New Questions Related to the Topological Degree

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To I. M. Gelfand with admiration.

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1 Topological degree and VMO

Degree theory for continuous maps has a long history and has been extensively studied, both from the point of view of analysis and topology. If $f \in C^0(S^n, S^n)$, $\deg f$ is a well-defined element of $\mathbb{Z}$, which is stable under continuous deformation. Starting in the early 1980s, the need to define a degree for some classes of discontinuous maps emerged from the study of some nonlinear PDEs (related to problems in liquid crystals and superconductors). These examples involved Sobolev maps in the limiting case of the Sobolev embedding; see Sections 2 and 3 below. (Topological questions for Sobolev maps strictly below the limiting exponent have been investigated in [15] and [14].) In these cases, the Sobolev embedding asserts only that such maps belong to the space VMO (see below) and *need not be continuous*.

In connection with degree for $H^{1/2}(S^1, S^1)$, L. Boutet de Monvel and O. Gabber suggested a concept of degree for maps in VMO(S^1, S^1) (see [2] and Section 3 below). In our joint work with L. Nirenberg [16], we followed up on their suggestion and established on firm grounds a degree theory for maps in VMO(S^n, S^n). Here is a brief summary of our contribution.

First, recall the definition of BMO (bounded mean oscillation), a concept originally introduced by F. John and L. Nirenberg in 1961. Let Ω be a smooth bounded open domain in $\mathbb{R}^n$, or a smooth, compact, n-dimensional Riemannian manifold (with or without boundary). An integrable function $f : \Omega \to \mathbb{R}$ belongs to BMO if

$$|f|_{\mathrm{BMO}} = \operatorname*{Sup}_{B\subset\Omega} \fint_B \fint_B |f(x) - f(y)|dxdy < \infty,$$

where the Sup is taken over all (geodesic) balls in Ω. It is easy to see that an equivalent seminorm is given by

$$\operatorname*{Sup}_{B\subset\Omega} \fint_B \left| f(x) - \fint_B f(y)dy \right| dx.$$

A very important subspace of BMO, introduced by L. Sarason, consists of VMO (vanishing mean oscillation) functions in the sense that

$$\lim_{|B|\to 0} \fint_B \fint_B |f(x) - f(y)|dxdy = 0.$$

It is easy to see that

$$\mathrm{VMO}(\Omega, \mathbb{R}) = \overline{C^0(\overline{\Omega}, \mathbb{R})}^{\mathrm{BMO}}.$$

The space VMO is equipped with the BMO seminorm $|f|_{\mathrm{BMO}}$. Clearly, $L^\infty \subset \mathrm{BMO}$. It is well known that BMO is strictly bigger than L^∞ (a standard example is $f(x) = |\log |x||$); however, as a consequence of the classical John–Nirenberg inequality,

$$\mathrm{BMO} \subset \bigcap_{p<\infty} L^p.$$

Thus BMO is "squeezed" between L^∞ and $\cap_{p<\infty} L^p$ and for many purposes serves as an interesting "substitute" for L^∞.

Concerning VMO, it is easy to see that $L^\infty \not\subset \mathrm{VMO}$, but, of course, $C^0 \subset \mathrm{VMO}$. A useful example showing that the inclusion is strict is the function

$$f(x) = |\log |x||^\alpha,$$

which belongs to VMO for every $\alpha < 1$. In some sense, VMO serves as a "substitute" for C^0. The Sobolev space $W^{1,n}$ provides an important class of VMO functions. Recall that for every $1 \le p < \infty$,

$$W^{1,p}(\Omega, \mathbb{R}) = \{f \in L^p(\Omega); \nabla f \in L^p(\Omega)\}.$$

Poincaré's inequality asserts that

$$\int_B \left| f - \fint_B f \right| \le C|B|^{1/n} \int_B |\nabla f|,$$

from which we deduce, using Hölder, that

$$\fint_B \left| f - \fint_B f \right| \leq C \left[\int_B |\nabla f|^n \right]^{1/n}$$

and thus $W^{1,n} \subset \mathrm{VMO}$.

Similarly, the fractional Sobolev space $W^{s,p}(\Omega)$ is contained in VMO for all $0 < s < 1$ and all $1 < p < \infty$ with $sp = n$ (the limiting case of the Sobolev embedding). Indeed, in the Gagliardo characterization, we have

$$W^{s,p}(\Omega) = \{f \in L^p(\Omega); \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} dxdy < \infty\}. \tag{1.1}$$

Clearly,

$$\begin{aligned} \int_B \int_B |f(x) - f(y)| dxdy &= \int_B \int_B \frac{|f(x) - f(y)|}{|x-y|^{(n/p)+s}} |x-y|^{(n/p)+s} dxdy \\ &\leq C|B|^{(1/p)+(s/n)} \int_B \int_B \frac{|f(x) - f(y)|}{|x-y|^{(n/p)+s}} dxdy. \end{aligned}$$

Using Hölder, we deduce that

$$\begin{aligned} &\int_B \int_B |f(x) - f(y)| dxdy \\ &\qquad \leq C|B|^{(1/p)+(s/n)+2-(2/p)} \left[\int_B \int_B \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} dxdy \right]^{1/p}, \end{aligned}$$

and thus when $sp = n$,

$$\fint_B \fint_B |f(x) - f(y)| dxdy \leq C \left[\int_B \int_B \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} dxdy \right]^{1/p},$$

which implies that $W^{s,p} \subset \mathrm{VMO}$.

