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On some questions of topology for S^1-valued fractional Sobolev spaces

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Abstract. The purpose of this paper is to describe the homotopy classes (i.e., path-connected components) of the space $W^{s,p}(\Omega; S^1)$. Here, $0 < s < \infty$, $1 < p < \infty$, Ω is a smooth, bounded, connected open set in $\mathbb{R}^N$ and

$$W^{s,p}(\Omega; S^1) = \{u \in W^{s,p}(\Omega; \mathbb{C});\, |u| = 1 \text{ a.e.}\}.$$

Our main results assert that $W^{s,p}(\Omega; S^1)$ is path-connected if $sp < 2$ while it has the same homotopy classes as $C^0(\bar{\Omega}; S^1)$ if $sp \geq 2$. We also present some results and open problems about density of smooth maps in $W^{s,p}(\bar{\Omega}; S^1)$.

Sobre algunas cuestiones topológicas para espacios de Sobolev fracionarios con valores en S^1

Resumen. El propósito de este artículo es describir las clases de homotópía (i.e., componentes conexas por arcos) del espacio $W^{s,p}(\Omega; S^1)$. Aquí, $0 < s < \infty$, $1 < p < \infty$, Ω es un abierto, regular, acotado y conexo de $\mathbb{R}^N$ y

$$W^{s,p}(\Omega; S^1) = \{u \in W^{s,p}(\Omega; \mathbb{C});\, |u| = 1 \text{ a.e.}\}.$$

Nuestros resultados principales establecen que $W^{s,p}(\Omega; S^1)$ es conexo por arcos si $sp < 2$ aunque tenga las mismas clases de homotopía que $C^0(\bar{\Omega}; S^1)$ si $sp \geq 2$. También presentamos algunos resultados y problemas abiertos sobre la densidad de aplicaciones regulares de $W^{s,p}(\bar{\Omega}; S^1)$.

1. Introduction

The purpose of this paper is to describe the homotopy classes (i.e., path-connected components) of the space $W^{s,p}(\Omega; S^1)$. Here, $0 < s < \infty$, $1 < p < \infty$, Ω is a smooth, bounded, connected open set in $\mathbb{R}^N$ and

$$W^{s,p}(\Omega; S^1) = \{ u \in W^{s,p}(\Omega; \mathbb{C});\ |u| = 1\, a.e.\}.$$

Our main results are

Theorem 1 *If $sp < 2$, then $W^{s,p}(\Omega; S^1)$ is path-connected.*

Theorem 2 *If $sp \geqslant 2$, then $W^{s,p}(\Omega; S^1)$ and $C^0(\bar{\Omega}; S^1)$ have the same homotopy classes in the sense of [7]. More precisely:*
a) each $u \in W^{s,p}(\Omega; S^1)$ is $W^{s,p}$-homotopic to some $v \in C^\infty(\bar{\Omega}; S^1)$;
b) two maps $u, v \in C^\infty(\bar{\Omega}; S^1)$ are C^0-homotopic if and only if they are $W^{s,p}$-homotopic.

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Here a simple consequence of the above results

Corollary 1 *If* $0 < s < \infty, 1 < p < \infty$ *and* Ω *is simply connected, then* $W^{s,p}(\Omega; S^1)$ *is path-connected.*

Indeed, when $sp < 2$ this is the content of Theorem 1. When $sp \geqslant 2$, we use a) of Theorem 2 to connect $u_1, u_2 \in W^{s,p}(\Omega; S^1)$ to $v_1, v_2 \in C^\infty(\bar{\Omega}; S^1)$; since Ω is simply connected, we may write $v_j = e^{i\varphi_j}$ for $\varphi_j \in C^\infty(\bar{\Omega}; \mathbb{R})$ and then we connect v_1 to v_2 via $e^{i[(1-t)\varphi_1 + t\varphi_2]}$.

When M is a compact connected manifold, the study of the topology of $W^{1,p}(\Omega; M)$ was initiated in Brezis - Li [7] (see also White [26] for some related questions). In particular, these authors proved Theorems 1 and 2 in the special case $s = 1$. The analysis of homotopy classes for an arbitrary manifold M and $s = 1$ was subsequently tackled by Hang - Lin [15]. The passage to $W^{s,p}$ introduces two additional difficulties:
a) when s is not an integer, the $W^{s,p}$ norm is not "local";
b) when $s \geqslant 2$ (or more generally $s > 1 + \frac{1}{p}$), gluing two maps in $W^{s,p}$ does not yield a map in $W^{s,p}$.

In our proofs, we exploit in an essential way the fact that the target manifold is S^1. (The case of a general target is widely open.) In particular, we use the existence of a lifting of $W^{s,p}$ unimodular maps when $s \geqslant 1$ and $sp \geqslant 2$ (see Bourgain - Brezis - Mironescu [4]). Another important tool is the following

Composition Theorem (Brezis - Mironescu [10]) *If* $f \in C^\infty(\mathbb{R}; \mathbb{R})$ *has bounded derivatives and* $s \geqslant 1$, *then* $\varphi \longmapsto f \circ \varphi$ *is continuous from* $W^{s,p} \cap W^{1,sp}$ *into* $W^{s,p}$.

Remark 1 A very elegant and straightforward proof of this Composition Theorem has been given by V. Maz'ya and T. Shaposhnikova [18].

A related question is the description, when $sp \geqslant 2$, of the homotopy classes of $W^{s,p}(\Omega; S^1)$ in terms of lifting. Here is a partial result

Theorem 3 *We have*
a) if $s \geqslant 1$, $N \geqslant 3$, *and* $2 \leqslant sp < N$, *then*

$$[u]_{s,p} = \{ue^{i\varphi}; \varphi \in W^{s,p}(\Omega; \mathbb{R}) \cap W^{1,sp}(\Omega; \mathbb{R})\};$$

b) if $sp \geqslant N$, *then*

$$[u]_{s,p} = \{ue^{i\varphi}; \varphi \in W^{s,p}(\Omega; \mathbb{R})\}.$$

Theorem 3 is due to Rubinstein - Sternberg [21] in the special case where $s = 1$, $p = 2$ and Ω is the solid torus in $\mathbb{R}^3$.

When $0 < s < 1$, $N \geqslant 3$ and $2 \leqslant sp < N$, there is no such simple description of $[u]_{s,p}$. For instance, using the "non-lifting" results in Bourgain - Brezis - Mironescu [4], it is easy to see that

$$[1]_{s,p} \underset{\neq}{\supset} \{e^{i\varphi}; \varphi \in W^{s,p}(\Omega; \mathbb{R})\}.$$

Here is an example: if $N = 3$, $\Omega = B_1$, $0 < s < 1$, $1 < p < \infty$, $2 \leqslant sp < 3$, then
a) $u(x) = e^{1/|x|^\alpha} \in [1]_{s,p}$;
b) there is no $\varphi \in W^{s,p}(B_1; \mathbb{R})$ such that $u = e^{i\varphi}$ for α satisfying $\frac{3-sp}{p} \leqslant \alpha < \frac{3-sp}{sp}$.

However, we conjecture the following result

Conjecture 1 *Assume that* $0 < s < 1$, $1 < p < \infty$, $N \geqslant 3$ *and* $2 \leqslant sp < N$. *Then*

$$[u]_{s,p} = u\overline{\{e^{i\varphi}; \varphi \in W^{s,p}(\Omega; \mathbb{R})\}}^{W^{s,p}}.$$

We will prove below (see Corollary 2) that "half" of Conjecture 1 holds, namely

$$[u]_{s,p} \supset u\overline{\{e^{i\varphi}; \varphi \in W^{s,p}(\Omega; \mathbb{R})\}}^{W^{s,p}}.$$

In a different but related direction, we establish some partial results concerning the density of $C^\infty(\bar{\Omega}; S^1)$ into $W^{s,p}(\Omega; S^1)$.

Theorem 4 *We have, for* $0 < s < \infty$, $1 < p < \infty$*:*
a) if $sp < 1$*, then* $C^\infty(\bar{\Omega}; S^1)$ *is dense in* $W^{s,p}(\Omega; S^1)$*;*
b) if $1 \leqslant sp < 2, N \geqslant 2$*, then* $C^\infty(\bar{\Omega}; S^1)$ *is not dense in* $W^{s,p}(\Omega; S^1)$*;*
c) if $sp \geqslant N$*, then* $C^\infty(\bar{\Omega}; S^1)$ *is dense in* $W^{s,p}(\Omega; S^1)$*;*
d) if $s \geqslant 1$ *and* $sp \geqslant 2$*, then* $C^\infty(\bar{\Omega}; S^1)$ *is dense in* $W^{s,p}(\Omega; S^1)$*.*

There is only one missing case for which we make the following

Conjecture 2 *If* $0 < s < 1, 1 < p < \infty, N \geqslant 3, 2 \leqslant sp < N$*, then* $C^\infty(\bar{\Omega}; S^1)$ *is dense in* $W^{s,p}(\Omega; S^1)$*.*

This problem is open even when Ω is a ball in $\mathbb{R}^3$. We will prove below the equivalence of Conjectures 1 and 2.

Parts of Theorem 4 were already known. Part a) is due to Escobedo [14]; so is part b), but in this case the idea goes back to Schoen - Uhlenbeck [24] (see also Bourgain - Brezis - Mironescu [5]). For $s = 1$, part c) is due to Schoen - Uhlenbeck [24]; their argument can be adapted to the general case (see, e.g., Brezis - Nirenberg [12] or Brezis - Li [7]). The only new result is part d). The proof relies heavily on the Composition Theorem and Theorems 2 and 3. We do not know any direct proof of d). We also mention that for $s = 1$ and $\Omega = B_1$, Theorem 4 was established by Bethuel - Zheng [3]. For a general compact connected manifold M and for $s = 1$, the question of density of $C^\infty(\bar{\Omega}; M)$ into $W^{1,p}(\Omega; M)$ was settled by Bethuel [1] and Hang - Lin [15].

Remark 2 In Theorems 2 and 4, one may replace Ω by a manifold with or without boundary. The statements are unchanged. However, the argument in the proof of Theorem 1 does not quite go through to the case of a manifold without boundary. Nevertheless, we make the following

Conjecture 3 *Let* Ω *be a manifold without boundary with* $\dim \Omega \geqslant 2$*. Then* $W^{s,p}(\Omega; M)$ *is path-connected for every* $0 < s < \infty, 1 < p < \infty$ *with* $sp < 2$*, and for every compact connected manifold* M*.*

Note that the condition $\dim \Omega \geqslant 2$ is necessary, since $W^{s,p}(S^1; S^1)$ is not path-connected when $sp \geqslant 1$.

Finally, we investigate the local path-connectedness of $W^{s,p}(\Omega; S^1)$. Our main result is

Theorem 5 *Let* $0 < s < \infty, 1 < p < \infty$*. Then* $W^{s,p}(\Omega; S^1)$ *is locally path-connected. Consequently, the homotopy classes coincide with the connected components and they are open and closed.*

The heart of the matter in the proof is the following

Claim. *Let* $0 < s < \infty, 1 < p < \infty$*. Then there is some* $\delta > 0$ *such that, if* $||u - 1||_{W^{s,p}} < \delta$*, then* u *may be connected to* 1 *in* $W^{s,p}$*.*

As a consequence of Theorem 5, we have

Corollary 2 *Let* $0 < s < 1, 1 < p < \infty$*. Then*

$$[u]_{s,p} \supset \overline{\{ue^{i\varphi}; \varphi \in W^{s,p}(\Omega; \mathbb{R})\}}^{W^{s,p}} = u\,\overline{\{e^{i\varphi}; \varphi \in W^{s,p}(\Omega; \mathbb{R})\}}^{W^{s,p}}.$$

Equality in Corollary 2 follows from the well-known fact that $W^{s,p} \cap L^\infty$ is an algebra. The inclusion is a consequence of the fact that, clearly, we have

$$[u]_{s,p} \supset \{ue^{i\varphi}; \varphi \in W^{s,p}(\Omega; \mathbb{R})\}$$

and of the closedness of the homotopy classes.

