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Asymptotics for the Ginzburg–Landau Equation in Arbitrary Dimensions

F. Bethuel

Analyse Numerique, Université P. et M. Curie, BC 187, 4, Place Jussieu, 75252 Paris Cedex 05, France

H. Brezis

Analyse Numerique, Université P. et M. Curie, BC 187, 4, Place Jussieu, 75252 Paris Cedex 05, France; and Department of Mathematics, Rutgers University, Hill Center, Busch Campus, 110 Frelinghuysen Road, Piscataway, New Jersey 08854

and

G. Orlandi

Dipartimento di Informatica, Università di Verona, Strada le Grazie, 37134 Verona, Italy

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Let Ω be a bounded, simply connected, regular domain of $\mathbb{R}^N$, $N \geqslant 2$. For $0 < \varepsilon < 1$, let $u_\varepsilon \colon \Omega \to \mathbb{C}$ be a smooth solution of the Ginzburg–Landau equation in Ω with Dirichlet boundary condition g_ε, i.e.,

$$\begin{cases} -\Delta u_\varepsilon = \dfrac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) & \text{in } \Omega, \\ u_\varepsilon = g_\varepsilon & \text{on } \partial\Omega. \end{cases} \qquad (GL)_\varepsilon$$

We are interested in the asymptotic behavior of u_ε as ε goes to zero under the assumption that $E_\varepsilon(u_\varepsilon) \leqslant M_0 \, |\log \varepsilon|$ and some conditions on g_ε which allow singularities of dimension $N-3$ on $\partial\Omega$.

I. INTRODUCTION

Let Ω be a bounded, simply connected, regular domain of $\mathbb{R}^N$, $N \geqslant 2$. For $0 < \varepsilon < 1$, let $u_\varepsilon \colon \Omega \to \mathbb{C}$ be a smooth solution of the Ginzburg–Landau equation in Ω, with Dirichlet boundary condition g_ε, i.e.,

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$$\begin{cases} -\Delta u_\varepsilon = \dfrac{1}{\varepsilon^2} u_\varepsilon (1-|u_\varepsilon|^2) & \text{in } \Omega, \\ u_\varepsilon = g_\varepsilon & \text{on } \partial\Omega. \end{cases} \tag{GL}_\varepsilon$$

We are interested in the asymptotic behavior of u_ε as ε goes to zero.

The case $N=2$ and $g_\varepsilon = g\colon \partial\Omega \to S^1$ smooth, independent of ε, has been extensively studied since the work of [Bethuel-Brezis-Hélein 1, 2]. The main result is the convergence of minimizers u_ε of the corresponding Ginzburg–Landau energy

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_\Omega (1-|u|^2)^2$$

to a limit u_* having a finite number $|d|$ of point singularities, where $d = \deg(g, \partial\Omega)$. Moreover, $E_\varepsilon(u_\varepsilon) = \pi\, |d|\, |\log \varepsilon| + O(1)$ as $\varepsilon \to 0$. This result was originally established for star-shaped domains in [Bethuel-Brezis-Hélein 2] and subsequently extended to general domains in [Struwe] (see also [Del Pino-Felmer] for a simple reduction of the general case to the star-shaped case).

If u_ε is a (nonminimizing) solution of $(GL)_\varepsilon$ and Ω is star-shaped, the same conclusion still holds except that the number of singular points of u_* is not necessarily $|d|$. There are two fundamental estimates in the proof:

$$E_\varepsilon(u_\varepsilon) \leqslant C\, |\log \varepsilon|, \tag{1}$$

$$\|u_\varepsilon\|_{W^{1,p}} \leqslant C_p \qquad \forall p < 2. \tag{2}$$

When Ω is *not* star-shaped, estimate (1) need *not* hold for general solutions of $(GL)_\varepsilon$ (see [Bethuel-Brezis-Hélein 2], Remark X.1). However, if one *assumes* (1) for a solution u_ε of $(GL)_\varepsilon$ then one may still prove that $u_\varepsilon \to u_*$ having a finite number of singularities (see [Bethuel] and also [Rivière 2]).

From now on we consider the case $N \geqslant 3$.

If $g_\varepsilon = g\colon \partial\Omega \to S^1$ is smooth and $\partial\Omega$ is simply connected, then one may write $g = \exp i\varphi_0$, where φ_0 is smooth. Minimizers u_ε of E_ε converge (at least in H^1) to $u_* = \exp i\varphi_*$, where φ_* is the harmonic extension of φ_0. The same conclusion holds when Ω is *star-shaped* and u_ε is *any* solution of $(GL)_\varepsilon$. This follows from the (easy) estimate

$$E_\varepsilon(u_\varepsilon) \leqslant C, \tag{3}$$

and the results in Section IV below. The validity of estimate (3) for $N \geqslant 3$ and g smooth is a basic difference between the case $N=2$ and $N \geqslant 3$; this is a consequence of Pohozaev identity, which takes a different form when $N=2$ and when $N \geqslant 3$.

The case of a simply connected domain is open:

Conjecture. Assume $\partial\Omega$ is simply connected, u_ε is a (nonminimizing) solution of $(GL)_\varepsilon$ with $g_\varepsilon = g\colon \partial\Omega \to S^1$ smooth, then (3) holds and $u_\varepsilon \to u_* = \exp i\varphi_*$.

Having in mind physical and geometrical problems involving singularities, we wish to handle cases where u_* admits singularities. Therefore, we assume that the Dirichlet data g is singular. (An alternative way of producing singularities is to consider a Neumann boundary condition, see [Almeida], [Almeida-Bethuel 1]; singularities can also be generated by an exterior field, see [Serfaty], [Sandier-Serfaty]).

In this paper, we will concentrate our attention on singularities of codimension two, and then it is natural to assume that the energy of u_ε blows up like $|\log \varepsilon|$:

H1. There exists a constant $M_0 > 0$ such that, for $0 < \varepsilon < 1$, the Ginzburg–Landau energy of u_ε is smaller than $M_0\, |\log \varepsilon|$, that is

$$E_\varepsilon(u_\varepsilon) \equiv \int_\Omega e_\varepsilon(u_\varepsilon) \equiv \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 + \frac{1}{4}\int_\Omega \frac{(1-|u_\varepsilon|^2)^2}{\varepsilon^2} \leqslant M_0\, |\log \varepsilon|. \tag{H1}$$

Such an assumption is automatically satisfied if u_ε is a minimizer of E_ε and $g_\varepsilon = g$ has simple singularities, e.g.,

$$N = 3, \qquad g(x) \simeq \frac{x - a_i}{|x - a_i|}$$

near its singularities $a_i \in \partial\Omega$. More generally, (H1) holds for minimizers if $g_\varepsilon = g \in H^{1/2}(\partial\Omega; S^1)$ ($N \geqslant 3$), see [Bourgain-Brezis-Mironescu 1, 2].

We make use of a second assumption which is more artificial but quite convenient. It has been introduced in [Lin-Rivière 1] and concerns the behavior of the boundary data $g_\varepsilon\colon \partial\Omega \to \mathbb{C}$.

H2. There exists a finite collection Σ of smooth $(N-3)$-dimensional submanifolds of $\partial\Omega$, such that

$$|g_\varepsilon(x)| = 1, \quad \text{if} \quad x \in \partial\Omega \quad \text{and} \quad d(x) = \operatorname{dist}(x, \Sigma) \geqslant \varepsilon, \quad |g_\varepsilon| \leqslant 1, \quad \forall x \in \partial\Omega, \tag{H2.1}$$

and

$$\text{for} \quad k = 1, 2 \qquad |\nabla_\top^k g_\varepsilon| \leqslant \frac{C_0}{\max(d, \varepsilon)^k} \qquad \text{on } \partial\Omega, \tag{H2.2}$$

where C_0 is some constant independent of ε. Here ∇_{T} denotes tangential differentiation.

When $N=3$ (so that Σ is a collection of points), and

$$g(x) \simeq \frac{x-a_i}{|x-a_i|}$$

near its singularities $a_i \in \partial\Omega$, then g_ε is a natural smooth approximation of g.

As a consequence of (H2.1), (H2.2) (with $k=1$) one deduces easily

$$\int_{\partial\Omega} |\nabla_{\mathrm{T}}\, g_\varepsilon|^2 \leqslant C\, |\log \varepsilon|, \tag{4}$$

$$\int_{\partial\Omega} |\nabla_{\mathrm{T}}\, g_\varepsilon|^p \leqslant C_p, \qquad \forall p<2, \tag{5}$$

and

$$\frac{1}{\varepsilon^2}\int_{\partial\Omega} (1-|g_\varepsilon|^2)^2 \leqslant C. \tag{6}$$

Here, and in what follows, the constants C and C_p depend on Ω, Σ, C_0 and M_0, but they are independent of ε; we emphasize the dependence of C_p on p, because it blows-up as $p \uparrow 2$.

Our main results are summarized in the following theorem.

Theorem 1. *Assume* $1 \leqslant p < \frac{N}{N-1}$ *and let* u_ε *be a solution of* (GL_ε) *satisfying* (H1)–(H2). *Then, for any* $0<\varepsilon<1$, *we have*

$$\int_\Omega |\nabla u_\varepsilon|^p \leqslant C_p. \tag{7}$$

For a subsequence $\varepsilon_n \to 0$, *there exist a map* $u_* \in W^{1,p}(\Omega)$ *and map* $g_* \in W^{1,p}(\partial\Omega)$ *such that*

(i) $|u_*|=1$ *on* Ω, $|g_*|=1$, $u_*=g_*$ *on* $\partial\Omega$;

(ii) $u_{\varepsilon_n} \to u_*$ *in* $W^{1,p}(\Omega)$, $g_{\varepsilon_n} \to g_*$ *in* $W^{1,p}(\partial\Omega)$;

(iii) $\operatorname{div}(u_* \times \nabla u_*)=0$ *in* Ω;

(iv) $e_{\varepsilon_n}(u_{\varepsilon_n})/|\log \varepsilon_n| \rightharpoonup \mu_*$ *as measures, where* μ_* *is a bounded measure on* $\bar{\Omega}$.

Set $\mathscr{S} = supp(\mu_*)$;

(v) $\mathscr{S}$ *is a closed subset of* $\bar{\Omega}$ *with* $\mathscr{H}^{N-2}(\mathscr{S}) < +\infty$;

(vi) *$u_* \in C^\infty(\Omega \setminus \mathscr{S})$, and for any ball $B(x_0, r)$ included in $\Omega \setminus \mathscr{S}$ there exists a function $\varphi_* \in C^\infty(B(x_0, r))$, such that $\Delta\varphi_* = 0$, $u_* = \exp(i\varphi_*)$;*

(vii) *$u_{\varepsilon_n} \to u_*$ in $C^k(K)$, for any compact subset K of $\Omega \setminus \mathscr{S}$;*

(viii) *$\mathscr{S}$ is $\mathscr{H}^{N-2}$-rectifiable;*

(ix) *μ_* is a stationary varifold.*

The case $N = 2$ (with slightly different assumptions on the boundary data: g is fixed and smooth) is treated in the book [Bethuel-Brezis-Hélein 2] (Chapter X, p. 101–136). See also [Bethuel], [Brezis].

The case $N \geqslant 3$ and u_ε minimizing has been extensively treated in [Rivière 1] (for $N = 3$), [Lin-Rivière 1], and also in [Sandier] (for $N = 3$), [Alberti-Baldo-Orlandi] and [Jerrard-Soner] via Γ-convergence arguments. In this case, it is proved moreover that $\mathscr{S}$ is area-minimizing. The case $N = 3$, and u_ε not minimizing has been studied in [Lin-Rivière 2].

Our proofs borrow many ingredients from the works quoted above (in particular [Bethuel-Brezis-Hélein 2], [Rivière 1], [Lin-Rivière 1], [Lin-Rivière 2]). We also use arguments of Geometric Measure Theory developed in [Ambrosio-Soner]. The first important tool in our proof is a variant of a monotonicity formula earlier used in [Rivière 1]. Such formulas play a central role in the theory of minimal surfaces, harmonic maps, and regularity theory for elliptic problems (see for instance [Giaquinta]).

The second important ingredient is the η-ellipticity theorem, which bounds $|u_\varepsilon|$ away from zero as soon as the local energy is bounded by $\eta\,|\log \varepsilon|$ with η small:

THEOREM 2. *Let $u = u_\varepsilon : B_1 \to \mathbb{C}$ be a solution of*

$$-\Delta u = \frac{1}{\varepsilon^2} u(1 - |u|^2) \qquad \text{in } B_1$$

for some $\varepsilon \in (0, 1/2)$. Assume

$$E_\varepsilon(u) \leqslant \eta\,|\log \varepsilon|.$$

Then

$$|u(0)| \geqslant 1 - K\eta^\alpha, \tag{8}$$

where $K > 0$ and $\alpha > 0$ depend only on N.

The name η-ellipticity is motivated as follows: once (8) holds we may write $u_\varepsilon = \rho_\varepsilon \exp i\varphi_\varepsilon$, with $\rho_\varepsilon = |u_\varepsilon|$ and the equation for the phase φ_ε becomes

$$\operatorname{div}(\rho_\varepsilon^2\, \nabla\varphi_\varepsilon) = 0,$$

which is uniformly elliptic.

In two dimensions, this type of result originated simultaneously in [Bethuel-Rivière] and [Struwe]. Though the proof of η-ellipticity in 2-d is simple and uses techniques introduced in [Bethuel-Brezis-Hélein 2], it turns out to be an extremely useful tool. It serves to analyze the local vorticity, provided the energy of the map is bounded by $K_0 |\log \varepsilon|$ with K_0 large, even for maps not satisfying $(GL)_\varepsilon$. It appears in a large number of papers dealing with 2-d variational methods ([Almeida-Bethuel 2], [Zhou-Zhou], [Bethuel-Saut]), or involving the more subtle functional of superconductivity ([Almeida], [Serfaty]). For surveys on these questions see [Bethuel] or [Rivière 2].

In higher dimension, the first η-ellipticity result was given in [Rivière 1] under the name "η-compactness" (for $N=3$ and minimizing maps), then in [Lin-Rivière 1] (for arbitrary dimension, minimizing maps), in [Lin-Rivière 2] for $N=3$, u_ε not necessarily minimizing, and finally in [Bethuel-Brezis-Orlandi] in the general case. In Section III we present a simplified proof of the [Bethuel-Brezis-Orlandi] result as well as a boundary version.

A key point in the proof of Theorem 1 is estimate (7). When $N=2$, $g_\varepsilon \equiv g$ and Ω is star-shaped this was proved in [Bethuel-Brezis-Hélein 2] as follows. First, Pohozaev identity provides immediately a uniform bound on the integral of the potential, namely

$$\frac{1}{\varepsilon^2}\int_\Omega (1-|u_\varepsilon|^2)^2 \leqslant C, \tag{9}$$

where C is independent of ε. Then, the derivation of (7) from (9) is explained in [Bethuel-Brezis-Hélein 2], Chapter X; it relies on a Hodge decomposition of $u_\varepsilon \times \nabla u_\varepsilon$ and the property $\operatorname{div}(u_\varepsilon \times \nabla u_\varepsilon) = 0$.

When $N=2$ and Ω is *not* star-shaped, the proof of (7) is more delicate. Instead of (9) one establishes the weaker form (10) below via a local Pohozaev identity (on a scale of order ε^α, $\alpha<1$) combined with an elementary maximal covering argument (the balls $B(a_i, R)$ are disjoint while the balls $B(a_i, 8R)$ cover $\bar{\Omega}$) as in [Struwe] and [Bethuel-Rivière]. From (10) one obtains (7) using exactly the same Hodge decomposition as above and the rest of the argument is unchanged.

We now return to dimension $N \geqslant 3$. The first and main step in the proof of (7) is

Proposition 1. *Fix $\beta \in (\frac{1}{2}, 1)$ and set*

$$A_{\varepsilon,\beta} = \{x \in \Omega;\ |u_\varepsilon(x)| \leqslant 1-\beta\},$$

where u_ε *is a solution of* $(GL)_\varepsilon$ *satisfying* (H1), (H2). *Then*

$$\frac{1}{\varepsilon^2}\int_{\Omega \cap A_{\varepsilon,\beta}} (1-|u_\varepsilon|^2)^2 \leqslant C_\beta, \tag{10}$$

where C_β *depends on* Ω, M_0, C_0 *and is independent of* ε.

The proof of Proposition 1 is for $N \geqslant 3$ quite involved and uses a number of ingredients:

— Local Pohozaev identities (as in [Bethuel-Rivière], [Bethuel], [Rivière 1], [Lin-Rivière 2]),

— Interior monotonicity formulas ([Chen-Struwe], [Chen-Lin], [Rivière 1], [Lin-Rivière 1]),

— Boundary monotonicity formulas ([Lin-Rivière 1]),

— Besicovitch covering theorem.

Finally, once Proposition 1 is established, the $W^{1,p}$ estimate as well as properties (i) to (vii) are proved by adapting the methods of [Bethuel-Brezis-Hélein 2]. In contrast with the 2-d case where the Hodge decomposition is fairly elementary, the case $N \geqslant 3$ requires a heavier machinery described in the Appendix, where we follow the presentation of [Iwaniec-Scott-Stroffolini] and [Giaquinta-Modica-Souček].

Combining the η-ellipticity with the $W^{1,p}$ bounds we are able to show concentration of energy on a singular set $\mathscr{S}$ of Hausdorff dimension $N-2$. Then, η-regularity asserts that $\{u_{\varepsilon_n}\}$ converges in strong norms locally away from $\mathscr{S}$.

Once the η-regularity has been established we are in a position to apply the beautiful theory of [Ambrosio-Soner], which yields immediately the geometric properties of $\mathscr{S}$ (statement (viii) and (ix) of Theorem 1).

We call the attention of the reader that some results in the paper are purely local, while others are truly global. For example, the results in Section II, III, IV are purely local. By contrast, the $W^{1,p}$ estimate is *not* local. Indeed, if for instance u_ε satisfies

$$-\Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon (1-|u_\varepsilon|^2) \qquad \text{in } B_1$$

and $E_\varepsilon(u_\varepsilon) \leqslant \eta\,|\log \varepsilon|$ with η small—or even $o(|\log \varepsilon|)$—it is *wrong* to infer that

$$\|u_\varepsilon\|_{W^{1,1}(B_{1/2})} \leqslant C$$

or any compactness property, even in L^1_{loc}, see [Brezis-Mironescu] and Remark III.4 below. The $W^{1,p}$ estimates in Section VI really uses the full information on the boundary condition g_ε.

The plan is the following:

II. Monotonicity formulas
III. The η-ellipticity
IV. Interior H^1 estimates imply C^k bounds
V. Proof of Proposition 1
VI. Global $W^{1,p}$ estimates, $1 \leqslant p < \frac{N}{N-1}$
VII. η-regularity
VIII. Convergence outside the singular set $\mathscr{S}$
IX. Properties of $\mathscr{S}$ and μ_*

II. MONOTONICITY FORMULAS

II.1. *Interior Monotonicity*

We first recall the standard Pohozaev identity:

LEMMA II.1. *Let $x_0 \in \Omega$, and $r > 0$ be such that $B_r(x_0) \subset \Omega$. Assume u is a solution of $(GL)_\varepsilon$, then*

$$\int_{B_r(x_0)} \frac{N-2}{2} |\nabla u|^2 + \frac{N}{4\varepsilon^2} (1-|u|^2)^2 = r \int_{\partial B_r(x_0)} \frac{|\nabla_{\mathrm{T}} u|^2}{2} - \frac{1}{2} \left|\frac{\partial u}{\partial n}\right|^2 + \frac{1}{4\varepsilon^2} (1-|u|^2)^2.$$

Let $R > 0$ be such that $B_r(x_0) \subset \Omega$. For $0 < r < R$, set

$$E_\varepsilon(x_0, r) = \frac{1}{2} \int_{B_r(x_0)} |\nabla u|^2 + \frac{1}{4\varepsilon^2} \int_{B_r(x_0)} (1-|u|^2)^2,$$

and

$$\tilde{E}_\varepsilon(x_0, r) = r^{2-N} E_\varepsilon(x_0, r).$$

Monotonicity formulas are concerned with quantities of the type $\tilde{E}_\varepsilon(x_0, r)$ (note that $r^{2-N} \int_{B_r} |\nabla u|^2$ is dimensionless). They play an important role in elliptic regularity theory (see [Giaquinta], [Schoen-Uhlenbeck 1, 2]). In the context of the Ginzburg–Landau equation for $\mathbb{R}^k$-valued maps they were introduced in [Chen-Struwe], [Chen-Lin], and used extensively in [Rivière 1], and then in [Lin-Rivière 1, 2].

LEMMA II.2 (Interior Monotonicity). *Assume u is a solution of $(GL)_\varepsilon$ in $B_R(x_0)$, then*

$$\frac{d}{dr} (\tilde{E}_\varepsilon(x_0, r)) = \frac{1}{r^{N-2}} \int_{\partial B_r(x_0)} \left|\frac{\partial u}{\partial n}\right|^2 + \frac{1}{r^{N-1}} \int_{B_r(x_0)} \frac{(1-|u|^2)^2}{2\varepsilon^2}, \qquad \text{for} \quad r < R.$$

Proof. First one has,

$$\frac{d}{dr}(E_\varepsilon(x_0, r)) = \int_{\partial B_r(x_0)} \frac{|\nabla u|^2}{2} + \frac{1}{4\varepsilon^2}\int_{\partial B_r(x_0)} (1-|u|^2)^2$$
$$= \int_{\partial B_r(x_0)} \frac{|\nabla_\top u|^2}{2} + \frac{1}{2}\left|\frac{\partial u}{\partial n}\right|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2.$$

Hence

$$\frac{d}{dr}(\tilde{E}_\varepsilon(x_0, r)) = -\frac{N-2}{r^{N-1}} E_\varepsilon(x_0, r) + \frac{1}{r^{N-2}}\int_{\partial B_r(x_0)} \frac{|\nabla_\top u|^2}{2} + \frac{1}{2}\left|\frac{\partial u}{\partial n}\right|^2$$
$$+\frac{1}{4\varepsilon^2}(1-|u|^2)^2$$
$$= -\left(\frac{N-2}{r^{N-1}}\int_{B_r}\frac{|\nabla u|^2}{2} + \frac{N-2}{4\varepsilon^2 r^{N-1}}\int_{B_r}(1-|u|^2)^2\right)$$
$$+\frac{1}{r^{N-2}}\int_{\partial B_r}\frac{|\nabla_\top u|^2}{2} + \frac{1}{2}\left|\frac{\partial u}{\partial n}\right|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2$$
$$= -\left(\frac{N-2}{r^{N-1}}\int_{B_r}\frac{|\nabla u|^2}{2} + \frac{N}{4\varepsilon^2 r^{N-1}}\int_{B_r}(1-|u|^2)^2\right)$$
$$+\frac{1}{2\varepsilon^2 r^{N-1}}\int_{B_r}(1-|u|^2)^2$$
$$+\frac{1}{r^{N-2}}\int_{\partial B_r}\frac{|\nabla_\top u|^2}{2} + \frac{1}{2}\left|\frac{\partial u}{\partial n}\right|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2.$$

Using Lemma II.1, we obtain

$$\frac{d}{dr}(\tilde{E}_\varepsilon(x_0, r)) = -\left[\frac{1}{r^{N-2}}\int_{\partial B_r}\frac{|\nabla_\top u|^2}{2} - \frac{1}{2}\left|\frac{\partial u}{\partial n}\right|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2\right]$$
$$+\frac{1}{2\varepsilon^2 r^{N-1}}\int_{B_r}(1-|u|^2)^2$$
$$+\frac{1}{r^{N-2}}\left[\int_{\partial B_r}\frac{|\nabla_\top u|^2}{2} + \frac{1}{2}\left|\frac{\partial u}{\partial n}\right|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2\right]$$
$$= \frac{1}{r^{N-2}}\int_{\partial B_r}\left|\frac{\partial u}{\partial n}\right|^2 + \frac{1}{r^{N-1}}\int_{B_r}\frac{(1-|u|^2)^2}{2\varepsilon^2},$$

which yields the result.

A straightforward consequence is

COROLLARY II.1. *Assume u is a solution of $(GL)_\varepsilon$ on $B_R(x_0) \subset \Omega$, then*

(i) *The function $r \mapsto \tilde{E}_\varepsilon(x_0, r)$ is nondecreasing in $(0, R)$.*

(ii) *In particular,*

$$\forall 0 < r < R, \qquad \tilde{E}_\varepsilon(x_0, r) \leqslant \tilde{E}_\varepsilon(x_0, R) \leqslant R^{2-N} E_\varepsilon(u_\varepsilon).$$

Moreover

$$\int_0^R \left[\frac{1}{r^{N-2}} \int_{\partial B_r(x_0)} \left| \frac{\partial u}{\partial n} \right|^2 + \frac{1}{r^{N-1}} \int_{B_r(x_0)} \frac{(1-|u|^2)^2}{2\varepsilon^2} \right] dr = \tilde{E}_\varepsilon(x_0, R) \tag{II.1}$$

and

$$\int_{B_R(x_0)} \frac{1}{r^{N-2}} \left(\left| \frac{\partial u}{\partial n} \right|^2 + \frac{1}{2(N-2)\varepsilon^2} (1-|u|^2)^2 \right) dx \leqslant \frac{N}{N-2} \tilde{E}_\varepsilon(x_0, R). \tag{II.2}$$

Formula (II.2) is obtained integrating by parts some terms in (II.1).

II.2. *Boundary Monotonicity Formulas*

Throughout the paper, we will use the following notation:

$$\check{B}_r(x_0) = B_r(x_0) \cap \Omega$$

$$E_\varepsilon(x_0, r) = \int_{\check{B}_r(x_0)} e_\varepsilon(u) = \int_{B_r(x_0) \cap \Omega} e_\varepsilon(u),$$

and

$$\tilde{E}_\varepsilon(x_0, r) = r^{2-N} E_\varepsilon(x_0, r).$$

Set $d_0 = \operatorname{dist}(x_0, \Sigma)$, for $x_0 \in \bar{\Omega}$. The following result can be easily deduced from [Lin-Rivière 1] (Lemma II.5, with $\alpha = \frac{1}{2}$).

LEMMA II.3. *We have, for every $x_0 \in \bar{\Omega}$, and any r such that*

$$0 < r < \inf\{R_1, d_0^2\},$$

$$\frac{d}{dr}\{e^{\Lambda r^{1/2}} \tilde{E}_\varepsilon(x_0, r)\} \geqslant r^{2-N} \int_{\partial B_r(x_0) \cap \Omega} \left| \frac{\partial u}{\partial n} \right|^2 + r^{1-N} \int_{\check{B}_r(x_0)} \frac{(1-|u|^2)^2}{2\varepsilon^2} - C,$$

where R_1, Λ and C are constants depending only on Ω, on Σ, on d_0 and on M_0, C_0 (from (H1), (H2)), *but not on r and ε.*

After integration, a straightforward consequence of Lemma II.3 is

COROLLARY II.2. *Let $x_0 \in \bar{\Omega}$, and $R > 0$ such that $R \leqslant \inf\{R_1, d_0^2\}$, where $d_0 = \operatorname{dist}(x_0, \Sigma)$. Then we have*

$$\tilde{E}_\varepsilon(x_0, r) \leqslant C(\tilde{E}_\varepsilon(x_0, R) + R), \qquad \textit{for every} \quad 0 < r < R, \tag{II.3}$$

where C is some constant independent of ε and x_0.

