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On a semilinear elliptic equation with inverse-square potential

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Abstract. We study the existence and nonexistence of solutions to a semilinear elliptic equation with inverse-square potential. The dividing line with respect to existence or nonexistence is given by a critical exponent, which depends on the strength of the potential.

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In this paper we study the existence and nonexistence of solutions $u \geq 0$ of the equation

$$-\Delta u = \frac{c}{r^2} u + u^p \tag{1}$$

in a ball $B_R = B(0, R)$ of $\mathbb{R}^N$, $N \geq 3$. Here $r = |x|$, $p > 1$ and the coefficient c satisfies the inequality $0 < c \leq c_0$, where $c_0 = (N-2)^2/4$ is the best constant in the Hardy inequality.

In this study an important role is played by the roots

$$\alpha = \alpha^{\pm} := (N-2)/2 \pm \sqrt{c_0 - c}$$

of the equation

$$\alpha^2 - (N-2)\alpha + c = 0. \tag{2}$$

Observe that $\alpha^+ > \alpha^- > 0$.

Our main result asserts that nontrivial solutions of equation (1) exist if and only if $p < p^+$ where

$$p^+ = 1 + 2/\alpha^-.$$

Theorem 1. *Let $0 \leq c \leq c_0$. For any $p \in (1, p^+)$, there exists a nontrivial solution to equation* (1) *with u^p and u/r^2 belonging to $L^1(B_R)$ and* (1) *holds in $\mathcal{D}'(B_R)$.*

The proof of Theorem 1 is straightforward and elementary, except for the limiting value $c = c_0$. The conclusion of Theorem 1 was known in many – but presumably not all – cases (see e.g. [12]). Concerning nonexistence we have

Theorem 2. *Let* $0 < c \le c_0$, $p \ge p^+$. *Assume* $u \in L^p_{loc}(B_R \setminus \{0\})$, $u \ge 0$ *satisfies*

$$-\Delta u - \frac{c}{r^2} u \ge u^p$$

in $\mathcal{D}'(B_R \setminus \{0\})$. *Then* $u \equiv 0$.

Theorem 2 is reminiscent of the nonexistence results of Brezis–Cabré [2] concerning the so-called very weak solutions to the inequality

$$-\Delta u \ge \frac{u^p}{r^2}, \quad u \ge 0, \quad u \in L^p_{loc}(B_R \setminus \{0\}),$$

for any $p > 1$. The nonexistence aspect in (1) when $p \ge p^+$ was first investigated by Pohozaev–Tesei [11]. However, the concept of solution used there was stronger; our concept is the weakest possible.

We also observe that Theorem 2 seems (formally) to contradict the Implicit Function Theorem since there is no solution of $-\Delta u = (c/|x|^2)u + u^p + t$, even when $t > 0$ is small. As observed in [1], this is due to the lack of an appropriate functional space in which to apply the IFT.

Proof of Theorem 1. Set $p^- = 1 + 2/\alpha^+$ and observe that

$$1 < p^- < \frac{N+2}{N-2} < p^+ \qquad \text{for any } 0 < c < c_0$$

and

$$\lim_{c \to 0} p^- = \frac{N}{N-2}, \quad \lim_{c \to c_0} p^- = \frac{N+2}{N-2},$$
$$\lim_{c \to 0} p^+ = +\infty, \quad \lim_{c \to c_0} p^+ = \frac{N+2}{N-2}.$$

We distinguish three cases:

Case 1: $\mathbf{0 \le c < c_0}$ *and* $\mathbf{p < \frac{N+2}{N-2}}$.

Here the existence of a positive solution $u \in H^1_0(B_R)$ of (1) is a standard and straightforward consequence of the Mountain Pass Theorem. In fact, one can find a radial solution by working in the class of radial functions.

Case 2: $\mathbf{0 \le c < c_0}$ *and* $\mathbf{p^- < p < p^+}$.

Here we have an explicit solution of (1) of the form $u = A/r^\beta$ with $\beta = 2/(p-1)$, $A > 0$ given by

$$A^{p-1} = -\beta^2 + (N-2)\beta - c > 0,$$

because α^-, α^+ are the roots of (2) and the restriction $\alpha^- < \beta < \alpha^+$ is equivalent to the condition $p^- < p < p^+$. Since $\beta < N-2$, u satisfies (1) in the sense of $\mathcal{D}'(B_R)$.

Case 3: $\mathbf{c = c_0}$ *and* $\mathbf{1 < p < p^+ = \frac{N+2}{N-2}}$.

