} h^{(k)}(x) = \frac{r^{(k)}(0)}{a + k} \,. \tag{25}$$

The change of variable $u = s/x$ applied to (24) yields

$$h^{(k)}(x) = x^{-a-k} \int_0^x s^{a+k-1} r^{(k)}(s) \, ds \,. \tag{26}$$

From this one concludes immediately that $h \in C^{k+1}(R_+)$ and that

$$h^{(k+1)}(x) = \frac{r^{(k)}(x)}{x} - \frac{a + k}{x^{a+k+1}} \int_0^x s^{a+k-1} r^{(k)}(s) \, ds \,. \tag{27}$$

The fact that $r \in C^k(\bar{R}_+)$ implies that given $\varepsilon > 0$ there exists a $\delta > 0$ such that, for $0 \leqq x < \delta$, we have $|r^{(k)}(x) - r^{(k)}(0)| < \varepsilon$. Multiplying both sides of (27) by x and using the triangle inequality we see that

$$\begin{aligned} |x h^{(k+1)}(x)| &\leqq |r^{(k)}(x) - r^{(k)}(0)| + \frac{a + k}{x^{a+k}} \int_0^x s^{a+k-1} \, |r^{(k)}(s) - r^{(k)}(0)| \, ds \\ &\leqq \varepsilon + \varepsilon = 2\varepsilon \qquad \text{for} \qquad 0 \leqq x < \delta \,. \end{aligned}$$

This completes the proof of (ii). We now return to the proof of (i) of Lemma 2. We differentiate (19) twice and get

$$\phi''(x; T) = -y_0''(x) \int_x^T z_0(s) \, ds + 2 y_0'(x) z_0(x) + y_0(x) z_0'(x) \,.$$

Applying Lemma 3 to the case where $r(x) = y_0(x)f(x)$ and $h(x) = y_0^2(x)z_0(x)$, the result $\lim_{x\to 0} x\phi''(x; T) = 0$ follows at once.

LEMMA 4.

(i) $\phi(x) = \lim_{T\to\infty} \phi(x; T) = -y_0(x)\int_x^\infty z_0(s)\, ds$

exists for all $x \in \bar{R}_+$,

(ii) $\|\phi\| \leqq \dfrac{\|f\|}{\lambda}$,

(iii) $(A - \lambda)\phi = f$ *and* $\lim_{x\to 0} x\phi''(x) = 0$,

(iv) $\|\phi'\| \leqq \dfrac{2\,\|f\|}{a}$.

Proof: Let

$$z_{01}(s) = s^{-a}\, y_0^{-2}(s)\int_0^s \tau^{a-1}\, y_0(\tau)\, d\tau ,$$

i.e., z_{01} is z_0 when $f \equiv 1$. Since, on $\bar{R}_+$, $y_0(x) \geqq 1$, we see that for $0 \leqq x \leqq T$

$$\left|\int_x^T z_{01}(s)\right| \leqq \left| y_0(x)\int_x^T z_{01}(s)\, ds\right| \leqq \frac{1}{\lambda}, \tag{28}$$

the last inequality being a consequence of Lemma 2 for the case $f \equiv 1$. Since $z_{01}(s) \geqq 0$ on $\bar{R}_+$, we conclude from (28) that

$$z_{01}(s) \in L_1(\bar{R}_+) \qquad \text{and} \qquad \int_0^\infty |z_{01}(s)|\, ds \leqq \frac{1}{\lambda}.$$

In general, $|z_0(s)| \leqq \|f\|\,|z_{01}(s)|$ and we have $z_0(s) \in L_1(\bar{R}_+)$ with

$$\int_0^\infty |z_0(s)|\, ds \leqq \frac{\|f\|}{\lambda}.$$

Thus,

$$\lim_{T\to\infty} \phi(x; T) = \lim_{T\to\infty} -y_0(x)\int_x^T z_0(s)\, ds = -y_0(x)\int_x^\infty z_0(s)\, ds ,$$

and this completes the proof of (i).

Let $\beta(T) = -\int_T^\infty z_0(s)\, ds$ and observe that $\lim_{T\to\infty} \beta(T) = 0$. One can represent $\phi(x)$ as

$$\phi(x) = \phi(x; T) + \beta(T)y_0(x) \qquad \text{for} \qquad 0 \leqq x \leqq T. \tag{29}$$

Since T can be chosen arbitrarily large, we conclude that $(A - \lambda)\phi = f$. We have already shown that $\lim_{x\to 0} x\phi''(x; T) = 0$ and so (29) implies that $\lim_{x\to 0} x\phi''(x) = 0$ also.

The inequality (ii) is an immediate consequence of the estimate

$$\max_{0\leqq x\leqq T} |\phi(x; T)| \leqq \frac{\|f\|}{\lambda}.$$

This completes the verification of (i), (ii) and (iii) and the only thing left to prove is (iv).