Moreover,

$$\int_0^R \frac{1}{r^{N-2}} \int_{\partial B_r(x_0) \cap \partial\Omega} \left|\frac{\partial u}{\partial n}\right|^2 + \frac{1}{r^{N-2}} \int_{\check{B}_r(x_0)} \frac{(1-|u|^2)^2}{2\varepsilon^2} \leqslant C(\tilde{E}_\varepsilon(x_0, R) + R) \tag{II.4}$$

and

$$\int_{\check{B}_r(x_0)} \frac{1}{r^{N-2}} \left(\left|\frac{\partial u}{\partial n}\right|^2 + \frac{(1-|u|^2)^2}{\varepsilon^2} \right) \leqslant C(\tilde{E}_\varepsilon(x_0, R) + R). \tag{II.5}$$

A second result from [Lin-Rivière 1] (Lemma II.6 there) will play a fundamental role.

Lemma II.4. *For any $x_0 \in \Omega$, we have*

$$\tilde{E}_\varepsilon(x_0, r) = r^{2-N} E_\varepsilon(x_0, r) = \frac{1}{r^{N-2}} \int_{B_r(x_0) \cap \Omega} e_\varepsilon(u) \leqslant M_1 \, |\log \varepsilon|,$$

where M_1 is a constant independent of ε, r and x_0.

The proof relies on the two monotonicity formulas above and a careful study of all integrals for x_0 near Σ.

II.3. *Consequences of the Monotonicity*

Consider, for $\mu > 0$, the sets

$$\Sigma_\mu = \{x \in \Omega;\ \mathrm{dist}(x, \Sigma) \leqslant \mu\},$$
$$K_\mu = \Omega \setminus \mathring{\Sigma}_\mu = \{x \in \Omega;\ \mathrm{dist}(x, \Sigma) \geqslant \mu\}.$$

Here we will take $\mu = \varepsilon^{1/8}$ and we apply Lemma II.3 with

$$\varepsilon^{1/2} \leqslant r \leqslant \varepsilon^{1/4}.$$

This yields

$$\frac{d}{dr}\left(e^{Ar^{1/2}} \tilde{E}_\varepsilon(x_0, r)\right) \geqslant r^{2-N} \int_{\partial B_r(x_0) \cap \Omega} \left|\frac{\partial u}{\partial n}\right|^2 + r^{1-N} \int_{\check{B}_r(x_0)} \frac{(1-|u|^2)^2}{\varepsilon^2} - C.$$

Integrating from $\varepsilon^{1/2}$ to $\varepsilon^{1/4}$, we obtain

$$\int_{\varepsilon^{1/2}}^{\varepsilon^{1/4}} r^{2-N} \int_{\partial B_r(x_0) \cap \Omega} \left|\frac{\partial u}{\partial n}\right|^2 + \int_{\varepsilon^{1/2}}^{\varepsilon^{1/4}} r^{1-N} \int_{\check{B}_r(x_0)} \frac{(1-|u|^2)^2}{\varepsilon^2} \leqslant C(\varepsilon^{1/4} + \tilde{E}_\varepsilon(x_0, \varepsilon^{1/4}))$$
$$\leqslant M_2 \, |\log \varepsilon|, \tag{II.6}$$

where $M_2 > 0$ is some constant, independent of ε, r, x_0. For the last inequality we have used Lemma II.4.

Set for $r > 0$ and $x_0 \in \bar{\Omega}$

$$I_r(x_0) = \frac{1}{r^{N-3}} \int_{\partial B_r(x_0) \cap \Omega} \left|\frac{\partial u}{\partial n}\right|^2 + \frac{1}{r^{N-2}} \int_{\check{B}_r(x_0)} \frac{(1-|u|^2)^2}{2\varepsilon^2}.$$

We deduce, from (II.6):

Proposition II.2. *Let $\mu = \varepsilon^{1/8}$, and $x_0 \in K_\mu$. There exists some radius $r(x_0) \in (\varepsilon^{1/2}, \varepsilon^{1/4})$ such that*

$$I_{r(x_0)}(x_0) \leqslant 4C\left(\frac{\varepsilon^{1/4} + \tilde{E}_\varepsilon(x_0, \varepsilon^{1/4})}{|\log \varepsilon|}\right) \leqslant 4M_2. \tag{II.7}$$

Proof. We argue by contradiction and assume that (for some ε) and every $r \in (\varepsilon^{1/2}, \varepsilon^{1/4})$

$$\frac{1}{r^{N-3}} \int_{\partial B_r(x_0) \cap \Omega} \left|\frac{\partial u}{\partial n}\right|^2 + \frac{1}{r^{N-2}} \int_{\check{B}_r(x_0)} \frac{(1-|u|^2)^2}{2\varepsilon^2} \geqslant 4C\left(\frac{\varepsilon^{1/4} + \tilde{E}_\varepsilon(x_0, \varepsilon^{1/4})}{|\log \varepsilon|}\right).$$

Dividing by r and integrating on the interval $(\varepsilon^{1/2}, \varepsilon^{1/4})$ we obtain

$$\int_{\varepsilon^{1/2}}^{\varepsilon^{1/4}} r^{2-N} \int_{\partial B_r(x_0) \cap \Omega} \left|\frac{\partial u}{\partial n}\right|^2 + \int_{\varepsilon^{1/2}}^{\varepsilon^{1/4}} \frac{1}{r^{N-1}} \int_{\check{B}_r(x_0)} \frac{(1-|u|^2)^2}{2\varepsilon^2} \geqslant 4C(\varepsilon^{1/4} + \tilde{E}_\varepsilon(x_0, \varepsilon^{1/4})),$$

a contradiction with (II.6).

Remark II.1. We call the attention of the reader to the fundamental estimate for the potential

$$\int_{B_{r(x_0)}(x_0) \cap \Omega} \frac{(1-|u|^2)^2}{2\varepsilon^2} \leqslant 4M_2 r(x_0)^{N-2}, \tag{II.8}$$

which is a consequence of (II.7). It will play a basic role in the proof of Proposition 1.

III. THE η-ELLIPTICITY

III.1. *η-Ellipticity in the Interior.*

Throughout this section, we assume that $\Omega = B_R(0) \subset \mathbb{R}^N$, $N \geqslant 2$, and that $u = u_\varepsilon\colon B_R(0) \to \mathbb{C}$ is a solution of the equation

$$-\Delta u = \frac{1}{\varepsilon^2} u(1-|u|^2) \qquad \text{in } B_R(0), \tag{III.1}$$

for some $\varepsilon \in (0, 1/2)$.

A typical result in this section is that, if

$$\tilde{E}_\varepsilon(0, R) = \frac{1}{R^{N-2}} E_\varepsilon(0, R) \leqslant \eta \left|\log \frac{\varepsilon}{R}\right|, \tag{III.2}$$

with η sufficiently small (less than a constant depending only on N), then

$$|u(0)| \geqslant \tfrac{1}{2}. \tag{III.3}$$

As already mentioned in the Introduction, this type of result was first considered in dimension two in [Bethuel-Brezis-Hélein 2], Lemma (IV.2), and then in [Bethuel-Rivière] and [Struwe]. In higher dimensions it was proved in [Rivière 1], [Lin-Rivière 1, 2], under various restrictive assumptions. The general case was settled in [Bethuel-Brezis-Orlandi], with more elementary arguments than in [Rivière 1], [Lin-Rivière 1], [Lin-Rivière 2](which uses Lorentz spaces). The original name given to this phenomenon in [Rivière 1] was "η-compactness," which is misleading because it suggests that the family of function (u_ε) is compact for some topology. In fact a construction from [Brezis-Mironescu] (see also Remark III.4 below) provides already for $N=2$, an example of a family (u_ε) satisfying (III.1) and

$$E_\varepsilon(0, R) = o(|\log \varepsilon|),$$

such that no subsequence of $\{u_\varepsilon\}$ converges on a set of positive measure. Hence, $\{u_\varepsilon\}$ is not relatively compact, even in L^1! We call it instead "η-ellipticity" for the following reason. If we apply the above mentioned result, we obtain in fact that (III.2) with a smaller η implies (see Proposition VII.1)

$$|u_\varepsilon(x)| \geqslant \frac{1}{2} \qquad \text{on } B\left(0, \frac{R}{2}\right). \tag{III.4}$$

We then may write $u = u_\varepsilon$ in terms of its modulus ρ and phase φ

$$u = \rho \exp(i\varphi) \qquad \text{in } B\left(0, \frac{R}{2}\right).$$

Equation (III.1) for ρ and φ becomes, in $B(0, \frac{R}{2})$,

$$\begin{cases} \operatorname{div}(\rho^2 \nabla\varphi) = 0 \\ -\Delta\rho + \rho\,|\nabla\varphi|^2 = \dfrac{1}{\varepsilon^2}\,\rho(1-\rho^2). \end{cases} \tag{III.5}$$

Hence condition (III.4) says that the first equation in (III.1) is uniformly elliptic.

The main result of this section is

THEOREM 2. *There exist constants $K>0$, and $\alpha>0$, depending only on N such that* (III.2) *(with arbitrary η) implies*

$$|u(0)| \geqslant 1-K\eta^{\alpha}. \tag{III.6}$$

Remark III.1. Another important point is the following: formulas (III.1) and (III.2) make sense for functions $u\colon B_R(0) \to \mathbb{R}^k$, for any k. The reader may wonder whether conclusion (III.3) also makes sense for any $k>2$. The answer is:

Yes in dimension $N=2$ (see part D below)

No in dimension $N>2$ (see part E below).

Remark III.2. In fact this type of problem can be imbedded in a more general setting: let $W\colon \mathbb{R}^k \to \mathbb{R}$, $W \geqslant 0$, be a smooth function, and let $M=\{W=0\}$. Assume M is a smooth manifold without boundary. Solutions of (III.1) are replaced by solutions of

$$-\Delta u = \frac{1}{\varepsilon^2} W'(u) \qquad \text{in} \quad \Omega \subset \mathbb{R}^N, \tag{III.1$'$}$$

and the Ginzburg–Landau energy is replaced by $E_\varepsilon(u)=\frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{\varepsilon^2}\int_\Omega W(u)$. The natural question is to find conditions under which (III.1′) and (III.2) with η small implies, e.g.,

$$\operatorname{dist}(u(0), M) \leqslant \delta, \qquad \text{for small} \quad \delta>0. \tag{III.3$'$}$$

We have not investigated that question.

Throughout this section K will denote "absolute" constants (depending possibly on N) that are independent of η, ε, R, etc...

We now turn to the proof of Theorem 2. After scaling, we may assume throughout the rest of Section III.1,

$$R=1.$$

We will also assume without loss of generality $\eta<\frac{1}{4}$. If there is no ambiguity, we will write B_r instead of $B_r(0)$. In fact we are going to prove that the conclusion of Theorem 2 holds for $0<\varepsilon<\varepsilon_N$, where $\varepsilon_N>0$, and depends only on N; if $\varepsilon_N \leqslant \varepsilon \leqslant 1$, the conclusion (III.6) still holds, but it is very easy to establish by standard arguments.

The proof of Theorem 2 is divided into four parts. The case $N=2$ will be studied in Part D.

In Part B and C we need $N \geqslant 3$ while Part A holds for $N \geqslant 2$.

Part A: Choosing a "Good" Radius r_0

LEMMA III.1. *Let $0<\delta<\frac{1}{16}$ and assume $0<\varepsilon<\delta^2$. There exists some constant $K>0$ such that if u is a solution of* (III.1), *then there exists some $r_0 \in (\varepsilon^{1/2}, \delta)$, depending on u, ε and δ, such that*

$$\frac{1}{r_0^{N-2}} \int_{B_{r_0}} \frac{(1-|u|^2)^2}{2\varepsilon^2} \leqslant K\eta\, |\log \delta|, \tag{III.7}$$

$$\int_{\delta r_0}^{r_0} \left[\frac{1}{r^{N-1}} \int_{B_r} \frac{(1-|u|^2)^2}{2\varepsilon^2} + \frac{1}{r^{N-2}} \int_{\partial B_r} \left| \frac{\partial u}{\partial n} \right|^2 \right] dr \leqslant K\eta\, |\log \delta|, \tag{III.8}$$

and

$$0 \leqslant \frac{E_\varepsilon(0, r_0)}{r_0^{N-2}} - \frac{E_\varepsilon(0, \delta r_0)}{(\delta r_0)^{N-2}} \leqslant K\eta\, |\log \delta|. \tag{III.9}$$

Proof. Choose an integer k such that

$$\varepsilon^{1/2} \left(\frac{\delta}{4} \right)^{-(k+1)} \leqslant \delta, \qquad \varepsilon^{1/2} \left(\frac{\delta}{4} \right)^{-(k+2)} > \delta. \tag{III.10}$$

For $j = 0, \ldots, k$ consider the intervals

$$I_j = \left(\left(\frac{\delta}{4} \right)^{-j} \varepsilon^{1/2}, \left(\frac{\delta}{4} \right)^{-j-1} \varepsilon^{1/2} \right).$$

Clearly, these intervals are disjoint and $\bigcup_{j=0}^k I_j \subset (\varepsilon^{1/2}, \delta)$. Hence, by Corollary II.1

$$\begin{aligned}\sum_{j=0}^k \int_{I_j} \left[\frac{1}{r^{N-2}} \int_{S_r} \left| \frac{\partial u}{\partial n} \right|^2 + \frac{1}{r^{N-1}} \int_{B_r} \frac{(1-|u|^2)^2}{2\varepsilon^2} \right] dr &\leqslant \frac{E_\varepsilon(0, \delta)}{\delta^{N-2}} \leqslant E_\varepsilon(0, 1) \\ &\leqslant \eta\, |\log \varepsilon|,\end{aligned}$$

where we have used the monotonicity formulas and (III.2) for the last inequality. Since

$$k+1 \geqslant \frac{1}{2} \frac{|\log \varepsilon^{1/2}|}{|\log \delta|},$$

we therefore deduce that there exists some $j_0 \in \{0, ..., k\}$ such that

$$\int_{I_{j_0}} \cdots \leqslant \eta \frac{|\log \varepsilon|}{k+1} \leqslant 4\eta \, |\log \delta|. \tag{III.11}$$

Set $\tilde{r}_0 = (\frac{\delta}{4}^{-j_0-1})\, \varepsilon^{1/2}$: (III.11) says that

$$\int_{\frac{\delta}{4}\tilde{r}_0}^{\tilde{r}_0} \left(\frac{1}{r^{N-1}} \int_{B_r} \frac{(1-|u|^2)^2}{\varepsilon^2} + \frac{1}{r^{N-2}} \int_{S_r} \left| \frac{\partial u}{\partial n} \right|^2 \right) \leqslant 4\eta \, |\log \delta|.$$

By the mean-value formula we then deduce that there exist some $r_0 \in [\frac{\delta}{4}\tilde{r}_0, \tilde{r}_0]$ such that (III.7) and (III.8) are satisfied. Finally, (III.9) follows from (III.8) and Lemma II.2.

Comment. Note that Lemma III.1 involves only scale invariant quantities. We will use heavily this fact in Step 3; we will apply the conclusions of Step 2 (which is stated on B_1) after a change of scale $x \to r_0 x$.

Part B: δ-Energy Decay

This step is the heart of the proof: it is specific to complex-valued Ginzburg–Landau equation. This is a new and basic estimate for the energy.

THEOREM 3. *Let u be a solution of* (III.1) *with $R = 1$, then*

$$\begin{aligned} E_\varepsilon(0, \delta) \leqslant K \Bigg\{ & \left[\int_{B_1} \frac{(1-|u|^2)^2}{\varepsilon^2} \right]^{\frac{1}{3}} E_\varepsilon(0, 1) \\ & + \left[\int_{B_1} \frac{(1-|u|^2)^2}{\varepsilon^2} \right]^{\frac{2}{3}} + \delta^N E_\varepsilon(0, 1) \Bigg\}. \end{aligned} \tag{III.12}$$

The starting point is the identity

$$4\, |u|^2 \, |\nabla u|^2 = 4\, |u \times \nabla u|^2 + |\nabla\, |u|^2|^2, \tag{III.13}$$

which holds for any map from $\mathbb{R}$ to $\mathbb{R}^k$; in the special case where $k = 2$, $|u(x_0)| \neq 0$, we may write near x_0

$$u(x) = \rho \exp(i\varphi),$$

and then

$$u \times \nabla u = \rho^2 \, \nabla\varphi,$$

i.e., $u \times \nabla u$ plays the role of the gradient of the phase. The advantage of the form (III.13) is that $u \times \nabla u$ is always globally well defined, while the phase need not to be well-defined when u vanishes somewhere.

The conclusion of the theorem is an estimate for the energy on B_δ. The difficult part is always the contribution of the phase, i.e., $u \times \nabla u$. In this part, we will make use of the following known estimates (see [Bethuel-Brezis-Hélein 1, 2], [Brezis]).

Lemma III.2. *Assume u verifies* (III.1) *on B_1. Then*

$$|u| \leqslant K, \qquad |\nabla u| \leqslant \frac{K}{\varepsilon} \qquad \textit{in } B_{1/2}.$$

Proof of Theorem 3. We divide the proof in several steps.

Step 1: *Hodge–de Rham decomposition of $u \times \nabla u$.*
As in [Bethuel-Brezis-Hélein 2], we observe that

$$\operatorname{div}(u \times \nabla u) = 0 \qquad \text{in } B_1, \tag{III.14}$$

that is $\sum_{i=1}^{N} \frac{\partial}{\partial x_i} (u \times \frac{\partial u}{\partial x_i}) = 0$; this holds because

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(u \times \frac{\partial u}{\partial x_i} \right) = \sum_{i=1}^{N} \frac{\partial u}{\partial x_i} \times \frac{\partial u}{\partial x_i} + u \times \Delta u = u \times \Delta u = 0,$$

by (III.1). In order to invoke Poincaré's Lemma it is more convenient to write (III.14) using the formalism of differential forms (see Appendix). This yields

$$d^*(u \times du) = 0 \qquad \text{in } B_1, \tag{III.15}$$

where $du = \sum_{i=1}^{N} \frac{\partial u}{\partial x_i} dx_i$ and d^* denotes the Hodge star operator, $d^* = \pm \bigstar\, d \bigstar$.

By the mean-value inequality, we may find some $r_1 \in [\frac{1}{4}, \frac{1}{2}]$ such that

$$\begin{aligned} \int_{\partial B_{r_1}} |\nabla u|^2 &\leqslant 8 \int_{B_1} |\nabla u|^2, \\ \int_{\partial B_{r_1}} (1 - |u|^2)^2 &\leqslant 8 \int_{B_1} (1 - |u|^2)^2. \end{aligned} \tag{III.16}$$

Let ξ be the solution of the auxiliary Neumann problem

$$\begin{cases} \Delta \xi = 0 & \text{in } B_{r_1} \\ \dfrac{\partial \xi}{\partial n} = u \times \dfrac{\partial u}{\partial n} & \text{on } \partial B_{r_1}. \end{cases}$$

Note that ξ exists since $\operatorname{div}(u \times \nabla u) = 0$ implies by integration $\int_{\partial B_{r_1}} (u \times \nabla u) \cdot n = 0$. Moreover, we have

$$\int_{B_{r_1}} |\nabla \xi|^2 \leqslant K \int_{B_1} \frac{|\nabla u|^2}{2} \leqslant K E_\varepsilon(u).$$

Since ξ is harmonic on B_{r_1}, we have by standard elliptic estimates, for $0 < \delta \leqslant r_1$,

$$\int_{B_\delta} |\nabla \xi|^2 \leqslant K \delta^N \int_{B_{r_1}} |\nabla \xi|^2 \leqslant K \delta^N \int_{B_1} |\nabla u|^2. \tag{III.17}$$

By construction we verify that

$$\operatorname{div}[(u \times \nabla u - \nabla \xi)\, 1_{B_{r_1}}] = 0, \qquad \text{in } \mathscr{D}'(\mathbb{R}^N),$$

where 1_A denotes the characteristic function of the set A. In the formalism of differential forms this becomes

$$d^*[(u \times du) - d\xi)\, 1_{B_{r_1}}] = 0 \qquad \text{in } \mathscr{D}'(\mathbb{R}^N).$$

By classical Hodge theory (see Appendix, Proposition A.7) there exists some 2- form φ on $\mathbb{R}^N$ such that $\varphi \in H^1_{loc}(\mathbb{R}^N)$ and

$$d^*\varphi = (u \times du - d\xi)\, 1_{B_{r_1}} \qquad \text{in } \mathbb{R}^N, \tag{III.18}$$

$$d\varphi = 0 \qquad \text{in } \mathscr{D}'(\mathbb{R}^N), \tag{III.19}$$

$$\|\nabla \varphi\|_{L^p(\mathbb{R}^N)} \leqslant K_p(\|\nabla u\|_{L^p(B_{r_1})} + \|\nabla \xi\|_{L^p(B_{r_1})}), \forall 1 < p < +\infty, \tag{III.20}$$

$$|\varphi(x)||x|^{N-1} \text{ tends to zero at infinity.} \tag{III.21}$$

We therefore have

$$u \times du = d^*\varphi + d\xi \qquad \text{in } B_{r_1}. \tag{III.22}$$

In order to bound the L^2-norm of $u \times du$ on B_δ, we turn next to estimates for $d^*\varphi$.

Step 2: *Improved estimates for* $\nabla \varphi$ *on* B_δ *when* $|u| \simeq 1$.
Let us first explain what is going on. Assume first that

$$|u| \equiv 1.$$

Then we claim

$$\Delta \varphi = 0 \qquad \text{in } \mathscr{D}'(\mathbb{R}^N \setminus \partial B_{r_1}),$$

i.e., all the components of the form φ are harmonic on $\mathbb{R}^N \setminus \partial B_{r_1}$. Indeed, recall that $-\Delta = d^*d + dd^*$, so that, since $d\varphi = 0$,

$$-\Delta\varphi = dd^*\varphi = d[(u \times du - d\xi)\, 1_{B_{r_1}}] \qquad \text{in } \mathbb{R}^N.$$

Therefore it suffices to check

$$d(u \times du) = 0 \qquad \text{on } B_{r_1}.$$

We have

$$d(u \times du) = \sum_{i<j} \pm 2(u_{x_i} \times u_{x_j})\, dx_i \wedge dx_j$$

If $|u| = 1$ and if u takes its values in $\mathbb{R}^2$, the vectors u_{x_i} and u_{x_j} are colinear, so that

$$u_{x_i} \times u_{x_j} = 0, \quad \forall i, j.$$

This would not be true if u takes its values in $\mathbb{R}^k$, $k \geqslant 3$. *This is the only place where we use the fact that u is complex-valued.*

Since φ is harmonic, for $0 < \varepsilon < \frac{1}{8}$,

$$\begin{aligned} \int_{B_\delta} |\nabla\varphi|^2 &\leqslant C\delta^N \int_{B_{r_1}} |\nabla\varphi|^2 \\ &\leqslant C\delta^N \int_{B_1} |\nabla u|^2. \end{aligned}$$

This shows that $\int_{B_\delta} |\nabla\varphi|^2$ has a good decay as δ goes to zero.

In our situation, u does *not* take its values in S^1. Instead we play on the "smallness" of the integral $\frac{1}{\varepsilon^2} \int_{B_1} (1-|u|^2)^2$, i.e. $|u|$ is close to 1, and we propose to use this fact in order to control $\int_{B_\delta} |\nabla\varphi|^2$. To make the estimate precise it is convenient to introduce a "smooth" projection of u on S^1.

Let $0 < \beta < \frac{1}{4}$, to be determined later and let $f\colon \mathbb{R}^+ \to (1, \frac{1}{1-\beta})$ be any smooth function such that

$$\begin{cases} f(t) = \dfrac{1}{t} & \text{if } \ t \geqslant 1-\beta \\ f(t) = 1 & \text{if } \ t \leqslant 1-2\beta \\ |f'| \leqslant 4 & \text{for any } \ t \in \mathbb{R}^+. \end{cases} \tag{III.23}$$

Define on $\mathbb{R}^N$ the function α as

$$\alpha(x) = \begin{cases} f^2(|u(x)|) & \text{in } B_{r_1} \\ 1 & \text{outside,} \end{cases}$$

so that

$$0 \leqslant \alpha - 1 \leqslant 4\beta \qquad \text{in } \mathbb{R}^N. \tag{III.24}$$

Note that

$$f^2(|u|)\, u \times du = f(|u|)\, u \times d(f(|u|)\, u),$$

hence

$$d(\alpha u \times du) = d(f^2(|u|)\, u \times du) = d(f(|u|)\, u \times d(f(|u|)\, u) \qquad \text{in } B_{r_1},$$

i.e.,

$$d(\alpha u \times du) = \sum_{i<j} 2(f(|u|)\, u)_{x_i} \times (f(|u|)\, u)_{x_j} dx_i \wedge dx_j.$$

Now we turn to φ. We have

$$d(\alpha d^* \varphi) = \omega_1 + \omega_2 + \omega_3 \qquad \text{in } \mathscr{D}'(\mathbb{R}^N),$$

where

$$\begin{aligned}
\omega_1 &= 1_{B_{r_1}} d(\alpha u \times du) = 1_{B_{r_1}} \sum_{i<j} 2(f(|u|)\, u)_{x_i} \times (f(|u|)\, u)_{x_j} dx_i \wedge dx_j, \\
\omega_2 &= \sigma_{\partial B_{r_1}} f(|u|)\, u \times du \wedge dr, \qquad (r = |x|), \\
\omega_3 &= d(-1_{B_{r_1}} \alpha\, d\xi) = d(1_{B_{r_1}} (1-\alpha)\, d\xi) - d(1_{B_{r_1}}\, d\xi) \\
&= d(1_{B_{r_1}} (1-\alpha)\, d\xi) + \sigma_{\partial B_{r_1}}\, dr \wedge d\xi \equiv \omega_{3,1} + \omega_{3,2}.
\end{aligned}$$

Here $\sigma_{\partial B_{r_1}}$ stands for the surface measure on ∂B_{r_1}. Finally we write

$$\begin{aligned}
-\Delta \varphi = dd^* \varphi &= d(\alpha d^* \varphi) + d((1-\alpha)\, d^* \varphi) \qquad \text{in } \mathscr{D}' \\
&= \omega_1 + \omega_2 + \omega_3 + \omega_4,
\end{aligned}$$

where

$$\omega_4 = d((1-\alpha)\, d^* \varphi).$$

Set

$$\varphi_i = G * \omega_i,$$

where $G(x) = c_N\,|x|^{2-N}$ is the fundamental solution of $-\Delta$ in $\mathbb{R}^N$. Since φ tends to zero at infinity by (III.21) and each φ_i tends to zero at infinity (because each ω_i has compact support), we conclude that

$$\varphi = \sum_{i=1}^{4} \varphi_i.$$

We now proceed to estimate separately each φ_i (we also make use of the obvious notation $\varphi_3 = \varphi_{3,1} + \varphi_{3,2}$).

Estimate for φ_4. We have

$$\int_{\mathbb{R}^N} |\nabla \varphi_4|^2 \leqslant K\beta^2 \int_{B_1} |\nabla u|^2. \tag{III.25}$$

Proof of (III.25). We have

$$-\Delta \varphi_4 = \omega_4 = d((1-\alpha)\, d^*\varphi).$$

Multiplying by φ_4 and integrating we obtain

$$\int_{\mathbb{R}^N} |\nabla \varphi_4|^2 \leqslant \|1-\alpha\|_{L^\infty}\, \|\nabla \varphi\|_{L^2}\, \|\nabla \varphi_4\|_{L^2},$$

and thus

$$\int_{\mathbb{R}^N} |\nabla \varphi_4|^2 \leqslant K\beta\, \|\nabla u\|_{L^2(B_1)}\, \|\nabla \varphi_4\|_{L^2(\mathbb{R}^N)},$$

by (III.17), (III.20), (III.24), which yields the result.

