REMARKS ON THE STRONG MAXIMUM PRINCIPLE

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1. INTRODUCTION

The strong maximum principle asserts that if \( u \) is smooth, \( u \geq 0 \) and \(-\Delta u \geq 0\) in a connected domain \( \Omega \subset \mathbb{R}^N \), then either \( u \equiv 0 \) or \( u > 0 \) in \( \Omega \). The same conclusion holds when \(-\Delta \) is replaced by \(-\Delta + a(x)\) with \( a \in L^p(\Omega), p > \frac{N}{2} \) (this is a consequence of Harnack’s inequality; see e.g. [6], and also [7, Corollary 5.3]. Another formulation of the same fact says that if \( u(x_0) = 0 \) for some point \( x_0 \in \Omega \), then \( u \equiv 0 \) in \( \Omega \). A similar conclusion fails, however, when \( a \not\in L^p(\Omega) \), for any \( p > \frac{N}{2} \). For instance, \( u(x) = |x|^2 \) satisfies \(-\Delta u + a(x)u = 0\) in \( B_1 \) with \( a = \frac{2N}{|x|^2} \not\in L^{N/2}(\Omega) \).

If \( u \) vanishes on a larger set, one may still hope to conclude, under some weaker condition on \( a \), that \( u \equiv 0 \) in \( \Omega \). Such a result was obtained by Bénilan-Brezis [2, Appendix D] (with a contribution by R. Jensen) in the case where \( a \in L^1(\Omega) \) and \( \text{supp } u \) is a compact subset of \( \Omega \). Their maximum principle has been further extended by Ancona [1], who proved Theorem 1 below.

We recall that a function \( v : \Omega \to \mathbb{R} \) is quasicontinuous if there exists a sequence of open subsets \( (\omega_n) \) of \( \Omega \) such that \( v|_{\Omega \setminus \omega_n} \) is continuous \( \forall n \geq 1 \) and \( \text{cap } \omega_n \to 0 \) as \( n \to \infty \), where \( \text{cap } \omega_n \) denotes the \( H^1 \)-capacity of \( \omega_n \).

**Theorem 1** ([1]). Assume \( \Omega \subset \mathbb{R}^N \) is an open bounded set. Let \( u \in L^1(\Omega) \), \( u \geq 0 \ a.e. \ in \ \Omega \), be such that \( \Delta u \) is a Radon measure on \( \Omega \). Then there exists \( \tilde{u} : \Omega \to \mathbb{R} \) quasicontinuous such that \( u = \tilde{u} \) a.e. in \( \Omega \).

Let \( a \in L^1(\Omega), a \geq 0 \ a.e. \ in \ \Omega \). If

\[-\Delta u + au \geq 0 \quad \text{in } \Omega, \]

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in the following sense
\[ \int_E \Delta u \leq \int_E au \quad \text{for every Borel set } E \subset \Omega, \quad (1.1) \]
and if \( \bar{u} = 0 \) on a set of positive \( H^1 \)-capacity in \( \Omega \), then \( u = 0 \) a.e. in \( \Omega \).

The proof given by Ancona is purely based on Potential Theory, while ours is more direct in the spirit of PDE's. We also discuss carefully the meaning of the condition \( -\Delta u + au \geq 0 \) in \( \Omega \).

The next two corollaries follow immediately from the theorem above:

**Corollary 2.** Let \( u \) and \( a \) be as in Theorem 1, and suppose (1.1) is satisfied. If \( u = 0 \) on a subset of \( \Omega \) with positive measure, then \( u = 0 \) a.e. in \( \Omega \). If \( u \) is continuous in \( \Omega \) and \( u = 0 \) on a subset of \( \Omega \) with positive \( H^1 \)-capacity, then \( u \equiv 0 \) in \( \Omega \).

**Corollary 3.** Let \( u \) and \( a \) be as in Theorem 1. Suppose that \( \Delta u \in L^1(\Omega) \). If
\[ -\Delta u + au \geq 0 \quad \text{a.e. in } \Omega \]
and \( u = 0 \) on a subset of \( \Omega \) with positive measure, then \( u = 0 \) a.e. in \( \Omega \).