As a first step we prove that

$$\max_{0\leqq x\leqq T} |\phi'(x; T)| \leqq \frac{2\,\|f\|}{a}.$$

We shall estimate $\phi'(x; T)$ for $0 \leqq x < T$ and $\phi'(T; T)$, separately. We have

$$\phi'(T; T) = \lim_{x\to T} \frac{\phi(x; T)}{x - T}.$$

We introduce the auxiliary function $\psi(x) = \phi(x; T) + l(T - x)$, where l is a fixed constant to be chosen below. An elementary computation shows that ψ satisfies the equation

$$x\psi''(x) + a\psi'(x) - \lambda\psi(x) = f(x) - al - \lambda l(T - x). \tag{30}$$

Choosing $l = \|f\|/a$, we conclude from this that

$$f(x) - al - \lambda l(T - x) \leqq 0 \qquad \text{for} \qquad 0 \leqq x \leqq T.$$

Now $\psi(T) = 0$ and it is clear from the equation (30) and our choice of l that ψ cannot have a negative minimum. Hence, $\psi(x) \geqq 0$ for $0 \leqq x \leqq T$. This implies that $\phi(x; T)/(x - T) \leqq l = \|f\|/a$, $0 \leqq x \leqq T$. Choosing $l = -\|f\|/a$, one concludes that ψ cannot have a positive maximum and therefore we obtain the lower bound $\phi(x; T)/(x - T) \geqq -l = -\|f\|/a$, $0 \leqq x \leqq T$. This yields $|\phi'(T; T)| \leqq \|f\|/a$. We next estimate $\phi'(x; T)$ for $0 \leqq x \leqq T$. Let $w(x) = \phi'(x; T)$, then w satisfies the equation

$$xw'(x) + aw(x) = f(x) + \lambda\phi(x; T). \tag{31}$$

The argument of Lemma 1 applied to (31) yields

$$\max_{0\leqq x\leqq T} |w(x)| \leqq \frac{1}{a} \max_{0\leqq x\leqq T} |f(x) + \lambda\phi(x; T)| < \frac{2\,\|f\|}{a}.$$

We have thus shown that

$$\max_{0 \leqq x \leqq T} |\phi'(x; T)| \leqq \frac{2 \|f\|}{a} .$$

From (29) we have $\phi'(x) = \phi'(x; T) + \beta(T) y_0'(x)$, and since $\lim_{T\to\infty} \beta(T) = 0$, we see that $\lim_{T\to\infty} \phi'(x; T) = \phi'(x)$, from which we deduce the estimate (iv).

DEFINITION. For given $\alpha \geqq 0$, $a > 0$, we set

$$\mu(\alpha, a) = \begin{cases} 0 & \text{if} \quad \alpha - a + 1 \leqq 0 , \\ \dfrac{\alpha(\alpha - a + 1)^2}{4(\alpha + 1)} & \text{if} \quad \alpha - a + 1 > 0 . \end{cases} \tag{32}$$

THEOREM 2. *Assume* $(1 + x)^\alpha f(x) \in B$ *and* $\lambda > \mu(\alpha, a)$. *Then for the function* ϕ *satisfying* (10), (11), (12) *of Theorem* 1 *the inequality*

$$\|(1 + x)^\alpha \phi(x)\| \leqq \frac{\|(1 + x)^\alpha f(x)\|}{\lambda - \mu(\alpha, a)} \tag{33}$$

holds.

Proof: One checks in a straight-forward manner that if $\alpha - a + 1 \leqq 0$, then

$$\left(\frac{\alpha(\alpha + 1)x}{(1 + x)^2} - \frac{\alpha a}{1 + x}\right) \leqq 0 \quad \text{for all} \quad x \geqq 0 .$$

Similarly, if $\alpha - a + 1 > 0$, then we have

$$\max_{x \geqq 0} \left(\frac{\alpha(\alpha + 1)x}{(1 + x)^2} - \frac{\alpha a}{1 + x}\right) = \mu(\alpha, a) .$$

Now let $\psi(x) = (1 + x)^\alpha \phi(x; T)$, $0 \leqq x \leqq T$, where $\phi(x; T)$ is defined as in (19). We note that $\lim_{x\to 0} x\psi''(x) = 0$ and $\psi(T) = 0$. Moreover, ψ satisfies the following equation:

$$x\psi''(x) + \psi'(x)\left\{a - \frac{2\alpha x}{1 + x}\right\} + \psi(x)\left\{\frac{\alpha(\alpha + 1)x}{(1 + x)^2} - \frac{\alpha a}{1 + x} - \lambda\right\} = (1 + x)^\alpha f(x) . \tag{34}$$

Proceeding as in Lemma 1 and choosing $\lambda > \mu(\alpha, a)$, we conclude that

$$\max_{0 \leqq x \leqq T} |\psi(x)| \leqq \frac{\|(1 + x)^\alpha f(x)\|}{\lambda - \mu(\alpha, a)} ,$$

i.e.,

$$\max_{0 \leqq x \leqq T} |(1 + x)^\alpha \phi(x; T)| \leqq \frac{\|(1 + x)^\alpha f(x)\|}{\lambda - \mu(\alpha, a)} .$$

To complete the proof we let $T \to \infty$.