One of the basic results in [16] is the following.

Theorem 1 (H. Brezis and L. Nirenberg [16]). *Every map $f \in \mathrm{VMO}(S^n, S^n)$ has a well-defined degree. Moreover,*

(a) *this degree coincides with the standard degree when f is continuous;*
(b) *the map $f \mapsto \deg f$ is continuous on $\mathrm{VMO}(S^n, S^n)$ under BMO-convergence.*

It is quite easy to define the VMO-degree. For any given measurable map $f : S^n \to S^n$ and $0 < \varepsilon < 1$, set

$$\bar{f}_\varepsilon(x) = \fint_{B_\varepsilon(x)} f(y) dy.$$

Next, we present an elementary lemma that is extremely useful.

Lemma 1. *If $f \in \mathrm{VMO}(S^n, S^n)$, then*

$$|\bar{f}_\varepsilon(x)| \to 1 \quad \text{as } \varepsilon \to 0, \quad \text{uniformly in } x \in S^n.$$

Proof. Set

$$\rho_\varepsilon(x) = \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |f(y) - f(z)| dy dz,$$

so that $\rho_\varepsilon(x) \to 0$ as $\varepsilon \to 0$, uniformly in $x \in S^n$, since $f \in$ VMO. Then observe that

$$1 - \rho_\varepsilon(x) \le |\bar{f}_\varepsilon(x)| \le 1. \qquad \square$$

If $f \in \mathrm{VMO}(S^n, S^n)$, we may now set

$$f_\varepsilon(x) = \frac{\bar{f}_\varepsilon(x)}{|\bar{f}_\varepsilon(x)|}, \quad x \in S^n, \quad 0 < \varepsilon < \varepsilon_0(f).$$

Using ε as a homotopy parameter, we see that deg f_ε is well defined and *independent of* ε for $\varepsilon > 0$ sufficiently small. This integer is, by definition, VMO-deg f. The proof of Theorem 1(a) is straightforward. For the proof of (b), we refer to [16].

The space $\mathrm{VMO}(S^n, S^n)$ is larger than $C^0(S^n, S^n)$. However, its structure, from the point of view of connected (or, equivalently, path-connected) components, is similar to $C^0(S^n, S^n)$. More precisely, there is a VMO version of the celebrated Hopf result.

Theorem 2. *The homotopy classes (i.e., the path-connected components) of* $\mathrm{VMO}(S^n, S^n)$ *are characterized by their VMO-degree.*

Remark 1. By contrast, it is *not* possible to define a degree for maps in $L^\infty(S^n, S^n)$. In fact, the space $L^\infty(S^n, S^n)$ is path-connected (see [16, Section I.5]).

2 Degree for $H^1(S^2, S^2)$ and beyond

In my earlier paper with J. M. Coron [12] (see also [9, 10]), we were led to a concept of degree for maps in $H^1(S^2, S^2)$. Our original motivation came from solving a nonlinear elliptic system, proposed in [17], which amounts to finding critical points of the Dirichlet integral

$$E(u) = \int_\Omega |\nabla u|^2$$

subject to the constraint

$$u \in H^1_\varphi(\Omega, S^2) = \{u \in H^1(\Omega; S^2); u = \varphi \text{ on } \partial\Omega\},$$

where Ω denotes the unit disc in $\mathbb{R}^2$ and $\varphi : \partial\Omega \to S^2$ is given (smooth). In the process of finding critical points, it is natural to investigate the connected components of $H^1_\varphi(\Omega, S^2)$, a question which is closely related to the study of the components of $H^1(S^2, S^2)$. The way we defined a degree for $H^1(S^2, S^2)$ was with the help of an *integral formula*. Recall that if $f \in C^1(S^n, S^n)$, Kronecker's formula asserts that

$$\deg f = \fint_{S^n} \det(\nabla f), \tag{2.1}$$

where $\det(\nabla f)$ denotes the $n \times n$ Jacobian determinant of f. When $n = 2$, the right-hand side of (2.1) still makes sense when f is not C^1, but merely in $H^1(S^2, S^2)$ because $\det(\nabla f) \in L^1$. We were able to prove (via a density argument) that the RHS in (2.1) belongs to $\mathbb{Z}$ and we took it as a definition of the H^1-degree of f. Similarly, one may use (2.1) to define a degree for every map $f \in W^{1,n}$. In view of the discussion in Section 1, we know that $W^{1,n} \subset \text{VMO}$ and thus any $f \in W^{1,n}(S^n, S^n)$ admits a VMO-degree in the sense of Section 1. Fortunately, the two definitions coincide. In fact, we have the following.