This case is a little more delicate : here we need the improved Hardy inequality which asserts that

$$\int_{B_R} |\nabla u|^2 \geq c_0 \int_{B_R} \frac{u^2}{r^2} + c_q \|u\|^2_{L^q(B_R)},$$

for any $1 \leq q < \frac{2N}{N-2}$ and $u \in C_0^\infty(B_R)$; see [5]. Let H be the Hilbert space obtained by completing $C_0^\infty(B_R)$ with respect to the scalar product

$$a(u,v) = \int_{B_R} \nabla u \cdot \nabla v - c_0 \int_{B_R} uv/r^2.$$

Clearly H is contained in every $L^q(B_R)$ with $1 \leq q < \frac{2N}{N-2}$ with continuous injection. Moreover the injection is compact. This fact is due to H. Brezis and the proof is presented in Lemmas 3.2, 3.3 of [7]. We may then use the Mountain Pass theorem in H and the Palais–Smale condition is satisfied.

Proof of Theorem 2. We will use the following lemma which can be seen as a fairly easy consequence of Theorem 7.7 in [9]. It is also closely related to a result in [4]. We provide a proof for completeness.

Lemma 1. *Let $\Sigma \subset\subset \Omega$ be a closed set of zero (newtonian) capacity and assume that $u, f \in L^1_{loc}(\Omega \setminus \Sigma)$ are two nonnegative functions such that*

$$-\Delta u \geq f \qquad \text{in } \mathcal{D}'(\Omega \setminus \Sigma).$$

Then $u, f \in L^1_{loc}(\Omega)$ and

$$-\Delta u \geq f \qquad \text{in } \mathcal{D}'(\Omega).$$

Furthermore given any smooth subdomain $\Omega' \subset\subset \Omega$, if $v \in L^1(\Omega')$ is the solution of

$$\begin{cases} -\Delta v = f & \text{in } \Omega' \\ v = 0 & \text{on } \partial\Omega', \end{cases}$$

in the sense that

$$\int v(-\Delta\phi) = \int f\phi \qquad \forall \phi \in C^2(\bar{\Omega}') \quad \text{such that} \quad \phi|_{\partial\Omega'} \equiv 0,$$

then

$$u \geq v \quad \text{a.e. in} \quad \Omega'. \tag{3}$$

Proof of Lemma 1. Let $u_k = \min(u,k)$, $k > 0$, which by Kato's lemma (see [10]) satisfies

$$-\Delta u_k \geq f_k \quad \text{in} \quad \mathcal{D}'(\Omega \setminus \Sigma), \tag{4}$$

where $f_k := f\chi_{\{u<k\}}$. Since $-\Delta u_k$ is a nonnegative distribution on $\Omega \setminus \Sigma$, it extends to a nonnegative measure on $\Omega \setminus \Sigma$. Since u_k is bounded, it follows from a Gagliardo–Nirenberg-type inequality that $u_k \in H^1_{loc}(\Omega \setminus \Sigma)$. We show next that in fact $u_k \in H^1_{loc}(\Omega)$. We first take a nonnegative function $\phi \in C_0^\infty(\Omega)$ and a sequence $\phi_n \in C_0^\infty(\Omega \setminus \Sigma)$ converging to ϕ in $H^1(\Omega)$. This is always possible since $\mathrm{cap}_\Omega(\Sigma) = 0$ (take e.g. $\phi_n = \phi(1-\chi_n)$ where $\chi_n = 1$ near Σ and $\|\chi_n\|_{H^1} \to 0 = \mathrm{cap}_\Omega(\Sigma)$). We then have, with $C_k = e^k$,

$$\begin{aligned}\int |\nabla u_k|^2 \phi_n^2 \le C_k \int e^{-u_k} |\nabla u_k|^2 \phi_n^2 &= -C_k \int \phi_n^2 \nabla(e^{-u_k}) \cdot \nabla u_k \\ &= C_k \left(2 \int e^{-u_k} \phi_n \nabla \phi_n \cdot \nabla u_k + \int e^{-u_k} \Delta u_k \phi_n^2 \right) \\ &\le 2C_k^2 \int e^{-u_k} |\nabla \phi_n|^2 + \frac{1}{2} \int e^{-u_k} |\nabla u_k|^2 \phi_n^2,\end{aligned}$$

so that

$$\int |\nabla (u_k \phi_n)|^2 \le C_k' \int |\nabla \phi_n|^2.$$

Passing to the limit as $n \to \infty$ in the above inequality implies that $u_k \in H^1_{loc}(\Omega)$.

We next show that

$$-\Delta u_k \ge f_k \qquad \text{in } \mathcal{D}'(\Omega). \tag{5}$$

Take ϕ and ϕ_n as above, so that by (4),

$$\int u_k(-\Delta \phi_n) \ge \int f_k \phi_n. \tag{6}$$

Now, as $n \to \infty$,

$$\int u_k(-\Delta \phi_n) = \int \nabla u_k \nabla \phi_n \to \int \nabla u_k \nabla \phi = -\int u_k \Delta \phi.$$

Passing to the limit in (6) as $n \to \infty$, we thus obtain (5).