THEOREM 3. *Assume* $f \in B^k$, $k \geqq 0$, *and let* ϕ *denote the unique solution to* $(A - \lambda)\phi = f$ *satisfying the condition* $\lim_{x\to 0} x\phi''(x) = 0$. *Then we have*

$$\phi \in C^{k+2}(R_+) \cap B^{k+1}, \qquad \|\phi\|_k \leqq \frac{\|f\|_k}{\lambda}, \tag{35}$$

$$x\phi''(x) \in B^k, \tag{36}$$

$$\lim_{x\to 0} x\phi^{(k+2)}(x) = 0. \tag{37}$$

Moreover, if we set $w_l(x) = \phi^{(l)}(x)$, *then*

$$xw_l''(x) + (a + l)w_l'(x) - \lambda w_l(x) = f^{(l)}(x), \qquad 0 \leqq l \leqq k. \tag{38}$$

Thus, $\phi^{(l)}(x)$ *satisfies equation* (8) *with* $a + l$ *in place of a, and hence*

$$\|\phi^{(l+1)}(x)\| = \|w_l'(x)\| \leqq \frac{2}{a + l}\|f^{(l)}(x)\|, \qquad 0 \leqq l \leqq k. \tag{39}$$

If, in addition, $(1 + x)^\alpha f^{(k)}(x) \in B$, *then*

$$(1 + x)^\alpha \phi^{(k)}(x) \in B \quad \text{and} \quad \|(1 + x)^\alpha \phi^{(k)}(x)\| \leqq \frac{\|(1 + x)^\alpha f^{(k)}(x)\|}{\lambda - \mu(\alpha, a + k)} \tag{40}$$

for $\lambda > \mu(\alpha, a + k)$.

Proof: Theorem 1 corresponds to the case $k = 0$. We proceed in the general case by mathematical induction. We assume then that Theorem 3 is true for the case $k = n$ and we shall show that it is true for the case $k = n + 1$.

If $f \in B^{n+1}$, then *a fortiori* $f \in B^n$ and hence, by our induction hypothesis, $\phi \in B^{n+1} \cap C^{n+2}(R_+)$. We also have from the equation (8) that

$$\phi''(x) = \frac{f(x) + \lambda\phi(x) - a\phi'(x)}{x} \in C^{n+1}(R_+)$$

and so

$$\phi \in C^{n+3}(R_+) \cap B^{n+1}.$$

This means that the equation $(A - \lambda)\phi = f$ may be differentiated $(n + 1)$ times to yield

$$x\phi^{(n+3)}(x) + (a + n + 1)\phi^{(n+2)}(x) - \lambda\phi^{(n+1)}(x) = f^{(n+1)}(x). \tag{41}$$

In terms of $w_{n+1}(x)$, this last equation can be written as

$$w_{n+1}''(x) + (a + n + 1)w_{n+1}'(x) - \lambda w_{n+1}(x) = f^{(n+1)}(x) \in B \tag{42}$$

Our induction hypothesis implies that $xw_n''(x) + a_n w_n'(x) - \lambda w_n(x) = f^{(n)}(x)$ and also that $\lim_{x\to 0} xw_n''(x) = 0$, $w_n \in B^1 \cap C^2(R_+)$. Thus,

$$w_n(x) = -y_n(x)\int_x^\infty z_n(s)\, ds\,,$$

where

$$z_n(s) = s^{-(a+n)}\, y_n^{-2}(s)\int_0^s \tau^{a+n-1}\, y_n(\tau) f^{(n)}(\tau)\, d\tau\,,$$

with $xy_n''(x) + (a+n)y_n'(x) - \lambda y_n(x) = 0$, or equivalently $y_n(x) = y_0^{(n)}(x)$. From the assumption $f^{(n)} \in B^1$ it follows, by Lemma 3, that $z_n \in C^1(\bar{R}_+) \cap C^2(R_+)$ and hence $w_n \in C^2(\bar{R}_+) \cap C^3(R_+)$ and also that

$$\lim_{x\to 0} xw_n^{(3)}(x) = \lim_{x\to 0} xw_{n+1}^{(2)}(x) = 0\,.$$

We may thus apply Theorem 1 to equation (42) (with a replaced by $a+n+1$) and conclude that

(i) $\|w_{n+1}\| \leqq \dfrac{\|f^{(n+1)}\|}{\lambda}$,

(ii) $\|w_{n+1}'\| = \|\phi^{(n+2)}\| \leqq \dfrac{2}{a+n+1}\, \|f^{(n+1)}\| \leqq \dfrac{2\,\|f^{(n+1)}\|}{a}$.

Estimate (i) together with the inductive hypothesis that $\|\phi\|_n \leqq \|f\|_n/\lambda$ imply that $\|\phi\|_{n+1} \leqq \|f\|_{n+1}/\lambda$ and this completes the proof of statements (35) through (39). Statement (40) is a consequence of Theorem 2 with $a+k$ replacing a.

THEOREM 4. *Assume* $f \in B_0^k$; *then* $\phi \in B_0^{k+1}$, ϕ *being the same as in Theorem* 3.