Lemma 2. *For every $f \in W^{1,n}(S^n, S^n)$,*

$$W^{1,n}\text{-deg } f = \text{VMO-deg } f.$$

Moreover, the components of $W^{1,n}(S^n, S^n)$ are characterized by their degree.

Using this concept of degree, we managed to prove in [12] that if φ is not a constant, then E achieves its minimum on two distinct components of $H^1_\varphi(\Omega, S^2)$. A very interesting question remains open.

Open Problem 1. *Does E admit a critical point in each component of $H^1_\varphi(\Omega, S^2)$ when φ is not a constant?*

Even the special case

$$\varphi(x, y) = (Rx, Ry, \sqrt{1 - R^2}), \quad 0 < R < 1, \quad x^2 + y^2 = 1,$$

is open.

It is also interesting to study the homotopy structure of $W^{1,p}(S^n, S^n)$ for values of $p \neq n$. This was done in my joint paper with Y. Li [14].

Theorem 3. *When $p > n$, the standard (C^0) degree of maps in $W^{1,p}$ is well defined and the components of $W^{1,p}$ are characterized by their degree. When $1 \leq p < n$, $W^{1,p}$ is path-connected.*

Following the earlier paper [15], we started to investigate with Y. Li [14] the homotopy structure of $W^{1,p}(M, N)$ when M and N are general Riemannian manifolds (M possibly with boundary, while $\partial N = \emptyset$). When $p \geq \dim M$, the homotopy structure of $W^{1,p}(M, N)$ is identical to that of $C^0(M, N)$. When $\dim M > 1$ and $1 \leq p < 2$, we proved in [14] that $W^{1,p}(M, N)$ is *always* path-connected. When p decreases from $\dim M$ to 2, the set $W^{1,p}(M, N)$ becomes larger and larger while various surprising phenomena may occur:

(a) Some homotopy classes persist below the Sobolev threshold $p = \dim M$, where maps need not belong to VMO.

(b) As p decreases, the set $W^{1,p}(M, N)$ increases, and in this process some of the homotopy classes "coalesce" as p crosses distinguished *integer* values—and usually there is a cascade of such levels where the homotopy structure undergoes "dramatic" jumps.
(c) As p decreases, new homotopy classes may "suddenly" appear at some (integral) levels; every map in these new classes must have "robust" singularities: they cannot be erased via homotopy.

We refer the interested reader to [14] and to the subsequent remarkable paper by F. B. Hang and F. H. Lin [18].

3 Degree for $H^{1/2}(S^1, S^1)$. Can one hear the degree of continuous maps?

Another important example that motivated my work with L. Nirenberg [16] was the concept of degree for maps in $H^{1/2}(S^1, S^1)$ due to L. Boutet de Monvel and O. Gabber (presented in [2, appendix]). The motivation in [2] came from a Ginzburg–Landau model arising in superconductivity. This $H^{1/2}$-degree also plays an important role in our study of the Ginzburg–Landau vortices with F. Bethuel and F. Hélein (see [1]). For example, it is at the heart of the proof of the following.

Lemma 3. *Let Ω be the unit disc in $\mathbb{R}^2$ and let φ be a smooth map from $\partial\Omega = S^1$ into S^1. Then*

$$[H^1_\varphi(\Omega, S^1) \neq \emptyset] \Leftrightarrow [\deg \varphi = 0].$$

The way Boutet de Monvel and Gabber originally defined a degree for $H^{1/2}(S^1, S^1)$ went as follows. First, observe that if $f \in C^1(S^1, \mathbb{C} \setminus \{0\})$, then the Cauchy formula asserts that

$$\deg f = \frac{1}{2i\pi} \int_{S^1} \frac{\dot{f}}{f}. \tag{3.1}$$

In particular, if $f \in C^1(S^1, S^1)$ we may write (3.1) as

$$\deg f = \frac{1}{2i\pi} \int_{S^1} \bar{f}\dot{f} = \frac{1}{2\pi} \int_{S^1} \det(f, \dot{f}) \tag{3.2}$$

(which is the simplest form of Kronecker's formula (2.1)). Then Boutet de Monvel and Gabber observed that the right-hand side of (3.2) still makes when f is not C^1, but merely in $H^{1/2}$. To do so, they interpret the RHS in (3.2) as a scalar product in the duality $H^{1/2} - H^{-1/2}$ ($\bar{f} \in H^{1/2}$, $\dot{f} \in H^{-1/2}$). Using a density argument, they prove that the RHS in (3.2) belongs to $\mathbb{Z}$ and they take it as definition for the $H^{1/2}$-degree of f. On the other hand, recall (see Section 1) that $H^{1/2}(S^1) \subset \mathrm{VMO}(S^1)$. Therefore, any $f \in H^{1/2}(S^1, S^1)$ admits a VMO-degree in the sense of Section 1, and, in fact, we have the following.

