

THE INTERPLAY BETWEEN ANALYSIS AND TOPOLOGY IN SOME NONLINEAR PDE PROBLEMS

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0. FROM THE LAPLACE EQUATION TO HARMONIC MAPS: A HISTORICAL PERSPECTIVE

Let us recall the formulation of the standard Dirichlet problem for the Laplace operator. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and let $\varphi : \partial\Omega \rightarrow \mathbb{R}$ be a given function. The problem is to find a (smooth) function $u : \Omega \rightarrow \mathbb{R}$ satisfying

$$(0.1) \quad \Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} = 0 \quad \text{in } \Omega,$$

$$(0.2) \quad u = \varphi \quad \text{on } \partial\Omega.$$

In a fundamental paper from 1890, H. Poincaré [49] (see also the lecture notes from a course of H. Poincaré [50] at the Sorbonne) gave the first complete proof of existence and uniqueness of a solution for problem (0.1) - (0.2) when $n = 2$ and $n = 3$:

Theorem 1 (H. Poincaré). *Given any $\varphi \in C^0(\partial\Omega)$, there is a unique classical solution $u \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ of (0.1) - (0.2).*

The “balayage” method introduced by H. Poincaré in his proof of Theorem 1 relies heavily on tools of Potential theory: maximum principle, Harnack’s inequality, explicit representation formulas for the Dirichlet problem in a ball (Poisson integral), etc.

In 1900, D. Hilbert [39], in a celebrated address, followed by a (slightly) more detailed paper in 1904, announced that he had solved the Dirichlet problem (0.1) - (0.2) via the **Dirichlet principle** which had been discovered by G. Green in 1833, with later contributions by C. F. Gauss (1837), W. Thomson (=Lord Kelvin) (1847) and G. Riemann (1853). Dirichlet’s principle asserts that any solution u of (0.1) - (0.2) is a **minimizer** of the Dirichlet integral

$$E(v) = \int_{\Omega} |\nabla v|^2 = \sum_{i=1}^n \int_{\Omega} \left(\frac{\partial v}{\partial x_i} \right)^2$$

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in the class of function $v : \Omega \rightarrow \mathbb{R}$ satisfying the boundary condition $v = \varphi$ on $\partial\Omega$. The existence of a minimizer for E had been taken for granted until 1870, when K. Weierstrass pointed out that a rigorous proof was lacking. It is easy to see that any (classical) solution of (0.1) - (0.2) is a minimizer for E . The touchy points are the following:

a) Prove directly, without invoking equations (0.1) - (0.2), that E admits a minimizer.

b) Prove that the minimizer (which belongs to the class of functions having finite energy) is smooth and satisfies (0.1) - (0.2).

The announcement of Hilbert turned out to be a little premature. Instead, it became a program which stimulated many people during the period 1900-1940: B. Levi, H. Lebesgue, L. Tonelli, R. Courant, S. L. Sobolev and many others.

In 1940, H. Weyl [60] completed Hilbert's program. By 1940 the Calculus of Variations had been placed on firm ground, and it provided a very fruitful link between PDE and Functional Analysis; for further discussions we refer the reader to the expository paper of H. Brezis and F. Browder [15].

Starting in the 1940's a number of mathematicians considered similar questions for **systems**. Namely, the unknown $u : \Omega \rightarrow \mathbb{R}$ becomes a vector-valued function $u : \Omega \rightarrow \mathbb{R}^k$ or more generally $u : \Omega \rightarrow N$ where $N (= N^k)$ is a k -dimensional manifold (usually without boundary). The domain $\Omega \subset \mathbb{R}^n$ could also be replaced by a manifold $M (= M^n)$ (with or without boundary). The most natural generalization of the Dirichlet principle to this setting is the problem of **harmonic maps**. The **energy** of a map $u : M \rightarrow N$ is defined by

$$E(u) = \int_M e(u) dV$$

where the energy density is defined in local coordinates by

$$e(u) = \sum_{i,j,\alpha,\beta} g_{ij}(u) \gamma^{\alpha\beta}(x) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta}$$

and $\gamma^{\alpha,\beta}$, g_{ij} are the metric tensors on M and N . The corresponding Euler-Lagrange equation associated with the minimization of E subject to the constraint $u : M \rightarrow N$ is a **system of nonlinear PDE's**:

$$(0.3) \quad -\Delta u = \Gamma(u)(\nabla u, \nabla u) \text{ in } M,$$

where Δ is the Laplace-Beltrami operator on M and Γ is a quadratic expression in ∇u related to the Christoffel symbols on N . The solutions of the equation (0.3), i.e., the critical points of E , are called the **harmonic maps**. When M has a boundary ∂M , the PDE (0.3) is usually coupled with a Dirichlet boundary condition

$$(0.4) \quad u = \varphi \quad \text{on } \partial M.$$

Basic questions (existence, regularity, etc.) for **systems of nonlinear PDE's**, and in particular harmonic maps, became the subject of intensive investigations after the middle of the 20th century. Here are some names of leading contributors for the period 1940 to the early 80's:

- starting in the 40's: C. Morrey;
- starting in the 60's: E. DeGiorgi, E. Giusti, M. Miranda, F. Almgren, H. Federer, J. Eells, J. H. Sampson, etc.;

- starting in the 70's: M. Giaquinta, R. Hamilton, S. Hildebrandt, R. Schoen, S. T. Yau, W. Jäger, etc.;

- starting in the early 80's: K. Uhlenbeck, L. Simon, J. Jost, L. Lemaire, B. White, etc.

The interested reader will find more information in the following books and review articles: C. Morrey [48], H. Federer [27], R. Hamilton [32], J. Eells and L. Lemaire [22], [23], S. Hildebrandt [40], M. Giaquinta [28], J. Jost [42], E. Giusti [31], R. Schoen and S. T. Yau [58].

Until the late 70's, the main motivation came from problems arising in Differential Geometry and some of the tools came from Geometric Measure theory. Starting in the mid 80's two new trends became highly visible:

On the one hand, the number of papers dealing with harmonic maps has increased enormously, giving an impression of chaotic growth. On the other hand, motivations coming from **physics** brought a refreshing wind of new problems and suggested new tools.

Here are a few topics in physics having connections with harmonic maps and some of the mathematicians involved in their study:

A) **Yang-Mills** equations and **gauge** theories (M. Atiyah, S. K. Donaldson, C. Taubes, K. Uhlenbeck, E. Witten, N. Seiberg, etc.)

B) **Liquid crystals** (J. Ericksen, D. Kinderlehrer, R. Hardt, F. H. Lin, H. Brezis, J.-M. Coron, E. Lieb, etc.)

C) **Superfluids, superconductors, Ginzburg-Landau models** (F. Bethuel, H. Brezis, F. Hélein, R. Hardt, F. H. Lin, T. Rivière, etc.)

D) **Skyrmions** (M. Esteban, E. Lieb, etc.)

Before getting into details, I would like to stress two major differences between the standard Dirichlet problem (0.1) - (0.2) and the theory of harmonic maps.

1) In the standard Dirichlet problem harmonic functions are smooth (C^∞) in Ω . By contrast, **harmonic maps have singularities**. In fact, they often correspond to visible physical observations: point or line "defects", "vortices", etc. Sometimes singularities occur because of **topological obstructions**: for example some (smooth) topologically nontrivial boundary conditions on $\partial\Omega$ do not admit smooth extensions inside Ω (see the discussion in Section 2 and Section 4). Sometimes singularities appear **even in the absence of topological obstructions** simply because solutions with singularities have lower energy (seems paradoxical, but see Section 1.2). Here, the key words are **partial regularity**, i.e., the singular set of (some) harmonic maps is often "small" (it consists of isolated points, lines, etc.).

2) The fact that one deals with maps from the manifold M into the manifold N makes the underlying function spaces much richer from the point of view of Topology. Therefore one may hope to find **multiple solutions** - ideally (at least) one in each homotopy class. On the other hand, the standard homotopy classes of $C^0(M, N)$ need not be the relevant tool when dealing with singular solutions. The natural function space is the Sobolev space $H^1(M, N)$ and **its connected component may be quite different from the ones of $C^0(M, N)$** . For example if $M = N = S^3$, the space $H^1(S^3, S^3)$ is path-connected while $C^0(S^3, S^3)$ admits connected components (classified by their degree). It is necessary to revisit Homotopy theory in the framework of Sobolev spaces, and this is a whole new field under investigation; see Section 4.

In this “labyrinth” of results I propose to concentrate on three very special cases. Each of them displays different features. Despite their simplicity these model cases illuminate general phenomena.