Proof: By equation (38) of Theorem 3, it is enough to prove that, if $f \in B_0$, then $\phi \in B_0^1$. Let f_n denote a sequence of $C^1(\bar{R}_+)$ functions with compact support such that $\lim_{n\to\infty} \|f_n - f\| = 0$, and let ϕ_n denote the corresponding solutions to $(A-\lambda)\phi_n = f_n$. Since $\lim_{\alpha\to 0} \mu(\alpha, a) = 0$, we can choose, for any $\lambda > 0$, an $\alpha > 0$ small enough so that $\mu(\alpha, a) < \lambda$. By Theorems 2 and 3 we know that $(1+x)^\alpha \phi_n(x)$ and $(1+x)^\alpha \phi_n'(x)$ are in B. Thus, $\phi_n \in B_0^1$. From (13) and (14) we have

$$\|\phi_n - \phi\| \leqq \frac{\|f_n - f\|}{\lambda}\,,$$

$$\|\phi_n' - \phi'\| \leqq \frac{2\,\|f_n - f\|}{a}\,.$$

Consequently, $\lim_{n\to\infty} \|\phi_n - \phi\|_1 = 0$ and since $\phi_n \in B_0^1$, we see that $\phi \in B_0^1$.

THEOREM 5. *Assume* $(1 + x)^\alpha f(x) \in B_0$, *for some* $\alpha \geqq 0$. *Then*

$$(1 + x)^\alpha \phi(x) \in B_0 ,$$

where ϕ *is the unique solution to* $(A - \lambda)\phi = f$, $\lambda > \mu(\alpha, a)$, *satisfying*

$$\lim_{x\to 0} x\phi''(x) = 0 .$$

Proof: Let f_n denote a sequence of continuous functions with compact support such that $\lim_{n\to\infty} \|(1 + x)^\alpha(f_n(x) - f(x))\| = 0$, and let ϕ_n denote the solutions corresponding to (8) with properties (10), (11), (12). Then, as is easily shown, $(1 + x)^\alpha\phi_n(x) \in B_0$, and so by (33) we have

$$\|(1 + x)^\alpha(\phi_n(x) - \phi(x))\| \leqq \frac{\|(1 + x)^\alpha(f_n(x) - f(x))\|}{\lambda - \mu(\alpha, a)} .$$

Hence, $\lim_{n\to\infty} \|(1 + x)^\alpha(\phi_n(x) - \phi(x))\| = 0$ and therefore $(1 + x)^\alpha\phi(x) \in B_0$.

3. Application to the Evolution Equation (2)

The results we have proved in Theorems 1–5 can be summarized in the following way:

Let $D_k(A)$ denote the set of functions with the properties

(43)
$$g \in C^{k+2}(R+) \cap B_0^{k+1} ,$$
$$Ag \in B_0^k ,$$
$$\lim_{x\to 0} xg^{(k+2)}(x) = 0 .$$

For all $f \in B_0^k$, there exists a unique $g \in D_k(A)$, denoted by $R(\lambda, A)f = g$, satisfying

(44) $(\lambda - A)g = f,$ so that $R(\lambda, A) = (\lambda - A)^{-1}$,

(45) $D_k(A)$ is dense in B_0^k,

(46) A maps $D_k(A)$ into B_0^k.

(47) For every $\lambda > 0$, $\lambda - A$ is a one-to-one map from $D_k(A)$ onto B_0^k with $\lambda \|R(\lambda, A)f\|_k \leqq \|f\|_k$.

Moreover, for $g \in D_k(A)$, we have

$$\|g'\|_k \leqq \frac{2}{a} \|Ag\|_k , \tag{48}$$

$$\|xg''(x)\|_k \leqq 3 \|Ag\|_k . \tag{49}$$

If $f \in B^k$ and $(1 + x)^\alpha f^{(k)}(x) \in B$, we define

$$\|f\|_{k,\alpha} = \|f\|_k + \|(1 + x)^\alpha f^{(k)}(x)\| . \tag{50}$$

Finally, we have

$$\|R(\lambda, A)f\|_{k,\alpha} \leqq \frac{\|f\|_{k,\alpha}}{\lambda - \mu(\alpha, a + k)}, \qquad \lambda > \mu(\alpha, a + k) , \tag{51}$$

for $f \in B_0^k$ such that $(1 + x)^\alpha f^{(k)}(x) \in B$.

Assertions (44), (46), (47) and (51) follow at once from Theorems 1–5, while (45) is obvious. We need to check only (48) and (49).

Let $Ag = f$; then $(A - \lambda)g = f - \lambda g$ and, by Theorem 3,

$$\|g'\|_k \leqq \frac{2 \|f - \lambda g\|_k}{a} .$$

This is true for every $\lambda > 0$ so letting $\lambda \to 0$ we obtain

$$\|g'\|_k \leqq \frac{2 \|f\|_k}{a} = \frac{2 \|Ag\|_k}{a} .$$

Since $xg''(x) = Ag(x) - ag'(x)$, we conclude that

$$\|xg''(x)\| \leqq \|Ag\|_k + a\|g'\|_k \leqq 3\|Ag\|_k .$$

Results (44) through (51) imply

THEOREM 6.