Lemma 4. *For every $f \in H^{1/2}(S^1, S^1)$,*

$$H^{1/2}\text{-deg } f = \text{VMO-deg } f.$$

Lemmas 2 and 4 show the unifying character of the VMO-degree, putting various concepts of degree (for continuous maps, for $W^{1,n}(S^n, S^n)$ maps, for $H^{1/2}(S^1, S^1)$ maps, etc.) under a common roof.

In 1996, I. M. Gelfand invited me to present at his seminar the VMO-degree theory we had just developed with Louis Nirenberg. He asked me to elaborate on the special case of the $H^{1/2}(S^1, S^1)$-degree. I wrote down Gagliardo's characterization of $H^{1/2}$ which, in this special case, takes the form

$$H^{1/2}(S^1) = \left\{ f \in L^2(S^1); \int_{S^1} \int_{S^1} \frac{|f(x) - f(y)|^2}{|x - y|^2} dxdy < \infty \right\}.$$

Since I. M. Gelfand was not fully satisfied with Gagliardo's formulation, I also wrote down the characterization of $H^{1/2}$ in terms of the Fourier coefficients (a_n) of f:

$$H^{1/2}(S^1) = \left\{ f \in L^2(S^1); \sum_{n=-\infty}^{+\infty} |n||a_n|^2 < \infty \right\}$$

(see also Lemma 5 below). At that point, I. M. Gelfand asked whether there is a connection between the degree and the Fourier coefficients. At first, I was surprised by his question, but I realized shortly afterwards that if one inserts the Fourier expansion

$$f(\theta) = \sum_{n=-\infty}^{+\infty} a_n e^{in\theta}$$

into (3.2), one finds

$$\deg f = \sum_{n=-\infty}^{+\infty} n|a_n|^2. \tag{3.3}$$

Formula (3.3) is easily justified when $f \in C^1(S^1, S^1)$. The density of $C^1(S^1, S^1)$ into $H^{1/2}(S^1, S^1)$ and the stability of degree under VMO-convergence (and thus under $H^{1/2}$-convergence) yield the following.

Theorem 4. *For every $f \in H^{1/2}(S^1, S^1)$,*

$$\text{VMO-deg } f = \sum_{n=-\infty}^{+\infty} n|a_n|^2. \tag{3.4}$$

Formula (3.4) raises some intriguing questions. First, however, we present a consequence of Theorem 4.

Corollary 1. *Let (a_n) be a sequence of complex numbers satisfying*

$$\sum_{n=-\infty}^{+\infty} |n||a_n|^2 < \infty, \tag{3.5}$$

$$\sum_{n=-\infty}^{+\infty} |a_n|^2 = 1, \tag{3.6}$$

and

$$\sum_{n=-\infty}^{+\infty} a_n \bar{a}_{n+k} = 0 \quad \forall k \neq 0. \tag{3.7}$$

Then

$$\sum_{n=-\infty}^{+\infty} n|a_n|^2 \in \mathbb{Z}. \tag{3.8}$$

Proof. Set

$$f(\theta) = \sum_{n=-\infty}^{+\infty} a_n e^{in\theta},$$

so that $f \in H^{1/2}(S^1, \mathbb{C})$. Moreover, we have

$$\int_{S^1} (|f(\theta)|^2 - 1) e^{ik\theta} d\theta = 0 \quad \forall k. \tag{3.9}$$

Indeed, for $k = 0$, (3.9) follows from (3.6), and for $k \neq 0$, (3.9) follows from (3.7). Thus we obtain

$$|f(\theta)| = 1 \quad \text{a.e.} \tag{3.10}$$

Applying Theorem 4, we find (3.8). □

Pedagogical Question. Is there an elementary proof of Corollary 1 that does not rely on Theorem 4?

Suppose now $f \in C^0(S^1, S^1)$ and $f \notin H^{1/2}$. Then the series

$$\sum_{n=-\infty}^{+\infty} |n||a_n|^2$$

is divergent. The LHS in (3.4) is well defined, but the RHS is not. It is natural to ask whether $\deg f$ may still be computed as a "principal value" of the series $\sum_{n=-\infty}^{+\infty} n|a_n|^2$ (which is not absolutely convergent). In [11] we raised the question of whether standard summation processes can be used to compute the degree of a general $f \in C^0(S^1, S^1)$. Let, for example,

$$\sigma_N = \sum_{n=-N}^{+N} n|a_n|^2$$

or

$$P_r = \sum_{n=-\infty}^{+\infty} n|a_n|^2 r^{|n|}, \quad 0 < r < 1.$$

Is it true that, for any $f \in C^0(S^1, S^1)$,

$$\deg f = \lim_{N\to+\infty} \sigma_N \quad \text{or} \quad \deg f = \lim_{r\downarrow 1} P_r?$$

J. Korevaar [20] has shown that the answer is negative. He has constructed interesting examples of maps $f \in C^0(S^1, S^1)$ of degree 0 such that σ_N (respectively, P_r) need not have a limit as $N \to \infty$ (respectively, $r \to 1$) or may converge to any given real number $\lambda \neq 0$, including $\pm\infty$. In view of this fact, we now propose a more "modest" question: Do the absolute values of the Fourier coefficients determine the degree? More precisely, we have the following.