Estimate for $\varphi_{3,1}$. As above,

$$\int_{\mathbb{R}^N} |\nabla \varphi_{3,1}|^2 \leqslant 4\beta^2 \int_{B_1} |\nabla u|^2. \tag{III.26}$$

Estimate for φ_2 *and* $\varphi_{3,2}$. Observe that φ_2 and $\varphi_{3,2}$ are harmonic on B_{r_1} (recall that $r_1 \in (1/4, 1/2)$). By standard elliptic estimates for harmonic functions (see, e.g., [Bethuel-Brezis-Orlandi], Appendix),

$$\|\nabla \varphi_2\|_{L^\infty(B_{1/8})} \leqslant K \left(\int_{\partial B_{r_1}} |\nabla u|^2 \right)^{1/2},$$

$$\|\nabla \varphi_{3,2}\|_{L^\infty(B_{1/8})} \leqslant K \left(\int_{\partial B_{r_1}} |\nabla \xi|^2 \right)^{1/2}.$$

Therefore

$$\int_{B_\delta} |\nabla \varphi_{3,2}|^2 + |\nabla \varphi_2|^2 \leqslant K\delta^N \int_{B_1} |\nabla u|^2, \qquad \forall 0 < \delta < \tfrac{1}{8}. \tag{III.27}$$

Estimate for φ_1. We start with the crucial observation that

$$|\omega_1| \leqslant K\beta^{-2} \frac{(1-|u|^2)^2}{\varepsilon^2} \qquad \text{in } B_1. \tag{III.28}$$

Proof of (III.28). We must distinguish the two regions

$$V_\beta = \{x \in B_1;\, |u(x)| \geqslant 1-\beta\}, \qquad W_\beta = \{x \in B_1;\, |u(x)| \leqslant 1-\beta\}.$$

Recall that

$$\omega_1 = 1_{B_{r_1}} d(\alpha u \times du) = 1_{B_{r_1}} \sum_{i<j} 2(f(|u|)\, u)_{x_i} \times (f(|u|)\, u)_{x_j}\, dx_i \wedge dx_j.$$

On V_β we have $f(|u(x)|) = \frac{1}{|u(x)|}$ and therefore

$$(f(|u|)\, u)_{x_i} \times (f(|u|)\, u)_{x_j} = 0, \qquad \text{for} \quad i \neq j.$$

On W_β we have, by Lemma III.2

$$|(f(|u|)\, u)_{x_i}| \leqslant \frac{K}{\varepsilon},$$

so that

$$|\omega_1| \leqslant \frac{K}{\varepsilon^2} \leqslant \frac{K}{\varepsilon^2} \beta^{-2}\beta^2 \leqslant \frac{K}{\varepsilon^2} \beta^{-2}(1-|u|)^2 \leqslant K\beta^{-2} \frac{(1-|u|^2)^2}{\varepsilon^2},$$

which yields (III.28).

The final *crucial estimate* is

$$\|\varphi_1\|_{L^\infty(\mathbb{R}^N)} \leqslant \frac{K}{\beta^2} E_\varepsilon(0, 1). \tag{III.29}$$

Indeed

$$\varphi_1(x) = \int_{\mathbb{R}^N} \frac{c_N}{|x-y|^{N-2}}\, \omega_1(y)\, dy = \int_{B_{r_1}} \frac{c_N}{|x-y|^{N-2}}\, \omega_1(y)\, dy,$$

so that

$$|\varphi_1(x)| \leqslant \frac{K}{\beta^2} \int_{B_{r_1}} \frac{(1-|u(y)|^2)^2}{\varepsilon^2 \, |x-y|^{N-2}} \, dy.$$

Assume $|x| \leqslant r_1$. Since $B_{r_1} \subset B_{1/2}(x)$ we have

$$|\varphi_1(x)| \leqslant \frac{K}{\beta^2} \int_{B_{1/2}(x)} \frac{(1-|u(y)|^2)^2}{\varepsilon^2 \, |x-y|^{N-2}} \, dy.$$

Next we invoke the monotonicity formula (II.2) centered at the point x, to assert that

$$\int_{B_{1/2}(x)} \frac{(1-|u(y)|^2)^2}{\varepsilon^2 \, |x-y|^{N-2}} \, dy \leqslant K E_\varepsilon \left(x, \frac{1}{2} \right) \leqslant K E_\varepsilon(0, 1).$$

Hence for every $x \in B_{r_1}$

$$|\varphi_1(x)| \leqslant K \beta^{-2} E_\varepsilon(0, 1).$$

Recall that $\Delta \varphi_1 = 0$ outside B_{r_1}, so that by the maximum principle

$$\|\varphi_1\|_{L^\infty(\mathbb{R}^N)} = \|\varphi_1\|_{L^\infty(B_{r_1})} \leqslant K \beta^{-2} E_\varepsilon(0, 1),$$

which is (III.29).

Going back to the equation

$$-\Delta \varphi_1 = \omega_1 \qquad \text{in } \mathbb{R}^N,$$

we conclude

$$\int_{\mathbb{R}^N} |\nabla \varphi_1|^2 \leqslant \|\varphi_1\|_{L^\infty(\mathbb{R}^N)} \int_{B_{r_1}} |\omega_1|,$$

so that

$$\int_{\mathbb{R}^N} |\nabla \varphi_1|^2 \leqslant K \beta^{-4} \int_{B_1} \frac{(1-|u|^2)^2}{\varepsilon^2} \, E_\varepsilon(0, 1). \tag{III.30}$$

Step 2 completed: The estimate for φ. We are combining all the estimates for φ_1, φ_2, $\varphi_{3,1}$, $\varphi_{3,2}$ and φ_4. This yields, for $0 < \delta < 1/8$,

$$\int_{\mathbb{R}^N} |\nabla \varphi|^2 \leqslant K(\beta^2 + \delta^N) \int_{B_1} |\nabla u|^2 + K \beta^{-4} \int_{B_1} \frac{(1-|u|^2)^2}{\varepsilon^2} \, E_\varepsilon(0, 1). \tag{III.31}$$

Step 3: *Improved estimates for* $\nabla(|u|^2)$ *on* B_δ. The equation for $|u|^2$ reads

$$-\Delta(1-|u|^2)+\frac{2(1-|u|^2)\,|u|^2}{\varepsilon^2}=2\,|\nabla u|^2.$$

Multiplying by $1-|u|^2$ and integrating on B_{r_1} we obtain

$$\int_{B_{r_1}} |\nabla\,|u|^2|^2+\frac{2(1-|u|^2)^2\,|u|^2}{\varepsilon^2}=2\int_{B_{r_1}} (1-|u|^2)\,|\nabla u|^2+\int_{\partial B_{r_1}} (1-|u|^2)\,\frac{\partial |u|^2}{\partial n}.$$

From (III.16) we deduce

$$\left|\int_{\partial B_{r_1}} (1-|u|^2)\,\frac{\partial\,|u|^2}{\partial n}\right|\leqslant K\varepsilon\left(\int_{B_1}\frac{(1-|u|^2)^2}{\varepsilon^2}\right)^{1/2}\left(\int_{B_1}|\nabla u|^2\right)^{1/2}.$$

On the other hand

$$\begin{aligned}\left|\int_{B_{r_1}} (1-|u|^2)\,|\nabla u|^2\right| &\leqslant K\int_{V_{\beta^2}}\beta^2\,|\nabla u|^2+K\beta^{-2}\int_{W_{\beta^2}}\frac{(1-|u|^2)^2}{\varepsilon^2}\\ &\leqslant K\beta^2\int_{B_1}|\nabla u|^2+K\beta^{-2}\int_{B_1}\frac{(1-|u|^2)^2}{\varepsilon^2}.\end{aligned}\tag{III.32}$$

Finally, we have

$$\begin{aligned}\int_{B_{r_1}}|\nabla\,|u|^2|^2 &\leqslant K\left(\beta^2\int_{B_1}|\nabla u|^2+\beta^{-2}\int_{B_1}\frac{(1-|u|^2)^2}{\varepsilon^2}\right.\\ &\quad\left.+\varepsilon\left(\int_{B_1}\frac{(1-|u|^2)^2}{\varepsilon^2}\right)^{1/2}\left(\int_{B_1}|\nabla u|^2\right)^{1/2}\right)\\ &\leqslant K\left(\beta^2\int_{B_1}|\nabla u|^2+\beta^{-2}\int_{B_1}\frac{(1-|u|^2)^2}{\varepsilon^2}\right).\end{aligned}\tag{III.33}$$

Step 4: *The final estimate. Proof of Theorem* 3 *completed.* We must prove that $\forall\delta<1/8$,

$$\begin{aligned}E_\varepsilon(0,\delta)&\leqslant K\left(\int_{B_1}\frac{(1-|u|^2)^2}{\varepsilon^2}\right)^{1/3}E_\varepsilon(0,1)\\ &\quad+K\left(\int_{B_1}\frac{(1-|u|^2)^2}{\varepsilon^2}\right)^{2/3}+K\delta^N E_\varepsilon(0,1).\end{aligned}$$

Proof. Recall that

$$4\,|u|^2\,|\nabla u|^2=4\,|u\times\nabla u|^2+|\nabla\,|u|^2|^2,$$

and thus

$$\begin{aligned}4\,|\nabla u|^2 &= 4\,|u\times\nabla u|^2+|\nabla\,|u|^2|^2+4(1-|u|^2)\,|\nabla u|^2\\ &\leqslant 8(|\nabla\varphi|^2+|\nabla\xi|^2)+|\nabla\,|u|^2|^2+4(1-|u|^2)\,|\nabla u|^2 \qquad \text{by (III.22).}\end{aligned}$$

Combining (III.31), (III.17), (III.33), and (III.32) we obtain

$$\begin{aligned}\int_{B_\delta}|\nabla u|^2 \leqslant K\Bigg(&\beta^2\int_{B_1}|\nabla u|^2+\beta^{-2}\int_{B_1}\frac{(1-|u|^2)^2}{\varepsilon^2}+\delta^N\int_{B_1}|\nabla u|^2\\ &+\beta^{-4}E_\varepsilon(0,1)\int_{B_1}\frac{(1-|u|^2)^2}{\varepsilon^2}\Bigg),\end{aligned}$$

which yields

$$\begin{aligned}\int_{B_\delta}e_\varepsilon(u) \leqslant K\left(\beta^2E_\varepsilon(0,1)+\beta^{-2}\int_{B_1}\frac{(1-|u|^2)^2}{\varepsilon^2}+\beta^{-4}E_\varepsilon(0,1)\int_{B_1}\frac{(1-|u|^2)^2}{\varepsilon^2}\right)\\ +K\delta^N\int_{B_1}|\nabla u|^2.\end{aligned}$$

Set

$$p_\varepsilon=\int_{B_1}\frac{(1-|u|^2)^2}{\varepsilon^2}$$

and choose $\beta=p_\varepsilon^{1/6}$, if $p_\varepsilon\leqslant(\frac{1}{8})^6$. We obtain

$$\int_{B_\delta}e_\varepsilon(u)\leqslant K(p_\varepsilon^{1/3}E_\varepsilon(0,1)+p_\varepsilon^{2/3})+K\delta^N\int_{B_1}|\nabla u|^2.$$

If $p_\varepsilon\geqslant(\frac{1}{8})^6$ the inequality is obvious, using only the first term on the right hand side.

Part C: Proof of Theorem 2 Completed when $N\geqslant 3$

We start with a solution u_ε of (III.1) on B_1 satisfying the estimate

$$E_\varepsilon(0,1)\leqslant\eta\,|\log\varepsilon|. \tag{III.34}$$

Recall that in Part A we have exhibited some $r_0\in(\varepsilon^{1/2},\delta)$, where δ is fixed but to be determined later, such that

$$\frac{1}{r_0^{N-2}}\int_{B_{r_0}}\frac{(1-|u|^2)^2}{\varepsilon^2}\leqslant K\eta\,|\log\delta|, \tag{III.35}$$

$$\frac{E_\varepsilon(0,r_0)}{r_0^{N-2}}-\frac{E_\varepsilon(0,\delta r_0)}{(\delta r_0)^{N-2}}\leqslant K\eta\,|\log\delta|. \tag{III.36}$$

We apply Theorem 3 to the function $u_\#(x)=u(r_0 x)$ on B_1. The equation for $u_\#$ is

$$-\Delta u_\# = \frac{u_\#(1-|u_\#|^2)}{{\varepsilon_\#}^2} \qquad \text{in } B_1,$$

where $\varepsilon_\# = \frac{\varepsilon}{r_0}$. Note in particular that $\varepsilon_\# \in (\frac{\varepsilon}{\delta}, \varepsilon^{1/2})$.

We apply Theorem 3 to $u_\#$, with the same δ as above. Here $E^\#_\varepsilon$ denotes the Ginzburg–Landau energy relative to $u_\#$.

$$E^\#_{\varepsilon_\#}(0,\delta) \leqslant K E^\#_{\varepsilon_\#}(0,1)\left(\int_{B_1} \frac{(1-|u_\#|^2)^2}{{\varepsilon_\#}^2}\right)^{1/3} + K\left(\int_{B_1} \frac{(1-|u_\#|^2)^2}{{\varepsilon_\#}^2}\right)^{2/3}$$
$$+ K\delta^N E^\#_{\varepsilon_\#}(0,1).$$

By scaling we have the identities

$$E^\#_{\varepsilon_\#}(0,1) = \tilde{E}_\varepsilon(0,1),$$
$$E^\#_{\varepsilon_\#}(0,\delta) = \frac{1}{r_0^{N-2}} E_\varepsilon(0,\delta r_0) = \delta^{2-N}\tilde{E}_\varepsilon(0,\delta r_0),$$

and

$$\int_{B_1} \frac{(1-|u_\#|^2)^2}{\varepsilon_\#^2} = \frac{1}{r_0^{N-2}} \int_{B_{r_0}} \frac{(1-|u|^2)^2}{\varepsilon^2}.$$

Going back to u, we find

$$\frac{1}{r_0^{N-2}} E_\varepsilon(0,\delta r_0) \leqslant K\tilde{E}_\varepsilon(0,r_0)\left(\frac{1}{r_0^{N-2}}\int_{B_{r_0}} \frac{(1-|u|^2)^2}{\varepsilon^2}\right)^{1/3}$$
$$+K\left(\frac{1}{r_0^{N-2}}\int_{B_{r_0}} \frac{(1-|u|^2)^2}{\varepsilon^2}\right)^{2/3}$$
$$+K\delta^N \tilde{E}_\varepsilon(0,r_0). \tag{III.37}$$

Using (III.36) and (III.35) we obtain

$$\tilde{E}_\varepsilon(0,r_0) \leqslant \tilde{E}_\varepsilon(0,\delta r_0) + K\eta\,|\log\delta|$$
$$\leqslant \frac{K}{\delta^{N-2}}\tilde{E}_\varepsilon(0,r_0)(\eta\,|\log\delta|)^{1/3} + \frac{K}{\delta^{N-2}}(\eta\,|\log\delta|)^{2/3}$$
$$+K\delta^2\tilde{E}_\varepsilon(0,r_0).$$

Hence

$$\tilde{E}_\varepsilon(0, r_0)\left(1 - K\left(\frac{\eta^{1/3}\,|\log \delta|^{1/3}}{\delta^{N-2}} + \delta^2\right)\right) \leqslant \frac{K}{\delta^{N-2}}\,(\eta\,|\log \delta|)^{2/3}.$$

Now we choose $\delta = \eta^{1/3N}$ if $\eta^{1/3N} > \varepsilon^2$, i.e. $\eta > \varepsilon^{6N}$ (otherwise, see later), then

$$\tilde{E}_\varepsilon(0, r_0)(1 - K\eta^{2/3N}\,|\log \eta|) \leqslant K\eta^{(N+2)/3N}\,|\log \eta|^{2/3}.$$

If $\eta \leqslant \eta_0$ (η_0 is an absolute constant) we have

$$\tilde{E}_\varepsilon(0, r_0) \leqslant K\eta^{(N+2)/3N}\,|\log \eta|^{2/3}. \tag{III.38}$$

In the case $\eta \leqslant \varepsilon^{6N}$, inequality (III.38) still holds, as a consequence of monotonicity and (III.34).

Finally, we invoke monotonicity once more to assert that, $\forall r \leqslant r_0$

$$\tilde{E}_\varepsilon(0, r) \leqslant \tilde{E}_\varepsilon(0, r_0) \leqslant K\eta^{(N+2)/3N}\,|\log \eta|^{2/3}.$$

In particular, for $r = \varepsilon$,

$$\frac{1}{\varepsilon^N}\int_{B_\varepsilon} (1 - |u|^2)^2 \leqslant \tilde{E}_\varepsilon(0, \varepsilon) \leqslant K\eta^{(N+2)/3N}\,|\log \eta|^{2/3}. \tag{III.39}$$

We now conclude immediately with the help of the following lemma

Lemma III.3. *Let u be a solution of* (III.1) *on B_1 with $N \geqslant 2$. Then*

$$1 - |u(0)| \leqslant K\left(\frac{1}{\varepsilon^N}\int_{B_\varepsilon} (1 - |u|^2)^2\right)^{1/(N+2)}.$$

Proof. Set $k = |u(0)|$ and assume that $k \leqslant 1$ (otherwise there is nothing to be proved). By Lemma III.2 we have

$$|u(x) - u(0)| \leqslant \frac{K}{\varepsilon}\,|x| \leqslant \frac{1-k}{2},$$

provided $|x| \leqslant \frac{\varepsilon(1-k)}{2K} \equiv \gamma$. Therefore $|u(x)| \leqslant \frac{1+k}{2}$ on B_γ. We distinguish two cases.

Case 1. $\gamma < \varepsilon$. Then

$$\int_{B_\gamma} (1 - |u|^2)^2 \leqslant \int_{B_\varepsilon} (1 - |u|^2)^2.$$

On the other hand

$$\int_{B_\gamma}(1-|u|^2)^2 \geqslant \int_{B_\gamma}(1-|u|)^2 \geqslant \left(\frac{1-k}{2}\right)^2 |B_\gamma| = K\varepsilon^N(1-k)^{N+2},$$

by definition of γ. Consequently

$$(1-k)^{N+2} \leqslant \frac{K}{\varepsilon^N}\int_{B_\varepsilon}(1-|u|^2)^2,$$

and the conclusion follows.

Case 2. $\gamma \geqslant \varepsilon$. Then

$$|u(x)| \leqslant \frac{1+k}{2} \qquad \text{in } B_\varepsilon,$$

and

$$\int_{B_\varepsilon}(1-|u|^2)^2 \geqslant \left(\frac{1-k}{2}\right)^2 |B_\varepsilon|.$$

Therefore

$$(1-k)^{N+2} \leqslant (1-k)^2 \leqslant \frac{K}{\varepsilon^N}\int_{B_\varepsilon}(1-|u|^2)^2,$$

and the conclusion of the lemma follows.

Part D: Theorem 2 *when* $N=2$

As we have already mentioned, the statement of Theorem 2 when $N=2$ still holds for any solution u of (III.1) with values into $\mathbb{R}^k$, $k \geqslant 2$.

We start again with a solution u_ε of (III.1), satisfying the estimate

$$E_\varepsilon(0,1) \leqslant \eta\,|\log\varepsilon|.$$

By Part A (with $\delta = \frac{1}{8}$), there exists some $r_0 \in (\varepsilon^{1/2}, \frac{1}{8})$ such that

$$\frac{1}{\varepsilon^2}\int_{B_{r_0}}(1-|u|^2)^2 \leqslant K\eta.$$

In particular

$$\frac{1}{\varepsilon^2}\int_{B_\varepsilon}(1-|u|^2)^2 \leqslant K\eta,$$

so that by Lemma III.3

$$1-|u(0)| \leqslant K\eta^{1/4},$$

and the proof is complete. We emphasize that the two ingredients Part A and Lemma III.3 are valid for any solution with values into $\mathbb{R}^k$. It is only Part B (and thus Part C) which requires the assumption $k=2$.

Part E. Some additional remarks.

Remark III.3. The conclusion of Theorem 2 fails when $N \geqslant 3$ and $k \geqslant 3$.

First, we assume that $N=3$ and $k=3$. Let u_ε be a minimizer of E_ε on B_1 with the boundary condition

$$u(x) = x \qquad \text{on } \partial B_1 .$$

Clearly

$$E_\varepsilon(u_\varepsilon) \leqslant E_\varepsilon\left(\frac{x}{|x|}\right) = \frac{1}{2}\int_{B_1} \left|\nabla\left(\frac{x}{|x|}\right)\right|^2 < +\infty,$$

so that (III.3) is fulfilled with $\eta = C\,|\log \varepsilon|^{-1}$.

If the conclusion of Theorem 2 holds we would deduce that $|u_\varepsilon| \to 1$ uniformly on $B_{1/2}$. On the other hand

$$u_\varepsilon(x) \to u_*(x) = \frac{x}{|x|} \qquad \text{strongly in } H^1(B_1),$$

by Theorem 7.1 in [Brezis-Coron-Lieb] (see also [Lin 1]). In particular, $u_\varepsilon \to u_*$ strongly in $H^1(S_r)$ for a.e. sphere $S_r = \partial B_r$.

In view of the stability of the degree for maps $h\colon S^2 \to S^2$ under strong H^1 convergence (via the Kronecker representation formula as in [Schoen-Uhlenbeck 2] or via VMO-degree as in [Brezis-Nirenberg]) we conclude that for a.e. $r \in (0, 1/2)$

$$\deg\left(\frac{u_\varepsilon}{|u_\varepsilon|}, S_r\right) \to 1 \qquad \text{as} \quad \varepsilon \to 0.$$

This is impossible because

$$\deg\left(\frac{u_\varepsilon}{|u_\varepsilon|}, S_r\right) = 0 \qquad \text{for all} \quad r \in (0, 1/2).$$

In the general case, $N \geqslant 3$ and $k \geqslant 3$ we use the u_ε above considered as a map from $B_1 \subset \mathbb{R}^N$ with values in $\mathbb{R}^k$, with u_ε independent of the variables $x_4, x_5, \ldots, x_N$.

Remark III.4. As we already mentioned in the Introduction, the assumptions in Theorem 2 do not guarantee compactness of the sequence $\{u_\varepsilon\}$, even in L^1_{loc}, even when $N = 3$, as long as $k \geqslant 2$. The lack of compactness is due to the phase. In [Brezis-Mironescu], the authors constructed a sequence u_ε of solutions of (III.1) satisfying

$$E_\varepsilon(u_\varepsilon) = o(|\log \varepsilon|),$$

and u_ε has no subsequence converging a.e. on a set of positive measure. For the convenience of the reader we recall the argument. Let

$$v_n = \exp(inx_1) \qquad \text{and} \qquad g_n = v_{n \mid \partial B_1}.$$

Let $u_{\varepsilon, n}$ be a minimizer of E_ε in $H^1_{g_n}$. Clearly

$$E_\varepsilon(u_{\varepsilon, n}) \leqslant E_\varepsilon(v_n) \leqslant Kn^2. \tag{III.40}$$

On the other hand, by [Bethuel-Brezis-Hélein 1] we know that for each fixed n, $u_{\varepsilon, n}$ tends to v_n in $L^\infty(B_1)$ as $\varepsilon \to 0$. We may then construct a sequence $\varepsilon_n \to 0$ such that

$$\|u_{\varepsilon_n, n} - v_n\|_{L^\infty} \leqslant \frac{1}{n} \tag{III.41}$$

and also

$$E_{\varepsilon_n}(u_{\varepsilon_n, n}) = o(|\log \varepsilon_n|).$$

It is well known that v_n converges weakly $\sigma(L^\infty, L^1)$ to zero, and hence $u_{\varepsilon_n, n} \to 0$ weakly $\sigma(L^\infty, L^1)$ by (III.41).

We now argue by contradiction. Suppose that $u_{\varepsilon_n, n}$ converge a.e. to a limit, say u, on a set A with $\text{meas}(A) > 0$. Necessarily $u = 0$ a.e. On the other hand, $|u_{\varepsilon_n, n}| \to 1$ as $n \to \infty$, uniformly. Hence $|u| = 1$ a.e. on A, a contradiction.

In this example, the noncompactness of the sequence u_ε is "generated" by the oscillations of the boundary conditions g_ε. The situation becomes totally different if one prescribes further assumptions on the boundary conditions g_ε. This is explained in Section VII.

III.2. η-Ellipticity at the Boundary

We are going to extend, in this section, the result of Theorem 2 to the case where $B_R(x_0)$ intersects the boundary of Ω. Throughout this section, x_0 will be a point in $\bar{\Omega}$, and $R > 0$ will be such that

$$0 \leqslant R \leqslant d_0^2 \equiv \operatorname{dist}(x_0, \Sigma)^2, \qquad R \leqslant R_1, \tag{III.42}$$

where R_1 is the constant in Lemma II.3 (monotonicity formula at the boundary). We are going to prove

THEOREM 2bis. *Let $\eta > 0$, $x_0 \in \Omega$ and $R > 0$ verifying* (III.42). *There exist constants $K > 0$, $\alpha > 0$ depending only on N, and $0 < \varepsilon_0 < 1$ depending only on η, Ω, M_0 and C_0 such that, if u_ε is a solution to $(GL)_\varepsilon$ verifying* (H1) *and* (H2) *with*

$$0 < \varepsilon < \inf\{\varepsilon_0, R^4\} \tag{III.43}$$

and

$$\tilde{E}_\varepsilon(u_\varepsilon) \leqslant \eta \left|\log \frac{\varepsilon}{R}\right|, \tag{III.44}$$

then

$$1 - |u(x_0)| \leqslant K\eta^\alpha. \tag{III.45}$$

The proof of Theorem 2 bis follows the same arguments as the proof of Theorem 2. For each part A, B, C we will briefly point out the modification to be made.

Part A. Choosing a Good Radius

We have the following variant of Lemma III.1.

LEMMA III.1bis. *Let $0 < \delta < \frac{1}{16}$ and assume*

$$0 < \varepsilon < \varepsilon_1 \equiv \inf\left\{R^4, \left(\frac{\delta}{R}\right)^4, e^{-\frac{R_1}{\eta}}\right\}. \tag{III.46}$$

Let u_ε be a solution of $(GL)_\varepsilon$ verifying (H1), (H2), (III.42), (III.44). *Then, there exists some $r_0 \in (\varepsilon^{1/2}, \varepsilon^{1/4})$ such that*

$$\frac{1}{r_0^{N-2}} \int_{\breve{B}_{r_0}(x_0)} \frac{(1-|u|^2)^2}{\varepsilon^2} \leqslant K\eta\, |\log \delta|, \tag{III.47}$$

$$0 \leqslant \tilde{E}(x_0, r_0) - \tilde{E}(x_0, \delta r_0) \leqslant K\eta\, |\log \delta|. \tag{III.48}$$

Proof. The argument is similar to the proof of Lemma III.1. By (III.46), $\varepsilon^{1/4} \leqslant R$. Choose $k \in \mathbb{N}$ such that

$$\varepsilon^{1/2}\left(\frac{\delta}{4}\right)^{-(k+1)} \leqslant \varepsilon^{1/4}, \qquad \text{and} \qquad \varepsilon^{1/2}\left(\frac{\delta}{4}\right)^{-(k+2)} > \varepsilon^{1/4}. \tag{III.49}$$

Consider for $j = 0, \ldots, k$ the intervals

$$I_j = \left(\left(\frac{\delta}{4}\right)^{-j} \varepsilon^{1/2}, \left(\frac{\delta}{4}\right)^{-(j+1)} \varepsilon^{1/2}\right).$$

These intervals are disjoint and $\bigcup_{j=0}^{k} I_j \subset (\varepsilon^{1/2}, \varepsilon^{1/4})$. Set

$$\gamma_j = \int_{I_j} \left[\frac{1}{r^{N-2}} \int_{\check{S}_r(x_0)} \left|\frac{\partial u}{\partial n}\right|^2 + \frac{1}{r^{N-1}} \int_{\check{B}_r(x_0)} \frac{(1-|u|^2)^2}{\varepsilon^2}\right].$$

By Corollary II.2, we have

$$\begin{aligned}\sum_{j=0}^{k} \gamma_j &\leqslant K(\tilde{E}_\varepsilon(x_0, \varepsilon^{1/4}) + \varepsilon^{1/4}) \\ &\leqslant K(\tilde{E}_\varepsilon(x_0, R) + R) \leqslant K(\tilde{E}_\varepsilon(x_0, R) + R_1),\end{aligned}$$

where we have invoked monotonicity again for the last inequalities. On the other hand, by (III.49),

$$k+1 \geqslant \frac{|\log \varepsilon^{1/4}|}{|\log \delta|} = \frac{1}{4} \frac{|\log \varepsilon|}{|\log \delta|}.$$

Therefore for some $j_0 \in \{0, \ldots, k\}$

$$\begin{aligned}\gamma_{j_0} &\leqslant K\,|\log \delta| \left(\frac{E_\varepsilon(x_0, R) + R_1}{|\log \varepsilon|}\right) \\ &\leqslant K\,|\log \delta| \left(\eta + \frac{R_1}{|\log \varepsilon|}\right) \leqslant K\eta\,|\log \delta|,\end{aligned}$$

by (III.44) and (III.46). The proof is then completed as in Lemma III.3.