(52) *A is the infinitesimal generator of a strongly continuous semigroup of contraction operators* $T(t) : B_0^k \to B_0^k$, *i.e.*,

$$\| T(t)g\|_k \leqq \|g\|_k, \quad g \in B_0^k, \quad \textit{with domain } D_k(A).$$

(53) *If* $g \in D_k(A)$, *then* $V(x, t) = T(t)g(x)$ *is the unique solution to equation* (2) *with the following properties*:

$$\sum_{l=0}^{k} \left\| \frac{\partial^l V(x, t)}{\partial x^l} \right\| = \|V(\cdot, t)\|_k \leqq \|g\|_k \quad \textit{for all} \quad t \geqq 0\,, \tag{54}$$

$$\max_{x \geqq 0} \left| \frac{\partial^{k+1} V(x, t)}{\partial x^{k+1}} \right| \leqq \frac{2}{a} \|Ag\|_k \qquad \textit{for all} \quad t \geqq 0\,, \tag{55}$$

$$\max_{x \geqq 0} x \left| \frac{\partial^{k+2} V(x, t)}{\partial x^{k+2}} \right| \leqq 3 \|Ag\|_k \qquad \textit{for all} \quad t \geqq 0\,, \tag{56}$$

Finally if $g \in B_0^k$ *and* $(1 + x)^\alpha g^k(x) \in B_0$, *then*

$$\max_{x \geqq 0} \left| (1 + x)^\alpha \frac{\partial^k T(t) g(x)}{\partial x^k} \right| \leqq e^{\mu t} \|g\|_{k,\alpha}\,, \qquad \mu = \mu(\alpha, a + k)\,. \tag{57}$$

Proof: We note that (44) through (47) assert that A satisfies the conditions of the Hille-Yosida theorem and hence propositions (52) through (54) are an immediate consequence of that result.

If $g \in D_k(A)$, then $T(t)g(x) \in D_k(A)$ and, therefore, by (48),

$$\begin{aligned} \left\| \frac{\partial^{k+1} T(t) g(x)}{\partial x^{k+1}} \right\| &\leqq \frac{2}{a} \| AT(t) g(x) \|_k \\ &= \frac{2}{a} \|T(t) Ag(x)\|_k \\ &\leqq \frac{2}{a} \|Ag\|_k\,. \end{aligned}$$

In a similar way (56) is a consequence of (49). The only thing left to prove is (57). We do this by showing that $e^{-\mu t} T(t)$ is a strongly continuous semigroup of contraction operators on the Banach space $F_0(k, \alpha)$, where

$$F_0(k, \alpha) = \{f : f \in B_0^k\,,\ (1 + x)^\alpha f^{(k)}(x) \in B_0\} \tag{58}$$

with norm $\|f\|_{k,\alpha}$ defined by (50). Set

$$\Omega = A - \mu\,, \tag{59}$$

where $\mu = \mu(\alpha,\ a + k)$; thus $\lambda - \Omega = (\mu + \lambda) - A$. It is easily checked, using Theorems 3 and 5, that Ω satisfies the conditions of the Hille-Yosida theorem and is therefore the infinitesimal generator of the contraction semigroup

$$e^{-\mu t} T(t) : F_0(k, \alpha) \to F_0(k, \alpha)\,.$$

This proves (57) because we are assuming there that $g \in F_0(k, \alpha)$ and hence, by what we have just shown,

$$\|e^{-\mu t}T(t)g(x)\|_{k,\alpha} \leqq \|g\|_{k,\alpha} \qquad \text{or} \qquad \|T(t)g(x)\|_{k,\alpha} \leqq e^{\mu t}\|g\|_{k,\alpha}$$

which implies (57).

The following special case is of interest in probability theory and numerical analysis and explains the introduction of the spaces $F_0(k, \alpha)$. Our original goal, it should be recalled, was to solve equation (1). If $g(x) = f(\sqrt{x}) \in D_k(A)$, then $U(x, t) = V(x^2, 2t)$ is the unique solution to equation (1). An elementary computation yields

$$U_{xxx}(x, t) = 8x^3 V_{xxx}(x^2, 2t) + 12xV_{xx}(x^2, 2t) . \tag{60}$$

For the applications to probability and numerical analysis (cf. [6], [8]) it is important to obtain bounds on $\|U_{xxx}(x, t)\|$. In estimate (57) we set $\alpha = \frac{3}{2}$ and $k = 3$ and note that $\frac{3}{2} - (a + 3) + 1 = -(a + \frac{1}{2}) < 0$ and hence $\mu(\frac{3}{2}, a + 3) = 0$. We are now ready to bound the right-hand side of (60) for $g \in D_3(A) \cap F_0(3, \frac{3}{2})$:

$$\begin{aligned} \|xV_{xx}(x^2, 2t)\| &\leqq \max_{0 \leqq x \leqq 1} \|V_{xx}(x^2, 2t)\| + \max_{1 \leqq x \leqq \infty} \|x^2 V_{xx}(x^2, 2t)\| \\ &\leqq \|g\|_3 + 3\,\|Ag\| , \end{aligned} \tag{61}$$

where the last inequality is a consequence of (54) and (56). Also,

$$\begin{aligned} \|x^3 V_{xxx}(x^2, 2t)\| &\leqq \|(1 + x^2)^{3/2}\, V_{xxx}(x^2, 2t)\| \\ &\leqq \|(1 + x)^{3/2}\, V_{xxx}(x, 2t)\| \\ &\leqq \|g\|_{3,3/2} , \qquad \text{by (57)} . \end{aligned} \tag{62}$$