The plan is the following:

0. From the Laplace equation to harmonic maps: a historical perspective

1. Model Case I : $M = B^3$ and $N = S^2$

2. Model Case II : $M = B^2$ and $N = S^2$

3. Model Case III : $M = B^2$ (or B^3) and $N = S^1$

4. Homotopy theory in the framework of Sobolev spaces

Here B^k denotes the unit ball in \mathbb{R}^k .

1. MODEL CASE I: $M = B^3$ AND $N = S^2$

In this section we take $\Omega = M = B^3$, the unit ball in \mathbb{R}^3 , and $N = S^2$, the unit sphere in \mathbb{R}^3 . This is a simplified model for some problems occurring in the theory of nematic liquid crystals (see e.g. the expository presentations in J. Ericksen [24], J. Ericksen and D. Kinderlehrer [25], M. Kléman [46] and the paper of R. Hardt, D. Kinderlehrer and F. H. Lin [34].)

1.1. The Dirichlet minimization problem. Given a smooth boundary condition $\varphi : \partial\Omega = S^2 \rightarrow S^2$, we investigate the minimization problem

$$(1.1) \quad \underset{\substack{u: \Omega \rightarrow S^2 \\ u = \varphi \text{ on } \partial\Omega}}{\text{Min}} E(u), \text{ where } E(u) = \int_{\Omega} |\nabla u|^2.$$

The natural functional space is

$$H_{\varphi}^1(\Omega, S^2) = \{u \in L^2(\Omega, S^2); \nabla u \in L^2 \text{ and } u = \varphi \text{ on } \partial\Omega\}.$$

It is easy to see that H_{φ}^1 is never empty; for example

$$u(x) = \varphi(x/|x|)$$

belongs to H^1 because in 3-d

$$\int_{\Omega} \left| \nabla \left(\frac{x}{|x|} \right) \right|^2 < \infty.$$

Note that, by contrast,

$$C_{\varphi}^0(\overline{\Omega}, S^2) \neq \phi \iff \deg(\varphi) \neq 0.$$

In particular, if $\deg(\varphi) \neq 0$, every $u \in H_{\varphi}^1(\Omega, S^2)$ **must** have at least one singularity. This is a typical situation where a **topological condition forces the creation of singularities**.

It is easy to see that the minimum in (1.1) is achieved and satisfies the harmonic map equation (0.3), which, in this special case, takes the form

$$(1.2) \quad -\Delta u = u|\nabla u|^2 \text{ in } \Omega.$$

Equation (1.2) is a **coupled system**:

$$(1.2') \quad -\Delta u_i = u_i |\nabla u|^2 \quad i = 1, 2, 3$$

where $|\nabla u|^2 = \sum_{i,j} \left(\frac{\partial u_i}{\partial x_j} \right)^2$. A weak solution $u \in H_{\varphi}^1(\Omega, S^2)$ of (1.2) is called a **weakly harmonic map**.

We now turn to the question of regularity. An important result is

Theorem 2 (R. Schoen and K. Uhlenbeck [55], [56], [57]). *Assume u is a minimizer of E in $H_\varphi^1(\Omega, S^2)$. Then u is smooth (C^∞) except at a finite number of points in Ω .*

A very general partial regularity result asserts that if u is a minimizing harmonic map from M^n to N^k , then the singular set Σ of u has Hausdorff dimension $\leq (n-3)$. When $n = 3$, the result is more precise: Σ consists of a finite number of points.

Shortly afterwards we investigated the shape of u near its singular points and proved

Theorem 3 (H. Brezis, J.-M. Coron and E. Lieb [18]). *Assume u is a minimizer of E in $H_\varphi^1(\Omega, S^2)$ and let $a \in \Omega$ be one of its singular points; then, as $x \rightarrow a$,*

$$(1.3) \quad u(x) \sim \pm R \left(\frac{x-a}{|x-a|} \right)$$

for some rotation R .

From the simple form of each singularity a_i we deduce that

$$(1.4) \quad \deg(u, a_i) = \pm 1.$$

By standard degree consideration we have

$$(1.5) \quad \deg(\varphi) = \sum_i \deg(u, a_i)$$

and therefore

$$(1.6) \quad |\deg(\varphi)| \leq \text{Number of singularities of } u.$$

This provides a **lower bound** for the number of singularities in terms of topological invariants. An interesting open direction is to obtain **upper bounds** for the number of singularities. The argument of Schoen-Uhlenbeck is too indirect to provide an explicit bound. An interesting estimate of F. Almgren and E. Lieb [2] asserts that

$$(1.7) \quad \text{Number of singularities of } u \leq C \int_{\partial\Omega} |\nabla\varphi|^2$$

for some universal constant C . Again, the argument is so involved that it is hard to keep track of the constant C .

Open Problem 1. Evaluate the best constant C in (1.7). Is it $C = 1/8\pi$?

Note that $1/8\pi$ is a natural candidate in view of the estimate

$$(1.8) \quad |\deg(\varphi)| \leq \frac{1}{8\pi} \int_{\partial\Omega} |\nabla\varphi|^2.$$

A special case of Open Problem 1 is whether the condition $\int_{\partial\Omega} |\nabla\varphi|^2 < 8\pi$ implies the smoothness of minimizers.

A major difference between the standard Laplace equation (0.1) and the equation of harmonic maps, e.g. (1.2), is that weak solutions of (0.1) are smooth while (1.2) seems to carry little information about the singular set of u (this is true in 3-d; however, the situation is quite different in 2-d; see Theorem 10 in Section 2). A striking construction of T. Rivière yields:

Theorem 4 (T. Rivière [51]). *Given any smooth $\varphi, \varphi \neq \text{Constant}$, there exists a weak solution $u \in H_\varphi^1(\Omega, S^2)$ of (1.2) such that*

$$\text{Sing}(u) = \overline{\Omega}$$

where $\text{Sing}(u)$ is the complement of the maximal open set in Ω on which u is continuous.

In fact, there seems to be a plethora of weakly harmonic maps (see also the construction in Section 1.2) and a natural question is

Open Problem 2. Given any closed set $\Sigma \subset \Omega$, does there exist a weak solution $u \in H_\varphi^1(\Omega, S^2)$ of (1.2), e.g. for some smooth φ , such that

$$\text{Sing}(u) = \Sigma?$$

Let us mention that there is an “intermediate” concept between minimizing harmonic maps and weakly harmonic maps. This is the notion of stationary harmonic maps: they are critical points of the energy E with respect to variations in the domain Ω . Partial regularity results for such maps have been obtained by L. C. Evans [26] and F. Bethuel [4].

1.2. The case where $\deg(\varphi) = 0$; minimal connections and relaxed energy.

Throughout this section we assume that $\varphi : \partial\Omega = S^2 \rightarrow S^2$ is smooth and that

$$(1.9) \quad \deg(\varphi) = 0.$$

Here, there is **no topological obstruction** to the smoothness of minimizers, and the initial guess would be that minimizers are smooth: why would they create a singularity at a “high energy cost” when they are not forced by the topology of φ ? It turns out that the intuition is wrong! Sometimes configurations **with singularities** have lower energy than smooth ones (for maps $u \in H_\varphi^1(\Omega, S^2)$), i.e., **singularities may save energy!** This remarkable phenomenon was first pointed out by R. Hardt and F. H. Lin:

Theorem 5 (R. Hardt and F. H. Lin [35]; see also H. Brezis [13]). *There exist smooth boundary data $\varphi : \partial\Omega \rightarrow S^2$ satisfying (1.9) such that any minimizing harmonic map in $H_\varphi^1(\Omega, S^2)$ must have singularities.*

In fact the construction shows that for any $\varepsilon > 0$ there exists a smooth $\varphi = \varphi_\varepsilon : \partial\Omega \rightarrow S^2$ satisfying (1.9) such that

$$(1.10) \quad \text{Min}_{u \in H_\varphi^1(\Omega, S^2)} \int |\nabla u|^2 < \varepsilon,$$

while for every $v \in H_\varphi^1(\Omega, S^2) \cap C^0(\overline{\Omega}, S^2)$ we have

$$(1.11) \quad \int_{\Omega} |\nabla v|^2 \geq 16\pi - \varepsilon.$$

Such an occurrence is sometimes called a **gap phenomenon** because

$$(1.12) \quad \text{Inf}_{u \in H_\varphi^1(\Omega, S^2)} E(u) < \text{Inf}_{u \in C_\varphi^1(\overline{\Omega}, S^2)} E(u).$$

An intriguing open problem in this kind of situation is

Open Problem 3. Given any smooth $\varphi : \partial\Omega = S^2 \rightarrow S^2$ with $\deg(\varphi) = 0$, does there exist a **smooth** harmonic map satisfying $u = \varphi$ in $\partial\Omega$?