The next corollary follows from Theorem 1 and Remark 3:

**Corollary 4.** Let \( u \) and \( a \) be as in Theorem 1. Suppose that \( au \in L^1_{\text{loc}}(\Omega) \). If
\[ -\Delta u + au \geq 0 \quad \text{in } \mathcal{D}'(\Omega), \]
i.e.,
\[ \int \Delta u \varphi - \int au \varphi \leq 0 \quad \forall \varphi \in C_0^\infty(\Omega), \varphi \geq 0 \text{ in } \Omega, \]
and \( u = 0 \) on a subset of \( \Omega \) with positive measure, then \( u = 0 \) a.e. in \( \Omega \).

**Remark 1.** In view of Corollary 4 above, it would seem natural to replace condition (1.1) in Theorem 1 by
\[ \int \Delta u \varphi \leq \int au \varphi \quad \forall \varphi \in C_0^\infty(\Omega), \varphi \geq 0 \text{ in } \Omega, \quad (1.2) \]
which makes sense even if \( au \notin L^1_{\text{loc}}(\Omega) \) (note that \( au \varphi \geq 0 \text{ a.e.} \), so that the right-hand side is always well-defined, possibly taking the value \(+\infty\)). However, the strong maximum principle is no longer true in general. See Remark 4.
There are several interesting questions related to Theorem 1:

**Open problem 1.** In the statement of Theorem 1, suppose in addition that \( \text{supp} \, u \subset \Omega \) is a compact set. Can one replace the assumption \( a \in L^{1}_{\text{loc}} \) by a weaker condition, for example \( a^{1/2} \in L^{1}_{\text{loc}} \) (or \( a^{1/2} \in L^{p}_{\text{loc}} \) for some \( p > 1 \)), and still conclude that \( u = 0 \) a.e. in \( \Omega \)?

Note that one cannot hope to go below \( L^{1}/2 \). For instance the \( C^{2} \)-function \( u \) given by

\[
    u(x) = \begin{cases} 
        (1 - |x|^{2})^{4} & \text{for } |x| \leq 1 \\
        0 & \text{for } |x| > 1
    \end{cases}
\]

satisfies \(-\Delta u + au \geq 0\) for some function \( a(x) \) such that \( a(x) \sim \frac{1}{(1-|x|^{2})^{2}} \) for \(|x| \lesssim 1\). Here, \( a^{\alpha} \in L^{1} \) \( \forall \alpha < 1/2 \), but \( a^{1/2} \not\in L^{1} \).

Here is another question:

**Open problem 2.** Assume \( u \in C^{0}, \, u \geq 0, \) and \( a \in L^{q}_{\text{loc}} \) for some \( q \geq 1, \) \( a \geq 0 \) a.e., satisfy (1.1). Suppose that \( u = 0 \) on a set \( E \) with \( \text{cap}_{1,2q}(E) > 0 \), where \( \text{cap}_{1,2q} \) refers to the capacity associated with the Sobolev space \( W^{1,2q} \). Can one conclude that \( u \equiv 0 \)?

Theorem 1 above shows that the answer is positive when \( q = 1 \). It is also true when \( q > \frac{N}{2} \) by the strong maximum principle mentioned above (note that if \( q > \frac{N}{2} \) and \( x_{0} \) is any point, then \( \text{cap}_{1,2q}(\{x_{0}\}) > 0 \)).

### 2. Some comments about condition (1.1)

Since in the statement of Theorem 1 it may happen that \( au \not\in L^{1}_{\text{loc}}(\Omega) \), and so \( au \) is not necessarily a distribution, one should be careful in order to give a precise meaning to the inequality

\[ \Delta u \leq au \quad \text{in } \Omega. \]

More generally, let \( \mu \) be a Radon measure on \( \Omega \) and \( f \) a measurable function, \( f \geq 0 \) a.e. in \( \Omega \). Here are two possible definitions for the inequality \( \mu \leq f \) in \( \Omega \):

**Definition 1.** We shall write \( \mu \leq_{1} f \) in \( \Omega \) if

\[
    \int_{E} d\mu \leq \int_{E} f \quad \text{for every Borel set } E \subset \Omega.
\]

**Definition 2.** We shall write \( \mu \leq_{2} f \) in \( \Omega \) if

\[
    \int \varphi \, d\mu \leq \int f \varphi \quad \forall \varphi \in C^{\infty}_{0}(\Omega), \ \varphi \geq 0 \ \text{in } \Omega.
\]
In the first definition, we view $f$ as the nonnegative measure $f \, dx$, while in the second one $f$ is treated as if it were a distribution.