Part B. δ-Energy Decay Near the Boundary

We have

THEOREM 3bis. *Let u_ε be a solution to* $(GL)_\varepsilon$ *verifying* (H1), (H2), $x_0 \in \bar{\Omega}$, *and* $r > 0$ *such that, for* $d_0 = dist(x_0, \Sigma)$,

$$r \leqslant \inf\{R_1, d_0^2\}, \tag{III.50}$$

where R_1 is the constant in Lemma II.3 *(monotonicity formula).*

Then, for all $0 < \delta < 1/4$,

$$E_\varepsilon(x_0, \delta r) \leqslant C\left(\frac{1}{r^{N-2}}\int_{\check{B}_r(x_0)} \frac{(1-|u|^2)^2}{\varepsilon^2}\right)^{1/3} (E_\varepsilon(x_0, r) + r^{N-1})$$
$$+ C\left(\frac{1}{r^{N-2}}\int_{\check{B}_r(x_0)} \frac{(1-|u|^2)^2}{\varepsilon^2}\right)^{2/3}$$
$$+ C\delta^N (E_\varepsilon(x_0, r) + r^{N-1}). \tag{III.51}$$

For the proof, we follow the same steps and arguments as in the proof of Theorem 3. We must however devote special care to the boundary conditions: for that purpose, we will use the result of the Appendix (in particular Propositions A.6 and A.7). Before we give the details of the modifications to be made in the proof of Theorem 3, we recall the following basic estimates (see [Bethuel-Brezis-Hélein 1, 2]),

$$|u_\varepsilon| \leqslant 1 \qquad \text{on } \bar{\Omega}, \qquad |\nabla u_\varepsilon| \leqslant \frac{C}{\varepsilon} \qquad \text{on } \bar{\Omega}, \tag{III.52}$$

for any solution u_ε of $(GL)_\varepsilon$ verifying (H1) and (H2). In order to simplify the presentation of the proof of Theorem 3 bis, we will assume throughout that near x_0, $\partial\Omega$ is flat (i.e. an $(N-1)$-dimensional hyperplane locally near x_0).

Changing possibly the coordinates, we may write

$$x_0 = (0, x_{0,N}) \in \mathbb{R}^{N-1} \times \mathbb{R}^+,$$

so that our assumption on $\partial\Omega$ can be rephrased as

$$\Gamma_r \equiv B_r(x_0) \cap \partial\Omega \subset \mathbb{R}^{N-1} \times \{0\}. \tag{III.53}$$

Finally, we will also assume throughout that

$$B_{r/8}(x_0) \cap \partial\Omega \neq \varnothing, \qquad \left(\text{i.e. } x_{0,N} \leqslant \frac{r}{8}\right). \tag{III.54}$$

Otherwise, the proof of (III.51) is an easy consequence of Theorem 3.

Proof of Theorem 3bis (assuming (III.53)).

Step 1: *Hodge–de Rham decomposition of* $u \times \nabla u$. Let $r_1 \in [r/4, r/2]$ be such that

$$r\int_{\check{S}_{r_1}} |\nabla u|^2 \leqslant 8\int_{\check{B}_r} |\nabla u|^2,$$
$$r\int_{\check{S}_{r_1}} (1-|u|^2)^2 \leqslant 8\int_{\check{B}_r} (1-|u|^2)^2, \tag{III.55}$$

where $\check{S}_r \equiv \check{S}_r(x_0) = \partial B_r(x_0) \cap \Omega$ and $\check{B}_r \equiv \check{B}_r(x_0) = B_r(x_0) \cap \mathbb{R}^N_+$. Likewise, consider

$$\Gamma_{r_1} = B_{r_1}(x_0) \cap \partial\Omega \subset \Gamma_r \subset \mathbb{R}^{N-1} \times \{0\},$$

so that

$$\partial \check{B}_{r_1}(x_0) = \check{S}_{r_1} \cup \Gamma_{r_1}, \qquad \check{S}_{r_1} \cap \Gamma_{r_1} = \emptyset.$$

(Note that $\Gamma_{r_1} = B^{N-1}_{r_2} \times \{0\}$, where $r_2^2 = r_1^2 - x_{0,N}^2$; in particular $r_2^2 \geqslant \frac{3}{4} r_1^2$.)

By (H2), $|g_\varepsilon| = 1$ on Γ_{r_1} and is smooth. Therefore we may write, on Γ_{r_1},

$$g_\varepsilon = \exp(i\Psi_\varepsilon),$$

where Ψ_ε is a smooth function defined on Γ_{r_1}, so that

$$g_\varepsilon \times \nabla g_\varepsilon = \nabla \Psi_\varepsilon \qquad \text{on } \Gamma_{r_1}, \tag{III.56}$$

where ∇ denotes the tangential gradient.

On the other hand, since $r_1 \leqslant r \leqslant d_0^2$ (by assumption (III.50)),

$$|\nabla g_\varepsilon| \leqslant \frac{C_0}{d} \leqslant \frac{C_0}{\sqrt{r_1}}.$$

Hence, since $|\nabla g_\varepsilon| = |\nabla \Psi_\varepsilon|$ on Γ_{r_1},

$$|\nabla \Psi_\varepsilon| \leqslant \frac{C_0}{\sqrt{r_1}} \qquad \text{on } \Gamma_{r_1}. \tag{III.57}$$

Next, we introduce the solution ξ defined on $\check{B}_{r_1}(x_0)$ of the elliptic problem

$$\begin{cases} \Delta \xi = 0 & \text{in } \check{B}_{r_1}(x_0) \\ \dfrac{\partial \xi}{\partial n} = u \times \dfrac{\partial u}{\partial n} & \text{on } \check{S}_{r_1}(x_0) \\ \xi = \Psi_\varepsilon & \text{on } \Gamma_{r_1}(x_0). \end{cases} \tag{III.58}$$

(The existence of ξ is standard). As in Step 1 of the proof of Theorem 3, we have

$$d^*([u \times du - d\xi]\, 1_{\check{B}_{r_1}}) = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^N_+). \tag{III.59}$$

We may now invoke Proposition A.8 of the Appendix, to assert the existence of a 2-form φ, defined on $\mathbb{R}^N_+$, such that $\varphi \in H^1_{loc}(\mathbb{R}^N_+)$,

$$d^*\varphi = [u \times du - d\xi]\, 1_{\check{B}_{r_1}} \qquad \text{in } \mathbb{R}^N_+, \tag{III.60}$$

$$d\varphi = 0 \qquad \text{in } \mathbb{R}^N_+, \tag{III.61}$$

$$\varphi_\top = 0 \qquad \text{on } \partial\mathbb{R}^N_+ = \mathbb{R}^{N-1} \times \{0\}, \tag{III.62}$$

$$\|\nabla\varphi\|_{L^p(R^N_+)} \leqslant K_p(\|\nabla u\|_{L^p(\check{B}_{r_1}(x_0))} + \|\nabla\xi\|_{L^p(\check{B}_{r_1}(x_0))}) \qquad \forall 1 < p < +\infty, \tag{III.63}$$

$$|\varphi(x)|\, |x|^{N-1} \text{ remains bounded as } |x| \to +\infty \qquad (|x| > r). \tag{III.64}$$

We have therefore

$$u \times du = d^*\varphi + d\xi \qquad \text{in } \check{B}_{r_1}(x_0). \tag{III.65}$$

In particular, since on Γ_{r_1}, $(u \times du)_\top = (g_\varepsilon \times dg_\varepsilon)_\top = (d\Psi)_\top = (d\xi)_\top$, we conclude

$$(d^*\varphi)_\top = 0 \qquad \text{on } \partial\mathbb{R}^N_+ = \mathbb{R}^N \times \{0\}. \tag{III.66}$$

Step 2: *Improved estimates for* $\nabla\varphi$ *on* $\check{B}_{\delta r}(x_0)$.

As in the proof of Theorem 3, we consider the function α defined on $\mathbb{R}^N_+$ by $\alpha(x) = (|f(|u(x)|)|)^2$ on $\check{B}_{r_1}$, and $\alpha(x) = 1$ outside. Then, we have

$$d(\alpha d^*\varphi) = \omega_1 + \omega_2 + \omega_3 \qquad \text{in } \mathscr{D}'(\mathbb{R}^N_+),$$

where

$$\begin{aligned}
\omega_1 &= 1_{\check{B}_{r_1}(x_0)} \sum_{i<j} 2(f(|u|)\, u)_{x_i} \times (f(|u|)\, u)_{x_j}\, dx_i \wedge dx_j, \\
\omega_2 &= \sigma_{\check{S}_{r_1}(x_0)}\, f(|u|)\, u \times du \wedge dr, \qquad (r = |x - x_0|), \\
\omega_3 &= d(1_{\check{B}_{r_1}(x_0)}(1-\alpha)\, d\xi) + \sigma_{\check{S}_{r_1}}\, dr \wedge d\xi - \sigma_{\Gamma_{r_1}}\, dx_N \wedge d\xi \\
&\equiv \omega_{3,1} + \omega_{3,2} + \omega_{3,3},
\end{aligned}$$

where σ stands for surface measure. Set also

$$\omega_4 = d((1-\alpha)\, d^*\varphi),$$

so that

$$\Delta\varphi = \omega_1 + \omega_2 + \omega_3 + \omega_4.$$

For $i = 1, 2, 3, 4$, consider the solution $\varphi_i \in H^1_{loc}(\mathbb{R}^N_+)$ of the problem

$$\begin{cases} \Delta\varphi_i = \omega_i & \text{in } \mathbb{R}^N_+ \\ (\varphi_i)_\top = 0, \quad (d^*\varphi_i)_\top = 0 & \text{on} \quad \partial\mathbb{R}^N_+ = \mathbb{R}^{N-1}\times\{0\} \\ |\varphi_i(x)| \to 0 & \text{as} \quad |x| \to +\infty. \end{cases} \tag{III.67}$$

Set $\Phi = \varphi - \sum_{i=1}^4 \varphi_i$. In view of the previous estimates,

$$\begin{aligned} &\Delta\Phi = 0 && \text{in } \mathbb{R}^N_+, \\ &(\Phi)_\top = 0, \quad (d^*\Phi)_\top = 0 && \text{on} \quad \partial\mathbb{R}^N_+ = \mathbb{R}^{N-1}\times\{0\}, \end{aligned}$$
$$|\Phi(x)|\,|x|^{N-1} \text{ remains bounded as } |x| \to +\infty,$$

so that, by Proposition A.3, $\Phi = 0$, i.e.

$$\varphi = \sum_{i=1}^4 \varphi_i\,. \tag{III.68}$$

We now proceed to estimate ξ, and then each φ_i separately.

Estimate for ξ. We claim that

$$\int_{\check{B}_{r_1}(x_0)} |\nabla\xi|^2 \leqslant C\left(\int_{\check{B}_{r_1}(x_0)} |\nabla u|^2 + r^{N-1}\right) \tag{III.69}$$

and that, for $0 < \delta < \frac{1}{4}$,

$$\int_{\check{B}_{\delta r}(x_0)} |\nabla\xi|^2 \leqslant C\delta^N\left(\int_{\check{B}_r(x_0)} |\nabla u|^2 + r^{N-1}\right). \tag{III.70}$$

Proof of (III.69) *and* (III.70). We may write $\xi = \xi_1 + \xi_2$, where ξ_1 is the solution of

$$\begin{cases} \Delta\xi_1 = 0 \quad \text{in } \check{B}_{r_1}(x_0) \\ \xi_1 = \Psi_\varepsilon \quad \text{on } \Gamma_{r_1}, \qquad \dfrac{\partial\xi_1}{\partial n} = 0 \quad \text{on } \check{S}_{r_1}, \end{cases}$$

and ξ_2 is the solution of

$$\begin{cases} \Delta\xi_2 = 0 \quad \text{in } \check{B}_{r_1}(x_0) \\ \xi_2 = 0 \qquad \text{on } \Gamma_{r_1}, \qquad \dfrac{\partial\xi_2}{\partial n} = u \times \dfrac{\partial u}{\partial n} \quad \text{on } \check{S}_{r_1}. \end{cases}$$

By standard estimates,

$$\|\nabla \xi_1\|_{L^\infty(\check{B}_r(x_0))} \leqslant \sup_{x \in \Gamma_{r_1}} |\nabla \Psi_\varepsilon|(x) \leqslant \frac{C}{\sqrt{r}},$$

so that for any $0 < \delta < 1$ such that $\delta r \leqslant r_1$,

$$\int_{\check{B}_{\delta r}(x_0)} |\nabla \xi_1|^2 \leqslant K \delta^N r^N \|\nabla \xi_1\|^2_{L^\infty} \leqslant C\, \delta^N r^{N-1}. \tag{III.71}$$

For ξ_2 we have, by (III.55),

$$\int_{\check{B}_{r_1}(x_0)} |\nabla \xi_2|^2 \leqslant K r_1 \int_{\check{S}_{r_1}(x_0)} |\nabla u|^2 \leqslant K \int_{\check{B}_{r_1}(x_0)} |\nabla u|^2, \tag{III.72}$$

and by standard elliptic estimates, for $0 < \delta < 1$ such that $\delta r \leqslant r_1$,

$$\begin{aligned} \int_{\check{B}_{\delta r}(x_0)} |\nabla \xi_2|^2 &\leqslant K \delta^N \int_{\check{B}_r(x_0)} |\nabla \xi|^2, \\ \int_{\check{B}_{\delta r}(x_0)} |\nabla \xi|^2 &\leqslant K \delta^N \int_{\check{B}_r(x_0)} |\nabla u|^2. \end{aligned} \tag{III.73}$$

Combining (III.71) and (III.72) we obtain (III.69). Likewise combining (III.71) and (III.73), we obtain (III.70).

Estimate for $\|\nabla \varphi\|_{L^2(\mathbb{R}^N_+)}$. In view of (III.63) (for $p = 2$) and (III.69), we are led to

$$\int_{\mathbb{R}^N_+} |\nabla \varphi|^2 \leqslant C \left(\int_{\check{B}_r(x_0)} |\nabla u|^2 + r^{N-1} \right). \tag{III.74}$$

Estimate for φ_4. We multiply the equation $-\Delta \varphi_4 = d((1-\alpha)\, d^* \varphi)$ by φ, and integrate on $\mathbb{R}^N_+$. Since $1 - \alpha = 0$ on $\partial \mathbb{R}^N_+$, integration by parts and computations similar to those for (III.25) and (III.74) yield

$$\int_{\mathbb{R}^N_+} |\nabla \varphi_4|^2 \leqslant C \beta^2 \left(\int_{\check{B}_r(x_0)} |\nabla u|^2 + r^{N-1} \right). \tag{III.75}$$

Estimate for $\varphi_{3,1}$. As in the proof of Theorem 3, we obtain, using (III.69),

$$\int_{\mathbb{R}^N_+} |\nabla \varphi_{3,1}|^2 \leqslant C \beta^2 \left(\int_{\check{B}_r(x_0)} |\nabla u|^2 + r^{N-1} \right). \tag{III.76}$$

Estimate for φ_2, $\varphi_{3,2}$, *and* $\varphi_{3,3}$. Using (III.69), we obtain as in the proof of Theorem 3, for $0<\delta<\frac{1}{4}$,

$$\int_{\check{B}_{\delta r}(x_0)} |\nabla\varphi_2|^2+|\nabla\varphi_{3,1}|^2+|\nabla\varphi_{3,3}|^2 \leqslant C\delta^N\left(\int_{\check{B}_r(x_0)} |\nabla u|^2+r^{N-1}\right). \tag{III.77}$$

Estimate for φ_1. We observe first that, as a consequence of (III.52), the same arguments as in the proof of Theorem 3 show that

$$|\omega_1|\leqslant C\,\frac{\beta^{-2}}{\varepsilon^2}\,(1-|u|^2)^2 \qquad \text{in } \check{B}_r(x_0). \tag{III.78}$$

Next, we claim that

$$\|\varphi_1\|_{L^\infty(\mathbb{R}^N_+)} \leqslant \frac{C}{\beta^2 r^{N-2}}\,(E_\varepsilon(x,r)+r^{N-1}). \tag{III.79}$$

Proof of (III.79). By Proposition A.3 of the Appendix, we have, for every $x\in\mathbb{R}^N_+$,

$$\begin{aligned}|\varphi_1(x)| &\leqslant \int_{\mathbb{R}^N_+} \frac{2c_N}{|x-y|}\,|\omega_1(y)|\,dy = \int_{\check{B}_{r_1}(x_0)} \frac{2c_N}{|x-y|}\,|\omega_1(y)|\,dy\\ &\leqslant \frac{C}{\beta^2}\int_{\check{B}_{r_1}(x_0)} \frac{(1-|u(y)|^2)^2}{\varepsilon^2\,|x-y|^{N-2}}\,dy.\end{aligned}$$

By the maximum principle, we deduce

$$\|\varphi_1\|_{L^\infty(\mathbb{R}^N_+)} = \sup_{x\in\check{B}_{r_1}(x_0)} |\varphi_1(x)|, \tag{III.80}$$

and, for $x\in\check{B}_{r_1}(x_0)$, $\check{B}_{r_1}(x_0)\subset B_{r/2}(x)$,

$$|\varphi_1(x)|\leqslant \frac{C}{\beta^2}\int_{\check{B}_{r/2}(x)} \frac{(1-|u(y)|^2)^2}{\varepsilon^2\,|x-y|^{N-2}}\,dy. \tag{III.81}$$

By inequality (II.5) of Corollary II.2, we have

$$\begin{aligned}\int_{\check{B}_{r/2}(x)} \frac{(1-|u(y)|^2)^2}{\varepsilon^2\,|x-y|^{N-2}}\,dy &\leqslant C\left(\tilde{E}_\varepsilon\left(x,\frac{r}{2}\right)+r\right)\\ &\leqslant C(\tilde{E}_\varepsilon(x_0,r)+r),\end{aligned} \tag{III.82}$$

where we have used, for the last inequality, the fact that $\check{B}_{r/2}(x)\subset\check{B}_r(x_0)$. Combining (III.80), (III.81), and (III.82) we deduce (III.79).

Finally, as in the proof of Theorem 3, we combine (III.78) and (III.79) to obtain

$$\int_{\mathbb{R}^N_+} |\nabla\varphi_1|^2 \leqslant \frac{C}{\beta^2}\left[\frac{1}{r^{N-2}}\int_{B_{r_1}(x_0)} \frac{(1-|u|^2)^2}{\varepsilon^2}\right](E_\varepsilon(x,r)+r^{N-1}). \quad \text{(III.83)}$$

Step 2 *completed.* Combining the estimates for φ_1, φ_2, $\varphi_{3,1}$, $\varphi_{3,2}$, $\varphi_{3,3}$, and φ_4 we are led to, for $0<\delta<1/4$ and $0<\beta<1/8$,

$$\int_{\check{B}_{\delta r}(x_0)} |\nabla\varphi|^2 \leqslant C(\beta^2+\delta^N)\left(\int_{\check{B}_r(x_0)} |\nabla u|^2+r^{N-1}\right) + C\beta^{-4}\left(\frac{1}{r^{N-2}}\int \frac{(1-|u|^2)^2}{\varepsilon^2}\right)(E_\varepsilon(x_0,r)+r^{N-1}). \quad \text{(III.84)}$$

Step 3: *Estimate for* $\nabla(|u|^2)$. The same argument as in the proof of Theorem 3 yields

$$\int_{\check{B}_{r_1}(x_0)} |\nabla(|u|^2)|^2 \leqslant C\left(\beta^2\int_{\check{B}_r(x_0)} |\nabla u|^2+\beta^{-2}\int_{\check{B}_r(x_0)} \frac{(1-|u|^2)^2}{\varepsilon^2}\right). \quad \text{(III.85)}$$

Step 4: Combining (III.84) and (III.85), we complete the proof of Theorem 3 bis as in the proof of Theorem 3.

Part C. Proof of Theorem 2bis *Completed when* $N\geqslant 3$.

The argument is similar to Part C of the proof of Theorem 2. In Part A, we found (provided $\varepsilon\leqslant\varepsilon_1$) an $r_0\in(\varepsilon^{1/2},\varepsilon^{1/4})$ such that (III.47) and (III.48) hold. Applying Theorem 3 bis to x_0 and $r=r_0$, we obtain

$$\begin{aligned} E_\varepsilon(x_0,\delta r_0) &\leqslant C\left(\frac{1}{r^{N-2}}\int_{\check{B}_{r_0}(x_0)} \frac{(1-|u|^2)^2}{\varepsilon^2}\right)^{1/3}(E_\varepsilon(x_0,r_0)+r_0^{N-1}) \\ &\quad + C\left(\frac{1}{r^{N-2}}\int_{\check{B}_{r_0}(x_0)} \frac{(1-|u|^2)^2}{\varepsilon^2}\right)^{2/3} + C\delta^N(E_\varepsilon(x_0,r_0)+r_0^{N-1}) \\ &\leqslant C(\eta\,|\log\delta|)^{1/3}(E_\varepsilon(x_0,r_0)+r_0^{N-1}) \\ &\quad + Cr_0^{N-2}(\eta\,|\log\delta|)^{2/3}+C\delta^N(E_\varepsilon(x_0,r_0)+r_0^{N-1}), \end{aligned}$$

where we have used (III.47). Hence, by (III.48),

$$\begin{aligned} \tilde{E}_\varepsilon(x_0,r_0) &\leqslant K\eta\,|\log\delta|+\tilde{E}_\varepsilon(x_0,\delta r_0) \\ &\leqslant K\eta\,|\log\delta|+K\delta^{2-N}(\eta\,|\log\delta|)^{2/3} \\ &\quad + C(\delta^{2-N}(\eta\,|\log\delta|)^{1/3}+\delta^2)(\tilde{E}_\varepsilon(x_0,r_0)+r_0), \end{aligned}$$

and therefore

$$\tilde{E}_\varepsilon(x_0, r_0)[1-C(\eta^{1/2}|\log\delta|^{1/3}\delta^{2-N}+\delta^2)] \leqslant K\delta^{2-N}(\eta|\log\delta|)^{2/3}+r_0 .$$

We choose $\delta=\eta^{\frac{1}{3N}}$. Then

$$\tilde{E}_\varepsilon(x_0, r_0)[1-C(\eta^{\frac{2}{3N}}|\log\eta|)] \leqslant K\eta^{\frac{N+2}{3N}}|\log\eta|^{2/3}+r_0 .$$

If $\eta\leqslant\eta_0$ (η_0 some constant), then, since $r_0\leqslant\varepsilon^{1/4}$,

$$\tilde{E}(x_0, r_0)\leqslant K\eta^{\frac{N+2}{3N}}|\log\eta|^{2/3}+r_0 .$$

Set $\varepsilon_0=\inf\{\varepsilon_1, K\eta^{\frac{N+2}{3N}}|\log\eta|^{2/3}\}$. If $\varepsilon<\varepsilon_0$, then

$$\tilde{E}(x_0, r_0)\leqslant K\eta^{\frac{N+2}{3N}}|\log\eta|^{2/3},$$

and we conclude as in the proof of Theorem 3.

IV. INTERIOR H^1 ESTIMATES IMPLY C^k BOUNDS

In this section, we assume that u_ε is a solution of

$$-\Delta u_\varepsilon=\frac{1}{\varepsilon^2}u_\varepsilon(1-|u_\varepsilon|^2) \qquad \text{in } B_1 \tag{IV.1}$$

and that

$$\int_{B_1}|\nabla u_\varepsilon|^2+\frac{(1-|u_\varepsilon|^2)^2}{\varepsilon^2}\leqslant L_0, \tag{IV.2}$$

for some (arbitrary) constant L_0 independent of ε. The goal is to control all the C^k norms of u_ε in $B_{1/2}$, independently of ε.

In the interesting situations where vorticity appears, estimate (IV.2) is *not* satisfied in all of Ω. However, away from a certain singular set $\mathscr{S}$, (IV.2) is valid, and u_ε is bounded in C^k away from $\mathscr{S}$. The main result in this Section, Theorem IV.1 below, will be used in Section VIII to establish C^k convergence of u_ε outside $\mathscr{S}$.

THEOREM IV.1. *Assume u_ε verifies* (IV.1) *and* (IV.2). *Then for every $k\in\mathbb{N}$,*

$$\|u_\varepsilon\|_{C^k(B_{1/2})}\leqslant C_k, \tag{IV.3}$$

$$\left\|\frac{1-|u_\varepsilon|^2}{\varepsilon^2}\right\|_{C^k(B_{1/2})}\leqslant C_k. \tag{IV.4}$$

Proof. Let $r_0 > r_1 > r_2 > r_3 \in (1/2, 7/8)$.

Step 1: $|u_\varepsilon| \to 1$ uniformly on $B_{7/8}$ as $\varepsilon \to 0$.

Proof. If $x \in B_{7/8}$ then $B(x, 1/8) \subset B_1$ and $E_\varepsilon(x, 1/8) \leqslant L_0$, i.e.

$$\tilde{E}_\varepsilon(x, 1/8) \leqslant 8^{N-2} L_0.$$

Therefore by Theorem 2

$$|u_\varepsilon(x)| \geqslant 1 - K\eta_\varepsilon^\alpha,$$

where K and α depend only on N, and

$$\eta_\varepsilon = \frac{8^{N-2} L_0}{|\log 8\varepsilon|}. \tag{IV.5}$$

This completes the proof of Step 1.