Another consequence of Theorem 5 is

Corollary 3 *Conjecture 1* $\Leftrightarrow$ *Conjecture 2.*

PROOF. By Corollary 2, we have

$$[u]_{s,p} \supset u\overline{\{e^{i\varphi}; \varphi \in W^{s,p}(\Omega;\mathbb{R})\}}^{W^{s,p}}.$$

We prove that the reverse inclusion follows from Conjecture 1. By Proposition 1 a) below, we may take $u = 1$. Let $v \in [1]_{s,p}$. By Theorem 5, there is some $\varepsilon > 0$ such that $||v - w||_{W^{s,p}} < \varepsilon \Rightarrow w \in [1]_{s,p}$. Let $(w_n) \subset C^\infty(\bar{\Omega}; S^1)$ be such that $w_n \to v$ in $W^{s,p}$ and $||w_n - v||_{W^{s,p}} < \varepsilon$. By Theorem 2 b), we obtain that w_n and 1 are homotopic in $C^0(\bar{\Omega}; S^1)$. Thus $w_n = e^{i\varphi_n}$ for some *globally* defined smooth φ_n. Hence

$$v \in \overline{\{e^{i\varphi}; \varphi \in W^{s,p}(\Omega;\mathbb{R})\}}^{W^{s,p}}.$$

Conversely, assume that Conjecture 2 holds. Let $u \in W^{s,p}(\Omega; S^1)$. By Theorem 2 a), there is some $w \in C^\infty(\bar{\Omega}; S^1)$ such that $w \in [u]_{s,p}$. By Proposition 1 b), we have $u\bar{w} \in [1]_{s,p}$. Thus $u\bar{w} \in \overline{\{e^{i\varphi}; \varphi \in W^{s,p}(\Omega;\mathbb{R})\}}^{W^{s,p}}$, so that clearly $u\bar{w} \in \overline{\{e^{i\varphi}; \varphi \in C^\infty(\bar{\Omega};\mathbb{R})\}}^{W^{s,p}}$.
Finally, $u \in \overline{\{we^{i\varphi}; \varphi \in C^\infty(\bar{\Omega};\mathbb{R})\}}^{W^{s,p}}$, i.e. u may be approximated by smooth maps.

In the same vein, we raise the following

Open Problem 1. *Let Ω be a manifold with or without boundary. Is $W^{s,p}(\Omega; M)$ locally path-connected for every s, p and every compact manifold M?*

The case $s = 1$ can be settled using the methods of Hang - Lin [15]. We will return to this question in a subsequent work; see Brezis - Mironescu [11].

The reader who is looking for more open problems may also consider the following

Open Problem 2. *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain. Assume $0 < s < \infty$, $1 < p < \infty$ and $1 \leqslant sp < 2$ (this is the range where $C^\infty(\bar{\Omega}; S^1)$ is not dense in $W^{s,p}(\Omega; S^1)$). Set*

$$\mathcal{R}_0 = \{u \in W^{s,p}(\Omega; S^1); u \textit{ is smooth except at a finite number of points}\}.$$

(Here, the number and location of singular points is left free). Is $\mathcal{R}_0$ dense in $W^{s,p}(\Omega; S^1)$?

Comment. $\mathcal{R}_0$ is known to be dense in $W^{s,p}(\Omega; S^1)$ in many cases, e.g.:
a) $s = 1$ and $1 \leqslant p < 2$; see Bethuel-Zheng [3]
b) $s = 1 - 1/p$ and $2 < p < 3$; see Bethuel [2]
c) $s = 1/2$ and $p = 2$; see Rivière [20].

The paper is organized as follows

2. Proof of Theorem 1

Case 1: $sp < 1$
When $sp < 1$, we have the following more general result

Theorem 6 *If $s > 0$, $1 < p < \infty$, $sp < 1$ and M is a compact manifold, then $W^{s,p}(\Omega; M)$ is path-connected.*

PROOF. Fix some $a \in M$. For $u \in W^{s,p}(\Omega; M)$, let

$$\tilde{u} = \begin{cases} u, & \text{in } \Omega \\ a, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Since $sp < 1$, we have $\tilde{u} \in W^{s,p}_{loc}(\mathbb{R}^N; M)$. Let $U(t,x) = \tilde{u}(x/(1-t))$, $0 \leqslant t < 1$, $x \in \Omega$ and $U(1,x) \equiv a$. Then clearly $U \in C([0,1]; W^{s,p}(\Omega; M))$ and U connects u to the constant a (here we use only $sp < N$).

Case 2: $1 < sp < 2, N \geqslant 2$
In this case one could adapt the tools developed in Brezis - Li [7], but we prefer a more direct approach.
Let $\varepsilon > 0$ be such that the projection onto $\partial\Omega$ be well-defined and smooth in the region $\{x \in \mathbb{R}^N;\ \text{dist}(x, \partial\Omega) < 2\varepsilon\}$. Let $\omega = \{x \in \mathbb{R}^N \setminus \bar{\Omega};\ \text{dist}(x, \partial\Omega) < \varepsilon\}$. We have $\partial\omega = \partial\Omega \cup \Lambda$, where $\Lambda = \{x \in \mathbb{R}^N \setminus \Omega;\ \text{dist}(x, \partial\Omega) = \varepsilon\}$.
Since $1 < sp < 2$, we have $1/p < s < 1 + 1/p$; thus, for $u \in W^{s,p}$ we have tr $u \in W^{s-1/p,p}$. Let $u \in W^{s,p}(\Omega; S^1)$. Fix some $a \in S^1$ and define $v \in W^{s-1/p,p}(\partial\omega; S^1)$ by

$$v = \begin{cases} \text{tr}\, u, & \text{on } \partial\omega \\ a, & \text{on } \Lambda. \end{cases}$$

We use the following extension result. (The first result of this kind is due to Hardt - Kinderlehrer - Lin [16]; it corresponds to our lemma when $\sigma = 1 - 1/p$, $p < 2$.)

Lemma 1 *Let $0 < \sigma < 1$, $1 < p < \infty$, $\sigma p < 1$. Then any $v \in W^{\sigma,p}(\partial\omega; S^1)$ has an extension $w \in W^{\sigma+1/p,p}(\omega; S^1)$.*

The proof is given in Appendix A; see Lemma A.1. It relies heavily on the lifting results in Bourgain - Brezis - Mironescu [4].
Returning to the proof of Case 2, with w given by Lemma 1, set

$$\tilde{u} = \begin{cases} u & \text{in } \Omega \\ w & \text{in } \omega \\ a & \text{in } \mathbb{R}^n \setminus (\Omega \cup \omega) \end{cases}$$

Clearly, $\tilde{u} \in W^{s,p}_{loc}(\mathbb{R}^N; S^1)$ and $\tilde{u}$ is constant outside some compact set. As in the proof of Theorem 6, we may use $\tilde{u}$ to connect u to a, since once more we have $sp < N$.

Case 3: $sp = 1,\ N \geqslant 2$
The idea is the same as in the previous case; however, there is an additional difficulty, since in the limiting case $s = 1/p$ the trace theory is delicate - in particular, tr $W^{1/p,p} \neq L^p$ (unless $p = 1$). Instead of trace, we work with a notion of "good restriction" developed in Appendix B; when $s = 1/2$, $p = 2$, the space of functions in $H^{1/2}$ having 0 as good restriction on the boundary coincides with the space $H^{1/2}_{00}$ of Lions - Magenes [17] (see Theorem 11.7, p. 72).
Our aim is to prove that any $u \in W^{1/p,p}(\Omega; S^1)$ can be connected to a constant $a \in S^1$.
Step 1: *we connect $u \in W^{1/p,p}(\Omega; S^1)$ to some $u_1 \in W^{1/p,p}(\Omega; S^1)$ having a good restriction on $\partial\Omega$*

Let $\varepsilon > 0$ be such that the projection Π onto $\partial\Omega$ be well-defined and smooth in the set $\{x \in \mathbb{R}^N;\ \text{dist}\ (x, \partial\Omega) < 2\varepsilon)\}$. For $0 < \delta < \varepsilon$, set $\Sigma_\delta = \{x \in \Omega;\ \text{dist}\ (x, \partial\Omega) = \delta\}$. By Fubini, for a.e. $0 < \delta < \varepsilon$, we have

$$u|_{\sum_\delta} \in W^{1/p,p}(\Sigma_\delta) \text{ and } \int_{\Sigma_\delta}\int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+1}}\, dy\, ds_x < \infty. \tag{1}$$

By Lemma B.5, this implies that u has a good restriction on Σ_δ, and that Rest $u|_{\sum_\delta} = u|_{\sum_\delta}$ a.e. on Σ_δ.

Let any $0 < \delta < \varepsilon$ satisfying (1). For $0 < \lambda < \delta$, let Ψ_λ be the smooth inverse of $\Pi|_{\sum_\lambda} : \Sigma_\lambda \to \partial\Omega$. Let also $\Omega_\lambda = \{x \in \Omega;\ \text{dist}\ (x, \partial\Omega) > \lambda\}$. Consider a continuous family of diffeomorphisms $\Phi_t : \bar{\Omega} \to \overline{\Omega_{t\delta}}$, $0 \leqslant t \leqslant 1$, such that $\Phi_0 = \text{id}$ and $\Phi_t|_{\partial\Omega} = \Psi_{t\delta}$. Then $t \mapsto u \circ \Phi_t$ is a homotopy in $W^{1/p,p}$. Moreover, if $u_t = u \circ \Phi_t$, then $u_0 = u$ and $u_1|_{\partial\Omega} = u|_{\sum_\delta} \circ \Psi_\delta|_{\partial\Omega}$. By (1), u_1 has a good restriction on $\partial\Omega$.

Step 2: *we extend u_1 to $\mathbb{R}^N$*

Let $\omega = \{x \in \mathbb{R}^N \setminus \bar{\Omega};\ \text{dist}\ (x; \partial\Omega) < \varepsilon\}$. As in Case 2, we fix some $a \in S^1$ and set

$$v = \begin{cases} u_1, & \text{on } \partial\Omega \\ a, & \text{on } \Lambda. \end{cases}$$

Clearly, $v \in W^{1/p,p}(\partial\omega)$, so that $v \in W^{\sigma,p}(\partial\omega)$ for $0 < \sigma < 1/p$. We fix any $0 < \sigma < 1/p$. By Lemma 1, there is some $w \in W^{\sigma+1/p,p}(\omega; S^1)$ such that $w|_{\partial\omega} = v$. We define

$$\tilde{u}_1 = \begin{cases} u_1, & \text{in } \Omega \\ w, & \text{in } \omega \\ a, & \text{in } \mathbb{R}^N \setminus (\Omega \cup \omega). \end{cases}$$

We claim that $\tilde{u}_1 \in W^{1/p,p}_{loc}(\mathbb{R}^N; S^1)$. Obviously, $\tilde{u} \in W^{1/p,p}_{loc}(\mathbb{R}^N \setminus \Omega)$. It remains to check that $\tilde{u}_1 \in W^{1/p,p}(\Omega \cup \omega)$. This is a consequence of

Lemma 2 *Let $0 < s < 1, 1 < p < \infty, sp \geqslant 1$ and $\rho > s$. Let $u_1 \in W^{s,p}(\Omega)$ and $w \in W^{\rho,p}(\omega)$. Assume that u_1 has a good restriction* Rest $u_1|_{\partial\Omega}$ *on $\partial\Omega$ and that* tr $w|_{\partial\Omega} =$ Rest $u_1|_{\partial\Omega}$. *Then the map*

$$\begin{cases} u_1, & \text{in } \Omega \\ w, & \text{in } \omega \end{cases}$$

belongs to $W^{s,p}(\Omega \cup \omega)$.

Clearly, in the proof of Lemma 2 it suffices to consider the case of a flat boundary. When $\Omega = (-1,1)^{N-1} \times (0,1)$ and $\omega = (-1,1)^{N-1} \times (-1,0)$, the proof of Lemma 2 is presented in Appendix B; see Lemma B.4.

Returning to Case 3 and applying Lemma 2 with $s = 1/p$, $\rho = \sigma + 1/p$, we obtain that $\tilde{u}_1 \in W^{1/p,p}_{loc}(\mathbb{R}^N)$. As in the two previous cases, this means that u_1 is $W^{1/p,p}$-homotopic to a constant.

Case 4: $1 \leqslant sp < 2,\ N = 1$

In this case, Ω is an interval. Recall the following result proved in Bourgain - Brezis - Mironescu [4] (Theorem 1): if Ω is an interval and $sp \geqslant 1$, then for each $u \in W^{s,p}(\Omega; S^1)$ there is some $\varphi \in W^{s,p}(\Omega; \mathbb{R})$ such that $u = e^{i\varphi}$. Recall also that, when $sp \geqslant N$, then $C^\infty(\mathbb{R}; \mathbb{R})$ functions f with bounded derivatives operate on $W^{s,p}$; that is, the map $\varphi \mapsto f \circ \varphi$ is continuous from $W^{s,p}$ into itself (see, e.g., Peetre [19] for $sp > N$, Runst - Sickel [23], Corollary 2 and Remark 5 in Section 5.3.7 or Brezis - Mironescu [9] when $sp = N$; this is also a consequence of the Composition Theorem). By combining these two results, we find that the homotopy $t \mapsto e^{i(1-t)\varphi}$ connects $u = e^{i\varphi}$ to 1.

The proof of Theorem 1 is complete.

3. Proof of Theorems 2 and 3

We start with some useful remarks. For $u \in W^{s,p}(\Omega; S^1)$, let $[u]_{s,p}$ denote its homotopy class in $W^{s,p}$.

Proposition 1 *Let* $0 < s < \infty,\ 1 < p < \infty$. *For* $u, v \in W^{s,p}(\Omega; S^1)$, *we have*
a) $u[v]_{s,p} = [uv]_{s,p}$;
b) $[u]_{s,p} = [v]_{s,p} \Leftrightarrow [u\bar{v}]_{s,p} = [1]_{s,p}$;
c) $[u]_{s,p}\,[v]_{s,p} = [uv]_{s,p}$.