**Remark 2.** If $\mu \leq 1$ in $\Omega$, then $\mu \leq 2$ in $\Omega$. However, the converse is not true in general. See Remark 4 below.

**Remark 3.** If we assume in addition that $f \in L^1_{\text{loc}}(\Omega)$, then $\mu \leq 2f$ in $\Omega$. However, the converse is not true in general. See Remark 4 below.

**Remark 4.** Theorem 1 above is no longer true in general (even for the case where $\Delta u \in L^1(\Omega)$) if we replace (1.1) by

$$-\Delta u + au \geq 2 \quad \text{in } \Omega,$$

i.e., if

$$\int u \Delta \varphi \leq \int au \varphi \quad \forall \varphi \in C_0^\infty(\Omega), \varphi \geq 0 \text{ in } \Omega.$$

In fact, let $N \geq 2$. Take $v \in L^1(\mathbb{R}^N)$, $v \geq 0$ a.e. in $\mathbb{R}^N$, such that $\text{supp } v \subset B_1$, $\Delta v \in L^1(\mathbb{R}^N)$, but $v$ is unbounded (this is possible since $N \geq 2$). In particular, there exists $b \in L^1(\mathbb{R}^N)$, $b \geq 0$ a.e. in $\mathbb{R}^N$, such that $bv \notin L^1(\mathbb{R}^N)$.

Let $(x_j) \subset B_1$ be a dense sequence in $B_1$ and, for each $j \geq 1$, let

$$\gamma_j := \min \left\{ \frac{1}{j}, \frac{1}{2} - \frac{|x_j|}{2} \right\}.$$

We define

$$u(x) := \sum_{j=1}^\infty \frac{1}{2^j \gamma_j^{N-2}} v \left( \frac{x - x_j}{\gamma_j} \right), \quad a(x) := \sum_{j=1}^\infty \frac{1}{2^j \gamma_j^{N-2}} b \left( \frac{x - x_j}{\gamma_j} \right).$$

Then

$$u \in L^1(\mathbb{R}^N), \quad u \geq 0 \text{ a.e. in } \mathbb{R}^N,$$

$$\Delta u \in L^1(\mathbb{R}^N),$$

$$a \in L^1(\mathbb{R}^N), \quad a \geq 0 \text{ a.e. in } \mathbb{R}^N,$$

and

$$\int u \Delta \varphi \leq \int au \varphi \quad \forall \varphi \in C_0^\infty(\Omega), \varphi \geq 0 \text{ in } \Omega,$$

(note that the integral in the right-hand side is either 0 or $+\infty$), but $\text{supp } u \subset \overline{B_1}$ and $u \not\equiv 0$ in $\mathbb{R}^N$. On the other hand, in view of Theorem 1, the inequality $\Delta u \leq_1 au$ is not satisfied.
From now on, we shall always consider the inequality \( \Delta u \leq au \) in the sense of Definition 1. In particular we shall omit the subscript 1 in the symbol \( \leq_1 \).

3. PROOF OF THE QUASICONTINUITY STATEMENT OF THEOREM 1

Before proving the first part of Theorem 1 (see Lemma 1 below), we make the following remark:

**Remark 5.** If \( v \in H^1_{\text{loc}}(\Omega) \), then there exists \( \tilde{v} : \Omega \to \mathbb{R} \) quasicontinuous such that \( v = \tilde{v} \) a.e. in \( \Omega \) (see e.g. [5]). In addition, \( \tilde{v} \) is well-defined modulo polar subsets of \( \Omega \), i.e., if \( \tilde{v}_1 \) and \( \tilde{v}_2 \) are two quasicontinuous functions such that \( \tilde{v}_1 = v = \tilde{v}_2 \) a.e. in \( \Omega \), then there exists a polar set \( P \subset \Omega \) such that \( \tilde{v}_1(x) = \tilde{v}_2(x) \) \( \forall x \in \Omega \setminus P \) (see [3]).