Step 2: δ-Energy decay.
Let $x \in B_{7/8}$ and $0 < r < 1/8$. Then, for any $0 \leqslant \delta \leqslant 1/2$,

$$\tilde{E}_\varepsilon(x, \delta r) \leqslant K(\delta^2 + \delta^{2-N}(\varepsilon + \eta_\varepsilon^\alpha))\, \tilde{E}_\varepsilon(x, r), \tag{IV.6}$$

where η_ε is defined in (IV.5). Here and below K denotes generic constants depending only on N. In particular, there exists $\delta_0 \in (0, 1/2)$, $\varepsilon_0 > 0$, such that for $\varepsilon < \varepsilon_0$, $\forall 0 < r < 1/8$,

$$\tilde{E}_\varepsilon(x, \delta_0 r) \leqslant \tfrac{1}{2} \tilde{E}_\varepsilon(x, r). \tag{IV.7}$$

Proof. We may always assume that ε is sufficiently small so that

$$|u_\varepsilon| \geqslant \tfrac{1}{2} \qquad \text{in } B_{7/8}.$$

We then write

$$u_\varepsilon = \rho_\varepsilon \exp(i\varphi_\varepsilon) \qquad \text{in } B_{7/8},$$

and we may assume moreover

$$\frac{1}{|B_{7/8}|} \int_{B_{7/8}} \varphi_\varepsilon \in [0, 2\pi).$$

First, we turn to the contribution of the phase φ_ε. We have

$$-\Delta\varphi_\varepsilon = -\operatorname{div}((1 - \rho_\varepsilon^2)\, \nabla\varphi_\varepsilon) \qquad \text{in } B(x, r) \subset B_{7/8}.$$

Let $\tilde{\varphi}_\varepsilon$ be the harmonic function defined on $B(x, r)$ verifying $\tilde{\varphi}_\varepsilon = \varphi_\varepsilon$ on $B(x, r)$. In particular, we have

$$\int_{B(x,r)} |\nabla \tilde{\varphi}_\varepsilon|^2 \leqslant \int_{B(x,r)} |\nabla \varphi_\varepsilon|^2 \tag{IV.8}$$

and

$$\int_{B(x,\delta r)} |\nabla \tilde{\varphi}_\varepsilon|^2 \leqslant K\delta^N \int_{B(x,r)} |\nabla \tilde{\varphi}_\varepsilon|^2 \leqslant K\delta^N \int_{B(x,r)} |\nabla \varphi_\varepsilon|^2. \tag{IV.9}$$

Multiplying the equation

$$-\Delta(\varphi_\varepsilon - \tilde{\varphi}_\varepsilon) = -\operatorname{div}((1-\rho_\varepsilon^2)\, \nabla \varphi_\varepsilon) \qquad \text{in } B(x, r)$$

by $\varphi_\varepsilon - \tilde{\varphi}_\varepsilon$, and integrating on $B(x, r)$, we obtain

$$\int_{B(x,\delta r)} |\nabla(\varphi_\varepsilon - \tilde{\varphi}_\varepsilon)|^2 \leqslant K\eta_\varepsilon^\alpha \int_{B(x,r)} |\nabla \varphi_\varepsilon|^2. \tag{IV.10}$$

Combining (IV.9) and (IV.10), we are led to

$$\int_{B(x,\delta r)} |\nabla \varphi_\varepsilon|^2 \leqslant K(\delta^N + \eta_\varepsilon^\alpha) \int_{B(x,r)} |\nabla u_\varepsilon|^2.$$

We turn to ρ_ε. The same computations as in the proof of Theorem 3, step 3, yield

$$\int_{B(x,r/2)} |\nabla \rho_\varepsilon|^2 + \frac{(1-\rho_\varepsilon^2)^2}{\varepsilon^2} \leqslant K(\varepsilon + \eta_\varepsilon^\alpha) \int_{B(x,r)} |\nabla u_\varepsilon|^2. \tag{IV.11}$$

Combining (IV.10) and (IV.11) we derive (IV.6), and (IV.7) follows from the fact that $\eta_\varepsilon \to 0$ as $\varepsilon \to 0$.

Step 3: There exists a constant $C > 0$, and $\theta_0 \in (0, 1)$ such that, for $\varepsilon < \varepsilon_0$,

$$\|u_\varepsilon\|_{C^{0,\theta_0}(B_{r_0})} \leqslant C. \tag{IV.12}$$

Proof. Iterating (V.7) with $r = 1/8$, we obtain, for $\varepsilon \leqslant \varepsilon_0$,

$$\tilde{E}_\varepsilon(x, \delta_0^k \tfrac{1}{8}) \leqslant (\tfrac{1}{2})^k\, \tilde{E}_\varepsilon(x, \tfrac{1}{8}) \leqslant C(\tfrac{1}{2})^k, \qquad \forall k \in \mathbb{N}.$$

For $r \in (0, 1/8)$, let $k \in \mathbb{N}$ be such that $\delta_0^{k+1} \leqslant 8r \leqslant \delta_0^k$, i.e.

$$k \leqslant \frac{|\log 8r|}{|\log \delta_0|} \leqslant k+1.$$

By monotonicity, we have, for $x \in B_{7/8}$,

$$\begin{aligned}\tilde{E}_\varepsilon(x, r) &\leqslant \tilde{E}_\varepsilon(x, \delta_0^k r) \leqslant C(\tfrac{1}{2})^{k+1} = C \exp((k+1) \log \tfrac{1}{2})\\ &\leqslant C \exp(\mu_0 \log(8r)) = C(8r)^{\mu_0}\\ &\leqslant C r^{\mu_0},\end{aligned}$$

where $\mu_0 = (\log \frac{1}{2})(\log \delta_0)^{-1} > 0$.

In particular, we have established that for some constants $C > 0$, for $\varepsilon < \varepsilon_0$, $\forall 0 < r < 1/8$,

$$\int_{B(x, r)} |\nabla u_\varepsilon|^2 \leqslant C r^{N-2+\mu_0}. \tag{IV.13}$$

In view of a classical theorem of Morrey (see, e.g., [Giaquinta]), (IV.13) implies (IV.12) with $\theta_0 = \mu_0/2$.

Step 4: There exists a constant $C > 0$ such that for $\varepsilon < \varepsilon_0$

$$\|u_\varepsilon\|_{C^{1, \theta_0}(B_{r_1})} \leqslant C. \tag{IV.14}$$

Proof. In view of (IV.12), we deduce that

$$\|\rho_\varepsilon^2\|_{C^{0, \theta_0}(B_{r_0})} \leqslant C.$$

Since φ_ε satisfies the equation

$$\operatorname{div}(\rho_\varepsilon^2 \nabla \varphi_\varepsilon) = 0 \qquad \text{in } B_{r_0},$$

which is uniformly elliptic with C^{0, θ_0} coefficient ρ_ε^2, it follows from Schauder theory (see for instance [Gilbarg-Trudinger], Theorem 8.3.2) that

$$\|\varphi_\varepsilon\|_{C^{1, \theta_0}(B_{r_1})} \leqslant C \|\varphi_\varepsilon\|_{C^{0, \theta_0}(B_{r_0})} \leqslant C,$$

by (IV.12).

From now on we proceed as in [Bethuel-Brezis-Hélein 1].

Step 5: We have

$$1 - \rho_\varepsilon^2 \leqslant C\varepsilon^2 \qquad \text{in } B_{r_2}.$$

Proof. Set $\xi_\varepsilon = 1 - \rho_\varepsilon$. Then we have

$$-\Delta \xi_\varepsilon + \frac{1}{\varepsilon^2} \rho_\varepsilon (1 + \rho_\varepsilon)\, \xi_\varepsilon = \rho_\varepsilon\, |\nabla \varphi_\varepsilon|^2 \qquad \text{in } B_1,$$

thus

$$-\Delta\xi_\varepsilon + \frac{1}{2\varepsilon^2}\xi_\varepsilon \leqslant C \qquad \text{in } B_{r_1}$$

by Step 4. Applying Lemma 2 in [Bethuel-Brezis-Hélein 1] we deduce that

$$\|\xi_\varepsilon\|_{L^\infty(B_{r_2})} \leqslant C\varepsilon^2.$$

Step 6: We have

$$\|\nabla\varphi_\varepsilon\|_{C^k_{loc}} \leqslant C,$$
$$\|1-\rho_\varepsilon\|_{C^k_{loc}} \leqslant C\varepsilon^2.$$

Proof. The proof is by induction on k. For $k=0$, this is a consequence of the estimates of Step 4 and 5. The passage from k to $k+1$ is done as in step B6 of Lemma 2 in [Bethuel-Brezis-Hélein 1].

COROLLARY IV.1. *Under the assumption of Theorem* IV.1, *we have, for some sequence* $\varepsilon_n \to 0$,

$$u_{\varepsilon_n} \to u_* = \exp(i\varphi_*) \qquad \textit{in } C^k_{loc}(B_1)$$

for every $k \in \mathbb{N}$, *where* φ_* *is some harmonic function. Moreover,*

$$\frac{1-|u_\varepsilon|^2}{\varepsilon^2} \to |\nabla u_*|^2 = |\nabla\varphi_*|^2 \qquad \textit{in } C^k_{loc}(B_1).$$

Remark IV.1. There is a version of Theorem IV.1 up to the boundary $\partial\Omega$ but *only* for $C^{1,\alpha}$ norms. C^2 convergence does not hold near the boundary since $\Delta u_\varepsilon = 0$ on $\partial\Omega$ away from Σ, while its limit satisfies $-\Delta u_* = u_*\,|\nabla u_*|^2$ on $\bar{\Omega}\setminus\mathcal{S}$, ($u_*$ and $\mathcal{S}$ are defined in Section VIII). This requires some work, the arguments are basically the same as Steps 1–4 above.

V. PROOF OF PROPOSITION 1

The proof of Proposition 1 relies in a crucial way on the subtle Besicovitch Covering Theorem. We give first a statement of this theorem, following the presentation of [Giaquinta-Modica-Souček], p. 30 (see also [Giaquinta], [Evans-Gariepy]).

THEOREM V.1 (Besicovitch Covering Theorem). *Let E be a subset of $\mathbb{R}^N$ and let $r: E \to \mathbb{R}$ be a positive bounded function defined on E. Then one can choose an at most countable family of points $\Lambda := \{x_i\}_{i \in \mathbb{N}}$ in E such that*

(i) $E \subset \bigcup_i \overline{B(x_i, r(x_i))}$

(ii) *The balls $\overline{B(x_i, \frac{1}{3} r(x_i))}$ are mutually disjoint*

(iii) *The balls $B(x_i, r(x_i))$, $x_i \in \Lambda$ can be distributed in $\zeta(N)$ families $\mathscr{B}_k$ of disjoint closed balls, where $\zeta(N)$ is a constant depending only on N.*

Next we turn back to our situation. For $0 < \varepsilon < 1$, let $\mu = \varepsilon^{1/8}$,

$$K_\mu = \{x \in \Omega;\ \mathrm{dist}(x, \Sigma) \geqslant \mu\},$$

and let E be the set

$$E = A_\beta \cap K_\mu = \{x \in \Omega;\ |u(x)| \leqslant 1 - \beta,\ \mathrm{dist}(x, \Sigma) > \varepsilon^{1/8}\}. \tag{V.1}$$

We apply Theorem V.1 to the set E and take as function r the function defined on K_μ in Proposition II.2.

Since E is bounded and

$$\varepsilon^{1/2} \leqslant r(x) \leqslant \varepsilon^{1/4}, \qquad \forall x \in E,$$

the family Λ is bounded. Applying Theorem V.1, we obtain

PROPOSITION V.1. *There exists a finite family $\Lambda = \{x_i\}_{1 \leqslant i \leqslant l}$, $l \in \mathbb{N}$, such that*

$$E = A_\beta \cap K_\mu \subset \bigcup_{i=1}^{l} B(x_i, r(x_i)), \qquad x_i \in E.$$

Moreover, the balls $B_i := B(x_i, r(x_i))$ can be distributed in $\zeta(N)$ families $\mathscr{B}_k$ of disjoint balls. We have therefore

$$E = A_\beta \cap K_\mu \subset \bigcup_{i=1}^{\zeta(N)} \left(\bigcup_{B_i \in \mathscr{B}_k} B_i \right). \tag{V.2}$$

We are now able to proceed to the proof of Proposition 1 (stated in the Introduction).

Proof of Proposition 1 *completed.* One proceeds in several steps:

Step 1: We have

$$\int_{A_\beta \cap K_\mu} \frac{(1 - |u|^2)^2}{\varepsilon^2} \leqslant C \left(\sum_{i=1}^{l} r_i^{N-2} \right), \tag{V.3}$$

where the constant C is independent on ε, and where $r_i = r(x_i)$.

Proof. We deduce from (V.2) that

$$\int_{A_\beta \cap K_\mu} \frac{(1-|u|^2)^2}{\varepsilon^2} \leqslant \left(\sum_{i=1}^{l} \int_{B(x_i, r_i)} \frac{(1-|u|^2)^2}{\varepsilon^2} \right).$$

In view of Proposition II.2 and the definition of r_i, we have

$$\int_{B(x_i, r_i)} \frac{(1-|u|^2)^2}{\varepsilon^2} \leqslant C r_i^{N-2},$$

and the conclusion follows.

Step 2: Let $x \in A_\beta \cap K_\mu$. Then for any $\varepsilon^{1/2} \leqslant r \leqslant \varepsilon^{1/4}$,

$$E_\varepsilon(x, r) \geqslant C\beta^{1/\alpha} r^{N-2} |\log \varepsilon|. \tag{V.4}$$

Proof. Let $\eta = r^{2-N} \frac{E_\varepsilon(x, r)}{|\log \varepsilon|}$. Since $x \in A_\beta$, $|u(x)| \leqslant 1-\beta$.
On the other hand, by Theorem 3 bis

$$1 - K\eta^\alpha \leqslant |u(x)| < 1-\beta,$$

hence

$$\beta \leqslant K\eta^\alpha,$$

i.e., $\eta \geqslant C\beta^{1/\alpha}$ and

$$r^{2-N} E_\varepsilon(x, r) \geqslant C\beta^{1/\alpha} \left| \log \frac{\varepsilon}{r} \right|.$$

Since $\varepsilon^{1/2} \leqslant r \leqslant \varepsilon^{1/4}$ the conclusion follows.

Step 3: *Upper bound for* $\sum_{i=1}^{l} r_i^{2-N}$. Since in each family $\mathscr{B}_k$, for $k=1$ to $\zeta(N)$, the balls B_i are disjoint we have

$$\int_{\bigcup_{B_i \in \mathscr{B}_k} B_i} e_\varepsilon(u) = \sum_{B_i \in \mathscr{B}_k} E_\varepsilon(x_i, r_i).$$

By step 2

$$E_\varepsilon(x_i, r_i) \geqslant C\beta^{1/\alpha} r_i^{N-2} |\log \varepsilon|,$$

and by (H1), for any $k = 1$ to $\zeta(N)$

$$\int_{\bigcup_{B_i \in \mathscr{B}_k} B_i} e_\varepsilon(u) \leqslant \int_\Omega e_\varepsilon(u) \leqslant M_0 |\log \varepsilon|.$$

Hence

$$C \sum_{B_i \in \mathscr{B}_k} \beta^{1/\alpha} r_i^{N-2} |\log \varepsilon| \leqslant M_0 |\log \varepsilon|,$$

so that for any k,

$$\sum_{B_i \in \mathscr{B}_k} \beta^{1/\alpha} r_i^{N-2} \leqslant C.$$

Thus, we obtain the upper bound

$$\sum_{i=1}^{l} r_i^{N-2} \leqslant C\zeta(N)\, \beta^{-1/\alpha}. \tag{V.5}$$

Step 4: We have

$$\int_{A_\beta \cap K_\mu} \frac{(1-|u|^2)^2}{\varepsilon^2} \leqslant C\beta^{-1/\alpha}. \tag{V.6}$$

Proof. Combine (V.3) and (V.5).

Step 5: We have, for $\Sigma_\mu = \Omega \setminus K_\mu$,

$$\int_{\Sigma_\mu} \frac{(1-|u|^2)^2}{\varepsilon^2} \leqslant C\varepsilon^{1/8} |\log \varepsilon| \to 0 \qquad \text{as} \quad \varepsilon \to 0. \tag{V.7}$$

Proof. For any $x \in \Omega \setminus K_\mu$, $\operatorname{dist}(x, \Sigma) \leqslant \varepsilon^{1/8}$. By standard covering, we may find points z_i, for $i = 1, \ldots, q_\varepsilon$, on Σ such that

$$B(z_i, \varepsilon^{1/8}) \cap B(z_j, \varepsilon^{1/8}) = \varnothing \qquad \text{if} \quad i \neq j, \tag{V.8}$$

$$\bigcup_{i=1}^{q_\varepsilon} B(z_i, 8\varepsilon^{1/8}) \supset K_\mu. \tag{V.9}$$

It follows from (V.8) that the number of points z_i, i.e. q_ε can be bounded as

$$0 \leqslant q_\varepsilon \leqslant Cr^{-(N-3)} \mathscr{H}^{N-3}(\Sigma), \qquad \text{where} \quad r = \varepsilon^{1/8}, \tag{V.10}$$

and from (V.9) we deduce that

$$\int_{K_\mu} \frac{(1-|u|^2)^2}{\varepsilon^2} \leqslant 4 \int_K e_\varepsilon(u) \leqslant C \sum_{i=1}^{q_\varepsilon} E_\varepsilon(z_i, 8r). \tag{V.11}$$

On the other hand, by Lemma II.4,

$$E_\varepsilon(z_i, 16r) \leqslant M_2 r^{N-2} |\log \varepsilon|.$$

Inserting this relation into (V.11), we obtain

$$\begin{aligned}\int_{K_\mu} \frac{(1-|u|^2)^2}{\varepsilon^2} &\leqslant C q_\varepsilon r^{N-2} \, |\log \varepsilon| \\ &\leqslant C r^{-N+3} r^{N-2} \, |\log \varepsilon|, \qquad \text{by (V.10)} \\ &\leqslant C r \, |\log \varepsilon| = C \varepsilon^{1/8} \, |\log \varepsilon|,\end{aligned}$$

which yields the result.

Step 6: *Proof of Proposition* 1 *completed.* Combining (V.6) with (V.7) we obtain (10) and complete the proof.

Remark V.1. Proposition 1 assumes that u_ε satisfies the boundary condition $u_\varepsilon = g_\varepsilon$ on $\partial\Omega$, with g_ε satisfying (H2). If we *drop* the boundary condition, there will be an interior version of Proposition 1.

PROPOSITION 1bis. *Let u_ε be a solution of*

$$-\Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) \qquad \textit{in } B_R .$$

For $\beta \in (\frac{1}{2}, 1)$, set

$$A_{\varepsilon, \beta} = \{x \in B_{\frac{R}{2}}, \, |u_\varepsilon(x)| \leqslant 1 - \beta\}.$$

Then

$$\int_{A_{\varepsilon, \beta}} \frac{(1-|u_\varepsilon|^2)^2}{\varepsilon^2} \leqslant C_\beta \left(\frac{E_\varepsilon(u_\varepsilon)}{|\log \frac{\varepsilon}{R}|} \right)^2,$$

where C_β depends only on β (and is independent of ε).

The proof of Proposition 1bis follows the same strategy as above—in fact, it is even easier because it does not involve any boundary condition.

As a consequence of Proposition 1bis, we may approach the boundary of Ω, and obtain the following

COROLLARY V.1. *Let u_ε be a solution of*

$$-\Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) \qquad \textit{in } \Omega.$$

Let $0 < \alpha < 1$, $\frac{1}{2} < \beta < 1$ and let

$$A_{\varepsilon, \alpha, \beta} = \{x \in \Omega, \, \mathit{dist}(x, \partial\Omega) \geqslant \varepsilon^\alpha, \, |u_\varepsilon(x)| \leqslant 1 - \beta\}.$$

Then

$$\int_{A_{\varepsilon,\alpha,\beta}} \frac{(1-|u_\varepsilon|^2)^2}{\varepsilon^2} \leqslant C_{\alpha,\beta}\left(\frac{E_\varepsilon(u_\varepsilon)}{|\log \varepsilon|}\right)^2,$$

where $C_{\alpha,\beta}$ depends on Ω, α, β but not on ε.

We emphasize once more that there is no assumption involving the boundary of Ω.

VI. GLOBAL ESTIMATES IN $W^{1,p}$, $1 \leqslant p < \frac{N}{N-1}$

VI.1. *Introduction*

In this section our first aim will be to establish for $1 \leqslant p < \frac{N}{N-1}$ the bound (7), that is

$$\int_\Omega |\nabla u_\varepsilon|^p \leqslant C_p,$$

for any solution u_ε to $(GL)_\varepsilon$ verifying (H1), (H2). As in [Bethuel-Brezis-Hélein 2], Section X, this is the main ingredient (together with η-ellipticity) in order to establish compactness properties of the set $\{u_\varepsilon\}_{0 \leqslant \varepsilon \leqslant 1}$ of solutions verifying (H1) and (H2).

As in Section III our starting point is once more the equation

$$d^*(u_\varepsilon \times du_\varepsilon) = 0 \qquad \text{in } \Omega, \tag{VI.1}$$

and again we will use extensively Hodge–de Rham decompositions. An important difference here is however that we must work on the whole domain (instead of small balls) and that the boundary conditions on $\partial\Omega$ will be used in a fundamental way (actually, they *must* to be used in order to establish (7), see remarks at the end of Section III).

Since our analysis involves many questions related to differential forms (some of them not so widely well known, in particular those concerning boundary conditions), we will collect in the Appendix some background material, which we will use in later investigation.

VI.2. *Linear Problems Associated to the Ginzburg–Landau System*

Let $u = u_\varepsilon$ be a solution of $(GL)_\varepsilon$ verifying (H1), (H2). In order to derive (7) we write, as in Section III

$$4\,|u|^2\,|\nabla u|^2 = 4\,|u \times \nabla u|^2 + |\nabla\,|u|^2|^2, \tag{VI.2}$$

and our first goal will be to prove for $1 \leqslant p < \frac{N}{N-1}$

$$\int_\Omega |u \times \nabla u|^p \leqslant C_p.$$

To that aim, we write first a first-order system of equations for the 1-form

$$\mu = u \times \nabla u = \sum_{i=1}^{N} u \times \frac{\partial u}{\partial x_i}\, dx_i.$$

VI.2.1. *An Elliptic System of First-Order Equations*

As in Section III, let $0 < \beta < 1/4$ to be determined later, and let $f\colon \mathbb{R}^+ \to [1, 1/(1-\beta)]$ be a function verifying (III.23), that is

$$\begin{cases} f(t) = \dfrac{1}{t} & \text{if} \quad t \geqslant 1-\beta \\ f(t) = 1 & \text{if} \quad t \leqslant 1-2\beta \\ |f'| \leqslant 4 & \text{for any} \quad t \in \mathbb{R}^+. \end{cases} \tag{VI.3}$$

In Ω, let α be the function defined by

$$\alpha(x) = f^2(|u(x)|) \qquad \text{in } \Omega, \tag{VI.4}$$

so that, as already observed,

$$1 \leqslant \alpha \leqslant 1 + 4\beta \qquad \text{in } \Omega. \tag{VI.5}$$

On Ω we consider also the 2-form ω defined by

$$d(\alpha u \times du) = \sum_{i<j} 2(f(|u|)\, u)_{x_i} \times (f(|u|)\, u)_{x_j}\, dx_i \wedge dx_j.$$

As in Section III we have

$$\omega(x) = 0 \qquad \text{if} \quad |u(x)| > 1-\beta \tag{VI.6}$$

and

$$|\omega(x)| \leqslant K\beta^{-2}\, \frac{(1-|u|^2)^2}{\varepsilon^2}. \tag{VI.7}$$

As an immediate, yet crucial consequence of Proposition 1, we obtain

COROLLARY VI.1. *We have*

$$\|\omega\|_{L^1} = \int_\Omega |\omega(x)| \leqslant K\beta^{-2} \equiv C_\beta. \tag{VI.8}$$

Remark that C_β is a constant depending only on β, Ω, etc., but not on ε.

At this stage we have now a system of two first-order equations for $\mu = u \times du$, namely

$$\begin{cases} d^*\mu = 0 & \text{in } \Omega \\ d(\alpha\mu) = \omega & \text{in } \Omega, \end{cases}$$

(compare with Corollary A.1 and the remarks thereafter), where ω is bounded in L^1 uniformly in ε. As we will see, this system, together with the boundary condition $\mu_\top = \alpha g \times dg$ on $\partial\Omega$ yields an L^p bound for μ. Perhaps the simplest way to derive this result is to use a Hodge–de Rham decomposition.

VI.2.2. *Hodge–de Rham Decomposition and Second Order Linear Elliptic Equations*

In view of Proposition A.7 of the Appendix (with $l = 1$), there is a function H defined on Ω and a 2-form Φ defined on Ω, such that

$$\alpha\mu = dH + d^*\Phi \qquad \text{in } \Omega, \tag{VI.9}$$

with

$$\begin{cases} d\Phi = 0 & \text{in } \Omega \\ H = 0 & \text{on } \partial\Omega \\ \Phi_\top = 0 & \text{on } \partial\Omega. \end{cases} \tag{VI.10}$$

We will show next that Φ verifies an elliptic equation involving ω.

LEMMA VI.1. *We have*

$$\begin{cases} -\Delta\Phi = \omega & \textit{in } \Omega \\ \Phi_\top = 0 & \textit{on } \partial\Omega \\ (d^*\Phi)_\top = A & \textit{on } \partial\Omega, \end{cases} \tag{VI.11}$$

where A is the 1*-form defined on $\partial\Omega$ as*

$$A = \alpha g \times dg. \tag{VI.12}$$

Proof. Since $-\Delta = dd^* + d^*d$, and since $d\Phi = 0$ on Ω, it follows

$$\begin{aligned} -\Delta\Phi &= dd^*\Phi + d^*d\Phi = dd^*\Phi \\ &= d(\alpha\mu - dH) = d(\alpha\mu) = \omega, \end{aligned}$$

so that we obtain the first equation.

Since $H = 0$ on $\partial\Omega$, $(dH)_\top = 0$ on $\partial\Omega$ and

$$(\alpha\mu)_\top = (dH)_\top + (d^*\Phi)_\top = (d^*\Phi)_\top \qquad \text{on } \partial\Omega,$$

hence the third relation is established.

Remark VI.1. Note that at this stage we have not used the equation for u, and in particular equation VI.11 for Φ holds for any map $u\colon \Omega \to \mathbb{R}^2$ (provided it is sufficiently regular). Similar Hodge–de Rham decompositions will turn out to be useful for instance in evolution problems, as we will show in forthcoming works. The time coordinate, in that case, must be included with the space coordinates.

Remark VI.2. The boundary data g enters directly in the definition of Φ.

Remark VI.3. In the case where $|u| \geqslant 1 - \beta$ on Ω, then we have $\Delta\Phi = 0$ on Ω, that is

$$dd^*\Phi = 0 \qquad \text{in } \Omega.$$

Applying Hodge–de Rham decomposition to $d^*\Phi$ we may assert that there exists some function φ, defined on Ω such that

$$d\varphi = d^*\Phi.$$

We then verify easiliy that for some constant C

$$u(x) = |u| \exp(i\varphi + C). \tag{VI.13}$$

In the general case, however, u must vanish, and ω is not zero and represents, through Eq. (VI.11), the obstruction (in particular the topological obstruction) to the lifting property (VI.13).

Remark VI.4. If $N = 2$ and $g_\varepsilon = g$ on $\partial\Omega$ (with $|g| = 1$) then Φ becomes

$$\begin{aligned} &\Phi = \Phi_{12}\, dx_1 \wedge dx_2 \\ &\omega = 2(f(|u|)\, u)_{x_1} \times (f(|u|)\, u)_{x_2}\, dx_1 \wedge dx_2\,. \end{aligned}$$

In this case the condition $\Phi_{\mathrm{T}} = 0$ is automatically verified and (VI.11) becomes

$$-\Delta\Phi_{12} = 2(f(|u|)\,u)_{x_1} \times (f(|u|)\,u)_{x_2} \qquad \text{in } \Omega, \tag{VI.14}$$

$$\frac{\partial\Phi_{12}}{\partial n} = g \times g_\tau \qquad \text{on } \partial\Omega. \tag{VI.15}$$

This type of equation and, in particular, the boundary condition (VI.15) have been studied and used extensively in [Bethuel-Brezis-Hélein 2]. This shows in particular that the equation (VI.11) for the 2-*form* Φ is the natural extension to higher dimension of the elliptic problems involving the *function* Φ_{12} in case $N = 2$.