Combining estimates (61) and (62) we conclude that

$$\|U_{xxx}(x, t)\| \leqq 2\,\|g\|_{3,3/2} + 3\,\|Ag\| \tag{63}$$

for $f(\sqrt{x}) = g(x) \in D_3(A) \cap F_0(3, \frac{3}{2})$.

4. Concluding Remarks

a. It should be noted that equation (2) does not fall under the theory of degenerate elliptic-parabolic equations as established by Kohn-Nirenberg [4] or O. Oleinik [7].

Finally we note also that equation (1) is not the only equation that can be reduced to (2) by a change of variable. Consider, for example, the equation

$$U_t(x, t) = c_1\, x^{2-\alpha}\, U_{xx}(x, t) + c_2\, x^{1-\alpha}\, U_x(x, t) ,$$

where $\alpha > 0$, $c_2 > (1 - \alpha)c_1$. The transformation $V(x, t) = U(x^{1/\alpha}, ct)$, where $c = (c_1 \alpha^2)^{-1}$, reduces it to $V_t = xV_{xx} + bV_x$, where $b = \alpha c(c_1(\alpha - 1) + c_2)$, and this is of course equation (2) (cf. [9]).

b. The problem of obtaining invariance principles for Markov chains converging to Markov processes has been studied by Skorokhod in [8]. In this paper he points out the importance of obtaining estimates of the form

$$\max_{0 \leqq t \leqq 1} \left\{ \max_{0 \leqq l \leqq 3} \left\| \frac{\partial^l T(t) g(x)}{\partial x^l} \right\| \right\} \leqq M . \tag{64}$$

It is clear that our Theorem 6 provides a complete answer to this question for the semigroups considered in this paper. The probability applications of the results obtained will appear in a separate publication.

Appendix

by Peter D. Lax

This appendix contains another proof of the differentiability of solutions of the singular parabolic equation

$$u_t = Gu , \qquad t \geqq 0 , \tag{A.1}$$

where

$$G = \tfrac{1}{2}D^2 + \frac{\gamma}{x} D , \qquad D = \frac{\partial}{\partial x} , \quad \gamma = \text{const.}, \quad x > 0 . \tag{A.2}$$

As shown in the body of this paper, the second order operator G (or rather its closure) satisfies the hypotheses of the Hille-Yoshida theorem in the space C of bounded, continuous functions on $R_+ = [0, \infty]$, normed by the maximum norm $\|u\|$. It follows that solutions of (A.1) form a one-parameter semigroup of contractions over C. This semigroup can be denoted symbolically by

$$u(t) = (\exp tG)u_0 , \qquad u_0 = u(0) , \ t \geqq 0 .$$

The contractive property is expressed by

$$\|\exp tG\| \leqq 1 , \qquad t \geqq 0 . \tag{A.3}$$

Since $\exp tG$ commutes with G, we have for any integer k

$$G^k u(t) = (\exp tG) G^k u_0 ; \tag{A.4}$$

the meaning of this relation is that wherever u_0 belongs to the domain of G^k, so does $u(t)$ and (A.4) holds. Applying (A.3) to (A.4) we see that

$$\|G^k u(t)\| \leqq \|G^k u_0\| . \tag{A.5}$$

Our main result is the following:

THEOREM A.1. *For* $\gamma > -\frac{1}{2}$ *and for any integer* k,

$$\|D^{2k} u\| \leqq c\|G^k u\| \tag{A.6}$$

for all u *in the domain of* G^k; *the constant* c *depends on* γ *and* k *but not on* u.

First a word about the domain of G^k; it certainly includes all bounded functions u with bounded and continuous derivatives up to order $2k$ which satisfy

$$(D^j u)(0) = 0\,, \qquad j = 1, 3, 5, \cdots, 2k-1\,. \tag{A.7}$$

Suppose u_0 belongs to this class of functions; then u_0 belongs to the domain of G^k, and by (A.4) so does $u(t)$ for $t > 0$. Combining (A.5) and (A.6), we obtain a bound for $\|D^{2k} u\|$ which is uniform in both t and x. This is the desired differentiability theorem for solutions u of (A.1) whose initial value u_0 is differentiable and satisfies (A.7).