**Notation.** Given \( k > 0 \), we denote by \( T_k : \mathbb{R} \to \mathbb{R} \) the truncation function

\[
T_k(s) := \begin{cases} 
  k & \text{if } s \geq k, \\
  s & \text{if } -k < s < k, \\
  -k & \text{if } s \leq -k.
\end{cases}
\]

The existence of a quasicontinuous function \( \tilde{u} : \Omega \to \mathbb{R} \) such that \( u = \tilde{u} \) a.e. in \( \Omega \) as in the statement of Theorem 1 is a consequence of Lemma 1 below (see [1]):

**Lemma 1.** Let \( \Omega \subset \mathbb{R}^N \) be an open set. Assume \( u \in L^1(\Omega) \) is such that \( \Delta u \) is a Radon measure on \( \Omega \). Then

\[
T_k(u) \in H^1_{\text{loc}}(\Omega) \quad \forall k > 0,
\]

and, for each open subset \( A \subset \subset \Omega \), there exists \( C_A > 0 \) so that

\[
\int_A |\nabla T_k(u)|^2 \leq k \left( \int_\Omega |\Delta u| + C_A \int_\Omega |u| \right) \quad \forall k > 0.
\]

Moreover, there exists \( \tilde{u} : \Omega \to \mathbb{R} \) quasicontinuous such that \( u = \tilde{u} \) a.e. in \( \Omega \).

**Proof.** We shall split the proof of Lemma 1 into two steps:

**Step 1.** Proof of (3.1) and (3.2). We first extend \( u \) to the whole \( \mathbb{R}^N \) so that \( u \equiv 0 \) outside \( \Omega \). Let \( \rho \in C^\infty_0(B_1) \) be a radial, nonnegative, mollifier. Set

\[
u(x) := \rho \ast u(x) = \int_\Omega \rho(x - y)u(y) \, dy \quad \forall x \in \Omega.
\]

For \( k > 0 \) fixed, we have \( T_k(u_\varepsilon) \in H^1(\Omega) \) and

\[
\nabla T_k(u_\varepsilon) = \nabla u_\varepsilon \chi_{[|u_\varepsilon| < k]},
\]

where \( \chi_{[|u_\varepsilon| < k]} \) denotes the characteristic function of the set \( [|u_\varepsilon| < k] \).
Given an open set $A \subset \Omega$, let $\varphi \in C_0^\infty(\Omega)$ be such that $0 \leq \varphi \leq 1$ in $\Omega$ and $\varphi \equiv 1$ on $A$. On the one hand, using (3.3) and integrating by parts, we have

$$
\int |\nabla T_k(u_\varepsilon)|^2 \varphi = \int \nabla T_k(u_\varepsilon) \cdot (\nabla u_\varepsilon) \varphi = - \int T_k(u_\varepsilon)(\Delta u_\varepsilon) \varphi - \int T_k(u_\varepsilon) \nabla u_\varepsilon \cdot \nabla \varphi. \tag{3.4}
$$

On the other hand,

$$
\int T_k(u_\varepsilon)\nabla u_\varepsilon \cdot \nabla \varphi = - \int u_\varepsilon \nabla T_k(u_\varepsilon) \cdot \nabla \varphi - \int u_\varepsilon T_k(u_\varepsilon) \Delta \varphi
$$

$$
= - \int T_k(u_\varepsilon)\nabla [T_k(u_\varepsilon)]^2 \nabla \varphi - \int u_\varepsilon T_k(u_\varepsilon) \Delta \varphi
$$

$$
= \frac{1}{2} \int [T_k(u_\varepsilon)]^2 \Delta \varphi - \int u_\varepsilon T_k(u_\varepsilon) \Delta \varphi
$$

$$
= \frac{1}{2} \int [T_k(u_\varepsilon)]^2 \Delta \varphi - \int u_\varepsilon T_k(u_\varepsilon) \Delta \varphi
$$

$$
= - \int T_k(u_\varepsilon)(u_\varepsilon - \frac{1}{2} T_k(u_\varepsilon)) \Delta \varphi \geq -k \int |u_\varepsilon| \Delta \varphi. \tag{3.5}
$$

It follows from (3.4) and (3.5) that

$$
\int_A |\nabla T_k(u_\varepsilon)|^2 \leq \int |\nabla T_k(u_\varepsilon)|^2 \varphi
$$

$$
\leq k \left( \int_{\text{supp} \varphi} |\Delta u_\varepsilon| + \|\Delta \varphi\|_{L^\infty} \int_{\text{supp} \varphi} |u_\varepsilon| \right).
$$

In particular, for every $0 < \varepsilon < \text{dist} (\text{supp} \varphi, \partial \Omega)$,

$$
\int_A |\nabla T_k(u_\varepsilon)|^2 \leq k \left( \int |\Delta u| + \|\Delta \varphi\|_{L^\infty} \int |u| \right).
$$

Letting $\varepsilon \downarrow 0$, we conclude that $T_k(u) \in H^1(A)$ and (3.2) holds with $C_A = \|\Delta \varphi\|_{L^\infty}$.