Next we derive the equation for the function H.

LEMMA VI.2. *The function H verifies the elliptic equation, with Dirichlet boundary conditions*

$$\begin{cases} \operatorname{div}(\alpha^{-1}\nabla H) = d^*((1-\alpha^{-1})\,d^*\Phi) & \text{in } \Omega \\ H = 0 & \text{on } \partial\Omega. \end{cases} \tag{VI.16}$$

Proof. First note that by (VI.5)

$$0 < \frac{1}{1+4\beta} \leqslant \alpha^{-1} \leqslant 1, \tag{VI.17}$$

so that we may multiply (VI.9) to assert

$$u \times du = \alpha^{-1}\,dH + \alpha^{-1}\,d^*\Phi.$$

Hence,

$$\begin{aligned} d^*(u \times du) &= d^*(\alpha^{-1}\,dH) + d^*(\alpha^{-1}\,d^*\Phi) \\ &= d^*(\alpha^{-1}\,dH) + d^*((\alpha^{-1} - 1)\,d^*\Phi), \end{aligned}$$

and the conclusion follows by Eq. (VI.1).

Remark VI.5. Note that here Eq. $(GL)_\varepsilon$ (and its consequence (VI.1)) enters in a crucial way in the derivation of (VI.16). Together with Proposition 1, it will actually be the only place where it will be used in order to bound $\|u \times du\|_{L^p}$ for $1 \leqslant p < \frac{N}{N-1}$.

VI.2.3. *Bounds on $\|\nabla H\|_{L^p}$, $\|\nabla\Phi\|_{L^p}$, for $1 \leqslant p < \frac{N}{N-1}$*

Since Φ enters directly in the equations (VI.16) for H, whereas the equation for Φ involves only ω and g, we will begin with estimates for Φ.

PROPOSITION VI.1. *We have, for* $1 \leqslant p < \frac{N}{N-1}$,

$$\|\nabla \Phi\|_{L^p} \leqslant C_{\beta, p},$$

where $C_{\beta, p}$ *is a constant depending on* β, p, Ω, g *but independent on* ε.

Proof. In view of Proposition A.2 of the Appendix, we have

$$\|\nabla \Phi\|_{L^p} \leqslant C_p(\|\omega\|_{L^1} + \|\alpha g \times dg\|_{L^1}),$$

and the conclusion follows from Corollary VI.1, together with the estimate

$$\|\alpha g \times dg\|_{L^1} \leqslant C,$$

which is an easy consequence of (H2).

We next turn to the function H. This is the stage where we must make a suitable choice for the parameter β.

PROPOSITION VI.2. *Let* $1 \leqslant p < \frac{N}{N-1}$. *There is some* $0 < \beta_0 < 1/4$ *(depending possibly only on* p *and* Ω*) and a constant* $C > 0$ *(depending possibly on* p *and* Ω *but not on* β_0*) such that, for* $0 < \beta < \beta_0$

$$\|\nabla H\|_{L^p(\Omega)} \leqslant C|\beta| \, \|\nabla \Phi\|_{L^p(\Omega)} \leqslant C_{\beta, p}.$$

Proof. We write the operator $\operatorname{div}(\alpha^{-1}\nabla)$ as a perturbation of the Laplacian, that is

$$\operatorname{div}(\alpha^{-1}\nabla H) = \Delta H + \operatorname{div}((\alpha^{-1} - 1)\, \nabla H) \qquad \text{in } \Omega.$$

Let Δ_0^{-1} denote the inverse of the Laplacian with Dirichlet boundary condition on Ω, i.e.,

$$\begin{aligned} \Delta_0^{-1} : W^{-1,p}(\Omega) &\to W_0^{1,p}(\Omega) \\ f &\mapsto \Delta_0^{-1} f = w, \end{aligned}$$

where w is the unique solution in $W^{1,p}(\Omega)$ of

$$\begin{cases} \Delta w = f & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $1 < p < +\infty$, standard elliptic estimates show that Δ_0^{-1} is a continuous linear operator from $W^{-1,p}(\Omega)$ to $W_0^{1,p}(\Omega)$. Set

$$\Psi = d^*((1 - \alpha^{-1})\, d^*\Phi).$$

By (VI.5) we have

$$|\alpha^{-1}-1| \leqslant 4\beta, \tag{VI.18}$$

so that, for some constant K,

$$\|\Psi\|_{W^{-1,p}(\Omega)} \leqslant K\beta \, \|\nabla\Phi\|_{L^p(\Omega)}.$$

We now write (VI.16) as

$$\begin{cases} \Delta H = \operatorname{div}(1-\alpha^{-1}) \, \nabla H + g & \text{in } \Omega \\ H = 0 & \text{on } \partial\Omega, \end{cases}$$

which can be reformulated as

$$H = \Delta_0^{-1}(\operatorname{div}(1-\alpha^{-1}) \, \nabla H) + \gamma, \tag{VI.19}$$

where $\gamma \in W_0^{1,p}(\Omega)$ is given by

$$\gamma = \Delta_0^{-1}(\Psi).$$

Finally, we define the linear operator $T: W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ by

$$Tv = \Delta_0^{-1}(\operatorname{div}(1-\alpha^{-1}) \, \nabla v),$$

for any $v \in W_0^{1,p}(\Omega)$. Clearly T is continuous. Moreover, by (VI.18) we have

$$\|T\| \leqslant K\beta. \tag{VI.20}$$

Equation (VI.19) can now be written as

$$(\mathrm{Id}-T) \, H = \gamma,$$

where Id denotes the identity map on $W_0^{1,p}(\Omega)$. We choose β_0 so small that

$$K\beta_0 \leqslant \tfrac{1}{2},$$

where K is the constant in (VI.20). Therefore, for $\beta \leqslant \beta_0$, $\mathrm{Id}-T$ is invertible and

$$\begin{aligned} \|H\|_{W^{1,p}} &\leqslant \|(\mathrm{Id}-T)^{-1}\| \, \|\gamma\|_{W^{1,p}} \\ &\leqslant K \, \|\Psi\|_{W^{-1,p}} \leqslant K\beta \, \|\nabla\Phi\|_{L^p}, \end{aligned}$$

which completes the proof.

VI.2.4. *L^p Estimates for $|u \times du|$, $1 \leqslant p < \frac{N}{N-1}$*

Let $1 < p < \frac{N}{N-1}$ be given, and choose $\beta = \beta_0$, where β_0 is the constant introduced in the proof of Proposition VI.6. With this choice of β, the function α is also determined. Since

$$u \times du = \alpha^{-1}\, dH + \alpha^{-1}\, d^*\Phi,$$

we deduce from the results of previous section,

PROPOSITION VI.3. *Let $1 \leqslant p < \frac{N}{N-1}$ be given. There is a constant K_p, depending only on Ω, p, C_0 and K_0 such that for any solution u_ε of $(GL)_\varepsilon$ verifying* (H1), (H2) *we have*

$$\int_\Omega |u_\varepsilon \times du_\varepsilon|^p \leqslant K_p.$$

VI.3. *Estimates for $|\nabla |u_\varepsilon||$, $1 \leqslant p < 2$*

We follow here closely the argument of [Bethuel-Brezis-Hélein 2], Lemma X.13. Let $1 \leqslant p < 2$ and set

$$\rho = |u_\varepsilon|.$$

The equation for ρ is

$$-\Delta\rho + \frac{1}{\rho^3}\,|u \times du|^2 = \frac{1}{\varepsilon^2}\,\rho(1-\rho^2) \qquad \text{in } \Omega. \tag{VI.21}$$

We are going to prove

PROPOSITION VI.4. *Let $1 \leqslant p < 2$. There exists a constant K_p and $0 < \alpha < 1$ depending only on p, Ω, K_0, C_0 such that, for $0 < \varepsilon < 1$,*

$$\int_\Omega |\nabla\rho|^p \leqslant K_0\varepsilon^\alpha.$$

Proof. We introduce the set

$$S = \{x \in \Omega,\ \rho(x) > 1 - \varepsilon^{1/2}\}$$

and the function

$$\bar{\rho} = \max\{\rho,\ 1 - \rho^{1/2}\},$$

so that $\rho = \bar{\rho}$ on S and

$$0 \leqslant 1 - \bar{\rho} \leqslant \varepsilon^{1/2} \qquad \text{in } \Omega. \tag{VI.22}$$

We multiply (VI.21) by $\bar{\rho} - 1$ and integrate over Ω:

$$\int_\Omega \nabla\rho\, \nabla\bar{\rho} + \int_\Omega \frac{\rho(1-\rho^2)(1-\bar{\rho})}{\varepsilon^2} = \int_\Omega \frac{1-\bar{\rho}}{\rho^3}\, |u \times du|^2 + \int_{\partial\Omega} \frac{\partial u}{\partial n}\,(1-\bar{\rho}),$$

$$\int_{\Omega \setminus S} |\nabla\rho|^2 \leqslant \varepsilon^{1/2} \int_\Omega \frac{1}{\rho^3}\, |u \times \nabla u|^2 + \int_{\partial\Omega} |\nabla\, |u||(1-\bar{\rho}).$$

From assumption (H1) we deduce that

$$\int_\Omega \frac{1}{\rho^3}\, |u \times \nabla u|^2 \leqslant K\, |\log \varepsilon|,$$

whereas it follows from (H2) that $1 - \bar{\rho}(x) = 0$ if $x \in \partial\Omega$, $\operatorname{dist}(x, \Sigma) \geqslant \varepsilon$, so that

$$\begin{aligned}\int_{\partial\Omega} |\nabla\, |u||\, (1-\bar{\rho}) &\leqslant \varepsilon^{1/2} \int_{\partial\Omega \,\cap\, \operatorname{dist}(x,\Sigma) \leqslant \varepsilon} |\nabla\, |u|| \\ &\leqslant K\varepsilon^{1/2} \operatorname{meas}\{x \in \partial\Omega,\ \operatorname{dist}(x, \Sigma) \leqslant \varepsilon\} \leqslant K\varepsilon^{2N-2/2} \leqslant K.\end{aligned}$$

Here we have used the fact that

$$|\nabla u|\ (x) \leqslant \frac{K}{\varepsilon} \qquad \forall x \in \bar{\Omega}.$$

Combining the previous relations we obtain

$$\int_S |\nabla\rho|^2 \leqslant K\varepsilon^{1/2}\, |\log \varepsilon|. \tag{VI.23}$$

Finally, since by (H1)

$$\int_\Omega (1-\rho^2)^2 \leqslant K\varepsilon^2\, |\log \varepsilon|$$

and since $(1-\rho^2) \geqslant \varepsilon^{1/2}$ on $\Omega \setminus S$, we obtain

$$|\Omega \setminus S| \leqslant K\varepsilon\, |\log \varepsilon|.$$

Hence

$$\begin{aligned}\int_{\Omega\setminus S} |\nabla\rho|^p &\leqslant \left(\int_\Omega |\nabla\rho|^2\right)^{p/2} |\omega\setminus S|^{1-p/2}\\ &\leqslant K\,|\log\varepsilon|^{p/2}\,|\Omega\setminus S|^{1-p/2}\\ &\leqslant K\varepsilon^{1-p/2}\,|\log\varepsilon|. \end{aligned} \tag{VI.25}$$

Combining (VI.24) with (VI.25) we deduce the result.

VI.4. *Proof of* (7) *Completed*

Since we have now proved L^p bounds for the gradient of the phase as well as the gradient of the modulus, it suffices to combine the two estimates to bound $\|\nabla u\|_{L^p}$. More precisely, we have

$$|u|\,|\nabla u| \leqslant |u\times\nabla u| + |\nabla\rho|$$

We distinguish the cases $|u|\geqslant 1/2$ and $|u|\leqslant 1/2$. Recall that

$$A_{1/2} = \{x\in\Omega,\ |u|(x)\leqslant \tfrac{1}{2}\},$$

so that

$$|\nabla u| \leqslant 2\,|u\times\nabla u| + |\nabla\rho| \qquad \text{in } \Omega\setminus A_{1/2}.$$

Hence, for $1\leqslant p<\frac{N}{N-1}$

$$\int_{\Omega\setminus A_{1/2}} |\nabla u|^p \leqslant C_p(\|u\times\nabla u\|^p_{L^p(\Omega)} + \|\nabla\rho\|^p_{L^p(\Omega)}) \leqslant K_p, \tag{VI.26}$$

by Propositions VI.3 and VI.4, where K_p depends only on p, Ω, K_0, C_0. On the other hand we have, by the bound

$$\begin{gathered}|\nabla u| \leqslant \frac{K}{\varepsilon},\\ \int_{A_{1/2}} |\nabla u|^p \leqslant K\varepsilon^{-p}\int_{A_{1/2}} (1-|u|^2)^2 \leqslant K\varepsilon^{2-p},\end{gathered} \tag{VI.27}$$

where we have used Proposition 1 for the last inequality.

Combining (VI.26) and (VI.27) we deduce

$$\int_\Omega |\nabla u_\varepsilon|^p \leqslant K_p, \qquad \text{for} \quad 1 \leqslant p < \frac{N}{N-1},$$

that is, (7) is established.

VII. η-REGULARITY

We first recall the main η-ellipticity assertion in Section III: if u_ε is a solution of

$$-\Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) \qquad \text{in } B_R \tag{VII.1}$$

and if

$$\tilde{E}_\varepsilon(u_\varepsilon, R) \leqslant \eta_0 \left| \log \frac{\varepsilon}{R} \right|, \tag{VII.2}$$

with η_0 sufficiently small (depending only on N), then

$$|u_\varepsilon(x)| \geqslant \tfrac{1}{2} \qquad \text{in } B_{R/2}. \tag{VII.3}$$

We emphasize once more that no boundary condition is assumed. We also recall that in the absence of restrictions on the boundary condition, there is no hope to infer from (VII.1) and (VII.2) (even with small η_0) any compactness for u_ε, e.g. in L^1 (see the example in [Brezis-Mironescu] which is also described in Remark III.4).

In the previous section, we established global $W^{1,p}$ estimates, $1 \leqslant p < \frac{N}{N-1}$, assuming $u_\varepsilon = g_\varepsilon$ on $\partial\Omega$, where g_ε satisfies (H2). Here we will show how to gain further regularity (uniformly in ε) in the region where (VII.2) holds with η_0 sufficiently small. This gain is established in two steps. First, we prove that (VII.2) with small η_0 implies

$$\tilde{E}_\varepsilon\left(u_\varepsilon, \frac{3R}{4}\right) \leqslant C, \qquad \text{independently of } \varepsilon, \tag{VII.4}$$

and (VII.4) combined with the analysis in Section IV yields

$$\|\nabla u_\varepsilon\|_{C^{0,\alpha}(B(x_0, R/2))} \leqslant C,$$

and even

$$\|u_\varepsilon\|_{C^k(B(x_0, R/2))} \leqslant C,$$

if $B(x_0, R)$ does not intersect the boundary $\partial\Omega$.

Here is the main result in this section:

PROPOSITION VII.1. *There exists constants $\eta_0 > 0$, $R_2 > 0$ depending possibly on Ω, Σ, M_0, C_0, but not on ε, such that if u_ε is a solution of* $(GL)_\varepsilon$ *verifying* (H1) *and* (H2), *and if $x_0 \in \bar{\Omega}$ and $R > 0$ are such that*

$$0 < R < R_2 \, dist^2(x_0, \Sigma), \tag{VII.5}$$

and

$$\tilde{E}_\varepsilon(x_0, R) \leqslant \eta_0 \left|\log \frac{\varepsilon}{R}\right|, \tag{VII.6}$$

then, for $\varepsilon < R/16$,

$$|u_\varepsilon(x)| \geqslant \frac{1}{2}, \qquad \forall x \in \check{B}\left(x_0, \frac{3R}{4}\right), \tag{VII.7}$$

and

$$\tilde{E}_\varepsilon\left(x_0, \frac{R}{2}\right) \leqslant C(x_0, R), \tag{VII.8}$$

where $C(x_0, R)$ is a constant depending on x_0, R, Ω, Σ, M_0 and C_0, but not on ε.

Proof. Let K and α be the constants in Theorem 2 bis. Let $\eta_1 > 0$ be such that

$$1 - K\eta_1^\alpha = \tfrac{1}{2}.$$

The constant R_2 is choosen in such a way that, if x_0 and R satisfy (VII.5), then

$$\frac{R}{4} < \inf\{R_1, \operatorname{dist}^2(x, \Sigma)\}, \tag{VII.9}$$

for every $x \in \check{B}(x_0, \frac{3R}{4})$, where R_1 is the constant in Lemma II.3.

If $x \in B(x_0, \frac{3R}{4})$, then $B(x, \frac{R}{4}) \subset B(x_0, R)$ and $E_\varepsilon(x, \frac{R}{4}) \leqslant E_\varepsilon(x_0, R)$, i.e.

$$\tilde{E}_\varepsilon\left(x, \frac{R}{4}\right) \leqslant 4^{N-2} \tilde{E}_\varepsilon(x_0, R).$$

If u_ε verifies (VII.6), it follows that

$$\begin{aligned} \tilde{E}_\varepsilon\left(x, \frac{R}{4}\right) &\leqslant 4^{N-2} \eta_0 \left|\log \frac{\varepsilon}{R}\right| \leqslant 4^{N-2} \eta_0 \left(\left|\log \frac{4\varepsilon}{R}\right| + |\log 4|\right) \\ &\leqslant 2 \cdot 4^{N-2} \eta_0 \left(\left|\log \frac{4\varepsilon}{R}\right|\right), \end{aligned} \tag{VII.10}$$

since $\varepsilon < \frac{R}{16}$. Choose $\eta_1 = 2 \cdot 4^{N-2} \eta_0$, so that (VII.10) yields

$$\tilde{E}_\varepsilon\left(x, \frac{R}{4}\right) \leqslant \eta_1 \left|\log \frac{4\varepsilon}{R}\right|.$$

In view of (VII.9), we may now invoke Theorem 2 bis to assert that

$$|u_\varepsilon(x)| \geqslant 1 - K\eta_1^\alpha = \tfrac{1}{2},$$

i.e. (VII.7) is established.

We turn next to (VII.8). Since $|u_\varepsilon| \geqslant 1/2$ on $\check{B}(x_0, \frac{3R}{4})$, we may write

$$u_\varepsilon = \rho_\varepsilon \exp(i\varphi_\varepsilon) \qquad \text{in } \check{B}\left(x_0, \frac{3R}{4}\right),$$

where $\rho_\varepsilon = |u_\varepsilon|$, and where the real-valued function φ_ε is defined on $\check{B}(x_0, \frac{3R}{4})$ up to an integer multiple of 2π. We may therefore impose the additional condition

$$\frac{1}{|\check{B}\,(x_0, \frac{3R}{4})|} \int_{\check{B}(x_0, \frac{3R}{4})} \varphi_\varepsilon \in [0, 2\pi). \tag{VII.11}$$

By (7) in the Introduction we have, on the other hand,

$$\int_{\check{B}(x_0, \frac{3R}{4})} |\nabla \varphi_\varepsilon|^p + |\nabla \rho_\varepsilon|^p \leqslant C_p \qquad \forall 1 \leqslant p < \frac{N}{N-1}, \tag{VII.12}$$

where C_p does not depend on ε. As already seen, φ_ε verifies the equation

$$\operatorname{div}(\rho_\varepsilon^2\, \nabla \varphi_\varepsilon) = 0 \qquad \text{in } \check{B}\left(x_0, \frac{3R}{4}\right), \tag{VII.13}$$

which is uniformly elliptic on $\check{B}(x_0, \frac{3R}{4})$, since $\rho_\varepsilon^2 \geqslant 1/4$. We may now invoke standard elliptic theory to assert, in view of (VII.11), (VII.12), and the fact that

$$|\nabla_\top \varphi_\varepsilon(x)| \leqslant C(x_0, R) \qquad \text{on } B\left(x_0, \frac{3R}{4}\right) \cap \partial\Omega,$$

by (H2), that we have the stronger H^1 bound on the smaller domain $\check{B}(x_0, \frac{5R}{8})$,

$$\int_{\check{B}(x_0, \frac{5R}{8})} |\nabla \varphi_\varepsilon|^2 \leqslant C(x_0, R), \tag{VII.14}$$

where $C(x_0, R)$ denotes a constant depending on x_0, R, Ω, Σ, M_0, C_0, but not on ε.

Finally, we turn to ρ_ε. recall that ρ_ε verifies the equation

$$-\Delta\rho_\varepsilon + \rho_\varepsilon\, |\nabla\varphi_\varepsilon|^2 = \frac{1}{\varepsilon^2}\, \rho_\varepsilon(1-\rho_\varepsilon^2). \tag{VII.15}$$

Let ζ be a smooth function on $\mathbb{R}^N$ such that $\operatorname{supp} \zeta \subset B(x_0, \frac{5R}{8})$, $\zeta \geqslant 0$, and $\zeta \equiv 1$ on $\check{B}(x_0, \frac{R}{2})$. We multiply equation (VII.15) by $(1-\rho_\varepsilon)\,\zeta$ and integrate on $\check{B}(x_0, \frac{5R}{8})$ (note that $1-\rho_\varepsilon \equiv 0$ on $\partial\Omega \cap B(x_0, \frac{5R}{8})$). We obtain

$$\begin{aligned}
&\int_{\check{B}(x_0, \frac{5R}{8})} \left[|\nabla\rho_\varepsilon|^2 + \frac{\rho_\varepsilon(1-\rho_\varepsilon^2)}{\varepsilon^2}\,(1+\rho) \right] \zeta \\
&\qquad \leqslant \int_{\check{B}(x_0, \frac{5R}{8})} [|\nabla\varphi_\varepsilon|^2\, \zeta + |\nabla\rho_\varepsilon|\, |\nabla\zeta|](1-\rho_\varepsilon).
\end{aligned} \tag{VII.16}$$

Therefore we deduce, by (VII.14) and (VII.12),

$$\int_{\check{B}(x_0, \frac{R}{2})} |\nabla\rho_\varepsilon|^2 + \frac{(1-\rho_\varepsilon^2)^2}{\varepsilon^2} \leqslant C(x_0, R). \tag{VII.17}$$

Combining (VII.17) and (VII.14) we obtain (VII.8).

VIII. CONVERGENCE OUTSIDE THE SINGULAR SET $\mathscr{S}$

VIII.1. *Extraction of Subsequences: u_* and μ_* Are Born!*

In the previous sections, we established the following bound, for solutions u_ε of $(GL)_\varepsilon$ verifying (H1) and (H2):

$$\int_\Omega |\nabla u_\varepsilon|^p \leqslant C_p, \qquad \text{for any} \quad 1 \leqslant p < \frac{N}{N-1}. \tag{VIII.1}$$

We emphasize once more that the constant C_p depends on p, Ω, Σ, C_0 and M_0, but is independent of ε and u_ε. By assumption (H1), we have the obvious bound

$$\int_\Omega \frac{e_\varepsilon(u_\varepsilon)}{|\log \varepsilon|} \leqslant M_0, \tag{VIII.2}$$

whereas, by (H2), we have

$$\int_{\partial\Omega} |\nabla_\top g_\varepsilon|^p \leqslant C_p, \qquad \text{for any} \quad 1 \leqslant p < \frac{N}{N-1}. \tag{VIII.3}$$

Set

$$\mu_\varepsilon = \frac{e_\varepsilon(u_\varepsilon)}{|\log \varepsilon|}.$$

In view of inequalities (VIII.1) to (VIII.3), given any sequence $\varepsilon_n \to 0$, we may extract a subsequence (still denoted ε_n), such that

$$u_{\varepsilon_n} \rightharpoonup u_* \quad \text{in } W^{1,p}(\Omega), \qquad \text{for any} \quad 1 \leqslant p < \frac{N}{N-1}; \tag{VIII.4}$$

$$g_{\varepsilon_n} \rightharpoonup g_* \quad \text{in } W^{1,p}(\partial\Omega), \qquad \text{for any} \quad 1 \leqslant p < \frac{N}{N-1}, \tag{VIII.5}$$

and the convergence is in $C^1(K)$ for any compact subset K of $\partial\Omega \setminus \Sigma$;

$$\mu_{\varepsilon_n} \rightharpoonup \mu_* \qquad \text{in the weak } * \text{ topology of} \quad \mathscr{M}(\bar{\Omega}) = [C(\bar{\Omega})]^*, \tag{VIII.6}$$

in other words

$$\int_\Omega \mu_{\varepsilon_n}(x)\, \zeta(x) \to \int_\Omega \mu_* \zeta, \qquad \forall \zeta \in C(\bar{\Omega}).$$

Set $\mathscr{S} = \operatorname{supp} \mu_*$.

In the above convergences, u_* denotes a map belonging to $W^{1,p}(\Omega, S^1)$ (i.e. $|u_*| = 1$ a.e.), for any $1 \leqslant p < \frac{N}{N-1}$, g_* belongs to $W^{1,p}(\partial\Omega, S^1) \cap C^1(\partial\Omega \setminus \Sigma)$, for any $1 \leqslant p < \frac{N}{N-1}$ and μ_* is a bounded positive measure on $\bar{\Omega}$. Note that by the trace theorem,

$$u_* = g_* \qquad \text{on } \partial\Omega, \tag{VIII.7}$$

and passing to the limit in the equation

$$\operatorname{div}(u_{\varepsilon_n} \times \nabla u_{\varepsilon_n}) = 0 \qquad \text{in } \Omega,$$

we deduce

$$\operatorname{div}(u_* \times \nabla u_*) = 0 \qquad \text{in } \Omega. \tag{VIII.8}$$

VIII.2. *$\mathscr{S}$ has $N-2$ Hausdorff Dimension*

The main result of this subsection is

THEOREM VIII.1. *We have*

$$\mathscr{H}^{N-2}(\mathscr{S}) < +\infty. \tag{VIII.9}$$

The main ingredient in the proof is the following

PROPOSITION VIII.1. *Let $x_0 \in \bar{\Omega}$, $R > 0$, such that $0 < R < R_2 \operatorname{dist}^2(x_0, \Sigma)$. Assume*

$$\mu_*(\check{B}(x_0, R)) < \eta_0 R^{N-2}, \tag{VIII.10}$$

where η_0 and R_2 are defined in Proposition VII.1. *Then we have*

$$\mu_*\left(\check{B}\left(x_0, \frac{R}{2}\right)\right) = 0, \qquad \text{i.e. } \check{B}\left(x_0, \frac{R}{2}\right) \subset \bar{\Omega} \setminus \mathscr{S} = \bar{\Omega} \setminus \operatorname{supp} \mu_*. \tag{VIII.11}$$

Proof. It follows from (VIII.10) that there exists some $n_0 \in \mathbb{N}$ such that for $n \geqslant n_0$

$$\tilde{E}_{\varepsilon_n}(x_0, R) \leqslant \eta_0 \left|\log \frac{\varepsilon_n}{R}\right|, \tag{VIII.12}$$

and therefore by Proposition VI.1 we obtain (for $n \geqslant n_0$)

$$\tilde{E}_{\varepsilon_n}(x_0, R) \leqslant C(x_0, R). \tag{VIII.13}$$

Dividing (VIII.13) by $|\log \varepsilon_n|$, (VIII.11) follows.