Open Problem 2 (Can one hear the degree of continuous maps?). *Let $f, g \in C^0(S^1, S^1)$ and let (a_n), (b_n) denote the Fourier coefficients of f and g, respectively. Assume that*

$$|a_n| = |b_n| \quad \forall n \in \mathbb{Z}. \tag{3.11}$$

Can one conclude that

$$\deg f = \deg g?$$

Answer the same question if one assumes only that $f, g \in \mathrm{VMO}(S^1, S^1)$.

Of course, the answer to Open Problem 2 is positive if, in addition, $f, g \in H^{1/2}(S^1, S^1)$. This is a consequence of Theorem 4. The answer is still positive in a class of functions strictly larger than $H^{1/2}$. The proof is based on the following.

Theorem 5. *For every $f \in W^{1/3,3}(S^1, S^1)$, we have*

$$\text{VMO-deg}\, f = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon^2} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} |a_n|^2 \frac{\sin^2 n\varepsilon}{n}. \tag{3.12}$$

Corollary 2. *Assume that $f, g \in W^{1/3,3}(S^1, S^1)$ satisfy* (3.11). *Then*

$$\text{VMO-deg}\, f = \text{VMO-deg}\, g.$$

Corollary 3 (J. P. Kahane [19]). *Assume that $f, g \in C^{0,\alpha}(S^1, S^1)$, with $\alpha > 1/3$, satisfy* (3.11). *Then*

$$\deg f = \deg g.$$

Note that $C^{0,\alpha} \subset W^{1/3,3}$ $\forall \alpha > 1/3$. (This is an obvious consequence of Gagliardo's characterization (1.1)). Thus Corollary 2 implies Corollary 3. Our proof of Theorem 5 is a straightforward adaptation of the ingenious argument of J. P. Kahane [19] for $C^{0,\alpha}$, $\alpha > 1/3$.

Remark 2. The conclusion of Theorem 5 holds if $f \in W^{1/p,p}(S^1, S^1)$ with $1 < p \leq 3$ (since $W^{1/p,p} \cap L^\infty \subset W^{1/3,3}$ $\forall p \leq 3$). (Note that when $1 < p \leq 2$ the conclusion of Theorem 5 is an immediate consequence of Theorem 4 since $\sum |n||a_n|^2 < \infty$. However, in the range $2 < p \leq 3$, the conclusion is far from obvious since the series $\sum |n||a_n|^2$ may be divergent.) It is interesting to point out that formula (3.12) *fails* if one assumes only $f \in W^{1/p,p}(S^1, S^1)$ with $p > 3$. In fact, J. P. Kahane [19] has constructed an example of a function $f \in C^{0,1/3}(S^1, S^1)$ such that $\deg f = 0$ while

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^2} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} |a_n|^2 \frac{\sin^2 n\varepsilon}{n} = \lambda,$$

where λ could be any real number $\lambda \neq 0$. The heart of the matter is the existence of a 2π-periodic function $\varphi \in C^{0,1/3}(\mathbb{R}, \mathbb{R})$ such that

$$\int_0^{2\pi} (\varphi(\theta + h) - \varphi(\theta))^3 d\theta = \sin h \quad \forall h.$$

This still leaves open the question whether Corollary 2 holds when $W^{1/3,3}$ is replaced by $W^{1/p,p}$, $p > 3$.

Taking $p \to 1$ in Remark 2 suggests that Theorem 5 holds for $f \in W^{1,1}$. This is indeed true, and there is even a stronger statement.

Theorem 6. *For every $f \in C^0(S^1, S^1) \cap \mathrm{BV}(S^1, S^1)$, we have*

$$\deg f = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \sum_{n=-\infty}^{+\infty} |a_n|^2 \sin n\varepsilon.$$

Consequently, we also have the following.

Corollary 4. *Assume $f, g \in C^0(S^1, S^1) \cap \mathrm{BV}(S^1, S^1)$ satisfy* (3.11). *Then*

$$\deg f = \deg g.$$

Remark 3. It was already observed by J. Korevaar in [20] that for every $f \in C^0 \cap \mathrm{BV}$, one has

$$\deg f = \lim_{N \to \infty} \sum_{n=-N}^{+N} n|a_n|^2,$$

which also implies Corollary 4.

Proof of Theorem 5. We follow the argument of J. P. Kahane [19], except that we work in the fractional Sobolev space $W^{1/3,3}$ instead of the smaller Hölder space $C^{0,\alpha}$, $\alpha > 1/3$. Set

$$d = \text{VMO-deg } f.$$

By [16, Theorem 3 (and Remark 10)], we may write

$$f(\theta) = e^{i(\varphi(\theta)+d\theta)}$$

for some $\varphi \in \text{VMO}(S^1, \mathbb{R})$. Applying [3, Theorem 1] and the uniqueness of the lifting in VMO, we know that $\varphi \in W^{1/3,3}$.