**Step 2.** We prove that, under the assumptions of the lemma, there exists a function $\tilde{u} : \Omega \to \mathbb{R}$ quasicontinuous such that $u = \tilde{u}$ a.e. in $\Omega$.

By (3.1) and Remark 5, for each $k > 0$ there exists $\widehat{T_k(u)} : \Omega \to \mathbb{R}$ quasicontinuous such that $T_k(u) = \widehat{T_k(u)}$ a.e. in $\Omega$.

Let $v_k := \frac{1}{k} T_k(u)$, so that

$$
v_k \to 0 \quad \text{in } L^q(\Omega) \quad \forall q \in [1, \infty)
$$

Then $v_k \to \tilde{u}$ a.e. in $\Omega$.
and, by (3.2),
\[ \int_A |\nabla v_k|^2 \to 0 \quad \forall A \subset \subset \Omega. \]
In particular, \( v_k \to 0 \) in \( H^1_{\text{loc}}(\Omega) \), which implies there exists a polar set \( P \subset \Omega \) such that
\[ \tilde{v}_k(x) = \frac{1}{k} \tilde{T}_k(u_k)(x) \to 0 \quad \forall x \in \Omega \setminus P. \]
We conclude that
\[ \text{cap} \left[ \left| T_k(u) \right| > \frac{k}{2} \right] = \text{cap} \left[ |\tilde{v}_k| > \frac{1}{2} \right] \to 0. \quad (3.6) \]
Set
\[ w(x) := \begin{cases} \sup_{k \in \mathbb{N}} \left\{ \tilde{T}_k(u)(x) \right\} & \text{if } \sup_{k \in \mathbb{N}} |\tilde{T}_k(u)(x)| < \infty, \\ 0 & \text{otherwise}, \end{cases} \quad (3.7) \]
so that \( w = u \) a.e. in \( \Omega \). By (3.6) and the quasicontinuity of the functions \( \tilde{T}_k(u) \), it is easy to see that \( w \) is quasicontinuous in \( \Omega \). This concludes the proof of the lemma.

4. A variant of Kato’s inequality when \( \Delta u \) is a Radon measure

We start with the following (see [1])

**Lemma 2.** Let \( \Omega \subset \mathbb{R}^N \) be an open set. Assume \( u \in L^1(\Omega), u \geq 0 \) a.e. in \( \Omega \), is such that \( \Delta u \) is a Radon measure on \( \Omega \). Then
\[ \Delta T_k(u) \quad \text{is a Radon measure} \quad \forall k > 0. \]
Moreover, for any \( a \in L^\infty(\Omega), a \geq 0 \) a.e. in \( \Omega \), we have
\[ \Delta T_k(u) - aT_k(u) \leq (\Delta u - au)^+ \quad \text{in } D'(\Omega). \quad (4.1) \]

**Proof.** We shall use the same notation as in the proof of Lemma 1. By the standard \( L^1 \)-version of Kato’s inequality (see [4]) we have (note that \( T_k|_{\mathbb{R}^+} \) is concave)
\[ \Delta T_k(u_\varepsilon) \leq t_k(u_\varepsilon)\Delta u_\varepsilon \quad \text{in } \Omega \quad \forall \varepsilon > 0, \quad (4.2) \]
where the function \( t_k : \mathbb{R}^+ \to \mathbb{R} \) is given by
\[ t_k(s) := \begin{cases} 1 & \text{if } 0 \leq s \leq k, \\ 0 & \text{if } s > k. \end{cases} \]
Since \( T_k(s) \geq t_k(s)s \quad \forall s \geq 0 \) and \( a \geq 0 \) a.e. in \( \Omega \), it follows from (4.2) that
\[ \Delta T_k(u_\varepsilon) - aT_k(u_\varepsilon) \leq t_k(u_\varepsilon)(\Delta u_\varepsilon - au_\varepsilon) \leq (\Delta u_\varepsilon - au_\varepsilon)^+ \quad \text{in } D'(\Omega). \]
In other words, we have

$$\int T_k(u_\varepsilon) \Delta \varphi - a T_k(u_\varepsilon) \varphi \leq \int (\Delta u_\varepsilon - a u_\varepsilon)^+ \varphi \quad \forall \varphi \in C_0^\infty(\Omega), \varphi \geq 0 \text{ in } \Omega. \quad (4.3)$$