The proof relies on two well-known facts: $W^{s,p} \cap L^\infty$ is an algebra; moreover, if $u_n \to u,\ v_n \to v$ in $W^{s,p}$ and $||u_n||_{L^\infty} \leqslant C,\ ||v_n||_{L^\infty} \leqslant C$, then $u_n\, v_n \to uv$ in $W^{s,p}$. Here is, for example, the proof of c) (using a)). Let first $u_1 \in [u]_{s,p},\ v_1 \in [v]_{s,p}$. If U, V are homotopies connecting u_1 to u and v_1 to v, then UV connects $u_1\, v_1$ to uv; thus $[u]_{s,p}\,[v]_{s,p} \subset [uv]_{s,p}$. Conversely, if $w \in [uv]_{s,p}$, then $w \in u[v]_{s,p}$ (by a)), so that $w\bar{u} \in [v]_{s,p}$. Therefore, $w = u(w\bar{u}) \in [u]_{s,p}\,[v]_{s,p}$.

We next recall the degree theory for $W^{s,p}$ maps; see Brezis - Li - Mironescu - Nirenberg [8] for the general case, White [25] when $s = 1$ or Rubinstein - Sternberg [20] for the space $H^1(\Omega; S^1)$ and Ω the solid torus in $\mathbb{R}^3$. Let $0 < s < \infty,\ 1 < p < \infty$ be such that $sp \geqslant 2$. Let $u \in W^{s,p}(S^1 \times \Lambda; S^1)$, where Λ is some open connected set in $\mathbb{R}^k$. Clearly, for a.e. $\lambda \in \Lambda$, $u(\cdot, \lambda) \in W^{s,p}(S^1; S^1)$. For any such λ, $u(\cdot, \lambda)$ is continuous, so that it has a winding number (degree) $\deg\big(u(\cdot, \lambda)\big)$. The main result in [8] asserts that, if $sp \geqslant 2$, then this degree is constant a.e. and stable under $W^{s,p}$ convergence.

In the particular case where $s \geqslant 1$, there is a formula

$$\deg\,(u(\cdot, \lambda)) = \frac{1}{2\pi} \int_{S^1} u\,(x, \lambda) \wedge \frac{\partial u}{\partial \tau}\,(x, \lambda)\, ds_x,$$

where $u \wedge v = u_1\, v_2 - u_2\, v_1$. It then follows that, if $s \geqslant 1$ and $sp \geqslant 2$, we have

$$\deg\,(u|_{S^1 \times \Lambda}) = \fint_{\Lambda} \fint_{S^1} u(x, \lambda) \wedge \frac{\partial u}{\partial \tau}\,(x, \lambda)\, ds_x d\lambda.$$

Clearly, the above result extends to domains which are diffeomorphic to $S^1 \times \Lambda$. In the sequel, we are interested in the following particular case: let Γ be a simple closed smooth curve in Ω and, for small $\varepsilon > 0$, let Γ_ε be the ε-tubular neighborhood of Γ. We fix an orientation on Γ.

Let $\Phi : S^1 \times B_\varepsilon \to \Gamma_\varepsilon$ be a diffeomorphism such that $\Phi|_{S^1 \times \{0\}} : S^1 \times \{0\} \to \Gamma$ be an orientation preserving diffeomorphism; here B_ε is the ball of radius ε in $\mathbb{R}^{N-1}$. Then we may define $\deg\,(u|_{\Gamma_\varepsilon}) = \deg\,(u \circ \Phi|_{S^1 \times B_\varepsilon})$; this integer is stable under $W^{s,p}$ convergence.

We now prove b) of Theorem 2, which we restate as

Proposition 2 *Let* $0 < s < \infty,\ 1 < p < \infty,\ sp \geqslant 2$. *Let* $u, v \in C^\infty(\bar{\Omega}; S^1)$. *Then* $[u]_{s,p} = [v]_{s,p}$ *if and only if u and v are* C^0*- homotopic.*

PROOF. Using Proposition 1, we may assume $v = 1$. Suppose first that $u \in C^\infty(\bar{\Omega}; S^1)$ and 1 are C^0-homotopic. Then u and 1 are $W^{s,p}$-homotopic. Indeed, when $s = 1$, this is proved in Brezis - Li [7], Proposition A.1; however, their proof works without modification for any s. We sketch an alternative proof: since u and 1 are C^0-homotopic, there is some $\varphi \in C^\infty(\bar{\Omega}; \mathbb{R})$ such that $u = e^{i\varphi}$. Then $t \mapsto e^{i\,(1-t)\,\varphi}$ connects u to 1 in $W^{s,p}$.

Conversely, assume that the smooth map u is $W^{s,p}$-homotopic to 1. By continuity of the degree, we then have $\deg\,(u|_{\Gamma_\varepsilon}) = 0$ for each Γ. Since u is smooth, we obtain

$$0 = \deg\,(u|_{\Gamma_\varepsilon}) = \deg\,(u|_\Gamma) = \frac{1}{2\pi} \int_\Gamma u \wedge \frac{\partial u}{\partial \tau} ds.$$

Thus the closed form $X = u \wedge Du$ has the property that $\int_\Gamma X \cdot \tau ds = 0$ for any simple closed smooth curve Γ. By the general form of the Poincaré lemma, there is some $\varphi \in C^\infty(\bar\Omega;\mathbb{R})$ such that $X = D\varphi$. One may easily check that $u = e^{i(\varphi+C)}$ for some constant C. Then $t \mapsto e^{i(1-t)(\varphi+C)}$ connects u to 1 in $C^0(\bar\Omega; S^1)$.

We now turn to the proof of the remaining assertions in Theorems 2 and 3.

Case 1: $sp \geqslant N$, $N \geqslant 2$

Step 1: *each* $u \in W^{s,p}(\Omega; S^1)$ *can be connected to a smooth map* $v \in C^\infty(\bar\Omega; S^1)$

This is proved in Brezis - Li [7], Proposition A.2, for $s = 1$ and $p \geqslant N$; their arguments apply to any s and any p such that $sp \geqslant N$. The main idea originates in the paper Schoen - Uhlenbeck [23]; see also Brezis - Nirenberg [12], [13].

Step 2: *we have* $[u]_{s,p} = \{ue^{i\varphi}; \varphi \in W^{s,p}(\Omega;\mathbb{R})\}$

Let $\varphi \in W^{s,p}(\Omega;\mathbb{R})$. Then $t \longmapsto ue^{i(1-t)\varphi}$ connects $ue^{i\varphi}$ to u in $W^{s,p}$. (Recall that, if $f \in C^\infty(\mathbb{R};\mathbb{R})$ has bounded derivatives and $sp \geqslant N$, then the map $\varphi \mapsto f \circ \varphi$ is continuous from $W^{s,p}$ into itself.) This proves "$\supset$". To prove the reverse inclusion, by Proposition 1, it suffices to show that $[1]_{s,p} \subset \{e^{i\varphi}; \varphi \in W^{s,p}(\Omega;\mathbb{R})\}$.

Let $v \in [1]_{s,p}$. For each $x \in \Omega$, let $B_x \subset \Omega$ be a ball containing x. We recall the following lifting result from Bourgain - Brezis - Mironescu [4] (Theorem 2): if U is simply connected in $\mathbb{R}^N$ and $sp \geqslant N$, then for each $w \in W^{s,p}(U; S^1)$ there is some $\psi \in W^{s,p}(U;\mathbb{R})$ such that $w = e^{i\psi}$. Thus, for each $x \in \Omega$ there is some $\varphi_x \in W^{s,p}(B_x;\mathbb{R})$ such that $v|_{B_x} = e^{i\varphi_x}$. Note that , in $B_x \cap B_y$, we have $\varphi_x - \varphi_y \in W^{s,p}(B_x \cap B_y; 2\pi\mathbb{Z})$. Therefore, $\varphi_x - \varphi_y \in VMO(B_x \cap B_y; 2\pi\mathbb{Z})$, since $sp \geqslant N$. It then follows that $\varphi_x - \varphi_y$ is constant a.e. on $B_x \cap B_y$; see Brezis - Nirenberg [12], Section I.5.

By a standard continuation argument, we may thus define a (multi-valued) argument φ for v in the following way: fix some $x_0 \in \Omega$. For any $x \in \Omega$, let γ be a simple smooth path from x_0 to x. Then, for $\varepsilon > 0$ sufficiently small, there is a unique function $\varphi^\gamma \in W^{s,p}(\gamma_\varepsilon;\mathbb{R})$ such that $v|_{\gamma_\varepsilon} = e^{i\varphi^\gamma}$ and $\varphi^\gamma|_{B_\varepsilon(x_0)} = \varphi_{x_0}|_{B_\varepsilon(x_0)}$; here, γ_ε is the ε-tubular neighborhood of γ. We then set

$$\varphi|_{B_\varepsilon(x)} = \varphi^\gamma|_{B_\varepsilon(x)}.$$

We actually claim that φ is single-valued. This follows from

Lemma 3 *Assume that* $0 < s < \infty$, $1 < p < \infty$, $sp \geqslant N$, $N \geqslant 2$. *If* $w \in W^{s,p}(S^1 \times B_1; S^1)$ *is such that* deg $(w|_{S^1 \times B_1}) = 0$, *then there is some* $\psi \in W^{s,p}(S^1 \times B_1)$ *such that* $w = e^{i\psi}$.

Here, B_1 is the unit ball in $\mathbb{R}^{N-1}$. The proof of Lemma 3 is presented in Appendix C; see Lemma C.1.

Returning to the claim that φ is single-valued, we have that deg $(v|_{\Gamma_\varepsilon}) = 0$ for each Γ, since $v \in [1]_{s,p}$. By Lemma 3, a standard argument implies that φ is single-valued.

The proof of Theorems 2 and 3 when $sp \geqslant N$ is complete.

Case 2: $s \geqslant 1$, $1 < p < \infty$, $N \geqslant 3$, $2 \leqslant sp < N$

Step 1: *we have* $[u]_{s,p} = \{ue^{i\varphi}; \varphi \in W^{s,p}(\Omega;\mathbb{R}) \cap W^{1,sp}(\Omega;\mathbb{R})\}$ For "$\supset$", we use the Composition Theorem mentioned in the Introduction, which implies that $t \mapsto ue^{i(1-t)\varphi}$ connects $ue^{i\varphi}$ to u in $W^{s,p}$.

For "$\subset$" it suffices to prove that $[1]_{s,p} \subset \{e^{i\varphi}; \varphi \in W^{s,p}(\Omega;\mathbb{R}) \cap W^{1,sp}(\Omega;\mathbb{R})\}$. We proceed as in Case 1, Step 2. Let $v \in [1]_{s,p}$. The corresponding lifting result we use is the following (see Bourgain - Brezis - Mironescu [4], Lemma 4): if $s \geqslant 1$, $sp \geqslant 2$ and U is simply connected in $\mathbb{R}^N$, then for each $w \in W^{s,p}(U;S^1)$ there is some $\psi \in W^{s,p}(U;\mathbb{R}) \cap W^{1,sp}(U;\mathbb{R})$ such that $w = e^{i\psi}$. As in Case 1, for each x there is some $\varphi_x \in W^{s,p}(B_x;\mathbb{R}) \cap W^{1,sp}(B_x;\mathbb{R})$ such that $v|_{B_x} = e^{i\varphi_x}$. Since $\varphi_x - \varphi_y \in W^{1,1}(B_x \cap B_y; 2\pi\mathbb{Z})$, we find that $\varphi_x - \varphi_y$ is constant ae. on $B_x \cap B_y$ (see [4], Theorem B.1.). These two ingredients allow the construction of a multi-valued phase $\varphi \in W^{s,p} \cap W^{1,sp}$ for v. To prove that φ is actually single-valued, we rely on

Lemma 4 *Assume that* $s \geqslant 1$, $1 < p < \infty$, $N \geqslant 3$, $2 \leqslant sp < N$. *If* $w \in W^{s,p}(S^1 \times B_1; S^1)$ *is such that* deg $(w|_{S^1 \times B_1}) = 0$, *then there is some* $\psi \in W^{s,p}(S^1 \times B_1;\mathbb{R}) \cap W^{1,sp}(S^1 \times B_1;\mathbb{R})$ *such that* $v = e^{i\psi}$.

The proof of Lemma 4 is given in Appendix C; see Lemma C.2.

The proof of Step 1 is complete.

Step 2: *assume* $s \geqslant 1$, $1 < p < \infty$, $sp \geqslant 2$*; then, for each* $u \in W^{s,p}(\Omega; S^1)$*, there is some* $v \in W^{s,p}(\Omega; S^1) \cap C^\infty(\Omega; S^1)$ *such that* $v \in [u]_{s,p}$.