Write

$$\int_0^{2\pi} f(\theta+h)\bar{f}(\theta)d\theta = 2\pi \sum_{n=-\infty}^{+\infty} |a_n|^2 e^{inh} = \int_0^{2\pi} e^{idh} e^{i(\varphi(\theta+h)-\varphi(\theta))} d\theta, \tag{3.13}$$

$$e^{idh} = 1 + idh + O(|h|^2), \tag{3.14}$$

and

$$\begin{aligned} e^{i(\varphi(\theta+h)-\varphi(\theta))} &= 1 + i(\varphi(\theta+h)-\varphi(\theta)) - \frac{1}{2}(\varphi(\theta+h)-\varphi(\theta))^2 \\ &\quad + O(|\varphi(\theta+h)-\varphi(\theta)|^3). \end{aligned} \tag{3.15}$$

Thus

$$\begin{aligned} \text{Im}[e^{idh} e^{i(\varphi(\theta+h)-\varphi(\theta))}] &= \text{Im}[(1+idh)e^{i(\varphi(\theta+h)-\varphi(\theta))}] + O(|h|^2) \\ &= (\varphi(\theta+h)-\varphi(\theta)) + dh + O|h|^2) \\ &\quad + O(|h||\varphi(\theta+h)-\varphi(\theta)|^2) \\ &\quad + O(|\varphi(\theta+h)-\varphi(\theta)|^3). \end{aligned} \tag{3.16}$$

Integrating (3.16) with respect to θ yields

$$\left| \sum_{n=-\infty}^{+\infty} |a_n|^2 \sin nh - dh \right| \le C|h|^2 + C\int_0^{2\pi} |\varphi(\theta+h)-\varphi(\theta)|^3 d\theta. \tag{3.17}$$

Next, integrating (3.17) with respect to h on $(0, 2\varepsilon)$ gives

$$\begin{aligned} &\left| \sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} |a_n|^2 \left(\frac{1-\cos 2n\varepsilon}{n} \right) - 2d\varepsilon^2 \right| \\ &\qquad \le C\varepsilon^3 + C\int_0^{2\varepsilon} dh \int_0^{2\pi} |\varphi(\theta+h)-\varphi(\theta)|^3 d\theta \end{aligned}$$

and therefore

$$\begin{aligned}\left|\frac{1}{\varepsilon^2}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}|a_n|^2\frac{\sin^2 n\varepsilon}{n}-d\right|&\\ &\le C\varepsilon+\frac{C}{\varepsilon^2}\int_0^{2\varepsilon}\int_0^{2\pi}|\varphi(\theta+h)-\varphi(\theta)|^3dhd\theta\\ &\le C\varepsilon+C\int_0^{2\varepsilon}\int_0^{2\pi}\frac{|\varphi(\theta+h)-\varphi(\theta)|^3}{|h|^2}dhd\theta,\end{aligned}\tag{3.18}$$

which implies (3.12) since $\varphi\in W^{1/3,3}$. □

Proof of Theorem 6. Since $f\in C^0\cap\mathrm{BV}$, the corresponding φ satisfies $\varphi\in C^0\cap\mathrm{BV}$. We return to (3.17) with $h=\varepsilon$,

$$\left|\frac{1}{\varepsilon}\sum_{h=-\infty}^{+\infty}|a_n|^2\sin n\varepsilon-d\right|\le C\varepsilon+\frac{C}{\varepsilon}\int_0^{2\pi}|\varphi(\theta+\varepsilon)-\varphi(\theta)|^3d\theta.\tag{3.19}$$

Next, we have

$$\int_0^{2\pi}|\varphi(\theta+\varepsilon)-\varphi(\theta)|d\theta\le\varepsilon\|\varphi\|_{\mathrm{BV}}.\tag{3.20}$$

Inserting (3.2) into (3.19) gives

$$\left|\frac{1}{\varepsilon}\sum_{n=-\infty}^{+\infty}|a_n|^2\sin n\varepsilon-d\right|\le C\varepsilon+C\operatorname*{Sup}_{\theta}\|\varphi(\theta+\varepsilon)-\varphi(\theta)\|^2_{L^\infty},\tag{3.21}$$

and the conclusion follows since $\varphi\in C^0$. □

Remark 4. It has been pointed out to me by J. P. Kahane that a slightly stronger conclusion holds in Theorem 5.

Theorem 5′. *For every* $f\in W^{1/3,3}(S^1,S^1)$, *we have*

$$\text{VMO-deg}\, f=\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\sum_{n=-\infty}^{+\infty}|a_n|^2\sin n\varepsilon.\tag{3.22}$$

Proof. Returning to (3.17), it suffices to verify that

$$\lim_{h\downarrow 0}\frac{1}{h}\int_0^{2\pi}|\varphi(\theta+h)-\varphi(\theta)|^3d\theta=0.\tag{3.23}$$

Set

$$I(t)=\int_0^{2\pi}|\varphi(\theta+t)-\varphi(\theta)|^3d\theta$$

so that

$$I^{1/3}(t_1+t_2) \le I^{1/3}(t_1) + I^{1/3}(t_2).$$

Thus

$$I(t_1+t_2) \le 4(I(t_1)+I(t_2)).$$

Consequently,

$$I(h) \le \frac{8}{h}\int_{h/2}^{h}(I(s)+I(h-s))ds = \frac{8}{h}\int_0^h I(s)ds \le 8h\int_0^h \frac{I(s)}{s^2}ds.$$

Since $\varphi \in W^{1/3,3}$, we know that

$$\int_0^{2\pi}\frac{I(s)}{s^2}ds < \infty$$

and (3.23) follows.