For $\lambda > 0$, let $\Omega_\lambda := \{ x \in \Omega : \text{dist} (x, \partial \Omega) > \lambda \}$. Thus, if $0 < \varepsilon < \lambda$, we get

$$\Delta u_\varepsilon - a u_\varepsilon = (\Delta u - a u)_\varepsilon + (au)_\varepsilon - a u_\varepsilon \leq \rho_\varepsilon * (\Delta u - a u)^+ + |(au)_\varepsilon - a u| + |a u - a u_\varepsilon| \text{ in } \Omega_\lambda.$$ 

Therefore, for any $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$ in $\Omega$, and $0 < \varepsilon < \text{dist} (\text{supp } \varphi, \partial \Omega)$, we may write

$$\int (\Delta u_\varepsilon - a u_\varepsilon)^+ \varphi \leq \int \rho_\varepsilon * (\Delta u - a u)^+ \varphi + \| \varphi \|_{L^\infty} \left\{ \| (au)_\varepsilon - a u \|_{L^1} + \| a \|_{L^\infty} \| u - u_\varepsilon \|_{L^1} \right\} \quad (4.4)$$

$$= \int (\rho_\varepsilon * \varphi)(\Delta u - a u)^+ + o(1).$$

Since $\rho_\varepsilon * \varphi \to \varphi$ uniformly in $\Omega$ and $(\Delta u - a u)^+$ is a Radon measure in $\Omega$, by letting $\varepsilon \downarrow 0$ in (4.3) and (4.4), we conclude that

$$\int T_k(u) \Delta \varphi - a T_k(u) \varphi \leq \int (\Delta u - a u)^+ \varphi \quad \forall \varphi \in C_0^\infty(\Omega), \varphi \geq 0 \text{ in } \Omega,$$

so that $T_k(u)$ is a Radon measure (take for instance $a \equiv 0$) and (4.1) holds.

**Lemma 3.** Let $\Omega \subset \mathbb{R}^N$ be an open set. Assume $u \in L^1(\Omega)$, $u \geq 0$ a.e. in $\Omega$, is such that $\Delta u$ is a Radon measure on $\Omega$. Let $a \in L^1(\Omega)$, $a \geq 0$ a.e. in $\Omega$. If

$$-\Delta u + a u \geq 0 \quad \text{in } \Omega,$$

in the following sense

$$\int_E \Delta u \leq \int_E a u \quad \text{for every Borel set } E \subset \Omega, \quad (4.5)$$

then

$$-\Delta T_k(u) + a T_k(u) \geq 0 \quad \text{in } \mathcal{D}'(\Omega) \quad \forall k > 0. \quad (4.6)$$

**Proof.** By the preceding lemma applied with $a_i := T_i(a)$, where $i$ is a positive integer, we know that

$$\Delta T_k(u) - a_i T_k(u) \leq (\Delta u - a_i u)^+ \quad \text{in } \mathcal{D}'(\Omega). \quad (4.7)$$
On the other hand, from (4.5) we get
\[ \int_E (\Delta u - a_i u) \leq \int_E (a - a_i) u \quad \text{for every Borel set } E \subset \Omega. \] (4.8)

Since \((a - a_i) u \geq 0 \ a.e. \ in \ \Omega\), (4.8) implies that
\[ 0 \leq \int_E (\Delta u - a_i u)^+ \leq \int_E (a - a_i) u \quad \text{for every Borel set } E \subset \Omega. \] (4.9)

Hence, \((\Delta u - a_i u)^+\) is a nonnegative measure which is absolutely continuous with respect to the Lebesgue measure. Therefore, we have
\[ (\Delta u - a_i u)^+ \in L^1(\Omega) \quad \forall i = 1, 2, \ldots \] (4.10)

We now return to (4.9) to conclude that
\[ 0 \leq (\Delta u - a_i u)^+ \leq (a - a_i) u \quad \text{a.e. in } \Omega. \]

In particular,
\[ (\Delta u - a_i u)^+ \downarrow 0 \ a.e. \ in \ \Omega \ \text{as } i \uparrow \infty. \] (4.11)