Consider the form $X = u \wedge Du$. Then $X \in W^{s-1,p}(\Omega) \cap L^{sp}(\Omega)$ (see Bourgain - Brezis - Mironescu [4], Lemmas D.1 and D.2). Let $\varphi \in W^{s,p}(\Omega; \mathbb{R}) \cap W^{1,sp}(\Omega; \mathbb{R})$ be any solution of $\Delta\varphi = \operatorname{div} X$ in Ω. By the Composition Theorem, we then have $e^{-i\varphi} \in W^{s,p}(\Omega; S^1)$, and thus $v = ue^{-i\varphi} \in W^{s,p}(\Omega; S^1)$. We claim that $v \in C^\infty(\Omega; S^1)$. Indeed, let B be any ball in Ω. Since $s \geqslant 1$ and $sp \geqslant 2$, there is some $\psi \in W^{s,p}(B; \mathbb{R}) \cap W^{1,sp}(B; \mathbb{R})$ such that $u|_B = e^{i\psi}$. It then follows that $X|_B = D\psi$. Thus $\Delta\varphi = \Delta\psi$ in B, i.e., $\psi - \varphi$ is harmonic in B. Since in B we have $v = ue^{-i\varphi} = e^{i(\psi-\varphi)}$, we obtain that $v \in C^\infty(B)$, so that the claim follows.

Using Step 1 and the equality $v = ue^{-i\varphi}$, we obtain that $v \in [u]_{s,p}$.

Step 3: *for each* $u \in W^{s,p}(\Omega; S^1)$*, there is some* $w \in C^\infty(\bar{\Omega}; S^1)$ *such that* $w \in [u]_{s,p}$.

In view of Step 2, it suffices to consider the case where $u \in W^{s,p}(\Omega; S^1) \cap C^\infty(\Omega; S^1)$. We use the same homotopy as in Step 1, Case 3, in the proof of Theorem 1: $t \mapsto u \circ \Phi_t$, where Φ_t is a continuous family of diffeomorphisms $\Phi_t : \bar{\Omega} \to \overline{\Omega_{t\delta}}$ such that $\Phi_0 = id$. Clearly, $v = u \circ \Phi_1 \in C^\infty(\bar{\Omega}; S^1)$.

The conclusions of Theorems 2 and 3 when $s \geqslant 1$, $1 < p < \infty$, $N \geqslant 3$, $2 \leqslant sp < N$ follow from Proposition 2 and Steps 1 and 3.

We now complete the proof of Theorem 2 with

Case 3: $0 < s < 1$, $1 < p < \infty$, $N \geqslant 3$, $2 \leqslant sp < N$

In this case, all we have to prove is that, for each $u \in W^{s,p}(\Omega; S^1)$, there is some $v \in C^\infty(\bar{\Omega}; S^1)$ such that $v \in [u]_{s,p}$. The ideas we use in the proof are essentially due to Brezis - Li [7] (see §1.3, "Filling" a hole).

We may assume that u is defined in a neighborhood $\mathcal{O}$ of $\bar{\Omega}$; this is done by extending u by reflections across the boundary of Ω- the extended map is still in $W^{s,p}$ since $0 < s < 1$. We next define a good covering of Ω: let $\varepsilon > 0$ be small enough; for $x \in \mathbb{R}^N$, we set

$$\mathcal{C}_N^x = \bigcup \{x + \varepsilon l + (0, \varepsilon)^N; l \in \mathbb{Z}^N \text{ and } x + \varepsilon l + (0, \varepsilon)^N \subset \mathcal{O}\}.$$

Define also $\mathcal{C}_j^x$, $j = 1, ..., N-1$, by backward induction : $\mathcal{C}_j^x$ is the union of faces of cubes in $\mathcal{C}_{j+1}^x$.

By Fubini, for a.e. $x \in \mathbb{R}^N$, we have $u|_{\mathcal{C}_j^x} \in W^{s,p}$, $j = 1, ..., N-1$, in the following sense: since $1/p < s < 1$, we have tr $u|_{\mathcal{C}_{N-1}^x} \in W^{s-1/p,p}$ for all x. However, for a.e. x, we have the better property tr $u|_{\mathcal{C}_{N-1}^x} = u|_{\mathcal{C}_{N-1}^x} \in W^{s,p}$. For any such x, we have tr $\left(u|_{\mathcal{C}_{N-1}^x}\right)|_{\mathcal{C}_{N-2}^x} \in W^{s-1/p,p}$, but once more for a.e. such x we have the better property tr $\left(u|_{\mathcal{C}_{N-1}^x}\right)|_{\mathcal{C}_{N-2}^x} = u|_{\mathcal{C}_{N-2}^x} \in W^{s,p}$, and so on. (See Appendix E for a detailed discussion).

We fix any x having the above property and we drop from now on the superscript x.

Step 1: *we connect* u *to some smoother map* u_1 *Let* $k = [sp]$*, so that* $2 \leqslant k \leqslant N-1$. Since $u|_{\mathcal{C}_k} \in W^{s,p}$ and $sp \geqslant k$, there is a neighborhood ω of $\mathcal{C}_k$ in $\mathcal{C}_{k+1}$ and an extension $\tilde{u} \in W^{s+1/p,p}(\omega; S^1)$ of $u|_{\mathcal{C}_k}$. This extension is first obtained in each cube $C \subset \mathcal{C}_{k+1}$ starting from $u|_{\partial C}$ (see Brezis - Nirenberg [12], Appendix 3, for the existence of such an extension). We next glue together all these extensions to obtain $\tilde{u}$; $\tilde{u}$ belongs to $W^{s+1/p,p}$ since $1/p < s + 1/p < 1 + 1/p$. Moreover, the explicit construction in [12] yields some $\tilde{u} \in C^\infty(\omega \backslash \mathcal{C}_k)$. We next extend $\tilde{u}$ to $\mathcal{C}_{k+1}$ in the following way: for each $C \subset \mathcal{C}_{k+1}$, let Σ_C be a convex smooth hypersurface in $C \cap \omega$. Since Σ_C is k-dimensional and $k \geqslant 2$, $\tilde{u}|_{\Sigma_C}$ may be extended smoothly in the interior of Σ_C as an S^1-valued map (here, we use the fact that $\pi_k(S^1) = 0$). Let $\tilde{u}_C$ be such an extension. Then the map

$$v = \begin{cases} \tilde{u}, & \text{outside the } \Sigma_C\text{'s} \\ \tilde{u}_C, & \text{inside } \Sigma_C \end{cases}$$

belongs to $W^{s+1/p,p}(\mathcal{C}_{k+1})$. To summarize, we have found some $v \in W^{s+1/p,p}(\mathcal{C}_{k+1}; S^1)$ such that $v|_{\mathcal{C}_k} = u|_{\mathcal{C}_k}$.

Pick any $s < s_1 < \min\{s + 1/p, 1\}$ and let p_1 be such that $s_1 p_1 = sp + 1$ (note that $1 < p_1 < \infty$). By Gagliardo - Nirenberg (see, e.g., Runst [22], Lemma 1, p.329 or Brezis - Mironescu [10], Corollary 3), we have $W^{s+1/p,p} \cap L^\infty \subset W^{s_1,p_1}$. Thus $v \in W^{s_1,p_1}(\mathcal{C}_{k+1})$.

We complete the construction of the smoother map u_1 in the following way: if $k = N-1$, then v is defined in $\mathcal{C}_N$ and we set $u_1 = v$; if $k < N-1$, we extend v to $\mathcal{C}_N$ with the help of

Lemma 5 *Let $0 < s_1 < \infty$, $1 < p_1 < \infty$, $1 < s_1 p_1 < N$, $[s_1 p_1] \leqslant j < N$. Then any $v \in W^{s_1,p_1}(\mathcal{C}_j; S^1)$ has an extension $u_1 \in W^{s_1,p_1}(\mathcal{C}_N; S^1)$ such that $u_1|_{\mathcal{C}_l} \in W^{s_1,p_1}$ for $l = j, ..., N-1$.*

When $s_1 = 1$, Lemma 5 is due to Brezis - Li [7], Section 1.3, "Filling" a hole; for the general case, see Lemma D.3 in Appendix D.

We summarize what we have done so far: if $k = [sp]$, then there are some s_1, p_1 such that $s < s_1 < 1$, $1 < p_1 < \infty$, $s_1 p_1 = sp + 1$ and a map $u_1 \in W^{s_1,p_1}(\mathcal{C}_N; S^1)$ such that $u_1|_{\mathcal{C}_j} \in W^{s_1,p_1}$, $j = k, ..., N-1$ and $u_1|_{\mathcal{C}_k} = u|_{\mathcal{C}_k}$. By Gagliardo - Nirenberg and the Sobolev embeddings, we have in particular $u_1|_{\mathcal{C}_j} \in W^{s,p}$, $j = k, ..., N-1$. Finally, u and u_1 are $W^{s,p}$- homotopic by

Lemma 6 *Let $0 < s < 1$, $1 < p < \infty$, $1 < sp < N, [sp] \leqslant j < N$. If $u|_{\mathcal{C}_l} \in W^{s,p}$, $u_1|_{\mathcal{C}_l} \in W^{s,p}, l = j, ..., N$, and $u|_{\mathcal{C}_j} = u_1|_{\mathcal{C}_j}$, then u and u_1 are $W^{s,p}$-homotopic.*

The case $s = 1$ is due to Brezis - Li [7]; the proof of Lemma 6 in the general case is presented in the Appendix D- see Lemma D.4.

Step 2: *induction on* $[sp]$.

If $k = [sp] = N-1$, we have connected in the previous step u to $u_1 \in W^{s_1,p_1}(\mathcal{C}_N; S^1)$, where $s < s_1 < 1$, $1 < p_1 < \infty$ and $s_1 p_1 = sp + 1 \geqslant N$. Using Case 1 (i.e., $sp \geqslant N$) from this section, u_1 may be connected in W^{s_1,p_1} (and thus in $W^{s,p}$, by Gagliardo - Nirenberg and the Sobolev embeddings) to some $v \in C^\infty(\bar{\Omega}; S^1)$. This case is complete.

If $k = [sp] = N-2$, then $[s_1 p_1] = N-1$. By the previous case, u_1 can be connected in W^{s_1,p_1} (and thus in $W^{s,p}$) to some $v \in C^\infty(\bar{\Omega}; S^1)$. Clearly, the general case follows by induction.

The proof of Theorems 2 and 3 is complete.

We end this section with two simple consequences of the above proofs; these results supplement the description of the homotopy classes.

Corollary 4 *Let $0 < s < \infty$, $1 < p < \infty$, $sp \geqslant 2$, $N \geqslant 2$. For $u, v \in W^{s,p}(\Omega; S^1)$, we have $[u]_{s,p} = [v]_{s,p} \Leftrightarrow \deg\ (u|_{\Gamma_\varepsilon}) = \deg\ (v|_{\Gamma_\varepsilon})$ for every Γ.*

Corollary 5 *Let $0 < s_1, s_2 < \infty$, $1 < p_1, p_2 < \infty$, $s_1 p_1 \geqslant 2$, $s_2 p_2 \geqslant 2$, $N \geqslant 2$. For $u, v \in W^{s_1,p_1}(\Omega; S^1) \cap W^{s_2,p_2}(\Omega; S^1)$, we have $[u]_{s_1,p_1} = [v]_{s_1,p_1} \Leftrightarrow [u]_{s_2,p_2} = [v]_{s_2,p_2}$.*

Clearly, Corollary 5 follows from Corollary 4. As for Corollary 4, let $u_1, v_1 \in C^\infty(\bar{\Omega}; S^1)$ be such that $[u_1]_{s,p} = [u]_{s,p}$ and $[v_1]_{s,p} = [v]_{s,p}$. Then, by Theorem 2 b),

$$[u]_{s,p} = [v]_{s,p} \Leftrightarrow [u_1]_{s,p} = [v_1]_{s,p} \Leftrightarrow [u_1]_{C^0} = [v_1]_{C^0} \Leftrightarrow \deg\,(u_1|_\Gamma) = \deg\,(v_1|_\Gamma), \quad \forall \Gamma. \tag{2}$$

Moreover, we have

$$\deg\,(u_1|_\Gamma) = \deg\,(v_1|_\Gamma) \Leftrightarrow \deg\,(u_1|_{\Gamma_\varepsilon}) = \deg\,(v_1|_{\Gamma_\varepsilon}) \Leftrightarrow \deg\,(u|_{\Gamma_\varepsilon}) = \deg\,(v|_{\Gamma_\varepsilon}), \forall \Gamma, \tag{3}$$

by standard properties of the degree.

We obtain Corollary 4 by combining (2) and (3).

4. Proof of Theorem 4

According to the discussion in the Introduction, we only have to prove part d). Let $s \geqslant 1$, $1 < p < \infty$, $N \geqslant 3$, $2 \leqslant sp < N$. Let $u \in W^{s,p}(\Omega; S^1)$. By Theorem 2 a), there is some $v \in C^\infty(\bar{\Omega}; S^1)$ such that $v \in [u]_{s,p}$. By Theorem 3 b), there is some $\varphi \in W^{s,p}(\Omega; \mathbb{R}) \cap W^{1,sp}(\Omega; \mathbb{R})$ such that $v = ue^{i\varphi}$. Let $(\varphi_n) \subset C^\infty(\bar{\Omega}; \mathbb{R})$ be such that $\varphi_n \to \varphi$ in $W^{s,p} \cap W^{1,sp}$. By the Composition Theorem, the sequence of smooth maps $(ve^{-i\varphi_n})$ converges to u in $W^{s,p}(\Omega; S^1)$. The proof of Theorem 4 is complete.