A program to tackle Open Problem 3 was proposed by F. Bethuel, H. Brezis and J.-M. Coron [7]. We have not been able to complete the program, but I feel that there are some tools which are useful in other contexts and I would like to present them here; they involve the notion of **relaxed energy** and also of **minimal connection**.

The first observation is that $H_\varphi^1(\Omega, S^2) \cap C^0(\overline{\Omega}, S^2)$ is **not** dense in H_φ^1 for the **strong** H^1 topology. This is an immediate consequence of the gap phenomenon (1.10) - (1.11). (A similar observation, without the boundary condition, had been made earlier by R. Schoen and K. Uhlenbeck [56].) There is, however, density for the **weak** topology :

Lemma 1 (F. Bethuel [4], [5]). *Given any $u \in H_\varphi^1(\Omega, S^2)$ there exists a sequence $v_n \in H_\varphi^1(\Omega, S^2) \cap C^0(\overline{\Omega}, S^2)$ such that*

$$v_n \rightharpoonup u \text{ weakly in } H^1.$$

We define the **relaxed energy** of any $u \in H_\varphi^1(\Omega, S^2)$ to be

$$(1.13) \quad E_{\text{rel}}(u) = \inf \left\{ \liminf_{n \rightarrow \infty} \int_{\Omega} |\nabla v_n|^2; v_n \in H_\varphi^1(\Omega, S^2) \cap C^0(\overline{\Omega}, S^2) \text{ and } v_n \rightharpoonup u \text{ weakly } H^1 \right\}.$$

The concept of relaxed energy plays a very natural role in the Calculus of Variations (see e.g. M. Giaquinta, G. Modica and J. Souček [30]). Clearly

$$E_{\text{rel}}(u) \geq E(u) \quad \forall u \in H_\varphi^1(\Omega, S^1).$$

It is a pleasant surprise that the relaxed energy E_{rel} has an **explicit** form involving the “topological singularities” of u . More precisely, given any u in $H_\varphi^1(\Omega, S^2)$, consider the vector field defined in local coordinates by

$$(1.14) \quad D(u) = (u \cdot u_y \wedge u_z, u \cdot u_z \wedge u_x, u \cdot u_x \wedge u_y)$$

(this vector field has been introduced in H. Brezis, J.-M. Coron and E. Lieb [18]) and the distribution $T(u)$ defined through its action on $C^\infty(\overline{\Omega}, \mathbb{R})$ -functions by the formula

$$(1.15) \quad \langle T(u), \zeta \rangle = - \int_{\Omega} D(u) \cdot \nabla \zeta + \int_{\partial\Omega} (\text{Jac } \varphi) \zeta.$$

Note that the distribution $T(u)$ acts, in fact, on $\text{Lip}(\Omega, \mathbb{R})$ -functions and also that

$$(1.16) \quad D(u) \cdot n = \text{Jac } \varphi \text{ on } \partial\Omega.$$

Define the nonnegative number

$$(1.17) \quad L(u) = \max_{\substack{\zeta \in \text{Lip}(\Omega, \mathbb{R}) \\ \|\nabla \zeta\|_{L^\infty} \leq 1}} \langle T(u), \zeta \rangle$$

and observe that $L(u) < \infty$ since

$$\int_{\partial\Omega} (\text{Jac } \varphi) = 4\pi \text{deg}(\varphi) = 0.$$

Theorem 6 (F. Bethuel, H. Brezis and J.-M. Coron [7]). *We have, for every $u \in H_\varphi^1(\Omega, S^2)$,*

$$(1.18) \quad E_{rel}(u) = E(u) + 8\pi L(u).$$

The distribution $T(u)$ and the number $L(u)$ have interesting interpretations in terms of “topological singularities”.

A straightforward computation (see H. Brezis, J.-M. Coron and E. Lieb [18]) shows that if $u \in C^2(\overline{\Omega}, S^2)$, then $T(u) = 0$. More generally, if $u \in H_\varphi^1(\Omega, S^2)$ and $u \in C^0(\overline{\Omega} \setminus \bigcup_{i=1}^m \{a_i\})$, then

$$(1.19) \quad T(u) = 4\pi \sum d_i \delta_{a_i}$$

where a_1, a_2, \dots, a_m are m points in Ω , and $d_i = \deg(u, a_i) =$ topological degree of u restricted to any small sphere around a_i . Here, the relation (1.19) is to be interpreted in the sense of distributions, i.e.,

$$-\int_{\Omega} D(u) \cdot \zeta + \int_{\partial\Omega} (\text{Jac } \varphi) \zeta = 4\pi \sum_i d_i \zeta(a_i)$$

for every $\zeta \in \text{Lip}(\Omega, \mathbb{R})$.

The distribution $T(u)$ is a simple but efficient tool which carries **information about the location** and the **topological strength of the singularities**. Note that T detects only “**topological singularities**”. For example an “**analytic singularity**” such as

$$u(x, y, z) = \left(\cos \frac{1}{r^\alpha} \sin \frac{1}{r^\beta}, \sin \frac{1}{r^\alpha} \sin \frac{1}{r^\beta}, \cos \frac{1}{r^\beta} \right),$$

with $r^2 = x^2 + y^2 + z^2$, $0 < \alpha < 1/2$, $0 < \beta < 1/2$, goes undetected and $T(u) = 0$.

Since $u = \varphi$ on $\partial\Omega$ we have

$$(1.20) \quad \sum d_i = -\deg(\varphi) = 0.$$

The points a_i with $d_i > 0$ (resp. $d_i < 0$) are called **positive** (resp. **negative**). We list the positive points with each a_i repeated d_i times in the list. Write this list as p_1, p_2, \dots, p_k , with each p_j one of the positive points. Likewise, list the negative points and write them as n_1, n_2, \dots, n_k . The points of degree zero, i.e., the “**non-topological singularities**”, are omitted from these two lists. Rewriting (1.19) we may now state

Theorem 7. *Assume $u \in H_\varphi^1(\Omega, S^2)$ and u is continuous on $\overline{\Omega}$ except at a finite number of points in Ω . Then*

$$(1.21) \quad T(u) = 4\pi \left(\sum_{i=1}^k \delta_{p_i} - \sum_{i=1}^k \delta_{n_i} \right),$$

and $L(u)$, defined by (1.17), can be represented as

$$(1.22) \quad L(u) = \underset{\sigma}{\text{Min}} \sum_{i=1}^k |p_i - n_{\sigma(i)}|$$

where the minimum in (1.22) is taken over all permutations σ of the integers $\{1, 2, \dots, k\}$.

Formula (1.22) is an immediate consequence of (1.17), (1.21), and the following general

Lemma 2 (H. Brezis, J.-M. Coron and E. Lieb [18]). *Assume X is a metric space. Let p_1, p_2, \dots, p_k and n_1, n_2, \dots, n_k be $2k$ points in X .*

Then

$$(1.23) \quad \begin{aligned} & \text{Max}_{\zeta} \left\{ \sum_{i=1}^k (\zeta(p_i) - \zeta(n_i)); \zeta \in \text{Lip}(\Omega, \mathbb{R}) \text{ and } |\zeta|_{\text{Lip}} \leq 1 \right\} \\ & = \text{Min}_{\sigma} \sum_{i=1}^k d(p_i, n_{\sigma(i)}), \end{aligned}$$

where $|\zeta|_{\text{Lip}} = \text{Sup}_{x \neq y} |\zeta(x) - \zeta(y)|/d(x, y)$.

L is called the **length of a minimal connection** connecting the positive points to the negative points.