It follows from (4.10) and (4.11) that
\[ (\Delta u - a_i u)^+ \to 0 \ \text{in } L^1(\Omega) \ \text{as } i \to \infty. \] (4.12)

Finally, for any \( \varphi \in C^\infty_0(\Omega), \varphi \geq 0 \ in \ \Omega, \) by (4.7) and (4.12) we have
\[ \int T_k(u) \Delta \varphi - aT_k(u) \varphi \leq \int T_k(u) \Delta \varphi - a_i T_k(u) \varphi \leq \int (\Delta u - a_i u)^+ \varphi \to 0 \]
as \( i \to \infty, \) so that (4.6) holds.

5. Proof of Theorem 1 completed

It follows from Lemma 1 in Section 2 that, under the hypotheses of the theorem, there exists \( \tilde{u} : \Omega \to \mathbb{R} \) quasicontinuous such that \( u = \tilde{u} \ a.e. \ in \ \Omega. \)

Let us assume that \( \tilde{u} = 0 \) on a set of positive capacity \( E \subset \Omega. \) We shall prove that \( u = 0 \ a.e. \ in \ \Omega. \)

We split the proof into two steps:

Step 1. Under the hypotheses of the theorem, if we assume in addition that \( u \in L^\infty(\Omega), \) then \( u = 0 \ a.e. \ in \ \Omega. \)

Since \( u \in L^\infty(\Omega), \) we have \( au \in L^1(\Omega). \) It follows from (1.1) and Remark 3 that
\[ -\Delta u + au \geq 0 \quad \text{in } \mathcal{D}'(\Omega). \]
Recall that for $\varepsilon, \lambda > 0$ we have defined $\Omega_\lambda := \{ x \in \Omega : \text{dist} (x, \partial \Omega) > \lambda \}$ and

$$u_\varepsilon(x) := \rho_\varepsilon \ast u(x) = \int_{\Omega} \rho_\varepsilon(x-y) u(y) \, dy \quad \forall x \in \Omega,$$

where $\rho \in C^\infty_0 (B_1)$, $\rho \geq 0$ in $B_1$, is a radial mollifier.

Using the above notation, for $0 < \varepsilon < \lambda$, we have in $\Omega_\lambda$ that

$$\Delta u_\varepsilon \leq (au)_\varepsilon = au_\varepsilon + [(au)_\varepsilon - au_\varepsilon] \leq au_\varepsilon + [(au)_\varepsilon - au_\varepsilon]^+ \quad (5.1)$$

Since $(au)_\varepsilon \rightarrow au$ in $L^1(\Omega)$, $u_\varepsilon \rightarrow u$ a.e. in $\Omega$ and $u$ is bounded,

$$f_\varepsilon \rightarrow 0 \quad \text{in } L^1(\Omega). \quad (5.2)$$

Let $\delta > 0$ be a fixed number. Multiplying (5.1) by $1/(u_\varepsilon + \delta)$, we get

$$\frac{\Delta u_\varepsilon}{u_\varepsilon + \delta} \leq a + \frac{f_\varepsilon}{\delta} \quad \text{in } \Omega_\lambda \quad \forall \varepsilon \in (0, \lambda). \quad (5.3)$$

We also remark that

$$\frac{\nabla u_\varepsilon}{(u_\varepsilon + \delta)^2} = -\nabla \left( \frac{1}{u_\varepsilon + \delta} \right) \quad \text{in } \Omega. \quad (5.4)$$

Let $\varphi \in C^\infty_0(\Omega)$ and $0 < \varepsilon < \text{dist}(\text{supp} \varphi, \partial \Omega)$. We now use (5.4), integration by parts, estimate (5.3) and Cauchy-Schwarz, to get

$$\int \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon + \delta)^2} \varphi^2 = -\int \nabla u_\varepsilon \cdot \nabla \left( \frac{1}{u_\varepsilon + \delta} \right) \varphi^2 = \int \frac{\Delta u_\varepsilon}{u_\varepsilon + \delta} \varphi^2 + \int \frac{2\varphi \nabla \varphi \cdot \nabla u_\varepsilon}{u_\varepsilon + \delta}$$

$$\leq \int (a + \frac{f_\varepsilon}{\delta}) \varphi^2 + \frac{1}{2} \int \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon + \delta)^2} \varphi^2 + 2 \int |\nabla \varphi|^2.$$