Proof of Theorem VIII.1 *completed.* We must prove that there is a constant $C > 0$ such that for any $\delta > 0$, $\mathscr{S}$ is covered by a finite number ν_δ of balls of radius δ, such that

$$\nu_\delta \cdot \delta^{N-2} \leqslant C. \tag{VIII.14}$$

Let $\{x_i\}_{i\in I}$ be a family of points in $\bar{\Omega}$ such that

$$B\left(x_i, \frac{\delta}{2}\right) \cap B\left(x_j, \frac{\delta}{2}\right) = \varnothing \qquad \text{for} \quad i \neq j, \tag{VIII.15}$$

$$\bar{\Omega} \subset \bigcup_{i\in I} B(x_i, \delta). \tag{VIII.16}$$

In view of (VIII.15), we have

$$\sharp I \leqslant C\delta^N, \tag{VIII.17}$$

where C depends on Ω. Next let $J \subset I$ be defined by

$$J = \{i \in I,\ \mu_*(B(x_i, 2\delta)) \geqslant \eta_0 \delta^{N-2}\},$$

where η_0 is defined in Proposition VII.1. Since by (H1) $\mu_*(\bar{\Omega}) \leqslant M_0$ and since a point in Ω may belong to at most K different balls $B(x_i, 2\delta)$ (where $K \in \mathbb{N}$ is a constant depending only on N), we have

$$\sharp J \leqslant K M_0 \delta^{2-N}. \tag{VIII.18}$$

On the other hand we claim that

$$\operatorname{supp} \mu_* \subset \bigcup_{i\in J} B(x_i, \delta). \tag{VIII.19}$$

Indeed, if $i \notin J$, then $\mu_*(B(x_i, 2\delta)) < \eta_0 \delta^{N-2}$ and therefore by Proposition VIII.1, we obtain $\mu_*(\check{B}(x_i, \delta)) = 0$, so that

$$\check{B}(x_i, \delta) \subset \bar{\Omega} \setminus \operatorname{supp} \mu_*, \qquad \text{for} \quad i \notin J,$$

and (VIII.19) follows.

Combining (VIII.18) and (VIII.19), we obtain (VIII.14) with $\nu_\delta = \sharp J$ and the proof of Theorem VIII.1 is completed.

We close this subsection with an additional remark which will be crucial in Section IX.

PROPOSITION VIII.2. *For $x \in \Omega$, the function*

$$r \mapsto \frac{\mu_*(B(x, r))}{r^{N-2}}$$

is nondecreasing. Set

$$\Theta_*(x) = \lim_{r\to 0^+} \frac{\mu_*(B(x, r))}{r^{N-2}}.$$

Then

$$\Theta_*(x) \geqslant \eta_0 \qquad \text{for any} \quad x \in \mathscr{S} \cap \Omega.$$

Proof. The first assertion is a straightforward consequence of the interior monotonicity formula, whereas the second assertion follows directly from Proposition VIII.1.

VIII.3. *Convergence on $\Omega \setminus \mathscr{S}$*

In this subsection we prove

THEOREM VIII.2. *$u_* \in C^\infty(\Omega \setminus \mathscr{S})$ and for every ball $B(x, R) \subset \Omega \setminus \mathscr{S}$ we have $u_* = \exp i\varphi_*$, where φ_* is harmonic.*

Moreover, for every compact subset K of Ω,

$$u_{\varepsilon_n} \to u_* \qquad \text{in } C^k(K), \quad \forall k \in \mathbb{N},$$

$$\frac{1-|u_{\varepsilon_n}|^2}{\varepsilon_n^2} \to |\nabla u_*|^2 \qquad \text{in } C^k(K), \quad \forall k \in \mathbb{N}.$$

Proof. Let $x \in \Omega$ and $\delta > 0$ such that $B(x, 2\delta) \subset \Omega \setminus \mathscr{S}$ and $2\delta < R_2 \operatorname{dist}^2(x, \Sigma)$. Then $\mu_*(B(x, 2\delta)) = 0$ and by Proposition VII.1

$$E_{\varepsilon_n}(x, \delta) \leqslant C(x, \delta) \qquad \forall n \in \mathbb{N},$$

where $C(x, \delta)$ is independent of ε. By Corollary IV.1,

$$u_* = \exp i\varphi_* \qquad \text{in } B(x, \delta),$$

where φ_* is harmonic, and

$$u_{\varepsilon_n} \to u_* \qquad \text{in } C^k_{loc}(B(x, \delta)), \quad \forall k \in \mathbb{N},$$

$$\frac{1-|u_{\varepsilon_n}|^2}{\varepsilon_n^2} \to |\nabla u_*|^2 \qquad \text{in } C^k_{loc}(B(x, \delta)), \quad \forall k \in \mathbb{N}.$$

The conclusion follows by a covering argument.

IX. PROPERTIES OF $\mathscr{S}$ AND μ_*

We finally complete the proof of Theorem 1 by the following result (see, e.g., [Simon] for definitions of rectifiable set and stationary varifold).

THEOREM IX.1. *$\mathscr{S}$ is a countably $\mathscr{H}^{N-2}$-rectifiable set, and μ_* is a stationary varifold in Ω.*

Remark IX.1. To be more precise, the statement of Theorem IX.1 means: $\mathscr{S}$ is rectifiable, and the varifold $V_* = V(\mathscr{S}, \Theta_*)$ (see [Simon], Chapter 4) is stationary.

Proof. Both statements are an immediate and direct consequence of the analysis carried out in [Ambrosio-Soner] in the parabolic case (any solution of $(GL)_\varepsilon$ is of course a stationary solution of the corresponding parabolic equation). In their proof, they made use of an additional assumption on Θ_* (Condition (1.4) there). This assumption is precisely the one established in Proposition VIII.2, and is therefore verified by solutions of $(GL)_\varepsilon$ satisfying (H1) and (H2).

Comment. Here we sketch some of the main ideas in the proof in [Ambrosio-Soner], applied to the elliptic case considered here.

The starting point is the identity

$$\int_\Omega \left(e_\varepsilon(u_\varepsilon)\, \delta_{ij} - \frac{\partial u_\varepsilon}{\partial x_i} \cdot \frac{\partial u_\varepsilon}{\partial x_j} \right) \frac{\partial X^i}{\partial x_j} = 0, \qquad \forall X \in [C_c^1(\Omega)]^N. \tag{IX.1}$$

This classical identity (see, e.g., [Hélein]) expresses the fact that the stress energy tensor field for solutions u_ε to the Ginzburg–Landau equation is divergence free.

Set

$$\alpha_{ij,\varepsilon} = \frac{1}{|\log \varepsilon|} \left(e_\varepsilon(u_\varepsilon)\, \delta_{ij} - \frac{\partial u_\varepsilon}{\partial x_i} \cdot \frac{\partial u_\varepsilon}{\partial x_j} \right).$$

Note that $\alpha_{ij,\varepsilon}$ is a symmetric matrix with trace larger than $(N-2)\,\mu_\varepsilon$, and a little linear algebra shows that its eigenvalues are less or equal to μ_ε. Moreover,

$$|\alpha_{ij,\varepsilon}| \leqslant N \mu_\varepsilon. \tag{IX.2}$$

Extracting possibly a further subsequence from ε_n we may then assert that

$$\alpha_{ij,\varepsilon_n} \to \alpha_{ij,*} \text{ in the sense of measures.}$$

In view of (IX.2) we have $|\alpha_{ij,*}| \leqslant N\mu_*$, therefore we may write

$$\alpha_{ij,*}(x) = A_{ij}(x)\, \mu_* \qquad \text{for } \mu_* \text{ } a.e.\ x \in \Omega,$$

where the matrix $A_{ij}(x)$ is symmetric, with trace larger than $N-2$ and eigenvalues less or equal to one. Passing to the limit in (IX.1) we obtain

$$\int_\Omega A_{ij}(x)\frac{\partial X^i}{\partial x_j}\,d\mu_*(x)=0 \qquad \forall X\in[C_c^1(\Omega)]^N. \tag{IX.3}$$

The crucial point is to show that the matrix $A_{ij}(x)$ represents the orthogonal projection on a $(N-2)$-dimensional subspace P_x of $\mathbb{R}^N$. By a blow-up argument (see Theorem 3.8c), Lemma 3.9 and Remark 3.10 in [Ambrosio-Soner]), using the fact that $\Theta_*(x)\geqslant\eta_0$ on $\mathscr{S}$, one concludes first that $A_{ij}(x)$ has at least two eigenvalues equal to zero. Then, since as already observed the trace of $A_{ij}(x)$ has to be larger than $N-2$ and its eigenvalues do not exceed 1, one deduces that $A_{ij}(x)$ has precisely $N-2$ eigenvalues equal to 1 and two eigenvalues equal to zero. This means that $A_{ij}(x)$ represent the orthogonal projection on the space P_x spanned by the eigenvectors corresponding to the eigenvalue 1. Hence P_x and μ_* define a varifold V_*. Formula (IX.3) asserts precisely that the first variation of V_* vanishes, i.e. V_* is stationary.

Next we invoke a classical theorem in [Allard] (see also [Simon], Theorem 42.4) which asserts that a varifold having locally bounded first variation (in particular, a stationary varifold) and positive density is rectifiable. In our case the positive density of V_* follows directly from Proposition VIII.2. Therefore $\mathscr{S}$ is rectifiable, P_x is the approximate tangent plane to $\mathscr{S}$ at x, and $V_*=V(\mathscr{S},\Theta_*)$.

Remark IX.2. Stationary varifolds are an extension of the notion of minimal surfaces. However, unlike minimal surfaces, they might have rather singular behavior. It is generally conjectured that for an m-dimensional stationary varifold the singular set has null $\mathscr{H}^m$ measure.

Remark IX.3. In case u_ε is minimizing for the Ginzburg–Landau energy it is proved that Θ_* is an integer multiple of π and $\mathscr{S}$ is area-minimizing (see [Lin-Rivière 1], [Sandier], [Jerrard-Soner], [Alberti-Baldo-Orlandi]).

APPENDIX

A. *Elliptic Problems Involving Differential Forms*

A.1. *Basic Definitions*

We will follow here essentially the presentation of [Iwaniec, Scott, Stroffolini] and [Giaquinta-Modica-Souček]. Let $N\in\mathbb{N}$ and $l\in\mathbb{N}$, we denote by $\Lambda^l \mathbb{R}^N$ the set of l-covectors on $\mathbb{R}^N$. If $I = (i_1, \ldots, i_l)$ is an ordered l-uple, $1 \leqslant i_1 < i_2 < \cdots < i_l \leqslant N$, we set

$$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_l}.$$

The set $\Lambda(l, N)$ of all distinct l-uples I yields a basis of $\Lambda^l \mathbb{R}^N$, so that we may write an element a of $\Lambda^l \mathbb{R}^N$ as

$$a = \sum_{I \in \Lambda(l, N)} a_I \, dx_I$$

where $a_I \in \mathbb{R}$. A canonical scalar product on $\Lambda^l \mathbb{R}^N$ is defined as

$$\langle a, b \rangle = \sum_{I \in \Lambda(l, N)} a_I b_I.$$

Recall also that the Hodge star operator $\bigstar$

$$\bigstar : \Lambda^l \mathbb{R}^N \to \Lambda^{N-l} \mathbb{R}^N$$

is the linear operator defined, for $a \in \Lambda^l \mathbb{R}^N$ by

$$a \wedge \varphi = \langle \bigstar a, \varphi \rangle \, dx_1 \wedge \cdots \wedge dx_N,$$

for any $\varphi \in \Lambda^{N-l} \mathbb{R}^N$. In particular

$$\bigstar 1 = dx_1 \wedge \cdots \wedge dx_N \qquad \bigstar \, dx_1 \wedge \cdots \wedge dx_N = 1$$

and

$$\bigstar\bigstar = (-1)^{l(N-l)} \, \mathrm{Id}_{\Lambda^l \mathbb{R}^N}. \tag{A.1}$$

Since for a and b in $\Lambda^l \mathbb{R}^N$, $\langle a, b \rangle = \langle \bigstar a, \bigstar b \rangle$ it follows that

$$a \wedge \bigstar b = \langle a, b \rangle \, dx_1 \wedge \cdots \wedge dx_N.$$

We turn now to differential forms. Let Ω be a smooth domain in $\mathbb{R}^N$. A differential l-form on Ω is a distribution on Ω with values in $\Lambda^l \mathbb{R}^N$. Therefore, every l-form ω on Ω may be written as

$$\omega(x) = \sum_{I \in \Lambda(l, N)} \omega_I(x) \, dx_I, \tag{A.2}$$

where the coefficients ω_I are distributions in $\mathscr{D}'(\Omega)$. We will denote $\mathscr{D}'(\Lambda^l \Omega)$ the set of l-forms. Similarly we denote $C^\infty(\Lambda^l \bar{\Omega})$ (respectively, $C_c^\infty(\Lambda^l \Omega)$) the set of l-forms with smooth (respectively, smooth with compact support in Ω) coefficients in $\bar{\Omega}$.

If α, β are two l-forms in $C_c^\infty(\Lambda^l\Omega)$we set

$$\langle \alpha, \beta\rangle = \int_\Omega \langle \alpha(x), \beta(x)\rangle \, dx_1 \cdots dx_N = \int_\Omega \alpha \wedge \star\beta. \tag{A.3}$$

Clearly $\langle\ ,\ \rangle$ defines a scalar product on $C_c^\infty(\Lambda^l\Omega)$ and we extend its definition by density to various situations, for instance $\alpha \in C_c^\infty(\Lambda^l\Omega)$, $\beta \in \mathscr{D}'(\Lambda^l\Omega)$...

With these notations the exterior differential d: $\mathscr{D}'(\Lambda^l\Omega) \to \mathscr{D}'(\Lambda^{l+1}\Omega)$ is expressed, if ω is given by (A.2), by

$$d\omega(x) = \sum_{k=1}^{N} \left(\sum_{1 \leqslant i_1 < \cdots < i_l \leqslant N} \frac{\partial \omega_{i_1 \cdots i_l}}{\partial x_k} \, dx_k \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_l} \right).$$

If $\omega_1 \in \mathscr{D}'(\Lambda^{l_1}\Omega)$ and $\omega_2 \in C_c^\infty(\Lambda^{l_2}\Omega)$, then we have the Leibnitz rule

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{l_1}\, \omega_1 \wedge d\omega_2 . \tag{A.4}$$

The formal adjoint of d, for the scalar product given by (A.3), is the Hodge-codifferential d^*, defined by

$$d^* = (-1)^{Nl+1} \star d \star \colon \mathscr{D}'(\Lambda^{l+1}\Omega) \to \mathscr{D}'(\Lambda^l\Omega),$$

so that if $\alpha \in C_c^\infty(\Lambda^l \mathbb{R}^N)$, $\beta \in C_c^\infty(\Lambda^{l+1}\mathbb{R}^N)$

$$\langle d\alpha, \beta\rangle = \langle \alpha, d^*\beta\rangle. \tag{A.5}$$

The operators d and d^* enjoy the important properties

$$d \circ d = 0, \qquad d^* \circ d^* = 0.$$

The Laplace operator for forms Δ is defined as

$$-\Delta = dd^* + d^*d \colon \mathscr{D}'(\Lambda^l\Omega) \to \mathscr{D}'(\Lambda^l\Omega).$$

If ω is given by (A.2), then Δ expresses in cartesian coordinates

$$\Delta\omega = \sum_{1 \leqslant i_1 < \cdots < \leqslant i_l \leqslant N} \Delta\omega_{i_1 \cdots i_l} \, dx_{i_1} \wedge \cdots \wedge dx_{i_l} . \tag{A.6}$$

Finally, we define the gradient $\nabla\omega$ of an l-form ω by

$$\nabla\omega = \left(\frac{\partial\omega}{\partial x_1}, \ldots, \frac{\partial\omega}{\partial x_N} \right) \in [\mathscr{D}'(\Lambda^l\Omega)]^N,$$

where for $i = 1, \ldots, N$

$$\frac{\partial \omega}{\partial x_i} = \sum_{1 \leqslant i_1 < \cdots < i_l \leqslant N} \frac{\partial \omega_{i_1 \cdots i_l}}{\partial x_i} \, dx_{i_1} \wedge \cdots \wedge dx_{i_l} .$$

If α and β are in $C^\infty(\Lambda^l \bar{\Omega})$ we set

$$\langle \nabla \alpha, \nabla \beta \rangle = \sum_{i=1}^{N} \left\langle \frac{\partial \alpha}{\partial x_i}, \frac{\partial \beta}{\partial x_i} \right\rangle .$$

If α and β belong to $C_c^\infty(\Lambda^l \Omega)$ we deduce from (A.5) that

$$-\langle \Delta \alpha, \beta \rangle = \langle d\alpha, d\beta \rangle + \langle d^*\alpha, d^*\beta \rangle,$$

whereas we deduce from (A.6) that

$$-\langle \Delta \alpha, \beta \rangle = -\sum_I \langle \Delta \alpha_I, \beta_I \rangle = \sum_I \langle \nabla \alpha_I, \nabla \beta_I \rangle = \langle \nabla \alpha, \nabla \beta \rangle .$$

Hence, we obtain

$$\langle d\alpha, d\beta \rangle + \langle d^*\alpha, d^*\beta \rangle = \langle \nabla \alpha, \nabla \beta \rangle . \tag{A.7}$$

A.2. *Restrictions to the Boundary*

Since Ω is assumed smooth, near every point $x_0 \in \partial\Omega$, we may construct a local system of coordinates $(\tilde{x}_1, \ldots, \tilde{x}_N)$ such that $\tilde{x}_N = 0$ on $\partial\Omega$ and such that the curves $\{\tilde{x}_i = c_i, i = 1, \ldots, N-1\}$ are orthogonal to $\partial\Omega$. Every differential form $\omega \in C^\infty(\Lambda^l \bar{\Omega})$ can therefore be written, in a neighborhood of x_0, as

$$\omega = \sum_{1 \leqslant i_1 < \cdots < i_l \leqslant N} \tilde{\omega}_{i_1 \cdots i_l} \, d\tilde{x}_{i_1} \wedge \cdots \wedge d\tilde{x}_{i_l} .$$

We decompose ω (in the neighborhood of x_0) as

$$\omega = \omega_\top + \omega_N ,$$

where

$$\omega_\top(x) = \sum_{1 \leqslant i_1 < \cdots < i_l < N} \tilde{\omega}_{i_1 \cdots i_l} \, d\tilde{x}_{i_1} \wedge \cdots \wedge d\tilde{x}_{i_l}$$

$$\omega_N(x) = \sum_{1 \leqslant i_1 < \cdots < i_l = N} \tilde{\omega}_{i_1 \cdots i_{l-1}, N} \, d\tilde{x}_{i_1} \wedge \cdots \wedge d\tilde{x}_N .$$

This decomposition does not depend on the specific choice of coordinates $(\tilde{x}_1, \ldots, \tilde{x}_N)$. Note that on $\partial\Omega$

$$(d\omega)_\top = d(\omega_\top),$$

and in particular, if $\omega_\top = 0$, then $(d\omega)_\top = 0$.

Orienting the surface $\partial\Omega$ according to the outward normal, Stokes formula (i.e., integration by parts) gives

$$\int_\Omega d\omega = \int_{\partial\Omega} \omega_\top . \tag{A.8}$$

Next, we set

$$C^\infty_\top(\Lambda^l\bar{\Omega}) = \{\omega \in C^\infty(\Lambda^l\bar{\Omega}),\ \omega_\top = 0 \text{ on } \partial\Omega\}.$$

Using integration by parts one may prove that relation (A.6) extends to forms in $C^\infty_\top(\Lambda^l\bar{\Omega})$, that is

LEMMA A.1. *Let α and β be two l-forms in $C^\infty_\top(\Lambda^l\bar{\Omega})$. Then we have the identity*

$$\langle d\alpha, d\beta\rangle + \langle d^*\alpha, d^*\beta\rangle = \langle \nabla\alpha, \nabla\beta\rangle. \tag{A.9}$$

A straightforward, yet important, consequence of Lemma A.1 is the following:

COROLLARY A.1. *Let $\omega \in C^\infty(\Lambda^l\bar{\Omega})$ such that ω verifies the system of first-order equations*

$$d\omega = 0 \quad \text{in } \Omega \tag{A.10}$$

$$d^*\omega = 0 \quad \text{in } \Omega \tag{A.11}$$

$$\omega_\top = 0 \quad \text{on } \partial\Omega.$$

Then $\omega = 0$ in Ω.

Remark A.1. In the case $N = 2$, $l = 1$ we have $\omega = \omega_1\, dx_1 + \omega_2\, dx_2$ and (A.10), (A.11) become

$$\frac{\partial\omega_2}{\partial x_1} - \frac{\partial\omega_1}{\partial x_2} = 0, \qquad \frac{\partial\omega_1}{\partial x_1} + \frac{\partial\omega_2}{\partial x_2} = 0.$$

We recognize above the Cauchy–Riemann relation for $\tilde{\omega} = \omega_1 - i\omega_2$, that is the previous relations can be rephrased as

$$\frac{\partial \tilde{\omega}}{\partial \bar{z}} = 0,$$

i.e., $\tilde{\omega}$ is holomorphic.

Remark A.2. In analogy with the previous situation, solutions to (A.10) and (A.11) are called *harmonic forms*. Note that a harmonic form ω verifies $\Delta\omega = 0$, but the converse is not necessarily true, i.e. there are forms verifying $\Delta\omega = 0$, but not (A.10) or (A.11).

A.3. *Sobolev Spaces*

Let $1 < p < +\infty$. Let $\omega \in \mathcal{D}'(\Lambda^l\Omega)$ be an integrable form. For $x \in \Omega$, set

$$|\omega(x)|^2 = \langle \omega(x), \omega(x) \rangle = \sum_I \omega_I^2(x),$$

and let

$$L^p(\Lambda^l\Omega) = \{\omega \in \mathcal{D}'(\Lambda^l\Omega), \text{ s.t. } |\omega| \in L^p(\Omega)\},$$

equipped with the L^p norm

$$\|\omega\|_{L^p} = \left(\int_\Omega |\omega|^p\right)^{\frac{1}{p}}.$$

Similarly, we define the Sobolev space $W^{1,p}(\Lambda^l\Omega)$

$$W^{1,p}(\Lambda^l\Omega) = \{\omega \in L^p(\Lambda^l\Omega), \nabla\omega \in [L^p(\Lambda^l\Omega)]^N\},$$

equipped with the norm

$$\|\omega\|_{W^{1,p}}^p = \|\omega\|_{L^p}^p + \|\nabla\omega\|_{L^p}^p.$$

Finally, we set

$$W_{\mathrm{T}}^{1,p}(\Lambda^l\Omega) = \{\omega \in W^{1,p}(\Lambda^l\Omega), \omega_{\mathrm{T}} = 0\}.$$

The above spaces are all Banach spaces. For $p = 2$ they are Hilbert spaces, in particular $W^{1,2}(\Lambda^l\Omega)$ (respectively, $W_{\mathrm{T}}^{1,2}(\Lambda^l\Omega)$) is a Hilbert space for the scalar product

$$\langle \alpha, \beta \rangle_1 = \langle \alpha, \beta \rangle + \langle \nabla\alpha, \nabla\beta \rangle.$$

The standard density results of smooth functions in Sobolev spaces extend to Sobolev spaces for forms, in particular we have

LEMMA A.2. *$C_c^\infty(\Lambda^l\Omega)$ (resp. $C_\top^\infty(\Lambda^l\bar{\Omega})$) is dense in $W^{1,p}(\Lambda^l\Omega)$ (resp. $W_\top^{1,p}(\Lambda^l\Omega)$).*

Likewise, Poincaré inequality holds for the space $W_\top^{1,p}(\Lambda^l\Omega)$. We have

LEMMA A.3. *If Ω is bounded and smooth, then there exists a constant C_p, depending only on Ω and p, such that for any $\omega \in W_\top^{1,p}(\Lambda^l\Omega)$,*

$$\|\omega\|_{L^p} \leqslant C_p \|\nabla\omega\|_{L^p}. \tag{A.13}$$

In particular, $\| \ \|_{\top,1,p}$ defined by

$$\|\omega\|_{\top,1,p} = \|\nabla\omega\|_{L^p} \qquad \forall\omega \in W_\top^{1,p}(\Lambda^l\Omega)$$

defines a norm on $W_\top^{1,p}(\Lambda^l\Omega)$ which is equivalent to $\| \ \|_{1,p}$.

In the case $p=2$, since $C_\top^\infty(\Lambda^l\bar{\Omega})$ is dense in $W_\top^{1,2}(\Lambda^l\Omega)$, (A.9) still holds for α and β in $W_\top^{1,2}(\Lambda^l\Omega)$ and therefore

$$\|\nabla\alpha\|_{L^2}^2 = \|d\alpha\|_{L^2}^2 + \|d^*\alpha\|_{L^2}^2, \qquad \forall\alpha \in W_\top^{1,2}(\Lambda^l\Omega). \tag{A.14}$$

A.4. *A Second Order Elliptic Equation: Existence, Uniqueness*

In the next sections we will often have to deal with the following situation. Let $1 \leqslant l \leqslant N$, $\omega \in L^2(\Lambda^l\Omega)$ and $A \in L^2(\Lambda^{l-1}\partial\Omega)$. We consider the elliptic problem

$$-\Delta\psi = \omega \qquad \text{in } \Omega \tag{A.15}$$

$$\psi_\top = 0 \qquad \text{on } \partial\Omega \tag{A.16}$$

$$(d^*\psi)_\top = A \qquad \text{on } \partial\Omega \tag{A.17}$$

Clearly, (A.16) corresponds to a Dirichlet type boundary condition, whereas, as we will see in a moment, (A.17) corresponds to a Neumann-type boundary condition.

First we will prove that this problem possesses a unique (weak) solution (in $W_\top^{1,2}(\Lambda^l\Omega)$). For this purpose, we first derive its variational formulation.

LEMMA A.4. *Assume μ and A are smooth. Then $\psi \in C_\top^\infty(\Lambda^l\bar{\Omega})$ is a solution to* (A.15), (A.16), (A.17) *if and only if for any $\xi \in C_\top^\infty(\Lambda^l\bar{\Omega})$ we have*

$$\langle d\psi, d\xi\rangle + \langle d^*\psi, d^*\xi\rangle = \langle\omega, \xi\rangle - \int_{\partial\Omega} A \wedge (\star\psi)_\top. \tag{A.18}$$

Proof. As usual, we multiply Eq. (A.15) by ξ and integrate by parts. We give a few details of the computations in order to illustrate the different operations we have introduced so far.

Since $-\Delta = dd^* + d^*d$, we have to compute $\langle dd^*\psi, \xi\rangle$ and $\langle d^*d\psi, \xi\rangle$. We have by (A.4)

$$\langle dd^*\psi, \xi\rangle = \int_\Omega dd^*\psi \wedge \star\xi = \int_\Omega d(d^*\psi \wedge \star\xi) - (-1)^l\, d^*\psi \wedge d(\star\xi).$$

By Stokes formula

$$\int_\Omega d(d^*\psi \wedge \star\xi) = \int_{\partial\Omega} (d^*\psi \wedge \star\xi)_\top = \int_{\partial\Omega} (d^*\psi)_\top \wedge (\star\xi)_\top$$

$$= \int_{\partial\Omega} A \wedge (\star\xi)_\top .$$

On the other hand, since $\star\star = (-1)^{l(N-l)}\,\mathrm{Id}$,

$$\int_\Omega d^*\psi \wedge d(\star\xi) = (-1)^{l(N-l)} \int_\Omega d^*\psi \wedge \star\star d(\star\xi)$$

$$= (-1)^{l(N-l)} \int_\Omega d^*\psi \wedge \star(-1)^{Nl+1}\, d^*\xi = (-1)^{1-l^2} \langle d^*\psi, d^*\xi\rangle.$$

Combining the previous identities we obtain

$$\langle dd^*\psi, \xi\rangle = \langle d^*\psi, d^*\xi\rangle + \int_{\partial\Omega} A \wedge (\star\xi)_\top .$$

Similarly, using the fact that $\xi_\top = 0$ on $\partial\Omega$ we obtain

$$\langle d^*d\psi, \xi\rangle = \langle d\psi, d\xi\rangle,$$

and equality (A.18) follows.