5. Proof of Theorem 5

We start this section with a discussion on the stability of the degree: recall that if $sp \geqslant 2$, then deg $(u|_{\Gamma_\varepsilon})$ is well-defined and stable under $W^{s,p}$ convergence. However, while the condition $sp \geqslant 2$ is optimal for the existence of the degree (see Brezis - Li - Mironescu - Nirenberg [8], Remark 1), the stability of the degree of $W^{s,p}$ maps holds under (the weaker assumption of) W^{s_1,p_1} convergence, where $s_1p_1 \geqslant 1$. This property and Corollary 4 suggest the following generalization of Theorem 5

Theorem 7 *Let $0 < s < \infty$, $1 < p < \infty$, $0 < s_1 < s$, $1 < p_1 < \infty$, $1 \leqslant s_1p_1 \leqslant sp$. Then for each $u \in W^{s,p}(\Omega; S^1)$ there is some $\delta > 0$ such that*

$$\{v \in W^{s,p}(\Omega; S^1)\,;\, ||v - u||_{W^{s_1,p_1}} < \delta\} \subset [u]_{s,p}.$$

Note that $W^{s,p}(\Omega; S^1) \subset W^{s_1,p_1}(\Omega; S^1)$, by Gagliardo - Nirenberg and the Sobolev embeddings, so that Theorem 5 follows from Theorem 7 when $sp \geqslant 2$ (when $sp < 2$, there is nothing to prove, by Theorem 1).

Proof of Theorem 7

Step 1: *reduction to special values of s, s_1, p, p_1.*

We claim that it suffices to prove Theorem 7 when

$$0 < s_1 < s < 1 - (N-1)/p,\ 1 < p < \infty,\ 1 < p_1 < \infty,\ sp = 2,\ s_1p_1 = 1, N \geqslant 2. \tag{4}$$

Indeed, assume Theorem 7 proved for all the values of s, s_1, p, p_1 satisfying (4). Let $0 < s_0 < \infty$, $1 < p_0 < \infty, N \geqslant 2$ be such that $s_0p_0 \geqslant 2$ (when $N = 1$ or $s_0p_0 < 2$, there is nothing to prove). Let $u \in W^{s_0,p_0}$ and let s, s_1, p, p_1 satisfy (4) and the additional condition $s < s_0$. By Gagliardo - Nirenberg and the Sobolev embeddings, there is some $\delta_0 > 0$ such that

$$\begin{aligned} M = \ &\{v \in W^{s_0,p_0}(\Omega; S^1)\,;\, ||v - u||_{W^{s_0,p_0}} < \delta_0\} \subset \\ &\{v \in W^{s,p}(\Omega; S^1)\,;\, ||v - u||_{W^{s_1,p_1}} < \delta\}. \end{aligned} \tag{5}$$

By the special case of Theorem 7, we have $v \in M \Rightarrow v \in [u]_{s,p}$. By Corollary 5, we obtain $M \subset [u]_{s_0,p_0}$, i.e., $[u]_{s_0,p_0}$ is open.

In conclusion, it suffices to prove Theorem 7 under assumption (4). Moreover, by Proposition 1 we may assume $u = 1$.

Step 2: *construction of a good covering.*

We fix a small neighborhood $\mathcal{O}$ of $\bar{\Omega}$. By reflections across the boundary of Ω, we may associate to each $u \in W^{s,p}(\Omega; S^1)$ an extension $\tilde{u} \in W^{s,p}(\mathcal{O}; S^1)$ satisfying

$$||\tilde{u} - \tilde{v}||_{W^{s,p}(\mathcal{O})} \leqslant C_1\ ||u - v||_{W^{s,p}(\Omega)} \tag{6}$$

and

$$||\tilde{u} - \tilde{v}||_{W^{s_1,p_1}(\mathcal{O})} \leqslant C_1 ||u - v||_{W^{s_1,p_1}(\Omega)}. \tag{7}$$

In this section, $C_1, C_2, ...$ denote constants independent of $u, v, ...$

We fix some small $\varepsilon > 0$. By Lemma E.2 in Appendix E, for each $v \in W^{s,p}(\Omega; S^1)$ there is some $x \in \mathbb{R}^N$ (depending possibly on v) such that the covering $\mathcal{C}_N^x$ has the properties

$$v|_{\mathcal{C}_j^x} \in W^{s,p},\ j = 1, ..., N-1 \tag{8}$$

and

$$||v|_{\mathcal{C}_1^x} - 1||_{W^{s_1,p_1}(\mathcal{C}_1^x)} \leqslant C_2||v-1||_{W^{s_1,p_1}(\mathcal{O})} \leqslant C_2C_1||v-1||_{W^{s_1,p_1}(\Omega)} \tag{9}$$

(the last inequality follows from (7)).

While x may depend on v, the covering $\mathcal{C}_N^x$ has two features independent of v:

$$\text{the number of squares in } \mathcal{C}_2^x \text{ has a uniform upper bound } K; \tag{10}$$

$$\text{if } C^1, C^2 \text{ are two squares in } \mathcal{C}_2^x \text{, there is a path of squares in } \mathcal{C}_2^x \text{ each one having an edge in common with its neighbours, connecting } C^1 \text{ to } C^2. \tag{11}$$

Step 3: *choice of δ.*

We rely on

Lemma 7 *Let $C = (0, \varepsilon)^2$ and $0 < s_1 < 1$, $1 < p_1 < \infty$, $s_1p_1 = 1$. Then for each $\delta_1 > 0$ there is some $\delta_2 > 0$ such that every map $v \in W^{s_1,p_1}(\partial C; S^1)$ satisfying*

$$||v-1||_{W^{s_1,p_1}(\partial C)} < \delta_2 \tag{12}$$

has a lifting $\varphi \in W^{s_1,p_1}(\partial C; \mathbb{R})$ such that

$$||\varphi||_{W^{s_1,p_1}(\partial C)} < \delta_1. \tag{13}$$

Clearly, in Lemma 7, C may be replaced by the unit disc. For the unit disc, the proof of Lemma 7 is given in Appendix C; see Lemma C.3. In particular, if (12) holds, then we have

$$||\varphi||_{L^1(\partial C)} < C_3\delta_1 \tag{14}$$

for some C_3 independent of the δ's. We now take δ_1 such that

$$\delta_1 < \pi\varepsilon/C_3. \tag{15}$$

With δ_2 provided by Lemma 7, we choose

$$\delta = \min\{\delta_2/C_0, \delta_2/C_1C_2\}. \tag{16}$$

Step 4: *construction of a global lifting for $v|_{\mathcal{C}_1^x}$.*

Let $v \in W^{s,p}(\Omega; S^1)$ satisfy $||v-1||_{W^{s_1,p_1}} < \delta$. Since $\delta \leqslant \delta_2/C_1C_2$, (9) implies that the conclusion of Lemma 7 holds for $v|_{\partial C}$ and every square C in $\mathcal{C}_2^x$. Thus, for every $C \in \mathcal{C}_2^x$, $v|_{\partial C}$ has a lifting φ_C satisfying (14) and $\varphi_C \in W^{s_1,p_1}(\partial C)$.

We claim that $\varphi_C \in W^{s,p}(\partial C)$. The statement being local, it suffices to prove that $\varphi_C \in W^{s,p}(L)$, where L is the union of three edges in ∂C. Since L is Lipschitz homeomorphic with an interval, by Theorem 1 in [4] there is some $\psi \in W^{s,p}(L)$ such that $v = e^{i\psi}$ in L (here we use $0 < s < 1$ and $sp = 2 \geqslant 1$). In L, we have $\psi - \varphi_C \in (W^{s,p} + W^{s_1,p_1})(L; 2\pi\mathbb{Z})$; thus $\psi - \varphi_C$ is constant a.e. in L (see [4], Remark B.3), so that the claim follows.

Since $sp > 1$ and $v|_{\mathcal{C}_1^x} \in W^{s,p}, \varphi_C \in W^{s,p}$, we may redefine $v|_{\mathcal{C}_1^x}$ and φ_C on null sets in order to have continuous functions. We claim that the function $\varphi(y) = \varphi_C(y)$, if $y \in C$ is well-defined on $\mathcal{C}_1^x$ (and thus continuous and $W^{s,p}$). By (11), it suffices to prove that, if C^1, C^2 are squares in $\mathcal{C}_2^x$ having the edge $\mathcal{E}$ in common, then $\varphi_{C^1} = \varphi_{C^2}$ on $\mathcal{E}$. Clearly, on $\mathcal{E}$ we have $\varphi_{C^2} = \varphi_{C^1} + 2l\pi$ for some $l \in \mathbb{Z}$. Thus

$$||\varphi_{C^1} + 2l\pi||_{L^1(\mathcal{E})} = ||\varphi_{C^2}||_{L^1(\mathcal{E})} < C_3\delta_1,$$

by (14). It follows that

$$2|l|\pi\varepsilon = ||2l\pi||_{L^1(\mathcal{E})} \leqslant ||\varphi_{C^1}||_{L^1(\mathcal{E})} + C_3\delta_1 < 2C_3\delta_1, \tag{17}$$

which implies $l = 0$ by (15) and (16).

In conclusion, $v|_{\mathcal{C}_1^x}$ has a global lifting $\varphi \in W^{s,p}(\mathcal{C}_1^x; \mathbb{R})$.

Step 5: *construction of a good extension w of $v|_{\mathcal{C}_1^x}$.*

Let $\varphi_2 \in W^{s+1/p,p}(\mathcal{C}_2^x; \mathbb{R})$ be an extension of φ, $\varphi_3 \in W^{s+2/p,p}(\mathcal{C}_3^x; \mathbb{R})$ an extension of φ_2, and so on; let $\varphi_N \in W^{s+(N-1)/p,p}(\mathcal{C}_N^x; \mathbb{R})$ be the final extension. Note that these extensions exist since $s < 1 + (N-1)/p$, so that trace theory applies. We set $w = e^{i\varphi_N} \in W^{s+(N-1)/p,p}(\mathcal{C}_N^x; S^1)$. Since $(s + (N-1)/p) \cdot p = N + 1 > N$, we obtain by Theorem 3 that $w \in [1]_{s+(N-1)/p,p}$. By Corollary 5, we also have $w \in [1]_{s,p}$.

We complete the proof of Theorem 7 by proving

Step 6: $w \in [v]_{s,p}$.

We rely on the following variant of Lemma 6

Lemma 8 *Let $0 < s < 1$, $1 < p < \infty$, $1 < sp < N$, $[sp] \leqslant j < N$. Let $v, w \in W^{s,p}(\mathcal{C}_N; S^1)$ be such that $v|_{\mathcal{C}_l} \in W^{s,p}, w|_{\mathcal{C}_l} \in W^{s,p}$, $l = j, ..., N-1$. Assume that $v|_{\mathcal{C}_j}$ and $w|_{\mathcal{C}_j}$ are $W^{s,p}$-homotopic. Then v and w are $W^{s,p}$-homotopic.*

The proof of Lemma 8 is given Appendix D; see Lemma D.5.

When $N \geqslant 3$, we are going to apply Lemma 8 with $j = 2$. In order to prove that $v|_{\mathcal{C}_2}$ and $w|_{\mathcal{C}_2}$ are $W^{s,p}$-homotopic, it suffices to find, for each $C \in \mathcal{C}_2$, a homotopy U_C from $v|_C$ to $w|_C$ preserving the boundary condition on ∂C; we next glue together these homotopies (this works since $0 < s < 1$). We construct U_C using the lifting: since $sp = 2 = \dim C$ and C is simply connected, by Theorem 2 in [4] there is some $\psi \in W^{s,p}(C; \mathbb{R})$ such that $v = e^{i\psi}$ in C. By taking traces, we find that $v|_{\partial C} = e^{i\mathrm{tr}\,\psi} = e^{i\varphi_C}$; thus tr $\psi - \varphi_C \in (W^{s-1/p,p} + W^{s,p})(\partial C; 2\pi\mathbb{Z})$. Therefore, tr $\psi - \varphi_C$ is constant a.e., by Remark B.3 in [4]. We may assume that tr $\psi = \varphi_C =$ tr φ_2. Then $t \longmapsto e^{i((1-t)\psi + t\varphi_2)}$ is the desired homotopy U_C.

When $N = 2$, the above argument proves directly (i.e., without the help of Lemma 8) that $w \in [v]_{s,p}$.

The proof of Theorem 7 is complete.

Appendix A An extension lemma

In this appendix, we investigate, in a special case, the question whether a map in $W^{\sigma,p}(\partial\omega; S^1)$ admits an extension in $W^{\sigma+1/p,p}(\omega; S^1)$.

Lemma A.1 *Let $0 < \sigma < 1$, $1 < p < \infty$, $\sigma p < 1$, $N \geqslant 2$. Let ω be a smooth bounded domain in $\mathbb{R}^N$. Then every $v \in W^{\sigma,p}(\partial\omega; S^1)$ has an extension $w \in W^{\sigma+1/p,p}(\omega; S^1)$.*

PROOF. We distinguish two cases: $\sigma \leqslant 1 - 1/p$ and $\sigma > 1 - 1/p$.