Note that the distribution $T(u)$ and the number $L(u)$ are well-defined, through the formulas (1.15) and (1.17), for **every** $u \in H^1_{\varphi}(\Omega, S^2)$. And then one has the following:

Theorem 7' (H. Brezis and P. Mironescu). *Given any $u \in H^1_{\varphi}(\Omega, S^2)$, there exist two infinite sequences p_1, p_2, \dots and n_1, n_2, \dots in Ω such that*

$$(1.24) \quad \sum_{i=1}^{\infty} |p_i - n_i| < \infty$$

and

$$(1.25) \quad T(u) = 4\pi \sum_{i=1}^{\infty} (\delta_{p_i} - \delta_{n_i}).$$

Note that the right hand side in (1.25) is **not a measure** but a **distribution** acting on $\text{Lip}(\Omega, \mathbb{R})$ through the formula

$$< \sum_i (\delta_{p_i} - \delta_{n_i}), \zeta > = \sum_i (\zeta(p_i) - \zeta(n_i)).$$

The proof follows closely an argument in J. Bourgain, H. Brezis and P. Mironescu [10] (see Theorem 1). Moreover, $L(u)$ (defined by (1.17)) has an interpretation in terms of a minimal connection:

$$(1.26) \quad \begin{aligned} L(u) & = \text{Max}_{\zeta} \left\{ \sum_{i=1}^{\infty} (\zeta(p_i) - \zeta(n_i)); |\zeta|_{\text{Lip}} \leq 1 \right\} \\ & = \text{Inf}_{(\tilde{p}_i), (\tilde{n}_i)} \left\{ \sum_{i=1}^{\infty} |\tilde{p}_i - \tilde{n}_i|; T(u) = \sum_{i=1}^{\infty} (\delta_{\tilde{p}_i} - \delta_{\tilde{n}_i}) \right\}. \end{aligned}$$

Useful related results are:

Theorem 8 (F. Bethuel [4], [5], F. Bethuel and X. Zheng [9]). *Let \mathcal{R} denote the class of maps in $H^1_{\varphi}(\Omega, S^2)$ which are smooth on $\overline{\Omega}$, except at a finite number of points. Then \mathcal{R} is dense in H^1_{φ} (in the strong H^1 topology).*

Theorem 9 (F. Bethuel [5]). *Let $u \in H^1_{\varphi}(\Omega, S^2)$. Then $L(u) = 0$ if and only if there is a sequence (u_n) in $C^1_{\varphi}(\overline{\Omega}, S^2)$ such that $u_n \rightarrow u$ in H^1 .*

One may also ask whether a kind of “converse” to Theorem 7' holds:

Open Problem 4. Given any sequence of points $(p_i), (n_i)$ in Ω with $\sum_{i=1}^{\infty} |p_i - n_i| < \infty$, can one find some $u \in H_{\varphi}^1(\Omega, S^2)$ such that

$$T(u) = 4\pi \sum_{i=1}^{\infty} (\delta_{p_i} - \delta_{n_i})?$$

What is the optimal bound for $\int |\nabla u|^2$ in terms of φ and the length of a minimal connection $L = \text{Max}_{\zeta} \{ \sum_{i=1}^{\infty} (\zeta(p_i) - \zeta(n_i)); |\zeta|_{\text{Lip}} \leq 1 \}$?

The dipole construction introduced in H. Brezis, J.-M. Coron and E. Lieb [18] (see also F. Bethuel [5] and T. Rivière [51]) should be a basic ingredient in the construction of u .

We refer to M. Giaquinta, G. Modica and J. Souček [30], R. L. Jerrard and H. M. Soner [41], and G. Alberti, S. Baldo and G. Orlandi [1] for related questions in a more general geometric setting.

We now return to Open Problem 3. The first observation is that E_{rel} (defined by (1.13) or by (1.18)) is lower-semicontinuous (l.s.c.) for the weak topology on H_{φ}^1 . Therefore,

$$(1.27) \quad \text{Min}_{u \in H_{\varphi}^1(\Omega, S^2)} E_{\text{rel}}(u) \quad \text{is achieved.}$$

In fact, for every $t \in [0, 1]$ the functional

$$E_t(u) = \int_{\Omega} |\nabla u|^2 + 8\pi t L(u)$$

defined on $H_{\varphi}^1(\Omega, S^2)$ is l.s.c. for the weak topology on H_{φ}^1 , and thus

$$(1.28) \quad \text{Min}_{u \in H_{\varphi}^1(\Omega, S^2)} E_t(u) \quad \text{is achieved.}$$

Moreover, any critical point of E_t on $H_{\varphi}^1(\Omega, S^2)$ (and in particular any minimizer) is a weakly harmonic map; i.e., it satisfies (1.2). In particular, this machinery allows one (in general) to construct a **family** of weakly harmonic maps with a given boundary condition φ . When $t = 0$, the minimizers of $E_0 = E$ usually have singularities. One may hope that by “turning on” the L -term in E_t , the singularities of opposite signs will attract each other and eventually coalesce as $t \rightarrow 1$. Unfortunately, little is known about the regularity of minimizers for E_t .

Open Problem 5. Let u be a minimizer for $E_t, 0 < t < 1$, in $H_{\varphi}^1(\Omega, S^2)$. Is u smooth in $\bar{\Omega}$ except at a finite number of points in Ω ? Let u be a minimizer for E_1 . Is u smooth in $\bar{\Omega}$? (Try first to show that $L(u) = 0$.)

Partial regularity results for E_t are discussed in M. Giaquinta, G. Modica and J. Souček [30].

2. MODEL CASE II: $M = B^2$ AND $N = S^2$

In this section we take $\Omega = M = B^2$, the unit disc in \mathbb{R}^2 , and $N = S^2$, the unit sphere in \mathbb{R}^3 . Given a smooth boundary condition $\varphi : \partial\Omega = S^1 \rightarrow S^2$, we

investigate the minimization problem

$$(2.1) \quad \text{Min}_{\substack{u: \Omega \rightarrow S^2 \\ u = \varphi \text{ in } \partial\Omega}} E(u), \text{ where } E(u) = \int_{\Omega} |\nabla u|^2.$$

There are two major differences with Model Case I:

1) Here, there is no topological obstruction to regularity. For **any** smooth Dirichlet data φ , the class $C_{\varphi}^{\infty}(\overline{\Omega}, S^2)$ is nonempty; the topological condition $\deg(\varphi) = 0$ encountered in Section 1 has no counterpart.

2) A remarkable result of F. Hélein asserts that **any** weak solution $u \in H_{\varphi}^1(\Omega, S^2)$ of (1.2) is smooth. This is true in a much more general setting: the target N can be a manifold (of any dimension) provided the domain Ω is 2-dimensional.

Theorem 10 (F. Hélein [37], [38]). *Assume $u \in H_{\varphi}^1(\Omega, N)$ is a weakly harmonic map, where N is any Riemannian manifold. Then u is smooth.*

Remark 1. The regularity result of F. Hélein is very delicate and sensitive to the **special structure** of the harmonic map equation (1.2) (or (0.3)). For other systems similar in appearance to (1.2) – with quadratic growth in ∇u – the same conclusion might fail: weak H^1 solutions are **not** smooth; see e.g. S. Hildebrandt [40]. There seems to be some kind of **compensation** phenomenon, possibly similar to the null-condition of S. Klainerman; see e.g. S. Klainerman and M. Machedon [44], [45]. Another well-known equation which has the property that weak H^1 solutions are smooth is the system $\Delta u = u_x \wedge u_y$ occurring in the study of surfaces of constant mean curvature (see Wente [59], H. Brezis and J.-M. Coron [17]). An interesting direction of research is

Open Problem 6. Let $u \in H^1(\Omega, N)$ be a weak solution of

$$-\Delta u = F(x, u, \nabla u) \text{ in } \Omega \subset \mathbb{R}^2.$$

For which nonlinearities F can one conclude that u is smooth?

Here, one of the intriguing directions of research is the existence of **multiple** solutions for (1.2). Such a question was originally raised by M. Giaquinta and S. Hildebrandt [29]. Clearly,

$$\text{Min}_{u \in H_{\varphi}^1(\Omega, S^2)} \int |\nabla u|^2 \text{ is achieved,}$$

and the range $u(\Omega)$ of a minimizer lies in the “small” spherical cap on S^2 enclosed by the curve $\varphi(\partial\Omega)$. M. Giaquinta and S. Hildebrandt asked whether there is more than one solution to (1.2). The answer is indeed positive.

Theorem 11 (H. Brezis and J.-M. Coron [16], J. Jost [42]). *Given any $\varphi \neq \text{constant}$, problem (1.2) admits at least two distinct solutions.*

The technique we used in H. Brezis and J.-M. Coron [16] illustrates (in a rather elementary setting) the need to work with **homotopy classes within the framework of Sobolev spaces** (more about this in Section 4). The heart of the matter is

Lemma 3. *The space $H_{\varphi}^1(\Omega, S^2)$ admits infinitely many connected components.*

In order to construct these components we use a “relative” degree. Fix a “reference” map $u_0 \in H_\varphi^1(\Omega, S^2)$, for example a minimizer for (2.1). Given any $u \in H_\varphi^1(\Omega, S^2)$ set

$$v = \begin{pmatrix} u \\ u_0 \end{pmatrix} : S^2 \rightarrow S^2;$$

here we identify one copy of Ω with the upper hemisphere S_+^2 and another copy of Ω with lower hemisphere S_-^2 . The two maps u and u_0 glue well since they agree on the hemisphere and thus $v \in H^1(S^2, S^2)$. The key point is

Lemma 4. *Maps in $H^1(S^2, S^2)$ have a well-defined degree.*

Lemma 4 is far from obvious since maps in $H^1(S^2, S^2)$ need **not** be continuous and one cannot rely on standard degree. One way of defining degree is via the classical Kronecker formula

$$(2.2) \quad \deg(w) = \frac{1}{4\pi} \int_{S^2} \text{Jac}(w)$$

for any $w \in C^1(S^2, S^2)$. If $w \in H^1(S^2, S^2)$, the right hand side in (2.2) makes sense (since $\text{Jac}(w)$ is the determinant of a 2×2 matrix with L^2 entries). An additional density argument due to R. Schoen and K. Uhlenbeck [56] allows one to conclude that the r.h.s. in (2.2) belongs to \mathbb{Z} for every $w \in H^1(S^2, S^2)$ (see Section 4).