It is not hard to show that G^k is the closure in the graph topology of the differential operator G^k, defined for bounded u and bounded continuous derivatives up to order $2k$, satisfying the boundary condition (A.7). Therefore, it suffices to prove (A.6) for such u.

We shall deduce Theorem A.1 for this class of u from a more general theorem: Let H be a differential operator of order n, with a regular singularity at $x = 0$, of the form

$$H = D^n + \frac{h_1}{x} D^{n-1} + \cdots + \frac{h_n}{x^n}\,; \tag{A.8}$$

$h_1, h_2, \cdots, h_n$ are constants. For any b,

$$Hx^b = P(b)x^{b-n}\,, \tag{A.9}$$

where P is a polynomial of degree n. The roots of P are called the *indices* of H; we shall denote them by $\gamma_1, \cdots, \gamma_n$.

THEOREM A.2. *Suppose all indices of* H *have real parts which are less than* n:

$$\mathcal{R}e\, \gamma_j < n\,, \qquad j = 1, \cdots, n\,. \tag{A.10}$$

Then there exists a constant c *such that*

$$\|D^n u\| \leqq c\|Hu\| \tag{A.11}$$

for all C^n *functions* u *defined on* R_+ *and satisfying*

$$(D^j u)(0) = 0\,, \qquad j = 0, 1, \cdots n-1\,. \tag{A.12}$$

Remark 1. Actually, we are going to prove inequality (A.11) for functions u defined on some finite interval $(0, a)$. By homogeneity, the constant c does not depend on the length of the interval.

Remark 2. If λ is an index of H, then x^λ is annihilated by H; if $\mathcal{R}e\,\lambda$ were greater than or equal to n, $u = x^\lambda$ would be a C^n solution satisfying the initial condition (A.12) but for which (A.11) is false. This shows that (A.10) is necessary for the validity of inequality (A.11).

Remark 3. Theorem A.2 is true also for operators H with variable coefficients h_j ; for, such an operator can be written as

$$H = H_0 + E\,,$$

where H_0 is of the form (A.8) with constant h_j , and E is of the form

$$E = \sum_{j=1}^{n} \frac{l_j(x)}{x^j} D^{n-j}\,, \qquad l_j(0) = 0\,.$$

Using the initial condition (A.12) we can express everything in terms of $D^n u$:

$$\frac{1}{x^j} D^{n-j}u(x) = \frac{1}{(j-1)}\frac{1}{x}\int_0^x \left(1 - \frac{y}{x}\right)^{j-1} D^n u(y)\, dy\,.$$

Substituting these into E and denoting by ε the maximum of $l_j(x)$ over $[0, a]$, we conclude that, over the interval $[0, a]$,

$$\|Eu\| \leqq O(\varepsilon)\|D^n u\|\,. \tag{A.13}$$

By Theorem A.2,

$$\|D^n u\| \leqq c\|H_0 u\|\ ;$$

since $H_0 = H - E$, we see, using the triangle inequality and (A.13), that

$$\|D^n u\| \leqq c\|Hu\| + cO(\varepsilon)\|D^n u\|\,. \tag{A.14}$$

Since $l_j(0) = 0$, $\varepsilon = \max_{0\leqq x\leqq a} |l_j(x)|$ is small for a small; if we choose a so small that $cO(\varepsilon) < \frac{1}{2}$, (A.14) implies the desired inequality (A.11). For larger values of a we can obtain an estimate for $D^n u$ in terms of Hu trivially, since H is a regular differential operator away from $x = 0$; of course the value of the constant c depends in this case on the length a of the interval.

Remark 4. Inequality (A.6) of Theorem A.1 is a special case of inequality (A.11) of Theorem A.2, with G^k in place of H and $n = 2k$. We have to verify that the indices of G^k have real parts less than $n = 2k$. Now,

$$Gx^b = \tfrac{1}{2}b(b-1) + \gamma b x^{b-2} = q(b)x^{b-2}\ ;$$

we deduce recursively that

$$G^k x^b = \prod_{i=0}^{k-1} q(b - 2i) x^{b-2k} .$$

Thus the indices of G^k are roots of

$$q(b - 2i) = 0 , \qquad i = 0, \cdots , k - 1 . \tag{A.15}$$

Since the roots of q are 0 and $1 - 2\gamma$, the roots of (A.15) are

$$2i \quad \text{and} \quad 1 - 2\gamma + 2i , \qquad i = 0, \cdots k - 1 . \tag{A.16}$$

The largest of these is $1 - 2\gamma + 2(k - 1)$; the requirement that this be less than $n = 2k$,

$$2k - 1 - 2\gamma < 2k ,$$

is clearly equivalent with $-\frac{1}{2} < \gamma$.