Case $\sigma \leqslant 1 - 1/p$**:** since $\sigma p < 1$, v may be lifted in $W^{\sigma,p}$ (see Bourgain - Brezis - Mironescu [4]), i.e. there is some $\psi \in W^{\sigma,p}(\partial\omega; \mathbb{R})$ such that $v = e^{i\psi}$. Let $\varphi \in W^{\sigma+1/p,p}(\omega; \mathbb{R})$ be an extension of ψ. Then $w = e^{i\varphi} \in W^{\sigma+1/p,p}(\omega; S^1)$ (since $\sigma + 1/p \leqslant 1$ and $x \mapsto e^{ix}$ is Lipchitz). Clearly, w has all the required properties.

Case $\sigma > 1 - 1/p$**:** the argument is similar, but somewhat more involved. The proof in [4] actually yields a lifting which is better than $W^{\sigma,p}$; more specifically, this lifting ψ belongs to $W^{t\sigma,p/t}$ for $0 < t \leqslant 1$, see Remark 2, p.41, in the above reference. On the other hand, since $\sigma > 1 - 1/p$, we have $t = p/(\sigma p + 1) < 1$.

For this choice of t, we obtain that v has a lifting $\psi \in W^{\sigma,p} \cap W^{1-1/(\sigma p+1),\sigma p+1}$. This ψ has an extension $\varphi \in W^{\sigma+1/p,p} \cap W^{1,\sigma p+1}$. By the Composition Theorem stated in the Introduction, the map $w = e^{i\varphi}$ belongs to $W^{\sigma+1/p,p}(\omega; S^1)$. Clearly, we have $\operatorname{tr} w = v$.

Remark A.1. The special case $p < 2$ and $\sigma = 1 - 1/p$ was originally treated by Hardt - Kinderlehrer - Lin [16] via a totally different method. Their argument extends to the case $p < 2$ and $\sigma p < 1$, but does not seem to apply when $p \geqslant 2$.

Appendix B Good restrictions

In this appendix, we describe a natural substitute for the trace theory when $s = 1/p$; it is known that the standard trace theory is not defined in this limiting case.

For simplicity, we consider mainly the case of a flat boundary. However, we state Lemma B.5 (used in the proof of Theorem 1) for a general domain. We start by introducing some

Notations: let $Q = (0,1)^{N-1}$, $\Omega_+ = Q \times (0,1)$, $\Omega_- = Q \times (-1,0)$, $\Omega = \Omega_+ \cup \Omega_- = Q \times (-1,1)$. If v is a function defined on Q, we set $\tilde{v}(x',t) = v(x)$ for $(x',t) \in \Omega$.

Lemma B.1 *Let $0 < s < 1$, $1 < p < \infty$. Then for $u \in W^{s,p}(\Omega_+)$ and for any function v defined on Q, the following assertions are equivalent:*

a) $v \in W^{s,p}(Q)$ and

$$I = \int_{\Omega_+} \frac{|u(x) - \tilde{v}(x)|^p}{x_N^{sp}}\, dx < \infty; \tag{B.1}$$

b) the map $w_1 = \begin{cases} u, & \text{in } \Omega_+ \\ \tilde{v}, & \text{in } \Omega_- \end{cases}$*, belongs to $W^{s,p}(\Omega)$;*

c) the map $w_2 = \begin{cases} u - \tilde{v}, & \text{in } \Omega_+ \\ 0, & \text{in } \Omega_- \end{cases}$ *belongs to $W^{s,p}(\Omega)$.*

PROOF. Recall that, if U is a smooth or cube-like domain, then an equivalent (semi-) norm on $W^{s,p}(U)$ is given by

$$f \longmapsto \left(\sum_{j=1}^{N} \int_0^\infty \int_{\{x \in U;\, x+te_j \in U\}} \frac{f(x+te_j) - f(x)|^p}{t^{sp+1}}\, dxdt \right)^{1/p} \tag{B.2}$$

(see, e.g., Triebel [25]).

Clearly, both b) and c) imply that $v \in W^{s,p}(Q)$. Conversely, for $v \in W^{s,p}(Q)$ we have to prove the equivalence of (B.1), b) and c). We consider the norm given by (B.2). Taking into account the fact that w_1, w_2 belong to $W^{s,p}$ in Ω_+ and Ω_-, we see that

$$w_1 \in W^{s,p}(\Omega) \Leftrightarrow J = \int_{\Omega_+} \int_{-1}^{0} \frac{|u(x) - \tilde{v}(x)|^p}{(x_N - t)^{sp+1}}\, dtdx < \infty \tag{B.3}$$

and

$$w_2 \in W^{s,p}(\Omega) \Leftrightarrow J < \infty. \tag{B.4}$$

The lemma follows from the obvious inequality

$$\frac{1 - 2^{-sp}}{sp} I \leqslant J \leqslant \frac{1}{sp} I.$$

We now assume in addition that $sp \geqslant 1$ and derive the following

Corollary B.1 *Let $0 < s < 1$, $1 < p < \infty$ be such that $sp \geqslant 1$. Then, for every $u \in W^{s,p}(\Omega_+)$ we have* *a) for each* $0 \leqslant t_0 < 1$*, there is at most one function* v *defined on* Q *such that the maps*

$$w_1^{t_0} = \begin{cases} u, & \text{in } Q \times (t_0, 1) \\ \tilde{v}, & \text{in } Q \times (-1, t_0) \end{cases}$$

and

$$w_2^{t_0} = \begin{cases} u - \tilde{v}, & \text{in } Q \times (t_0, 1) \\ 0, & \text{in } Q \times (-1, t_0) \end{cases}$$

belong to $W^{s,p}(\Omega)$*;*
b) for a.e. $0 \leqslant t_0 < 1$*, the function* $v = u(\cdot, t_0)$ *has the property that* $w_1^{t_0}, w_2^{t_0} \in W^{s,p}(\Omega)$.

(As usual, the uniqueness of v is understood a.e.)

The above corollary suggests the following

Definition: *let* $0 < s < 1$, $1 < p < \infty$, $sp \geqslant 1$, $0 \leqslant t_0 < 1$*. Let* $u \in W^{s,p}(\Omega_+)$ *and let* v *be a function defined on* Q*. Then* v *is the downward good restriction of* u *to* $\{x_N = t_0\}$ *if* $w_1^{t_0}, w_2^{t_0} \in W^{s,p}(\Omega)$*; we then write* $v =$ *Rest* $u|^-_{x_N=t_0}$*. Similarly, for* $0 < t_0 < 1$ *we may define an upward good restriction Rest* $u|^+_{x_N=t_0} = v$ *as the unique function* v *defined on* Q *satisfying the two equivalent conditions*
a) $W_1^{t_0} = \begin{cases} \tilde{v}, & \text{in } Q \times (t_0, 1) \\ u, & \text{in } Q \times (0, t_0) \end{cases} \in W^{s,p}(\Omega_+)$
and
b) $W_2^{t_0} = \begin{cases} 0, & \text{in } Q \times (t_0, 1) \\ u - \tilde{v}, & \text{in } Q \times (0, t_0) \end{cases} \in W^{s,p}(\Omega_+)$.
If v *is both an upward and a downward good restriction, we call it a good restriction and we write* $v =$ *Rest* $u|_{x_N=t_0}$.

Corollary B.2 *Let* $0 < s < 1$, $1 < p < \infty$, $sp \geqslant 1$*. Let* $u \in W^{s,p}(\Omega_+)$*. Then, for a.e.* $0 < t_0 < 1$*, we have* Rest $u|_{x_N=t_0} = u(\cdot, t_0)$.

Remark B.1 If $sp > 1$, then functions $u \in W^{s,p}(\Omega_+)$ have traces for *all* $0 \leqslant t_0 \leqslant 1$. However, these traces need not be good restrictions. Here is an example: For $N = 2$, one may prove that the map $x \mapsto (x - 1/2e_1)/|x - 1/2e_1|$ belongs to $W^{s,p}(\Omega)$ if $0 < s < 1$,
$1 < p < \infty, sp < 2$. However, if $sp > 1$, its trace

$$\text{tr } u|_{x_2=0} = \begin{cases} 1, & \text{if } x_1 > 1/2 \\ -1, & \text{if } x_1 < 1/2 \end{cases}$$

does not belong to $W^{s,p}(0,1)$, so that it is not a good restriction.

Remark B.2 In the limiting case $s = 1/p$, functions in $W^{s,p}$ do not have traces. However, they do have good restrictions a.e.

Here is yet another simple consequence of Lemma B.1

Corollary B.3 *Let* $0 < s < 1$, $1 < p < \infty$, $sp \geqslant 1$*. Let* $u_\pm \in W^{s,p}(\Omega_\pm)$ *be such that* Rest $u_+|^-_{x_N=0} =$ Rest $u_-|^+_{x_N=0}$*. Then the map* $w = \begin{cases} u_+, & \text{in } \Omega_+ \\ u_-, & \text{in } \Omega_- \end{cases}$ *belongs to* $W^{s,p}$.

The following results explain the connections between good restrictions and traces.

Lemma B.2 *Let* $0 < s < 1$, $1 < p < \infty$, $sp > 1$*. Let* $u \in W^{s,p}(\Omega_+)$*. Assume that there exists* $v =$ Rest $u|^-_{x_N=0}$*. Then* $v = \text{tr } u|_{x_N=0}$.

PROOF. Let $w = \begin{cases} u - \tilde{v}, & \text{in } \Omega_+ \\ 0, & \text{in } \Omega_-. \end{cases}$ By Lemma B.1, we have $w \in W^{s,p}(\Omega)$. By trace theory and continuity of the trace, we have $0 = \text{tr } w|_{x_N=0}$, so that $\text{tr } u|_{x_N=0} = v$.

Lemma B.3 *Let* $0 < s < 1$, $1 < p < \infty$, $sp \geqslant 1$. *Let* $u \in W^{s+1/p,p}(\Omega_+)$. *Then, considered as a* $W^{s,p}$ *function,* u *has a good downward restriction to* $\{x_N = 0\}$ *which coincides with* $\operatorname{tr} u|_{x_N=0}$.

PROOF. Let $v = \operatorname{tr} u|_{x_N=0}$. Then $v \in W^{s,p}(Q)$, by the trace theory. By Lemma B.1, it remains to prove that

$$\int_{\Omega_+} \frac{|u(x) - \tilde{v}(x)|^p}{x_N^{sp}}\, dx < \infty. \tag{B.5}$$

Assume first that $s + 1/p = 1$. Then (B.5) follows from the well-known Hardy inequality

$$\int_Q \int_0^1 \frac{|u(x',t) - u(x',0)|^p}{t^p} dt dx \leqslant C \|Du\|_{L^p}^p,\ \forall u \in W^{1,p}(\Omega_+). \tag{B.6}$$

Consider now the case where $s + 1/p \neq 1$. Let $\sigma = s + 1/p$. We are going to prove that

$$\int_{\Omega_+} \frac{|u(x) - \tilde{v}(x)|^p}{x_N^{sp}}\, dx \leqslant C \|u\|_{W^{\sigma,p}}^p \tag{B.7}$$

for some convenient equivalent (semi-) norm on $W^{\sigma,p}$. It is useful to consider the norm

$$f \mapsto \left(\sum_{j=1}^N \int_0^\infty \int_{\{x \in U;\, x+te_j \in U,\, x+2te_j \in U\}} \frac{|f(x+2te_j) - 2f(x+te_j) + f(x)|^p}{t^{\sigma p+1}} dx dt \right)^{1/p} \tag{B.8}$$

(see, e.g., Triebel [24]).

For any $x' \in Q$ such that $u_{x'} = u(x', \cdot) \in W^{\sigma,p}(0,1)$, the map

$$f_{x'}(t) = \begin{cases} u(x',t), & \text{if } t > 0 \\ v(x'), & \text{if } t < 0 \end{cases}$$

belongs to $W^{\sigma,p}(-1,1)$, by standard trace theory. Moreover, for any such x' we have

$$\|f_{x'}\|_{W^{\sigma,p}(-1,1)}^p \leqslant C \|u_{x'}\|_{W^{\sigma,p}(0,1)}^p, \tag{B.9}$$

i.e.

$$\int_0^\infty \int_{\{h \in (-1,1);\, h+t \in (-1,1),\, h+2t \in (-1,1)\}} \frac{|f_{x'}(h+2t) - 2f_{x'}(h+t) + f_{x'}(h)|^p}{t^{\sigma p+1}}\, dh dt \leqslant$$
$$C \int_0^\infty \overrightarrow{\{h \in (0,1);\, h+t \in (0,1),\, h+2t \in (0,1)\}} \int \frac{|u_{x'}(h+2t) - 2u_{x'}(h+t) + u_{x'}(h)|^p}{t^{\sigma p+1}} dh dt.$$

In particular,

$$I = \int_0^{1/2} \int_{-2t}^{-t} \frac{|f_{x'}(h+2t) - 2f_{x'}(h+t) + f_{x'}(h)|^p}{t^{\sigma p+1}} dh dt \leqslant C \|u_{x'}\|_{W^{\sigma,p}}^p. \tag{B.10}$$

Since

$$I \geqslant C \int_0^{1/3} \frac{|u(x',t) - v(x')|^p}{t^{\sigma p}} dt = C \int_0^{1/3} \frac{|u(x',t) - v(x')|^p}{t^{sp+1}} dt, \tag{B.11}$$

we find that

$$\int_0^{1/3} \frac{|u(x',t) - v(x')|^p}{t^{sp+1}} dt \leqslant C \|u_{x'}\|_{W^{\sigma,p}}^p. \tag{B.12}$$

On the other hand, we clearly have

$$\int_{1/3}^1 \frac{|u(x',t) - v(x')|^p}{t^{sp+1}} dt \leqslant C \|u_{x'}\|_{L^p}^p + C|v(x')|^p. \tag{B.13}$$

By combining (B.12), (B.13) and integrating with respect to x', we obtain (B.7). The proof of Lemma B.3 is complete.