One may then decompose $H_\varphi^1(\Omega, S^2)$ into disjoint classes

$$H_\varphi^1(\Omega, S^2) = \bigcup_{k \in \mathbb{Z}} \mathcal{E}_k$$

where

$$\mathcal{E}_k = \{u \in H_\varphi^1(\Omega, S^2); \deg \begin{pmatrix} u \\ u_0 \end{pmatrix} = k\}.$$

It is tempting to minimize the energy in **each** class \mathcal{E}_k . Here we encounter a new difficulty which is common in “critical” problems involving “bubbling of spheres” (as in J. Sacks and K. Uhlenbeck [54]): “lack of compactness”, etc.; namely, $\inf_{\mathcal{E}_k} E$ need **not** be achieved, the reason being that a minimizing sequence converges only **weakly** in H^1 but the class \mathcal{E}_k defined via (2.2) is **not** closed under weak convergence. To overcome this difficulty we use a method similar to the one introduced by Th. Aubin [3] for the Yamabe problem. This device works (assuming $\varphi \not\equiv \text{Constant}$) and yields the existence of a minimizer for E in **one** of the classes \mathcal{E}_{+1} or \mathcal{E}_{-1} . The existence of more solutions for the equation (1.2) is not known even when φ has a simple explicit form. Consider the Dirichlet data

$$(2.3) \quad \varphi(x, y) = (Rx, Ry, \sqrt{1 - R^2}) \text{ for } (x, y) \in \partial\Omega$$

with $0 < R < 1$. In this case one can write down explicitly two solutions:

$$\underline{u}(x, y) = \frac{2\lambda}{\lambda^2 + r^2}(x, y, \lambda) + (0, 0, -1)$$

and

$$\bar{u}(x, y) = \frac{2\mu}{\mu^2 + r^2}(x, y, -\mu) + (0, 0, 1)$$

with $(x, y) \in \Omega$, $r^2 = x^2 + y^2$, $\lambda = \frac{1}{R} + \sqrt{\frac{1}{R^2} - 1}$ and $\mu = \frac{1}{R} - \sqrt{\frac{1}{R^2} - 1}$.

It is not difficult to see that \underline{u} is the absolute minimizer of E in $H_\varphi^1(\Omega, S^2)$. Using \underline{u} as the reference map u_0 , we see that $\bar{u} \in \mathcal{E}_{-1}$ and that \bar{u} is the minimizer in the class \mathcal{E}_{-1} . In addition, one can show that besides these two, **no other minimizer exists in any of the classes \mathcal{E}_k** .

Open Problem 7. Are there other solutions of (1.2) besides \underline{u} and \bar{u} ?

Either way, the answer to Open Problem 7 would be illuminating. A negative answer (only 2 solutions) might possibly shed some light on the important questions of whether solutions of some nonlinear systems inherit the symmetry of the data. A positive answer (more than 2 solutions) might involve the development of new techniques for finding (nonminimizing) critical points in variational problems with lack of compactness.

3. MODEL CASE III: $M = B^2$ (OR B^3) AND $N = S^1$

In this section we first take $\Omega = M = B^2$ and later $\Omega = M = B^3$. N will always be S^1 , the unit circle in \mathbb{R}^2 .

3.1 $M = \Omega = B^2$ and $N = S^1$. Given a smooth boundary condition $\varphi : \partial\Omega = S^1 \rightarrow S^1$, we investigate the minimization problem

$$(3.1) \quad \text{Min}_{\substack{u: \Omega \rightarrow S^1 \\ u = \varphi \text{ on } \partial\Omega}} E(u), \text{ where } E(u) = \int_{\Omega} |\nabla u|^2.$$

A new feature occurs, which was not present in Model Cases I and II.

a) If $\deg(\varphi) = 0$, there is **no** topological obstruction: $C_\varphi^\infty(\bar{\Omega}, S^1)$ is not empty. In this case we will see (in Theorem 12) that problem (3.1) is quite easy.

b) However, if $\deg(\varphi) \neq 0$, there is a topological obstruction to regularity since $C_\varphi^0(\bar{\Omega}, S^1)$ is **empty**. In some sense, the situation is “worse” than in Model Case II! Recall that, there, if $\deg(\varphi) \neq 0$, $C_\varphi^0(\bar{\Omega}, S^2)$ was empty, but at least the class $H_\varphi^1(\Omega, S^2)$ was **not** empty. Here, if $\deg(\varphi) \neq 0$, both $C_\varphi^0(\bar{\Omega}, S^1)$ and the natural energy class $H_\varphi^1(\Omega, S^1)$ are empty (see Lemma 5). Thus problem (3.1) makes no sense.

In fact, for many years, this case was considered “off-limits”; to quote R. Hamilton [32]: “If there is a topological obstruction to extending [the boundary condition] h , then $\mathcal{M}_h(X, Y)$ is empty and nothing more can be said.” At the beginning of the 90’s the “taboo” was broken. The incentive came from problems in physics, specifically in the theory of superfluids and superconductors. Their mathematical modeling resembles (3.1), and the “solutions” in the physics literature involve point vortices (in $2 - d$) or line vortices (in $3 - d$). The new idea is to tackle the “impossible” problem (3.1) via a **Ginzburg - Landau mechanism**; namely, the constraint $u : \Omega \rightarrow S^1$ (which is at the heart of the topological obstruction) is relaxed and replaced by a Ginzburg - Landau “penalty”. More precisely, one considers **all** functions $u : \Omega \rightarrow \mathbb{R}^2$, but the standard energy $\int_{\Omega} |\nabla u|^2$ is replaced by the **Ginzburg -**

Landau energy

$$(3.2) \quad E_\varepsilon(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{1}{4\varepsilon^2} \int_{\Omega} (|u|^2 - 1)^2$$

with a small parameter $\varepsilon > 0$. The strategy is to study the approximate problem

$$(3.3) \quad \text{Min}_{u \in H_\varphi^1(\Omega, \mathbb{R}^2)} E_\varepsilon(u)$$

and to analyze the limiting behavior of minimizers u_ε as $\varepsilon \rightarrow 0$. This program was initiated in 1992 by F. Bethuel, H. Brezis and F. Hélein [8] and was followed by many subsequent developments.

Here are some precise statements.

Lemma 5. *Let $\varphi : \partial\Omega = S^1 \rightarrow S^1$ be a smooth map. Then*

$$H_\varphi^1(\Omega, S^1) \neq \emptyset \iff \text{deg}(\varphi) = 0.$$

The proof of Lemma 5 involves the $H^{1/2}$ -degree theory for maps of S^1 into itself (see H. Brezis and L. Nirenberg [21]); it stresses once more the need to study Homotopy theory in the framework of Sobolev spaces (see Section 4).

When $\text{deg}(\varphi) = 0$, we have a complete and simple answer to problem (3.1).

Theorem 12. *Given any smooth map $\varphi : \partial\Omega = S^1 \rightarrow S^1$ with $\text{deg}(\varphi) = 0$, there exists a unique minimizer u of problem (3.1) – in fact a unique solution $u \in H_\varphi^1(\Omega, S^1)$ of the corresponding PDE (1.2). More precisely, $u = e^{i\psi}$ where ψ is the solution of*

$$\begin{aligned} \Delta\psi &= 0 && \text{in } \Omega \\ \psi &= \psi_0 && \text{on } \partial\Omega \end{aligned}$$

and ψ_0 is a smooth lifting of φ , i.e., $e^{i\psi_0} = \varphi$ on $\partial\Omega$.

The proof of Theorem 12 relies heavily on the theory of lifting for Sobolev maps into S^1 initiated by F. Bethuel and X. Zheng [9] and completed by J. Bourgain, H. Brezis and P. Mironescu [10].

When $\text{deg}(\varphi) \neq 0$ the situation is much more complicated. Here is a typical result:

Theorem 13 (F. Bethuel, H. Brezis and F. Hélein [8]). *For a subsequence $\varepsilon_n \rightarrow 0$, $u_{\varepsilon_n} \rightarrow u_*$ pointwise on Ω , except at a finite number of distinct points a_1, a_2, \dots, a_d in Ω , where $d = |\text{deg}(\varphi)|$. Moreover u_* is smooth on $\bar{\Omega}$, except at the points $(a_i), |u_*| = 1$ on Ω , and u_* satisfies the equation of harmonic maps (1.2) on $\Omega \setminus \bigcup_{i=1}^d \{a_i\}$. Moreover, if $\text{deg}(\varphi) > 0$, then, near each a_i ,*

$$(3.3) \quad u_*(x) \sim R \frac{(x - a_i)}{|x - a_i|} \text{ as } x \rightarrow a_i$$

for some rotation R .