4 New estimates for the degree

Going back to (3.3), we see that for every $f \in C^1(S^1, S^1)$,

$$|\deg f| \le \sum |n||a_n|^2. \tag{4.1}$$

Combining (4.1) with Gagliardo's characterization (1.1) of $H^{1/2}$, we find that

$$|\deg f| \le C\int_{S^1}\int_{S^1}\frac{|f(x)-f(y)|^2}{|x-y|^2}dxdy. \tag{4.2}$$

In fact, the sharp estimate

$$|\deg f| \le \frac{1}{4\pi^2}\int_{S^1}\int_{S^1}\frac{|f(x)-f(y)|^2}{|x-y|^2}dxdy \tag{4.3}$$

is an immediate consequence of (4.1) and the following.

Lemma 5. *For every $f \in H^{1/2}$, one has*

$$\int_{S^1}\int_{S^1}\frac{|f(x)-f(x)|^2}{|x-y|^2}dxdy = 4\pi^2\sum_{n=-\infty}^{+\infty}|n||a_n|^2. \tag{4.4}$$

Proof. Write

$$\int_{S^1}\int_{S^1}\frac{|f(x)-f(y)|^2}{|x-y|^2}dxdy = \int_0^{2\pi}\int_0^{2\pi}\frac{|\sum a_n e^{in\theta}-\sum a_n e^{in\psi}|^2}{|e^{i\theta}-e^{i\psi}|^2}d\theta d\psi$$

$$= \int_0^{2\pi} \frac{d\gamma}{|e^{i\gamma}-1|^2} \int_0^{2\pi} \left| \sum a_n (1-e^{in\gamma}) e^{in\theta} \right|^2 d\theta$$
$$= 2\pi \sum |a_n|^2 \int_0^{2\pi} \frac{|e^{in\gamma}-1|^2}{|e^{i\gamma}-1|^2} d\gamma.$$

However, for $|n| \geq 1$,

$$\frac{|e^{in\gamma}-1|^2}{|e^{i\gamma}-1|^2} = (e^{i(n-1)\gamma} + \cdots + 1)(e^{-i(n-1)\gamma} + \cdots + 1),$$

and thus

$$\int_0^{2\pi} \frac{|e^{in\gamma}-1|^2}{|e^{i\gamma}-1|^2} d\gamma = 2\pi |n|.$$

Inserting this into the previous equality yields (4.4). □

Remark 5. Inequality (4.3) can be viewed as an estimate for the "least amount of $H^{1/2}$-energy" necessary to produce a map $f : S^1 \to S^1$ with assigned degree. More precisely, we have

$$\operatorname*{Inf}_{\substack{f:S^1\to S^1 \\ \deg f = n}} \int_{S^1} \int_{S^1} \frac{|f(x)-f(y)|^2}{|x-y|^2} dx\, dy = 4\pi^2 |n|, \tag{4.5}$$

and the Inf in (4.5) is achieved when $f(\theta) = e^{in\theta}$. The existence of a minimizer for similar problems where the standard $H^{1/2}$ norm is replaced by equivalent norms (e.g., the trace of an H^1 norm on the disc with variable coefficients) is a very delicate question because of "lack of compactness"; we refer to [21].

Remark 6. Estimate (4.2) serves as a building block in the study of the least $H^{1/2}$-energy of maps $u : S^2 \to S^1$ with prescribed singularities. Such a question has been investigated in [5]. More precisely, recall that

$$\|u\|^2_{H^{1/2}(S^2)} = \int_{S^2} \int_{S^2} \frac{|u(x)-u(y)|^2}{|x-y|^3} dxdy.$$

Given points $\Sigma = \{p_1, p_2, \ldots, p_k\} \cup \{n_1, n_2, \ldots, n_k\}$, consider the class of maps

$$A = \{u \in C^1(S^2 \setminus \Sigma, S^1); \deg(u, p_i) = +1 \text{ and } \deg(u, n_i) = -1 \ \forall i\}.$$

Theorem 7 (Bourgain–Brezis–Mironescu [5]). *There exist absolute constants C_1, $C_2 > 0$ such that*

$$C_1 L(\Sigma) \leq \operatorname*{Inf}_{u \in A} \|u\|^2_{H^{1/2}(S^2)} \leq C_2 L(\Sigma), \tag{4.6}$$

where $L(\Sigma)$ is the length of a minimal connection connecting the points (p_i) to the points (n_i).