A simple consequence of Lemma B.3 is the following

Lemma B.4 *Let $0 < s < 1$, $1 < p < \infty$, $sp \geqslant 1$ and $\rho > s$. Let $u_1 \in W^{s,p}(\Omega_+)$ and $u_2 \in W^{\rho,p}(\Omega_-)$. Assume that u_1 has a good downward restriction $v = \text{Rest } u_1|^-_{x_N=0}$ and that $v = \text{tr } u_2|_{x_N=0}$. Then the map*

$$w = \begin{cases} u_1, & \text{in } \Omega_+ \\ u_2, & \text{in } \Omega_- \end{cases}$$

belongs to $W^{s,p}(\Omega)$.

PROOF. Let $u_3 \in W^{s+1/p,p}(\Omega_-)$ be an extension of v. Then $w = w_1 + w_2$, where

$$w_1 = \begin{cases} u_1, & \text{in } \Omega_+ \\ u_3, & \text{in } \Omega_- \end{cases}$$

and

$$w_2 = \begin{cases} 0, & \text{in } \Omega_+ \\ u_2 - u_3, & \text{in } \Omega_-. \end{cases}$$

By Lemma B.3 and the assumption $v = \text{Rest } u_1|^-_{x_N=0}$, we have $\text{Rest } u_1|^-_{x_N=0} = \text{Rest } u_3|^+_{x_N=0}$. By Corollary B.3, we find that $w_1 \in W^{s,p}(\Omega)$. It remains to prove that $w_2 \in W^{s,p}(\Omega)$. Let $\sigma = \min\{\rho, s + 1/p, 1\}$. Then $w_2 \in W^{\sigma,p}(\Omega)$, by standard trace theory. Thus $w_2 \in W^{s,p}(\Omega)$.

We conclude this section by stating the following precised form of Corollary B.1, b) in the case of a general boundary. We use the same notations as in the proof of Theorem 1, Case 4.

Lemma B.5 *em Let $u \in W^{1/p,p}(\Omega)$. Then*
a) for a.e. $0 < \delta < \varepsilon$ we have

$$u|_{\Sigma_\delta} \in W^{1/p,p}(\Sigma_\delta) \text{ and } \int_{\Sigma_\delta}\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+1}} dy ds_x < \infty; \tag{B.14}$$

b) for any such δ, u has a good restriction to Σ_δ which coincides (a.e. on Σ_δ) with $u|_{\Sigma_\delta}$.

Appendix C Global lifting

In this appendix, we investigate the existence of a global lifting in some domains with non-trival topology.

Lemma C.1 *Let $0 < s < \infty$, $1 < p < \infty$, $sp \geqslant N, N \geqslant 2$. Let $u \in W^{s,p}(S^1 \times B_1; S^1)$ be such that* $\deg(u|_{S^1\times B_1}) = 0$. *Then there is some $\varphi \in W^{s,p}(S^1 \times B_1; S^1)$ such that $u = e^{i\varphi}$.*

Here, B_1 is the unit ball in $\mathbb{R}^{N-1}$.
PROOF. Let $v : \mathbb{R}\times B_1 \to S^1$, $v(t,x) = u(e^{it}, x)$. Then $v \in W^{s,p}_{loc}(\mathbb{R}\times B_1; S^1)$, where "loc" refers only to the variable t. By Theorem 2 in Bourgain - Brezis - Mironescu [4], there is some $\psi \in W^{s,p}_{loc}(\mathbb{R}\times B_1;\mathbb{R})$ such that $v = e^{i\psi}$. We claim that ψ is 2π-periodic in the variable t. Indeed, for a.e. $x \in B_1$, we have $u \in W^{s,p}(S^1 \times \{x\}; S^1)$ and $\deg(u|_{S^1\times\{x\}}) = 0$. In particular, for any such x the map $u|_{S^1\times\{x\}}$ has a continuous lifting η_x. On the other hand, for a.e. $x \in B_1$ we have $\psi_x = \psi(\cdot, x) \in W^{s,p}_{loc}(\mathbb{R}\times\{x\};\mathbb{R})$. Thus, with $\lambda_x(t) = \eta_x(e^{it})$, we find that for a.e. $x \in B_1$ the function $\psi_x - \lambda_x$ is continuous and $2\pi\mathbb{Z}$ -valued; therefore it is a constant. Since λ_x is 2π-periodic, so is ψ_x for a.e. $x \in B_1$. We obtain that ψ is 2π-periodic in the variable t. Thus the map $\varphi : S^1 \times B_1 \to \mathbb{R}$, $\varphi(e^{it}, x) = \psi(t, x)$ is well-defined and belongs to $W^{s,p}(S^1 \times B_1;\mathbb{R})$. Moreover, we clearly have $u = e^{i\varphi}$.

In the same vein, we have

Lemma C.2 *Let $s \geqslant 1$, $1 < p < \infty$, $N \geqslant 3$, $2 \leqslant sp < N$. Let $u \in W^{s,p}(S^1 \times B_1; S^1)$ be such that* $\deg\ (u|_{S^1 \times B_1}) = 0$. *Then there is some $\varphi \in W^{s,p}(S^1 \times B_1; \mathbb{R}) \cap W^{1,sp}(S^1 \times B_1; \mathbb{R})$ such that $u = e^{i\varphi}$.*

The proof is similar to that of Lemma C.1; one has to use Lemma 4 in [4] instead of Theorem 2 in [4].

Lemma C.3 *Let $1 < p < \infty$ and $\delta_1 > 0$. Then there is some $\delta_2 > 0$ such that every $v \in W^{1/p,p}(S^1; S^1)$ satisfying $\|v - 1\|_{W^{1/p,p}(S^1)} < \delta_2$ has a global lifting $\varphi \in W^{1/p,p}(S^1; \mathbb{R})$ such that $\|\varphi\|_{W^{1/p,p}(S^1)} < \delta_1$.*

PROOF. Recall that if I is an interval, then every $w \in W^{1/p,p}(I; S^1)$ has a lifting $\psi \in W^{1/p,p}(I; \mathbb{R})$ (see Bourgain - Brezis - Mironescu [4], Theorem 1). Moreover, this lifting may be chosen to be (locally) continuous with respect to w, i.e. for every $w_0 \in W^{1/p,p}(I; S^1)$ there is some $\delta_0 > 0$ such that in the set

$$\{w; \|w - w_0\|_{W^{1/p,p}(I;S^1)} < \delta_0\}$$

there is a lifting $w \mapsto \psi$ continuous for the $W^{1/p,p}$ norm. (This assertion can be established using the same argument as in Step 7 of the proof of Theorem 4 in Brezis, Nirenberg [12]; it can also be derived from the explicit construction of ψ in the proof of Theorem 1 in [4]; see also Boutet de Monvel, Berthier, Georgescu, Purice [6] when $p = 2$).

Let $I = [-2\pi, 2\pi]$. To each $v \in W^{1/p,p}(S^1; S^1)$ we associate the map $w \in W^{1/p,p}(I; S^1)$, $w(t) = v(e^{it})$. By the above considerations, for every $\delta_3 > 0$ there is some $\delta_4 > 0$ such that, if $\|v - 1\|_{W^{1/p,p}(S^1)} < \delta_4$, then w has a lifting ψ such that $\|\psi\|_{W^{1/p,p}(I)} < \delta_3$. We claim that ψ is 2π-periodic if δ_3 is small enough. Indeed, the function $\xi(t) = \psi(t - 2\pi) - \psi(t)$ belongs to $W^{1/p,p}([0, 2\pi]; 2\pi\mathbb{Z})$, so that ξ is constant a.e. (see [4], Theorem B.1). Since $\|\xi\|_{L^1} \leqslant \|\psi\|_{L^1} < C\delta_3$, we have $\xi = 0$ (i.e. ψ is 2π-periodic) if $C\delta_3 < 2\pi$.

Thus, for δ_3 small enough, the map $\varphi(e^{it}) = \psi(t)$ is well-defined, belongs to $W^{1/p,p}$ and satisfies $\|\varphi\|_{W^{1/p,p}(S^1)} < \delta_1$ and $u = e^{i\varphi}$.

Appendix D Filling a hole - the fractional case

We adapt to fractional Sobolev spaces the technique of Brezis - Li [7], Section 1.3.

The first two results are preparations for the proofs of Lemmas 5,6 and 8 (see Lemmas D.3, D.4 and D.5 below).

Lemma D.1 *Let $0 < s < 1$, $1 < p < \infty$, $1 < sp < N$. Let $C = (-1, 1)^N$ and $u \in W^{s,p}(\partial C)$. Then $\tilde{u} \in W^{s,p}(C)$; here, $\tilde{u}(x) = u(x/|x|)$ and $|\ |$ is the L^∞ norm in $\mathbb{R}^N$. Moreover, the map $u \mapsto \tilde{u}$ is continuous from $W^{s,p}(\partial C)$ into $W^{s,p}(C)$.*

PROOF. Clearly, we have $\|\tilde{u}\|_{L^p(C)} \leqslant C_0 \|u\|_{L^p(\partial C)}$. Thus it suffices to prove, for the Gagliardo semi-norms in $W^{s,p}$, the inequality

$$\|\tilde{u}\|^p_{W^{s,p}(C)} \leqslant C_1(\|u\|^p_{W^{s,p}(\partial C)} + \|u\|^p_{L^p(\partial C)}). \tag{D.1}$$

We have

$$\int_C \int_C \frac{|\tilde{u}(x) - \tilde{u}(y)|^p}{|x - y|^{N+sp}} dx dy = \int_0^1 \int_0^1 \int_{\partial C} \int_{\partial C} \frac{|u(x) - u(y)|^p}{|\tau x - \sigma y|^{N+sp}} \tau^{N-1} \sigma^{N-1} ds_x ds_y d\tau d\sigma. \tag{D.2}$$

We claim that

$$I = \int_0^1 \int_0^1 \frac{\tau^{N-1} \sigma^{N-1}}{|\tau x - \sigma y|^{N+sp}} d\tau d\sigma \leqslant C_2 / |x - y|^{N+sp}. \tag{D.3}$$

Indeed,

$$\begin{aligned} I = \int_0^1 \int_0^{1/\tau} \frac{\tau^{N-1} (\lambda\tau)^{N-1}}{|\tau x - \lambda\tau y|^{N+sp}} d\lambda d\tau \quad &= \\ \int_0^1 \int_0^{1/\tau} \tau^{N-sp-1} \frac{\lambda^{N-1}}{|x - \lambda y|^{N+sp}} d\lambda d\tau \leqslant I_1 + I_2&, \end{aligned} \tag{D.4}$$

where $I_1 = \int_0^1 \int_0^2$ and $I_2 = \int_0^1 \int_2^\infty$.

On the one hand, we have

$$\begin{aligned} I_1 &= \int_0^1 \int_0^2 \tau^{N-sp-1} \frac{\lambda^{N-1}}{|x-\lambda y|^{N+sp}} d\lambda d\tau \\ &\leqslant C_3 \int_0^1 \int_0^2 \tau^{N-sp-1} \frac{\lambda^{N-1}}{|x-y|^{N+sp}} d\lambda d\tau \leqslant C_4/|x-y|^{N+sp}. \end{aligned} \tag{D.5}$$

On the other hand, we have

$$\begin{aligned} I_2 &= \int_0^1 \int_2^\infty \tau^{N-sp-1} \frac{\lambda^{N-1}}{|x-\lambda y|^{N+sp}} d\lambda d\tau \\ &\leqslant C_5 \int_0^1 \int_2^\infty \tau^{N-sp-1} \frac{\lambda^{N-1}}{\lambda^{N+sp}} d\lambda d\tau = C_5 \int_0^1 \int_2^\infty \tau^{N-sp-1} \lambda^{-sp-1}\, d\lambda d\tau \leqslant C_6. \end{aligned} \tag{D.6}$$

We obtain (D.3) by combining (D.4), (D.5) and (D.6). Finally, (D.1) follows from (D.2) and (D.3).

The proof of Lemma D.1 is complete.