The limit u_* can be viewed as a “minimizing harmonic map” in some generalized sense even though its energy $\int_\Omega |\nabla u_*|^2$ is infinite. In fact $\int_\Omega |\nabla u_\varepsilon|^2 \sim 2\pi d \log(1/\varepsilon)$ as $\varepsilon \rightarrow 0$. Infinite energies do not seem to bother physicists: one can “renormalize” the energy and keep only a **finite** quantity which plays an important role in locating the position of the singularities. We refer to the book of F. Bethuel, H. Brezis and F. Hélein [8] for a detailed discussion. We also point out that formula (3.3) implies

$$(3.4) \quad \text{deg}(u_*, a_i) = +1 \quad \forall i$$

if $\text{deg}(\varphi) > 0$ (otherwise $\text{deg}(u_*, a_i) = -1, \forall i$). Property (3.4) is consistent with the physical observation that all vortices have the same orientation. This is in contrast

to the situation described in Section 1, where singularities of opposite orientation do coexist. In view of (3.4) the count of the number of singularities is very easy.

Another important feature is that the limit u_* seems to be a kind of “**canonical**” solution for the “impossible” problem (3.1). It is in some sense “intrinsic” and does not depend on the Ginzburg-Landau approximation. There are several ways of relaxing the topological obstruction by introducing an ε -approximation: for example by drilling holes of size ε in Ω , or by considering the energy $\int_{\Omega} |\nabla u|^{2-\varepsilon}$. They all yield the **same limit** u_* (see F. Bethuel, H. Brezis and F. Hélein [8], and R. Hardt and F. H. Lin [36]).

3.2 $M = \Omega = B^3$ and $N = S^1$. Here one investigates problem (3.1) with a boundary condition $\varphi : \partial\Omega = S^2 \rightarrow S^1$ **having singularities**. An important example is the case where φ is smooth on $\partial\Omega$ except at a finite number of singularities a_i on $\partial\Omega$, and that, near each a_i ,

$$(3.5) \quad \varphi(z) \sim \left(\frac{z - a_i}{|z - a_i|} \right)^{d_i}, \quad z \in \partial\Omega,$$

modulo rotation. For simplicity one assumes that $\partial\Omega$ is flat near each a_i , so that the right hand side in (3.5) belongs to S^1 . Clearly

$$\sum d_i = 0,$$

and hence, one can list the points (a_i) as $(p_i), (n_i)$ (see Section 1). Here again one can prove that the natural energy class $H_{\varphi}^1(\Omega, S^1)$ is empty (this is already true locally, near a singularity; once more the proof uses the $H^{1/2}$ -degree theory for maps of S^1 into itself). So that problem (3.1) is impossible, and it is approached via the Ginzburg-Landau mechanism. The main result is the following:

Theorem 14 (T. Rivière [52], F. H. Lin and T. Rivière [47]). *For a subsequence $\varepsilon_n \rightarrow 0, u_{\varepsilon_n} \rightarrow u_*$ pointwise on Ω , except along a minimal connection Σ connecting in Ω the points p_i to the points n_i . Moreover u_* is smooth in $\bar{\Omega} \setminus \Sigma$ and u_* satisfies the equation of harmonic maps (1.2) in $\Omega \setminus \Sigma$.*

J. Bourgain, H. Brezis and P. Mironescu [12] have investigated the case of a general boundary condition $\varphi \in H^{1/2}(\partial\Omega, S^1)$.

An interesting

Open Direction 8. Investigate other situations (for general manifolds M and N) where the boundary condition $\varphi : \partial M \rightarrow N$ generates an obstruction and where the natural energy space $H_{\varphi}^1(M, N)$ is empty.

4. HOMOTOPY THEORY IN THE FRAMEWORK OF SOBOLEV SPACES

Let M and N be two compact Riemannian manifolds. One of the central questions in Topology is the study of homotopy classes of $C^0(M, N)$, i.e., the connected (or equivalently path-connected) components of the metric space $C^0(M, N)$. In other words, homotopy classes are the equivalence classes corresponding to the equivalence relation: $f \sim g$ if there exists a path $h(t) \in C^0([0, 1]; C^0(M, N))$ connecting f to g , i.e., $h(0) = f$ and $h(1) = g$. For example, when $M = N = S^n$, it is well-known that $f \sim g$ if and only if $\deg(f) = \deg(g)$.

In this section, I would like to present some new results concerning the study of homotopy classes when the space $C^0(M, N)$ is replaced by the Sobolev space $W^{1,p}(M, N)$.

Throughout this section M and N are smooth, connected, compact, oriented, Riemannian manifolds. We will always assume that $\partial N = \emptyset$, but M may or may not have a boundary; in particular the case where M is a domain in \mathbb{R}^n is of interest. The reader is encouraged to keep in mind elementary manifolds, such as spheres, balls and their products; the results are already of interest for such a simple situation.

Let us start with a simple observation about the scale of spaces $C^k(M, N)$ equipped with the metric

$$d(f, g) = \sum_{|\alpha| \leq k} \sup_{x \in M} d(D^\alpha f(x), D^\alpha g(x)).$$

In principle, for each k , one may introduce a new equivalence relation:

$$f \sim g \text{ in } C^k \iff f, g \text{ belong to the same path-connected component of } C^k;$$

i.e., there exists a homotopy $h(t) \in C([0, 1]; C^k(M, N))$ such that $h(0) = f$ and $h(1) = g$.

In fact, such a notion has no interest because of the “standard”

Lemma 6. *Let $f, g \in C^k(M, N)$; then $f \sim g$ in C^k if and only if $f \sim g$ in C^0 .*

The proof consists of smoothing the given homotopy $h \in C([0, 1]; C^0(M, N))$. For this purpose we may assume that $N \subset \mathbb{R}^K$ is an isometric embedding. Then $\rho_\varepsilon \star h$ does not take its values into N , but $\rho_\varepsilon \star h$ is uniformly close to h as $\varepsilon \rightarrow 0$ (because h is continuous). And then one may project $\rho_\varepsilon \star h$ back onto N for ε small.

As a consequence of Lemma 6 we see that the components of $C^k(M, N)$ shrink as k increases, but they “do not change their shape”. By contrast, we will see that the situation is **totally different** in the scale of Sobolev spaces $W^{1,p}$.

Let $1 \leq p < \infty$ be a real number. Let M and N be as above and $N \subset \mathbb{R}^K$. The spaces $W^{1,p}(M, \mathbb{R})$ and $W^{1,p}(M, \mathbb{R}^K)$ are defined as usual and equipped with the standard norm $\|f\|_{W^{1,p}}$. Set

$$W^{1,p}(M, N) = \{f \in W^{1,p}(M, \mathbb{R}^K); f(x) \in N \text{ a.e.}\},$$

equipped with the distance

$$d(f, g) = \|f - g\|_{W^{1,p}(M, \mathbb{R}^K)}.$$

As a metric space, $W^{1,p}(M, N)$ admits connected components and also path-connected components. In fact, they coincide because of the following result which is implicit in the work of F. B. Hang and F. H. Lin [33].

Proposition 1. *Given $f \in W^{1,p}(M, N)$, there exists $\varepsilon > 0$ (depending on f) such that*

$$d(g, f) < \varepsilon \implies g \sim f \text{ in } W^{1,p}(M, N).$$

We say that $g \sim f$ in $W^{1,p}$ if there is a path $h(t) \in C([0, 1]; W^{1,p}(M, N))$ such that $h(0) = f$ and $h(1) = g$.