In particular u_k is superharmonic in Ω and given almost any $x \in \Omega$ and any ball $B \subset \Omega$ centered at x, we have

$$u_k(x) \ge \frac{1}{|B|} \int_B u_k(y)\, dy. \tag{7}$$

Now, since $u \in L^1_{loc}(\Omega \setminus \Sigma)$ (and $|\Sigma| = 0$), $u_k \to u$ a.e. in Ω as $k \to \infty$ and u is finite almost everywhere. By Fatou's lemma we then conclude from (7) that for almost every ball B,

$$\int_B u < \infty,$$

which means that $u \in L^1_{loc}(\Omega)$. Using this information, we can now easily pass to the limit in (5) and conclude that $f \in L^1_{loc}(\Omega)$ and that

$$-\Delta u \ge f \qquad \text{in } \mathcal{D}'(\Omega).$$

It only remains to prove (3). We let ρ_n be a standard smooth mollifier and let $u_n = u * \rho_n$, $f_n = f * \rho_n$ so that for n large enough $-\Delta u_n \geq f_n$ and $u_n \geq 0$ in Ω'. By the maximum principle

$$u_n \geq v_n \qquad \text{in } \Omega',$$

where v_n solves

$$\begin{cases} -\Delta v_n = f_n & \text{in } \Omega' \\ \quad v_n = 0 & \text{on } \partial\Omega'. \end{cases}$$

As $n \to \infty$, $u_n \to u$ in $L^1(\Omega')$, $f_n \to f$ in $L^1(\Omega')$ and (by Lemma 4 in [3]) $v_n \to v$ in $L^1(\Omega')$, which yields the desired conclusion. □

We can now turn to the

Proof of Theorem 2. We argue by contradiction and assume that $u \not\equiv 0$. By Lemma 1, $u \in L^p_{loc}(B_R)$, $u/r^2 \in L^1_{loc}(B_R)$ and by the mean-value formula for superharmonic functions, given $R' \in (0, R)$, there exists $\epsilon > 0$ such that $u \geq \epsilon$ a.e. in $B_{R'}$. Let $\lambda := \epsilon^p/2 > 0$ and v_0 be the solution of

$$\begin{cases} -\Delta v_0 = \lambda & \text{in } B_{R'} \\ \quad v_0 = 0 & \text{on } \partial B_{R'}. \end{cases}$$

Once more by Lemma 1, we have

$$0 \leq v_0 \leq u. \tag{8}$$

Next, for $n \geq 1$, define inductively v_n by

$$\begin{cases} -\Delta v_n = \dfrac{c}{|x|^2} v_{n-1} + \dfrac{1}{2} v_{n-1}^p + \lambda & \text{in } B_{R'} \\ \quad v_n = 0 & \text{on } \partial B_{R'} \end{cases}$$

In order to have a well-defined solution v_n (in the sense of Lemma 4 in [3]) it suffices to prove that $f := \frac{c}{|x|^2} v_{n-1} + \frac{1}{2} v_{n-1}^p \in L^1(B_{R'})$. For $n = 1$, this follows from (8) and Lemma 1 which implies that $\frac{c}{|x|^2} u + \frac{1}{2} u^p \in L^1(B_{R'})$. Assume now that $v_{n-1} \in L^1(B_{R'})$ is well-defined. Using the maximum principle, it is easy to see that

$$0 \leq v_0 \leq v_1 \leq \cdots \leq v_{n-1} \leq u,$$

whence $f \in L^1(B_{R'})$ and by the maximum principle again, $0 \leq v_{n-1} \leq v_n \leq u$.

By monotone convergence, letting $v := \lim_{n \to \infty} v_n$, we have that

$$\begin{cases} -\Delta v = \dfrac{c}{|x|^2} v + \dfrac{1}{2} v^p + \lambda & \text{in } B_{R'} \\ \quad v = 0 & \text{on } \partial B_{R'}, \end{cases}$$

in the sense that given any $\phi \in C^2(\bar{B}_{R'})$ such that $\phi|_{\partial B_{R'}} \equiv 0$,

$$\int v(-\Delta\phi) = \int \frac{c}{|x|^2} v\phi + \frac{1}{2}\int v^p \phi + \lambda \int \phi.$$

This contradicts Theorem 1 of [6]. $\square$

Remark 1. Theorems 1 and 2 extend to more general situations – for example, when u^p is replaced by $|x|^{-\beta} u^q$. Assume $0 < c \le c_0$ and set $q^+ = 1 + \frac{2-\beta}{\alpha^-}$, where α^- is as above. The conclusions of Theorems 1 and 2 remain valid with p^+ replaced by q^+.

Remark 2. The argument presented in the proof of Theorem 2 may be used to provide a slightly simpler proof of Theorem 1 in [2].

Remark 3. Theorem 2 can be extended to problems of the type

$$-\Delta u = \frac{c}{\operatorname{dist}(x,\Sigma)^2} u + u^p,$$

where $c > 0$ is a small constant, Σ is a smooth compact manifold of codimension $k \ge 3$ and p is larger than some critical exponent, which can be computed explicitly in terms of k and c. The argument is the same as in the proof of Theorem 2 except that the result of [6] is replaced by a result from [8].

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