Actually, we have to prove inequality (A.6) under the initial conditions (A.7) which requires that the odd derivatives of u up to order $2k - 1$ vanish at $x = 0$, whereas (A.11) will be derived under the assumption (A.12) that all derivatives of u up to order $2k$ vanish at $x = 0$. To fill this gap we subtract from u an even polynomial p of degree less than $2k$ such that

$$v = u - p$$

satisfies all the boundary conditions (A.12). Thus, by Theorem A.2,

$$\|D^{2k} v\| \leqq c\|G^k v\| . \tag{A.17}$$

Since p is of degree less than $2k$, $D^{2k} p = 0$ and so

$$D^{2k} v = D^{2k} u .$$

Similarly, since by (A.16) all even integers less than $2k$ are indices of G^k, $G^k p = 0$ and so

$$G^k v = G^k u .$$

Thus the desired inequality (A.7) follows from (A.17)

Remark 5. Theorem 2 is trivial; we make the exponential substitution

$$x = e^{\xi} .$$

We introduce the abbreviation

$$Hu = f \tag{A.18}$$

and denote by ω and ϕ the functions

$$u(e^{\xi}) = \omega(\xi), \qquad f(e^{\xi}) = \phi(\xi) . \tag{A.19}$$

Using

$$\frac{\partial}{\partial x} = \frac{1}{x}\frac{\partial}{\partial \xi},$$

we see that

$$\phi(\xi) = (Hu)(e^{\xi}) = \frac{1}{x^n}\Gamma\omega, \tag{A.20}$$

where Γ is a differential operator with *constant* coefficients. By definition of the indices λ_s, x^{λ_j} is annihilated by H; therefore, $e^{\lambda_j \xi}$ is annihilated by Γ. Next we construct Green's function γ for the transpose Γ^{tr} of Γ, defined as that solution of

$$\Gamma^{\text{tr}}\gamma = 0 \tag{A.21}$$

which satisfies

$$\begin{aligned} \left(\frac{\partial}{\partial \xi}\right)^{j}\gamma(0) &= 0, \qquad j = 0, \cdots, n-2, \\ \left(\frac{\partial}{\partial \xi}\right)^{n-1}\gamma(0) &= 1. \end{aligned} \tag{A.22}$$

Assuming for convenience that the γ_j are distinct, γ is a linear combination:

$$\gamma(\xi) = \sum a_j e^{-\lambda_j \xi}. \tag{A.23}$$

Multiplying (A.20) by $x^n = e^{n\eta}$, we obtain

$$\Gamma\omega(\eta) = e^{n\eta}\phi(\eta).$$

We multiply this equation by $\gamma(\eta - \xi)$ and integrate with respect to η from $-\infty$ to ξ; integrating by parts we obtain

$$\omega(\xi) = \int_{-\infty}^{\xi} \gamma(\eta - \xi)e^{n\eta}\phi(\eta)\,d\eta. \tag{A.24}$$

In deriving this formula we use the fact that γ satisfies the differential equation (A.21) and the initial conditions (A.22). We also use the fact that, according to (A.12), $u(x)$ has a zero of order n at $x = 0$, and so $\omega(\eta)$ is $O(e^{n\eta})$ as $\eta \to -\infty$; on the other hand, we have assumed that $\mathscr{R}e\,\lambda_j < n$, which by (A.23) guarantees that $\gamma(\eta - \xi)$ and its derivatives are $O(e^{-n\eta})$ as $\eta \to -\infty$. Combining these facts we conclude that the contribution of the boundary terms at $-\infty$ is indeed zero.

Now we switch back to $y = e^{\eta}$ and $x = e^{\xi}$ and, using (A.23) and the notation

$$g(y) = \gamma(-\log y) = \sum a_j y^{\lambda_j}, \tag{A.25}$$

we deduce from (A.24) that

$$u(x) = \int_0^x g\left(\frac{x}{y}\right) y^{n-1} f(y)\, dy\,. \tag{A.26}$$

It follows from (A.22) that

$$(D^j g)(1) = 0 \qquad \text{for} \qquad j = 0, 1, \cdots, n-2\,, \tag{A.27}$$

$$(D^{n-1} g)(1) = 1\,.$$

Differentiating (A.26) n times and using (A.27), we get

$$(D^n u)(x) = f(x) + \int_0^x g^{(n)}\left(\frac{x}{y}\right) f(y) \frac{dy}{y}\,.$$

Introducing $y/x = t$ we obtain

$$(D^n u)(x) = f(x) + \int_0^1 g^{(n)}\left(\frac{1}{t}\right) \frac{1}{t} f(xt)\, dt\,. \tag{A.28}$$

We claim that

$$S = \int_0^1 \left| g^{(n)}\left(\frac{1}{t}\right) \frac{1}{t} \right| dt < \infty\,; \tag{A.29}$$

if this is so we can conclude from (A.28) that

$$\|D^n u\| \leqq (1 + S)\|f\|\,.$$

Since $f = Hu$, this proves (A.11) with $c = 1 + S$.

We turn to the proof of (A.29); by definition (A.25), $g(s)$ is a linear combination of the powers s^{λ_j}. Therefore, $g^{(n)}(1/t)/t$ is a linear combination of the powers

$$t^{n-\lambda_j-1}\,.$$

Since, by assumption, $\mathcal{R}e\ \lambda_j < n$, it follows that the real part of each of the above exponents is less than -1, and so each of the above powers is integrable over $[0, 1]$. This completes the proof of Theorem A.2.

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