A simple but useful example is the case $M = N = S^2$ which was already mentioned in Section 2. More generally, let us examine the space $W^{1,p}(S^2, S^2)$, $1 \leq p < \infty$, from the point of view of its components. One has to consider 3 different cases:

a) **Case $p > 2$.** It is not difficult to prove, following the same idea as in Lemma 6 and using the Sobolev embedding, $W^{1,p}(S^2, S^2) \subset C^0(S^2, S^2)$,

Lemma 7. *Let $f, g \in W^{1,p}(S^2, S^2)$; then $f \sim g$ in $W^{1,p}(S^2, S^2)$ if and only if $f \sim g$ in C^0 . In particular, the homotopy classes of $W^{1,p}(S^2, S^2)$ can be classified using their standard degree.*

b) **Case $p = 2$.** This is a very interesting case because it is a **limiting case** for the Sobolev embedding. $W^{1,2}$ is **not** contained in C^0 , so that the standard notion of degree is not well-defined. One may nevertheless still define a degree, as we already mentioned in Section 2, using the following strategy:

Step 1. If $f \in C^1(S^2, S^2)$, Kronecker's formula asserts that

$$(4.1) \quad \deg(f) = \frac{1}{4\pi} \int_{S^2} \det(\nabla f).$$

When $f \in W^{1,2}(S^2, S^2)$ the integral in (4.1) still makes sense. In order to prove that the right-hand side in (4.1) is an integer, one relies on

Step 2. This is an important observation due to Schoen–Uhlenbeck [56]. If $f \in W^{1,2}(S^2, S^2)$, then $\rho_\varepsilon \star f$ does **not** take its values into S^2 , and $\rho_\varepsilon \star f$ does **not** converge uniformly to f as $\varepsilon \rightarrow 0$ (otherwise f would be continuous). However, one can prove that $|(\rho_\varepsilon \star f)(x)| \rightarrow 1$ **uniformly** in x . This is a consequence of the fact that $f \in VMO$ (see H. Brezis and L. Nirenberg [21]) which, in turn, follows from Poincaré's inequality

$$\int_B \left| f - \int_B f \right| \leq |B|^{1/2} \int_B |\nabla f|.$$

As a result one may consider

$$f_\varepsilon(x) = P((\rho_\varepsilon \star f)(x))$$

where P is the projection on S^2 , which is well-defined near S^2 . Then $f_\varepsilon \in C^\infty(S^2, S^2)$ and $f_\varepsilon \rightarrow f$ in $W^{1,2}$.

Since $\deg(f_\varepsilon) \in \mathbb{Z}$ and

$$\deg(f_\varepsilon) \rightarrow \frac{1}{4\pi} \int_{S^2} \det(\nabla f) \text{ as } \varepsilon \rightarrow 0$$

we conclude that

$$\frac{1}{4\pi} \int_{S^2} \det(\nabla f) \in \mathbb{Z}.$$

We now **set**, for every $f \in W^{1,2}$

$$\deg(f) = \frac{1}{4\pi} \int_{S^2} \det(\nabla f).$$

It was subsequently observed in H. Brezis and L. Nirenberg [21] that $\deg(f)$ can still be defined for every $f \in W^{1,2}$ **without** using formula (4.1), only Step 2. Indeed, if $f \in W^{1,2}(S^2, S^2)$, consider f_ε as above (for $\varepsilon < \varepsilon_0$). Then $\deg(f_\varepsilon)$ is independent of ε , for $\varepsilon \in (0, \varepsilon_0)$, because one may use ε itself as the homotopy parameter, i.e., connect f_{ε_1} and f_{ε_2} via the homotopy $h(t) = f_{t\varepsilon_1 + (1-t)\varepsilon_2}$. We take **as the definition** of $\deg(f)$ the integer $\deg(f_\varepsilon)$ (for $0 < \varepsilon < \varepsilon_0$).

Proposition 2. *Assume $f, g \in W^{1,2}(S^2, S^2)$. Then $f \sim g$ in $W^{1,2}$ if and only if $\deg(f) = \deg(g)$. Consequently, $W^{1,2}(S^2, S^2)$ still admits infinitely many homotopy classes and they are classified using degree.*

c) **Case $p < 2$.** This case was not considered in H. Brezis and J.-M. Coron [16]. But in H. Brezis and L. Nirenberg [21] we observed that no degree theory (in any reasonable sense) can be defined. Indeed, the identity map can be homotoped to a constant map! This is done as follows. Fix any point $a \in \mathbb{R}^3$ with $|a| = 2$. Consider the path

$$h(t)(x) = \frac{x - ta}{|x - ta|}, \quad t \in [0, 1], x \in S^2.$$

Clearly h is smooth for $t \in [0, 1/2) \cup [1/2, 1]$ and $x \in S^2$. The only difficulty occurs at $t = 1/2$ because $a/2 \in S^2$. However, it is not difficult to check that $h(1/2) \in W^{1,p}(S^2, S^2)$ for every $p < 2$ and moreover

$$h(t) \in C([0, 1]; W^{1,p}(S^2, S^2))$$

for every $p < 2$. Finally it is clear that $h(1)$ has degree zero, and thus it can be homotoped to a constant in the C^0 sense (and thus in the C^1 sense by Lemma 6, and hence in the $W^{1,p}$ sense). Putting all this together we see that $h(0) = Id$ is homotopic to a constant in every $W^{1,p}$, $p < 2$.

We went one step further in H. Brezis and Y. Li [19] and proved

Theorem 15. *The space $W^{1,p}(S^2, S^2)$ is path-connected for every $p < 2$.*

We now have a complete picture for $W^{1,p}(S^2, S^2)$. When $p < 2$, $W^{1,p}(S^2, S^2)$ consists of one piece. At $p = 2$, $W^{1,2}(S^2, S^2)$ splits into infinitely many pieces. As p increases from 2 to ∞ , these pieces “shrink” but “do not change their shape”.

Exactly the same type of conclusion holds for $W^{1,p}(S^n, S^n)$. When $p < n$, $W^{1,p}(S^n, S^n)$ is path-connected. At $p = n$, a degree theory is well-defined, and thus $W^{1,n}(S^n, S^n)$ splits into infinitely many pieces. As p increases from n to ∞ , these pieces shrink without changing their shape.

At this stage, one would be inclined to believe that this phenomenon is typical. When $p < \dim M$, $W^{1,p}(M, N)$ is path-connected. As p increases from $\dim M$ to ∞ , $W^{1,p}(M, N)$ admits path-connected components similar to the ones of $C^0(M, N)$, and they shrink without changing their shape. The second assertion (for $p \geq \dim M$) is indeed true. However, the first assertion (for $p < \dim M$) is totally wrong. This was first pointed out in an important paper of B. White [61] and rediscovered a few years later by J. Rubinstein and P. Sternberg [53]: namely, “some topology” still survives for $W^{1,p}(M, N)$, even when $p < \dim M$. In fact $W^{1,p}(M, N)$ may have a very rich structure from the point of view of homotopy classes when $p < \dim M$. I will present later some striking examples.

Theorem 16 (B. White [61], J. Rubinstein and P. Sternberg [53]). *Let $M = \Omega \times S^1$ where Ω is the unit disc in \mathbb{R}^2 ; i.e., Ω is a solid torus in \mathbb{R}^3 . Let $N = S^1$. Then any $f \in W^{1,2}(M, N)$ admits a well-defined degree (stable under $W^{1,2}$ convergence).*

More precisely, write

$$f(x, \lambda) : S^1 \times D \rightarrow S^1;$$

then

$$\varphi(\lambda) = \deg f(\cdot, \lambda),$$

which is well-defined for a.e. $\lambda \in D$ (since $f(\cdot, \lambda) \in W^{1,2}(S^1, S^1)$ for a.e. $\lambda \in D$), is in fact a constant (a.e.).

Remark 2. It is quite surprising that a degree may still be defined even though $p = 2 < 3 = \dim M$. I should point out, however, that the same conclusion **fails** for $p \in [1, 2)$. Note that when $p \in [1, 2)$, $\varphi(\lambda)$ is still well-defined for a.e. $\lambda \in D$. However, φ is **not** constant in general.

Remark 3. The conclusion that φ is a constant (a.e.) may be related to a variety of results which have emerged in recent years in works of J. Bourgain, H. Brezis and P. Mironescu [10] [11] (see also H. Brezis [14]) about conditions implying that a given measurable function is constant. Here is such a typical result.

Theorem 17. *Let $G \subset \mathbb{R}^N$ be a connected open set and let $\varphi : G \rightarrow \mathbb{Z}$ be a measurable function. Assume that*

$$\int_G \int_G \frac{|\varphi(\lambda) - \varphi(\mu)|^p}{|\lambda - \mu|^{N+1}} d\lambda d\mu < \infty$$

(any $p \geq 1$). Then φ is a constant (a.e.).

An extension of Theorem 16 to higher dimensions is given by

Theorem 18 (H. Brezis, Y. Li, P. Mironescu and L. Nirenberg [20]). *Let $M = \Omega = S^n \times \Lambda$, where $n \geq 1$, and $\Lambda \subset \mathbb{R}^k$ is any open connected set with $k \geq 1$. If $p \geq n + 1$, any $f \in W^{1,p}(M, N)$ has a well-defined degree. More precisely, write*

$$f(x, \lambda) : S^n \times \Lambda \rightarrow S^n;$$

then

$$\varphi(\lambda) = \deg f(\cdot, \lambda),$$

which is well-defined for a.e. $\lambda \in \Lambda$ (since $p > n$), is in fact a constant.