Therefore,

$$\frac{1}{2} \int \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon + \delta)^2} \varphi^2 \leq \int (a + \frac{f_\varepsilon}{\delta}) \varphi^2 + 2 \int |\nabla \varphi|^2.$$

Since

$$\nabla \log \left( \frac{u_\varepsilon}{\delta} + 1 \right) = \frac{\nabla u_\varepsilon}{u_\varepsilon + \delta},$$

the estimate above may be rewritten as

$$\frac{1}{2} \int \left| \nabla \log \left( \frac{u_\varepsilon}{\delta} + 1 \right) \right|^2 \varphi^2 \leq \int (a + \frac{f_\varepsilon}{\delta}) \varphi^2 + 2 \int |\nabla \varphi|^2. \quad (5.5)$$
We now let $\varepsilon \downarrow 0$ in (5.5). It follows from (5.2) that (see also Lemma 1)
\[
\log \left( \frac{u}{\delta} + 1 \right) \in H^1_{\text{loc}}(\Omega) \quad \forall \delta > 0
\]
and
\[
\frac{1}{2} \int \left| \nabla \log \left( \frac{u}{\delta} + 1 \right) \right|^2 \varphi^2 \leq \int (a\varphi^2 + 2|\nabla \varphi|^2) \quad \forall \varphi \in C^\infty_0(\Omega). \tag{5.6}
\]
Let $E \subset \Omega$ be a set of positive capacity such that $\tilde{u} = 0$ on $E$. Without any loss of generality, we may assume that $E \subset \Omega_\lambda$ for some $\lambda > 0$ sufficiently small.

Assume $\omega \subset \subset \Omega$ is an open connected set containing $E$. Let $\varphi_0 \in C^\infty_0(\Omega)$ be a fixed test function such that $\varphi \equiv 1$ on $\omega$.

By (5.6), we have
\[
\int_\omega \left| \nabla \log \left( \frac{u}{\delta} + 1 \right) \right|^2 \leq 2 \int (a\varphi_0^2 + 2|\nabla \varphi_0|^2). \tag{5.7}
\]
Since the quasicontinuous representative
\[
\overline{\log \left( \frac{u}{\delta} + 1 \right)} = \log \left( \frac{\tilde{u}}{\delta} + 1 \right)
\]
of $\log \left( \frac{u}{\delta} + 1 \right)$ equals 0 on $E \subset \Omega$ with $\text{cap} E > 0$, it follows from a variant of Poincaré’s inequality (easily proved by contradiction) that there exists $C > 0$ (depending only on $E$ and $\Omega$) such that
\[
\int_\omega \log^2 \left( \frac{u}{\delta} + 1 \right) \leq C \int_\omega \left| \nabla \log \left( \frac{u}{\delta} + 1 \right) \right|^2 \quad \forall \delta > 0. \tag{5.8}
\]
(5.7) and (5.8) yield
\[
\int_\omega \log^2 \left( \frac{u}{\delta} + 1 \right) \leq 2C \int (a\varphi_0^2 + 2|\nabla \varphi_0|^2) \quad \forall \delta > 0. \tag{5.9}
\]
In particular, the integral in the left-hand side remains bounded as $\delta \downarrow 0$.

On the other hand,
\[
\log^2 \left( \frac{u}{\delta} + 1 \right) \rightarrow +\infty \quad \text{a.e. in } \omega \setminus [u = 0] \text{ as } \delta \downarrow 0. \tag{5.10}
\]
By (5.9) and (5.10), we conclude that $u = 0$ a.e. in $\omega$. Since $\omega$ is an arbitrary connected neighborhood of $E$ in $\Omega_\lambda$ for all $\lambda > 0$ small, we conclude that $u = 0$ a.e. in $\Omega$.

**Step 2.** Proof of Theorem 1 completed.

From Lemma 3, we know that $\Delta T_1(u)$ is a Radon measure and
\[
-\Delta T_1(u) + aT_1(u) \geq 0 \quad \text{in } D'(\Omega).
\]
In addition, \( \tilde{T}_1(u) = T_1(\tilde{u}) = 0 \) on \( E \subset \Omega \) with \( \text{cap} E > 0 \).

By Step 1, we have \( T_1(u) = 0 \) a.e. in \( \Omega \), and so \( u = 0 \) a.e. in \( \Omega \).

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