Lemma D.2 *Let $0 < s < 1$, $1 < p < \infty$, $1 < sp < N$. Let $v, w \in W^{s,p}(C; S^1)$ be such that $v|_{\partial C} = w|_{\partial C} \in W^{s,p}(\partial C)$. Then, there is a homotopy $U \in C^0([0,1]; W^{s,p}(C; S^1))$ such that $U(0,\cdot) = v$, $U(1,\cdot) = w$ and $U(t,\cdot)|_{\partial C} = v|_{\partial C}, \forall t \in [0,1]$.*

PROOF. Let $u = v|_{\partial C}$. It clearly suffices to prove the lemma in the special case $w = \tilde{u}$. In this case, let, for $0 \leqslant t < 1$,

$$U(t,x) = \begin{cases} v(x/(1-t)), & \text{if } |x| \leqslant 1-t \\ \tilde{u}(x), & \text{if } 1-t < |x| \leqslant 1; \end{cases}$$

set $U(1,\cdot) = \tilde{u}$. Clearly, $U \in C^0([0,1); W^{s,p}(C; S^1))$. It remains to prove that $U(t,\cdot) \to \tilde{u}$ as $t \to 1$. Let

$$f(x) = \begin{cases} v(x), & \text{if } |x| \leqslant 1 \\ \tilde{u}(x), & \text{if } |x| > 1 \end{cases}$$

and $g = f - \tilde{u}$. Then f, $\tilde{u} \in W^{s,p}_{loc}(\mathbb{R}^N)$, so that $g \in W^{s,p}_{loc}(\mathbb{R}^N)$. Since $g = 0$ outside C, we actually have $g \in W^{s,p}(\mathbb{R}^N)$. Thus

$$\begin{aligned} &\|U(t,\cdot) - \tilde{u}\|^p_{W^{s,p}(C)} = \|g(\cdot/(1-t))\|^p_{W^{s,p}(C)} \leqslant \\ &\|g(\cdot/(1-t))\|^p_{W^{s,p}(\mathbb{R}^N)} = (1-t)^{N-sp}\|g\|^p_{W^{s,p}(\mathbb{R}^N)} \to 0 \end{aligned}$$

as $t \to 1$. The proof of Lemma D.2 is complete.

We introduce a useful notation: let $u \in W^{s_1,p_1}(\mathcal{C}_k)$, where $0 < s_1 < 1$, $1 < p_1 < \infty$, $1 < s_1 p_1 < N$. We extend, for each $C \in \mathcal{C}_{k+1}$, $u|_{\partial C}$ to C as in Lemma D.1. Let $\tilde{u}$ be the map obtained by gluing these extensions. We next extend $\tilde{u}$ to $\mathcal{C}_{k+2}$ in the same manner, and so on, until we obtain a map defined in $\mathcal{C}_N$; call it $H_k(u)$.

Lemma D.3 *Let $0 < s_1 < 1$, $1 < p_1 < \infty$, $1 < s_1 p_1 < N$, $[s_1 p_1] \leqslant j < N$. Then every $v \in W^{s_1,p_1}(\mathcal{C}_j; S^1)$ has an extension $u_1 \in W^{s_1,p_1}(\mathcal{C}_N; S^1)$ such that $u_1|_{\mathcal{C}_l} \in W^{s_1,p_1}$ for $l = j, ..., N-1$.*

PROOF. We take $u_1 = H_j(v)$. We may use repeatedly Lemma D.1, since for $l = j+1, ..., N$ we have $1 < s_1 p_1 < l$.

Lemma D.4 *Let $0 < s < 1$, $1 < p < \infty$, $1 < sp < N$, $[sp] \leqslant j < N$. If $u|_{\mathcal{C}_l} \in W^{s,p}, u_1|_{\mathcal{C}_l} \in W^{s,p}, l = j, ..., N-1$, and $u|_{\mathcal{C}_j} = u_1|_{\mathcal{C}_j}$, then u and u_1 are $W^{s,p}$-homotopic.*

PROOF. We argue by backward induction on j. If $j = N-1$, then for each $C \in \mathcal{C}_N$ Lemma D.2 provides a $W^{s,p}$-homotopy of $u|_C$ and $u_1|_C$ preserving the boundary condition. By gluing together these homotopies we find that u and u_1 are $W^{s,p}$-homotopic (here we use $1/p < s < 1$). Suppose now that the conclusion of the lemma holds for $j+1$; we prove it for j, assuming that $j \geqslant [sp]$. By assumption, u and $H_{j+1}(u|_{\mathcal{C}_{j+1}})$ are $W^{s,p}$-homotopic, and so are u_1 and $H_{j+1}(u_1|_{\mathcal{C}_{j+1}})$. It suffices therefore to prove that $v = H_{j+1}(u|_{\mathcal{C}_{j+1}})$ and $v_1 = H_{j+1}(u_1|_{\mathcal{C}_{j+1}})$ are $W^{s,p}$-homotopic. For each $C \in \mathcal{C}_{j+1}$, we have $v|_{\partial C} = v_1|_{\partial C} = u|_{\partial C} = u_1|_{\partial C}$. By Lemma D.2, $v|_C$ and $v_1|_C$ are connected by a homotopy preserving the trace on ∂C. Gluing together these homotopies, we find that $v|_{\mathcal{C}_{j+1}}$ and $v_1|_{\mathcal{C}_{j+1}}$ are $W^{s,p}$-homotopic. If U connects $v|_{\mathcal{C}_{j+1}}$ to $v_1|_{\mathcal{C}_{j+1}}$, then Lemma D.1 used repeatedly implies that $t \mapsto H_{j+1}(U(t))$ connects in $W^{s,p}(\mathcal{C}_N; S^1)$ the map $H_{j+1}(v|_{\mathcal{C}_{j+1}})$ to $H_{j+1}(v_1|_{\mathcal{C}_{j+1}})$, i.e., v to v_1.

The proof of Lemma D.4 is complete.

Lemma D.5 *Let $0 < s < 1$, $1 < p < \infty$, $1 < sp < N$, $[sp] \leqslant j < N$. Let $v, w \in W^{s,p}(\mathcal{C}_N; S^1)$ be such that $v|_{\mathcal{C}_l} \in W^{s,p}, w|_{\mathcal{C}_l} \in W^{s,p}, l = j, ..., N-1$. Assume that $v|_{\mathcal{C}_j}$ and $w|_{\mathcal{C}_j}$ are $W^{s,p}$-homotopic. Then v and w are $W^{s,p}$-homotopic.*

PROOF. By Lemma D.4, v and $H_j(v|_{\mathcal{C}_j})$ (respectively w and $H^j(w|_{\mathcal{C}_j})$) are $W^{s,p}$-homotopic. If U connects $v|_{\mathcal{C}_j}$ to $w|_{\mathcal{C}_j}$ in $W^{s,p}$, then as in the proof of Lemma D.4, we obtain that $t \mapsto H_j(U(t))$ connects $H_j(v|_{\mathcal{C}_j})$ to $H_j(w|_{\mathcal{C}_j})$ in $W^{s,p}$. Thus v and w are $W^{s,p}$-homotopic.

Appendix E Slicing with norm control

In this section, we prove the existence of good coverings for $W^{s,p}$ maps. The arguments are rather standard.

Without loss of generality, we may consider maps defined in $\mathbb{R}^N$. Throughout this section, we assume $\varepsilon = 1$, i.e. we consider a covering with cubes of size 1. We start by introducing some useful notations: for $x \in C^N = (0,1)^N$ and for $j = 1, ..., N-1$, let

$$C_j = \bigcup \left\{ \sum_{k=1}^{j} t_k\, e_{i_k} \; + \sum_{l=1}^{N-j} \lambda_l e_{j_l};\; t_k \in \mathbb{R}, \lambda_l \in \mathbb{Z}, \{e_{i_k}\} \cup \{e_{j_l}\} = \{e_1, ... e_N\} \right\}$$

and $C_j(x) = x + C_j$. (With the notations introduced in Section 3, we have $C_j(x) = \mathcal{C}_j^x$ when $\Omega = \mathbb{R}^N$). For a fixed set $\Lambda \subset \{1, .., N\}$ such that $|\Lambda| = j$, let also

$$C_j^{\Lambda} = \left\{ \sum_{i \in \Lambda} t_i e_i + \sum_{j \notin \Lambda} \lambda_j e_j;\; t_i \in \mathbb{R}, \lambda_j \in \mathbb{Z} \right\},$$

so that

$$C_j = \cup \{C_j^{\Lambda}; \Lambda \subset \{1, ..., N\}, |\Lambda| = j\},$$

and with obvious notations

$$C_j(x) = \cup \{C_j^{\Lambda}(x); \Lambda \subset \{1, ..., N\}, |\Lambda| = j\}.$$

Instead of considering a fixed (semi-) norm on $W^{s,p}, 0 < s < 1, 1 < p < \infty$, it is convenient to consider a family of equivalent norms

$$|f|_j^p = \sum_{\substack{\Lambda \subset \{1, ..., N\} \\ |\Lambda| = j}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^j} \frac{|f(x + \sum_{i \in \Lambda} t_i e_i) - f(x)|^p}{|t|^{j+sp}} dt dx$$

(see, e.g., Triebel [24]). An obvious computation yields, for the usual Gagliardo (semi-) norm on $C_j^{\Lambda}(x)$,

Lemma E.1 *Let $0 < s < 1$, $1 < p < \infty$ and $u \in W^{s,p}$. Then*

$$\sum_{\substack{\Lambda \subset \{1,...,N\} \\ |\Lambda| = j}} \int_{C^N} \|u\|^p_{W^{s,p}(C_j^\Lambda(x))} dx \leqslant |u|^p_j \tag{E.1}$$

for some C independent of u.

We next define the norm $\|u\|_{W^{s,p}(C_j(x))}$ by the formula

$$\|u\|^p_{W^{s,p}(C_j(x))} = \sum_{C \in C_{j+1}(x)} \|u\|^p_{W^{s,p}(\partial C)}.$$

Lemma E.2 *Let $0 < s < 1$, $1 < p < \infty$. Then, for $u \in W^{s,p}$, we have*
a) for a.e. $x \in C^N$, $u|_{C_j(x)} \in W^{s,p}_{loc}$, $j = 1, ..., N-1$;
b) there is a fat set (i.e., with positive measure) $A \subset C^N$ such that

$$\|u\|^p_{W^{s,p}(C_j(x))} \leqslant C\, |u|^p_j, \quad \forall x \in A. \tag{E.2}$$

Remark E.1. Here, $u|_{C_j(x)}$ are restrictions, not traces. However, when $sp > 1$ we may replace restrictions by traces, by a standard argument. We obtain

Corollary E.1 *Let $0 < s < 1$, $1 < p < \infty$, $sp > 1$. Let $u \in W^{s,p}$. Then, for a.e. $x \in C^N$,* tr $u|_{C_{N-1}(x)} \in W^{s,p}$*. Moreover, for a.e. $x \in C^N$,* tr $u|_{C_{N-1}(x)}$ *has a trace on $C_{N-2}(x)$ which belongs to $W^{s,p}$, and so on.*

PROOF OF LEMMA E.2. In order to avoid long computations, we treat only the case $j = 1, N = 2$. The general case does not bring any additional difficulty. Let $C \in C_1(x)$; denote its lower (resp. upper, left, right) edge by C^l (resp. C^u, C^L, C^R). By (E.1), we have $u|_{C^l} \in W^{s,p}$ for a.e. $x \in C^2$ and, for x in a fat set, $\sum_{C \in C_1(x)} \|u\|^p_{W^{s,p}(C^l)} \leqslant$ const. $|u|^p_1$. Similar statements hold for the other edges.

It remains to control the cross - integrals in the Gagliardo norm, e.g. to prove

$$I = \int_{C^2} \sum_{C \in C_1(x)} \int_{C^l} \int_{C^L} \frac{|u(y) - u(z)|^p}{|y - z|^{2+sp}} dy dz \leqslant \text{ const. } \|u\|^p_{W^{s,p}} \tag{E.3}$$

(here, we take the usual Gagliardo norm in $W^{s,p}(\mathbb{R}^2)$). We have

$$\begin{aligned} I &= \int_{C^2} \sum_{m \in \mathbb{Z}^2} \int_0^1 \int_0^1 \frac{|u(x + m_1 e_1 + m_2 e_2 + \tau e_1) - u(x + m_1 e_1 + m_2 e_2 + \sigma e_2)|^p}{|\tau e_1 - \sigma e_2|^{2+sp}} d\sigma d\tau dx \\ &= \int_{\mathbb{R}^2} \int_0^1 \int_0^1 \frac{|u(y + \tau e_1) - u(y + \sigma e_2)|^p}{|\tau e_1 - \sigma e_2|^{2+sp}} d\sigma d\tau dy \\ &= \int_{\mathbb{R}^2} \int_0^1 \int_0^1 \frac{|u(z) - u(z - \tau e_1 + \sigma e_2)|^p}{|\tau e_1 - \sigma e_2|^{2+sp}} d\sigma d\tau dz \\ &\leqslant \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|u(z + h) - u(z)|^p}{|h|^{2+sp}} dh dz = \|u\|^p_{W^{s,p}}. \end{aligned}$$

The proof of Lemma E.2 is complete.

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