Theorem 7 is the $H^{1/2}$-version of an earlier result [13] concerning H^1 maps from S^3 into S^2 with singularities that had been motivated by questions arising in liquid crystals with point defects, while the analysis in [5] has its source in the Ginzburg–Landau model for superconductors. It is the LHS inequality in (4.6), which is related to (4.2). The RHS inequality in (4.6) comes from a "brute force" construction called the "dipole construction."

Remark 7. An immediate consequence of (4.3) is the estimate

$$|\deg f| \le \frac{1}{2\pi^2}\int_{S^1}\int_{S^1}\frac{|f(x)-f(y)|^p}{|x-y|^2} \quad \forall f \in C^1(S^1,S^1), \quad \forall p \in (1,2). \tag{4.7}$$

Estimate (4.7) deteriorates as $p \downarrow 1$ since the RHS in (4.7) tends to $+\infty$ unless f is constant (see [4]). It would be desirable to improve the constant $(1/2\pi^2)$ and establish that

$$|\deg f| \le C_p\int_{S^1}\int_{S^1}\frac{|f(x)-f(y)|^p}{|x-y|^2}dxdy \quad \forall f \in C^1(S^1,S^1), \quad \forall p \in (1,2). \tag{4.8}$$

with a constant $C_p \sim (p-1)$ as $p \downarrow 1$. In the limit as $p \downarrow 1$, one should be able to recover (in the spirit of [4]) the obvious inequality

$$|\deg f| \le \frac{1}{2\pi}\int |\dot f|. \tag{4.9}$$

Inequality (4.8) is also valid for $p > 2$, but it cannot be deduced from (4.3) and its proof requires much work.

Theorem 8 (Bourgain–Brezis–Mironescu [6]). *For every $p > 1$, there is a constant C_p such that for any (smooth) $f : S^1 \to S^1$,*

$$|\deg f| \le C_p\int_{S^1}\int_{S^1}\frac{|f(x)-f(y)|^p}{|x-y|^2} = C_p\|f\|^p_{W^{1/p,p}}. \tag{4.10}$$

There is an estimate stronger than (4.10).

Theorem 9 (Bourgain–Brezis–Mironescu [7]). *For any $\delta > 0$ sufficiently small, there is a constant C_δ such that, $\forall f \in C^0(S^1,S^1)$,*

$$|\deg f| \le C_\delta \iint_{\substack{S^1\ S^1\\ [|f(x)-f(y)|>\delta]}}\frac{1}{|x-y|^2}dxdy. \tag{4.11}$$

Remark 8. In Bourgain–Brezis–Nguyen [8], it was proved that (4.11) holds for any $\delta < 2^{1/2}$. This was later improved by H.-M. Nguyen [22], who established the bound

$$|\deg f| \le C \iint_{\substack{S^1\ S^1\\ [|f(x)-f(y)|>3^{1/2}]}}\frac{1}{|x-y|^2}dxdy; \tag{4.12}$$

Nguyen [22] has also constructed examples showing that (4.11) fails for any $\delta > 3^{1/2}$.

Open Problem 3. *What is the behavior of the best constant* C_δ *in* (4.11) *as* $\delta \downarrow 0$? *Is there a more precise estimate of the form*

$$|\deg f| \le C\delta \int_{S^1}\int_{S^1}_{[|f(x)-f(y)|>\delta]} \frac{1}{|x-y|^2}dxdy \tag{4.13}$$

with C independent of δ, *for all* $\delta < 3^{1/2}$?

In the spirit of [4], one might then be able to recover (4.9) as $\delta \to 0$.

Higher-dimensional analogues

Theorem 9 can be extended to higher dimensions.

Theorem 10 (Bourgain–Brezis–Mironescu [8]). *Let* $n \ge 1$. *For any* $\delta \in (0, 2^{1/2})$, *there is a constant* C_δ *such that* $\forall f \in C^0(S^n, S^n)$,

$$|\deg f| \le C \int_{S^n}\int_{S^n}_{[|f(x)-f(y)|>\delta]} \frac{1}{|x-y|^{2n}}dxdy. \tag{4.14}$$

A more refined version of Theorem 10 was obtained by H.-M. Nguyen [22]. He proved that (4.14) holds for any $\delta < [2 + 2/(n+1)]^{1/2}$ and that this range of δs is optimal for all dimensions n. From Theorem 10 we may, of course, recover the earlier estimate of Bourgain–Brezis–Mironescu [6]: $\forall n \ge 1$, $\forall p > n$, $\forall f \in W^{n/p,p}(S^n, S^n)$,

$$|\deg f| \le C(p,n) \int_{S^n}\int_{S^n} \frac{|f(x)-f(y)|^p}{|x-y|^{2n}}dxdy = C(p,n)\|f\|^p_{W^{n/p,p}}. \quad (4.15?)$$

In a different direction, it might be interesting to estimate other topological invariants in terms of fractional Sobolev norms. One of the simplest examples could be the following.

Open Problem 4. *Does one have*

$$|\text{Hopf-degree } f| \le C_p \int_{S^3}\int_{S^3} \frac{|f(x)-f(y)|^p}{|x-y|^6} \quad \forall p > 3, \quad \forall f \in C^1(S^3, S^2)?$$

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