Remark 4. Theorem 18 shows again that some topology still “persists” much below the critical Sobolev exponent $p = \dim M$. The condition $p \geq n + 1$ is usually **much weaker** than the condition $p \geq \dim M = n + k$, especially when k is large.

Going back to the study of the path-connected components for $W^{1,p}(M, N)$, here is a complement to Theorem 18, which gives a **complete classification** of the homotopy classes of $W^{1,p}$ when the parameter space Λ is a ball.

Theorem 19 (H. Brezis and Y. Li [19]). *Let $M = \Omega = S^n \times \Lambda$ where $n \geq 1$ and Λ is the unit ball in \mathbb{R}^k (any $k \geq 1$). Let $f, g \in W^{1,p}(M, N)$. If $p \geq n + 1$, then $f \sim g$ in $W^{1,p}$ if and only if $\deg(f) = \deg(g)$ (where \deg is meant in the sense of Theorem 18). If $p < n + 1$, then, always, $f \sim g$ in $W^{1,p}$; i.e., $W^{1,p}$ is path-connected.*

At this stage we decided with Yanyan Li to initiate a general investigation of the homotopy classes of $W^{1,p}(M, N)$ for a general pair of manifolds M and N and for a general $p \geq 1$.

Here is a first (somewhat surprising) result.

Theorem 20 (H. Brezis and Y. Li [19]). *For any pair M and N with $\dim M \geq 2$ and for any $p \in [1, 2)$, $W^{1,p}(M, N)$ is path-connected.*

We also introduced in H. Brezis and Y. Li [19] a concept which plays a very important role in identifying the possible “jumps” in homotopy type for $W^{1,p}$ as p varies. Here it is.

Let $p > 1$. Let $0 < \varepsilon < p - 1$. Clearly

$$W^{1,p+\varepsilon}(M, N) \subset W^{1,p-\varepsilon}(M, N).$$

Given $f \in W^{1,p}(M, N)$, we denote by $[f]_p$ its homotopy class in $W^{1,p}$. We denote by $W^{1,p}/\widetilde{\sim}$ the quotient of $W^{1,p}$ by the equivalence relation $f \widetilde{\sim} g$ (meaning $f \sim g$ in $W^{1,p}$).

Obviously, if $f, g \in W^{1,p+\varepsilon}$ and $f \sim g$ in $W^{1,p+\varepsilon}$, then $f \sim g$ in $W^{1,p-\varepsilon}$. As a result we have a canonical map

$$i_{p+\varepsilon,p-\varepsilon} : W^{1,p+\varepsilon}/\widetilde{\sim}_{p+\varepsilon} \rightarrow W^{1,p-\varepsilon}/\widetilde{\sim}_{p-\varepsilon}.$$

Definition. We say that a **change of topology** occurs at p if for every $\varepsilon \in (0, p - 1)$, the map $i_{p+\varepsilon,p-\varepsilon}$ is **not** bijective. We denote by $CT(M, N)$ the set of p 's for which a change of topology occurs at p .

If a change of topology occurs at p , two things may happen: either $i_{p+\varepsilon,p-\varepsilon}$ is not injective or $i_{p+\varepsilon,p-\varepsilon}$ is not surjective (or both!). Roughly speaking, **not injective** means that there are 2 maps $f, g \in W^{1,p+\varepsilon}$ such that f and g are **not** homotopic in $W^{1,p+\varepsilon}$ while $f \sim g$ in $W^{1,p-\varepsilon}$. In other words, one component (at least) of $W^{1,q}(M, N)$ **splits** into two (or more) distinct components as q **increases** from $p - \varepsilon$ to $p + \varepsilon$. Another viewpoint is to say that two distinct components of $W^{1,q}$ have **coalesced** as q decreases from $p + \varepsilon$ to $p - \varepsilon$. This is a common situation, which we have already encountered above. For example, a change of topology for $W^{1,p}(S^2, S^2)$ occurs at $p = 2$, because $i_{2+\varepsilon,2-\varepsilon}$ is **not injective**: any two maps $f, g \in W^{1,2+\varepsilon}$ are homotopic in $W^{1,2-\varepsilon}$, while they need not be in $W^{1,2+\varepsilon}$ (unless their degree is the same).

On the other hand, the fact that $i_{p+\varepsilon,p-\varepsilon}$ is **not surjective** means that a new component of $W^{1,q}$ appears (out of nowhere!) as q decreases from $p + \varepsilon$ to $p - \varepsilon$. This situation is more unusual, and it does not occur in the example $W^{1,p}(S^n, S^n)$. It means that some map $f \in W^{1,p-\varepsilon}$ **cannot be homotoped** in $W^{1,p-\varepsilon}$ to any $g \in W^{1,p+\varepsilon}$. In particular, such a map f cannot be smooth and it cannot be homotoped in $W^{1,p-\varepsilon}$ to any smooth map, so it must have a rather “robust” singularity!

We may now reformulate Theorem 15 and Theorem 19 using the above notion.

Theorem 21. *We have $CT(S^2, S^2) = \{2\}$ or more generally $CT(S^n, S^n) = \{n\}$ for any $n \geq 2$.*

Theorem 22. *We have $CT(S^n \times \Lambda, S^n) = \{n + 1\}$, where Λ is the unit ball in \mathbb{R}^k (for any $k \geq 1$).*

We also observed in H. Brezis and Y. Li [19] that a change in topology may occur for **several** values of p , not just one, as in the above examples. Here is such a situation, with a “**cascade of mergings**”.

Theorem 23. *We have $CT(S^1 \times S^2, S^1 \times S^2) = \{2, 3\}$.*

This is somewhat natural because we have here (at least) two invariants: write $f = (f_1(x, y), f_2(x, y))$ and set $d_1 = \deg f_1(\cdot, y)$ and $d_2 = \deg f_2(x, \cdot)$. Note that d_1 is well-defined and independent of $y \in S^2$ when $f \in W^{1,p}$ and $p \geq 2$ (by Theorem 18), while d_2 is well-defined and independent of $x \in S^1$ only when $p \geq 3$ (again by Theorem 18). Some new invariants appear, as p increases from 1 to ∞ , when crossing the values $p = 2$ and $p = 3$.

We also proposed in H. Brezis and Y. Li [19] two conjectures:

Conjecture 1. $CT(M, N)$ consists only of integers; i.e., change of topology for $W^{1,p}(M, N)$ occurs only when p is an integer.

Conjecture 2. Any map $f \in W^{1,p}(M, N)$ can always be connected in $W^{1,p}(M, N)$ to a smooth map (for any p and for any M, N).

Both conjectures were solved in a beautiful piece of work by F. B. Hang and F. H. Lin [33].

The answer to Conjecture 1 is positive:

Theorem 24 (F. B. Hang and F. H. Lin [33]). $CT(M, N)$ consists only of integers.

Theorem 24 is an immediate consequence of the following remarkable result which provides a “reduction” of the study of homotopy classes for $W^{1,p}$ to more classical concepts in Topology. Given $p \geq 2$, let $M^{[p]-1}$ be the $([p] - 1)$ -skeleton of M .

Theorem 25 (F. B. Hang and F. H. Lin [33]). Let $f, g \in W^{1,p}(M, N)$. Then $f \sim g$ in $W^{1,p}$ if and only if $f|_{\Sigma} \sim g|_{\Sigma}$ in C^0 for every generic Σ in $M^{[p]-1}$.

Note that $\dim \Sigma \leq [p] - 1$, and for a generic Σ , $f|_{\Sigma} \in W^{1,p}(\Sigma)$, while $p > \dim \Sigma$ (since $[p] \geq \dim \Sigma + 1$). Thus for a generic Σ , $f|_{\Sigma} \in C^0$, by the Sobolev embedding.

Concerning Conjecture 2, we had presented in H. Brezis and Y. Li [19] several cases where Conjecture 2 is true: for example when $\dim M = 2$ (for any N), or $\dim M = 3$ and $\partial M \neq \emptyset$ (for any N), or $N = S^1$ (for any M). However, F. B. Hang and F. H. Lin [33] found some situations where Conjecture 2 fails. Here is such an example:

Theorem 26 (F. B. Hang and F. H. Lin [33]). Let $M = \mathbb{RP}^3$ and $N = \mathbb{RP}^2$. Then $CT(M, N) = \{2, 3\}$. Moreover there are maps $f \in W^{1,p}(M, N)$, with $p \in (2, 3)$, such that f cannot be connected in $W^{1,p}$ to any smooth map.

Open Direction 9. Compute $CT(M, N)$ when M and N are simple manifolds, e.g. spheres, products of spheres, product of spheres with balls, etc.

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