In view of (A.18) and of the density of $C^\infty_\top(\Lambda^l\bar{\Omega})$ in $W^{1,2}_\top(\Lambda^l\Omega)$ we will say that $\psi \in W^{1,2}_\top(\Lambda^l\Omega)$ is a weak solution of (A.15) to (A.17) if and only if (A.18) holds for any $\xi \in W^{1,2}_\top(\Lambda^l\Omega)$. Applying Riesz theorem together with Lemma A.3 and equality (A.4) we obtain

PROPOSITION A.1. *For any $\omega \in L^2(\Lambda^l\Omega)$, $A \in L^2(\Lambda^{l-1}\partial\Omega)$ equation* (A.15), (A.16), (A.17) *possesses a unique weak solution in $W^{1,2}_\top(\Lambda^l\Omega)$.*

In order to see that the standard elliptic theory (for functions) applies to our problem, we will discuss in the next subsection the nature of the boundary conditions (A.16), (A.17).

A.5. *Comments about the Boundary Conditions*

In order to understand the nature of the boundary conditions we will consider first the case where Ω is locally a half space, i.e., $\Omega = \mathbb{R}^N_+ = \mathbb{R}^{N-1} \times [0, +\infty)$.

(A) *The case* $\Omega = \mathbb{R}^N_+ = \mathbb{R}^{N-1} \times [0, +\infty) = \{x_N \geqslant 0\}$ *(locally)*. Let $1 \leqslant l \leqslant N-1$ and let $\Lambda(l, N)$ be the set of ordered l-uples. We decompose $\Lambda(l, N)$ into two disjoint subsets $\Lambda^1(l, N)$ and $\Lambda^2(l, N)$, where

$$\Lambda^1(l, N) = \Lambda(l, N-1) = \{(i_1, ..., i_l) \in \Lambda(l, N), i_l \leqslant N-1\}$$

and

$$\begin{aligned} \Lambda^2(l, N) &= \{(i_1, ..., i_l) \in \Lambda(l, N), i_l = N\} \\ &= \{(i_1, ..., i_{l-1}, N), (i_1, ..., i_{l-1}) \in \Lambda(l-1, N-1)\}, \end{aligned}$$

so that

$$\Lambda(l, N) = \Lambda^1(l, N) \cup \Lambda^2(l, N), \qquad \Lambda^1(l, N) \cap \Lambda^2(l, N) = \emptyset.$$

For every l-form ψ on Ω we have in coordinates

$$\psi = \sum_{I \in \Lambda(l, N)} \psi_I \, dx_I.$$

With these notations we obtain

$$\psi_\top = \sum_{I \in \Lambda^1(l, N)} \psi_I \, dx_I, \qquad \psi_N = \sum_{I \in \Lambda^2(l, N)} \psi_I \, dx_I.$$

Condition (A.16) can therefore be rephrased as the Dirichlet condition

$$\psi_I(x) = 0 \quad \text{on } \partial\Omega, \qquad \forall I \in \Lambda^1(l, N),$$

or

$$\psi_{i_1, ..., i_l}(x) = 0 \quad \text{on } \partial\Omega, \qquad \text{if} \quad 1 \leqslant i_1 < \cdots < i_l \leqslant N-1.$$

Likewise we may express (A.17) in coordinates to obtain (the computations are slightly more involved) the Neumann boundary condition

$$\frac{\partial \psi_{I', N}}{\partial x_N} = \frac{\partial \psi_{I', N}}{\partial n} = \varepsilon_{I'} A_{I'} \quad \text{on } \partial\Omega, \qquad \forall I' \in \Lambda(l-1, N-1),$$

where $\varepsilon_{I'} = \pm 1$, that is

$$\frac{\partial \psi_{i_1, \dots, i_{l-1}, N}}{\partial n} = \pm A_{i_1, \dots, i_{l-1}}, \qquad \forall 1 \leqslant i_1 < \cdots < i_{l-1} \leqslant N-1.$$

In other words, we see that each component ψ_I verifies one of the standard elliptic problems, namely $-\Delta \psi_I = 0$, in Ω, $\psi_I = 0$ on $\partial\Omega$, if $I \in \Lambda^1(l, N)$, $-\Delta \psi_I = 0$, in Ω,

$$\frac{\partial \psi_{I', N}}{\partial n} = \varepsilon_{I'} A_{I'} \qquad \text{on } \partial\Omega,$$

if $I = (I', N) \in \Lambda^2(l, N)$. Hence standard elliptic estimates for the Laplacian (with either Dirichlet or Neumann boundary conditions) apply to ψ.

(B) *The general case.* In case Ω is a smooth bounded domain, we may reduce the study to the previous case introducing, locally near the boundary, curvilinear coordinates as in Section A.2. The first step in this classical construction is to cover $\bar{\Omega}$ (which is compact) by a finite number m of balls B_k such that, for some number $m_0 < m$ we are in one of the following situations:

(i) if $k \leqslant m_0$, $B_k \subset \Omega$

(ii) if $k > m_0$, $B_k \cap \Omega$ is diffeomorphic to $B^+ = B_1(0) \cap \mathbb{R}^N_+$.

Next let $\{\chi_k\}_{1 \leqslant k \leqslant m}$ be a partition of unity subordinate to the covering $\{B_k\}_{1 \leqslant k \leqslant m}$ of Ω, i.e. such that $\chi_k \in C^\infty_c(B_k)$, $\chi_k \geqslant 0$ and

$$\sum_{k=1}^m \chi_k = 1 \qquad \text{in } \Omega.$$

In case (ii), i.e. $k > m_0$, we may construct a C^∞ diffeomorphism $\phi_k : B^+ \to B_k \cap \Omega$ such that the following conditions hold

$$\begin{aligned} \phi_k(\{x_N = 0\} \cap B_1) &= \partial\Omega \cap B_k, \\ \phi_k(\partial B_1 \cap \{x_N > 0\}) &= \partial B_k \cap \Omega, \end{aligned} \tag{A.19}$$

$$\text{on} \{x_N = 0\} \cap B_1, \qquad \frac{\partial \phi_k(x)}{\partial x_N} \text{ is orthogonal to } \partial\Omega \cap B_k, \tag{A.20}$$

$$\left\langle \frac{\partial \phi_k(x)}{\partial x_N}, \frac{\partial \phi_k(x)}{\partial x_i} \right\rangle = 0 \quad \text{on} \{x_N = 0\} \cap \operatorname{supp} \chi_k \qquad \forall 1 \leqslant i \leqslant N-1, \tag{A.21}$$

where $\langle\, , \rangle$ denotes the standard scalar product on $\mathbb{R}^N$. This last relation expresses the fact that the normal to $\partial\Omega$ be pulled back by ϕ_k to a normal vector on ∂B^+, on $\{x_N = 0\} \cap \operatorname{supp} \chi_k$.

Let $\psi_k = \chi_k \psi$, so that $\psi = \sum_{k=1}^m \psi_k$. Note that

$$\Delta\psi_k = \chi_k \, \Delta\psi + \nabla\chi_k \cdot \nabla\psi + \psi \, \Delta\chi_k,$$

i.e.,

$$\Delta\psi_k = \chi_k \omega + \nabla\chi_k \cdot \nabla\psi + \psi \, \Delta\chi_k \qquad \text{in } \Omega.$$

We distinguish now the cases (i) and (ii).

Case (i), *i.e.,* $k \leqslant m_0$. We obtain the elliptic problem for ψ

$$\begin{cases} \Delta\psi_k = \chi_k \omega + \nabla\chi_k \cdot \nabla\psi + \psi \Delta\chi_k & \text{in } \Omega \\ \psi_k = 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{A.22}$$

i.e., a standard Dirichlet problem.

Case (ii), *i.e.,* $k > m_0$. Here the analysis is slightly more involved. We consider the pull-back $\tilde{\psi}_k$ on B^+ for ψ_k by the diffeomorphism ϕ_k, that is

$$\tilde{\psi}_k = \phi_k^*(\psi_k) \qquad \text{in } B^+. \tag{A.23}$$

Note that one recovers easily ψ_k from $\tilde{\psi}_k$ by the inverse pull-back

$$\psi_k = (\phi_k^{-1})^* \, (\tilde{\psi}_k), \tag{A.24}$$

so that, since ϕ_k^{-1} is smooth, any estimate on $\tilde{\psi}_k$ gives a similar estimate for ψ_k, and we may work on B^+. We consider also the pull-back $g^k = (g_{ij}^k)$, $1 \leqslant i, j \leqslant N$ of the standard euclidean metric (δ_{ij}), $1 \leqslant i, j \leqslant N$ on $B_k \cap \Omega$ by ϕ_k, i.e.

$$g_{ij}^k = \left\langle \frac{\partial\phi_k}{\partial x_i}, \frac{\partial\phi_k}{\partial x_j} \right\rangle \qquad \text{in } B^+.$$

Let $\gamma^k = (\gamma_{ij}^k)$ be the inverse matrix of (g_{ij}^k), and let

$$\eta^k = |\det(g_{ij}^k)|.$$

The Laplace operator Δ_{g^k} with respect to the metrics g^k is defined as

$$\Delta_{g^k} f = \sum_{i \leqslant j} \frac{\partial}{\partial x_i}\left(\gamma_{ij}^k \sqrt{\eta^k} \, \frac{\partial f}{\partial x_j}\right), \qquad \forall f \in C^2(B^+),$$

that is, $\Delta_{g^k} f$ is an elliptic operator of the second order in divergence form with smooth coefficients.

We are now able to write the equations for $\tilde{\psi}_k$. As usual, we use coordinates on B^+ and write

$$\tilde{\psi}_k = \sum_{I \in \Lambda(l, N)} \tilde{\psi}_I \, dx_I \qquad \text{in } B^+.$$

Then $\tilde{\psi}_I$ verifies

$$\begin{cases} -\Delta_g \tilde{\psi}_I = [\phi_k^*(\chi_k \omega + \nabla \chi_k \cdot \nabla \psi + \psi \, \Delta \chi_k)]_I & \text{in } B^+ \\ \tilde{\psi}_I = 0 \quad \text{on } \partial B^+ \cap \{x_N \geqslant 0\} \\ \tilde{\psi}_I = 0 \quad \text{if } I \in \Lambda(l, N-1) \text{ on } B^+ \cap \{x_N = 0\} \\ \dfrac{\partial \tilde{\psi}_{I', N}}{\partial x_N} = \varepsilon_{I'} g_{NN} [\phi_k^*(A)]_{I'} \quad \text{if } I' \in \Lambda(l-1, N-1), \text{ on } B^+ \cap \{x_N = 0\}. \end{cases} \tag{A.25}$$

Note that the last relation is in particular a consequence of (A.21), that is, for each $I' \in \Lambda(l-1, N-1)$, $I' = (i_1, ..., i_{l-1})$, we have $\forall s = 1, ..., l-1$,

$$g_{i_s N} = 0, \gamma_{i_s, N} = 0 \qquad \text{on} \quad \operatorname{supp} \chi_k \cap \{x_N = 0\},$$

and

$$\gamma_{NN} = g_{NN}^{-1}.$$

We see that the system (A.25) is very similar to that studied in Section A.4. Therefore standard elliptic estimates may be applied to assert higher regularity results, for instance

Lemma A.5. *Let* $1 < p < +\infty$. *If* $\mu \in L^p(\Lambda^l(\Omega))$, *then* $\psi \in W^{2,p}(\Lambda^l(\Omega))$ *and*

$$\|\psi\|_{W^{2,p}} \leqslant C \|\mu\|_{L^p}.$$

A.6. *Estimates Involving L^1-norms of the Data*

We are going to establish in this Section, estimates *à la Stampacchia* for problem (A.15), (A.16), (A.17) (similar estimates have been established in [Baldo-Orlandi]). More precisely, we have the following

Proposition A.2. *Let* $1 \leqslant p < \frac{N}{N-1}$. *There exists a constant* $C = C(p, \Omega)$ *depending only on* p *and* Ω *such that, for any solution* ψ *of problem* (A.15), (A.16), (A.17), *we have*

$$\|\psi\|_{W^{1,p}} \leqslant C(p, \Omega)(\|\mu\|_{L^1}(\Omega) + \|A\|_{L^1}(\partial\Omega)).$$

The proof of Proposition A.2 involves a duality argument of Stampacchia. More precisely, we are going to use in the proof the following:

LEMMA A.6. *Let $q \geqslant 2$ and let $h = (h_1, ..., h_N) \in [L^q(\Lambda^l \Omega)]^N$. Assume moreover that h has compact support in Ω. Let $\zeta \in W^{1,2}_{\mathrm{T}}(\Lambda^l \Omega)$ be the weak solution of*

$$\begin{cases} \Delta\zeta = \sum_{i=1}^{N} \dfrac{\partial}{\partial x_i} h_i & \text{in } \Omega \\ \zeta_{\mathrm{T}} = 0 & \text{on } \partial\Omega \\ (d^*\zeta)_{\mathrm{T}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{A.26}$$

that is $\zeta \in W^{1,2}_{\mathrm{T}}(\Lambda^l \Omega)$ is the unique solution to the variational problem

$$\int_\Omega \langle d\zeta, d\xi \rangle + \langle d^*\zeta, d^*\xi \rangle = \int_\Omega \langle h, \nabla\xi \rangle = \int_\Omega \sum_{i=1}^{N} h_i \frac{\partial}{\partial x_i} \xi \qquad \forall \xi \in W^{1,q}_{\mathrm{T}}(\Lambda^l \Omega). \tag{A.27}$$

Then $\zeta \in W^{1,q}_{\mathrm{T}}(\Lambda^l \Omega)$ and there exists a constant $C(q, \Omega)$ such that

$$\|\zeta\|_{W^{1,q}} \leqslant C(q, \Omega) \, \|h\|_{L^q}.$$

We postpone the proof of Lemma A.3 and show how it implies Proposition A.2.

Proof of Proposition A.2. Let $q > N$ be such that $\frac{1}{p} + \frac{1}{q} = 1$, let $E = [C_c^\infty(\Lambda^l \Omega)]^N$. By density of $C_c^\infty(\Lambda^l \Omega)$ in $L^q(\Lambda^l \Omega)$ and by duality we have

$$\|\nabla\psi\|_{L^p(\Omega)} = \sup \left\{ \int_\Omega \langle h, \nabla\psi \rangle, h \in E, \|h\|_{L^q} = 1 \right\}. \tag{A.29}$$

For $h \in E$, $\|h\|_{L^q} = 1$, let ζ be the solution of (A.27) and take $\xi = \psi$ as a test function in (A.27). This yields

$$\int_\Omega \langle h, \nabla\psi \rangle = \int_\Omega \langle d\zeta, d\psi \rangle + \langle d^*\zeta, d^*\psi \rangle.$$

On the other hand, taking $\xi = \zeta$ as a test function in (A.18) we are led to

$$\int_\Omega \langle d\zeta, d\psi \rangle + \langle d^*\zeta, d^*\psi \rangle = \int_\Omega \langle \mu, \zeta \rangle + \int_{\partial\Omega} A \wedge (\bigstar\zeta)_{\mathrm{T}}.$$

Combining the two equations we obtain

$$\int_\Omega \langle h, \nabla\psi\rangle = \int_\Omega \langle \mu, \zeta\rangle + \int_{\partial\Omega} A \wedge (\star\zeta)_\top . \tag{A.30}$$

Since $q > N$, $W^{1,q}(\Lambda^l\Omega) \hookrightarrow L^\infty(\Lambda^l\Omega)$, and

$$\|\zeta\|_{L^\infty(\Omega)} \leqslant C\, \|\zeta\|_{W^{1,q}(\Omega)}.$$

It follows from Lemma A.3 that

$$\|\zeta\|_{W^{1,p}(\Omega)} \leqslant C\, \|h\|_{L^q(\Omega)} = C,$$

hence $\|\zeta\|_{L^\infty(\Omega)} \leqslant C$. Going back to (A.30) this yields

$$\int_\Omega \langle h, \nabla\psi\rangle \leqslant C(\|\omega\|_{L^1(\Lambda^l\Omega)} + \|A\|_{L^1(\Lambda^{l-1}\partial\Omega)}).$$

Equality (A.29) yields then the conclusion.

Proof of Lemma A.6. Since $q \geqslant 2$, it follows from (A.27) that (A.26) has a unique weak solution $\zeta \in W^{1,2}_\top(\Lambda^l\Omega)$ with

$$\|\zeta\|_{1,2} \leqslant C\, \|h\|_{L^2} \leqslant C\, \|h\|_{L^q}. \tag{A.31}$$

We use next the construction introduced in Section A.5B to reduce the problem to a standard elliptic equation. For $1 \leqslant k \leqslant m$ we set

$$\zeta_k = \chi_k \zeta,$$

so that (A.31) implies

$$\|\zeta_k\|_{1,2} \leqslant C\, \|h\|_{L^q} \qquad \forall 1 \leqslant k \leqslant m.$$

We distinguish next two cases.

Case 1. $k \leqslant m_0$. We may write $\zeta_k = \zeta_k^0 + \zeta_k^1$, where ζ_k^0 is the solution of the problem

$$\begin{cases} \Delta\zeta_k^0 = \chi_k \displaystyle\sum_{i=1}^N \frac{\partial h}{\partial x_i} & \text{in } B_k \\ \zeta_k^0 = 0 & \text{on } \partial B_k, \end{cases}$$

and ζ_k^1 is the solution of

$$\begin{cases} \Delta\zeta_k^1 = \nabla\chi_k \cdot \nabla\zeta_k + \zeta_k \Delta\chi_k & \text{in } B_k \\ \zeta_k^1 = 0 & \text{on } \partial B_k . \end{cases}$$

Standard elliptic estimates yield

$$\|\zeta_k^0\|_{1,q} \leqslant C\,\|h\|_{L^q}$$

and

$$\|\zeta_k^1\|_{2,2} \leqslant \|\zeta_k\|_{1,2} \leqslant C\,\|h\|_{L^q}.$$

In the case $q \leqslant 2^* = \frac{2N}{N-2}$, we have $W^{2,2} \hookrightarrow W^{1,q}$ and therefore we are led to

$$\|\zeta_k\|_{1,q} \leqslant C\,\|h\|_{L^q}. \tag{A.32}$$

If $q > 2^*$ we obtain the same conclusion using a bootstrap argument.

Case 2. $k > m_0$. In this case we consider $\tilde{\zeta}_k = \phi^*(\zeta_k)$ and we write the equations for $\tilde{\zeta}_k$ (see (A.25)). Arguing as before we obtain

$$\|\tilde{\zeta}_k\|_{1,q} \leqslant C\,\|h\|_{L^q},$$

Since $\zeta_k = (\phi^{-1})^*\,(\tilde{\zeta}_k)$ we deduce similarly the bound (A.32).

Since (A.32) holds for any k, the Lemma is established.

A.7. *Elliptic problems on* $\mathbb{R}^N_+$

Let $\mathbb{R}^N_+ = \mathbb{R}^{N-1} \times (0, +\infty)$. In the proof of Theorem 3 bis (η-ellipticity at the boundary) we use the following

PROPOSITION A.3. *Let $\omega \in L^1(\Lambda^l \mathbb{R}^N_+)$ with compact support $\mathcal{K} \in \mathbb{R}^N_+$. Let ϕ be a solution of*

$$\begin{cases} -\Delta\phi = \omega & \textit{in } \mathbb{R}^N_+ \\ \phi_{\top} = 0 & \textit{on } \mathbb{R}^{N-1} \times \{0\} \\ (d^*\phi)_{\top} = 0 & \textit{on } \mathbb{R}^{N-1} \times \{0\}. \end{cases}$$

Assume moreover that $|\phi(x)| \to 0$ as $|x| \to +\infty$. Then, for $I \in \Lambda^1(l, N)$,

$$\phi_I = G_D^+ * \omega_I,$$

and for $I \in \Lambda^2(l, N)$,

$$\phi_I = G_N^+ * \omega_I,$$

where G_D^+ (respectively, G_N^+) denote the fundamental solution of $-\Delta$ on $\mathbb{R}^N_+$ with homogeneous Dirichlet (respectively, Neumann) boundary conditions on $\partial\mathbb{R}^N_+ = \mathbb{R}^{N-1} \times (0, +\infty)$. In particular, $\forall x \in \mathbb{R}^N_+$,

$$|\phi(x)| \leqslant 2c_n \int_{\mathbb{R}^N_+} \frac{|\omega|\,(y)}{|x-y|^{N-2}}\,dy.$$

A.8. *Hodge–de Rham Decomposition*

The Hodge–de Rham decomposition asserts that every l-form μ on the simply connected domain Ω can be decomposed as

$$\mu = dH + d^*\Phi,$$

where H is a $(l-1)$-form on Ω and Φ represents a $(l+1)$-form. In general there is no uniqueness of such a decomposition. We may therefore impose auxiliary conditions, in particular on the boundary. We have

PROPOSITION A.4. *Let $1 < p < +\infty$, and $\mu \in L^p(\Lambda^l \Omega)$. There exists a unique $H \in W^{1,p}(\Lambda^{l-1}\Omega)$ and a unique $\Phi \in W^{1,p}(\Lambda^{l+1}\Omega)$ such that*

$$\begin{cases} \mu = dH + d^*\Phi & & \text{in } \Omega, \\ d^*H = 0, & d\Phi = 0 & \text{in } \Omega, \\ H_\top = 0, & \Phi_\top = 0 & \text{on } \partial\Omega. \end{cases} \tag{A.33}$$

Moreover there exists a constant $C > 0$ such that

$$\|H\|_{W^{1,p}} + \|\Phi\|_{W^{1,p}} \leqslant \|\mu\|_{L^p}.$$

Proof. (A) Existence. Let ψ be the solution of

$$\begin{cases} -\Delta\psi = \mu & \text{in } \Omega \\ \psi_\top = 0 & \text{on } \partial\Omega \\ (d^*\psi)_\top = 0 & \text{on } \partial\Omega. \end{cases}$$

By Lemma A.5 we have $\psi \in W^{2,p}(\Lambda^l\Omega)$. Set

$$\begin{cases} H = d^*\psi \\ \Phi = d\psi. \end{cases}$$

Since $\Delta\psi = d(d^*\psi) + d^*(d\psi) = dH + d^*\Phi$ we verify that H and Φ satisfy (A.33).

(B) Uniqueness. By linearity it suffices to verify that if H and Φ verify (A.33) with $\mu = 0$, then $H = 0$, $\Phi = 0$.

If $\mu = 0$ we have

$$\Delta H = d^*dH = d^*(dH + d^*\Phi) = d^*\mu = 0,$$

so that H verifies

$$\begin{cases} -\Delta H = 0 & \text{in } \Omega \\ H_{\top} = 0 & \text{on } \partial\Omega \\ (d^*H)_{\top} = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence $H = 0$ by Proposition A.1. Similarly one proves that $\Phi = 0$.

A.9. *Hodge–de Rham Decomposition on* $\mathbb{R}^N$

In this subsection, we consider the case where μ is in $\mathscr{D}'(\Lambda^l \mathbb{R}^N)$ with compact support $\mathscr{K}$ in $\mathbb{R}^N$, i.e. $\mu \in \mathscr{E}'(\Lambda^l \mathbb{R}^N)$.

Consider the l-form ψ defined by

$$\psi = G * \mu, \tag{A.34}$$

where

$$G = c_N \frac{1}{|x|^{N-2}}$$

is the fundamental solution of $-\Delta$ in $\mathbb{R}^N$. By construction, we have

$$-\Delta\psi = \mu, \tag{A.35}$$

i.e.,

$$d(d^*\psi) + d^*(d\psi) = \mu.$$

We set as before

$$H = d^*\psi, \qquad \Phi = d\psi \qquad \text{in } \mathbb{R}^N.$$

Note that, in view of (A.34), ψ, H, and Φ are smooth on $\mathbb{R}^N \setminus \mathscr{K}$. We obtain, therefore

PROPOSITION A.5. *Let $\mu \in \mathscr{D}'(\Lambda^l \mathbb{R}^N)$, with compact support $\mathscr{K} \in \mathbb{R}^N$. Then there exist $H \in \mathscr{D}'(\Lambda^{l-1} \mathbb{R}^N)$, $\Phi \in \mathscr{D}'(\Lambda^{l+1} \mathbb{R}^N)$, smooth on $\mathbb{R}^N \setminus \mathscr{K}$ such that*

$$\mu = dH + d^*\Phi \qquad \text{in } \mathscr{D}'(\Lambda^l \mathbb{R}^N),$$
$$d^*H = 0, \qquad d\Phi = 0 \qquad \text{in } \mathscr{D}'(\Lambda^l \mathbb{R}^N).$$

Note that for the previous Hodge–de Rham decomposition we have not uniqueness, since ψ is not the unique solution to (A.35).

In order to obtain uniqueness we must impose conditions at infinity (i.e., as $|x| \to +\infty$) on H and Φ. Since in this paper we only have to deal with functions, we restrict our attention to the case where μ is a bounded measure. We have

PROPOSITION A.6. *Let $\mu \in \mathscr{D}'(\Lambda^l \mathbb{R}^N)$ be a bounded measure with compact support $\mathscr{K}$. Then there exists a unique $H \in \mathscr{D}'(\Lambda^{l-1} \mathbb{R}^N)$, a unique $\Phi \in \mathscr{D}'(\Lambda^{l+1} \mathbb{R}^N)$, smooth on $\mathbb{R}^N \setminus \mathscr{K}$, such that*

$$\mu = dH + d^*\Phi \qquad \text{in } \mathscr{D}'(\Lambda^l \mathbb{R}^N),$$

$$d^*H = 0, \qquad d\Phi = 0 \qquad \text{in } \mathscr{D}'(\Lambda^l \mathbb{R}^N),$$

and there exists $R > 0$, $K > 0$, such that

$$|H(x)|\,|x|^{N-1} \leqslant C, \qquad |\Phi(x)|\,|x|^{N-1} \leqslant C, \qquad \text{for } \; |x| \geqslant R. \tag{A.37}$$

Proof. Clearly H and Φ, given by (A.36), verify (A.37). For the uniqueness it suffices to prove that the solution H_0, Φ_0 to the homogeneous problem

$$dH_0 + d^*\Phi_0 = 0, \qquad d^*H = 0, \qquad \text{and} \qquad d\Phi = 0 \text{ in } \mathbb{R}^N,$$

verifying (A.37) are $H_0 = 0$, $\Phi_0 = 0$. In view of (A.38),

$$\Delta H_0 = 0, \qquad \Delta \Phi_0 = 0,$$

and the conclusion is therefore a classical result for harmonic functions.

Finally, we have

PROPOSITION A.7. *Let $\mu \in \mathscr{D}'(\Lambda^l \mathbb{R}^N)$ be a bounded measure with compact support $\mathscr{K}$, such that*

$$d^*\mu = 0 \qquad \text{in } \mathscr{D}'(\Lambda^l \mathbb{R}^N).$$

Then, there exists a unique $\Phi \in \mathscr{D}'(\Lambda^{l+1} \mathbb{R}^N)$, smooth on $\mathbb{R}^N \setminus \mathscr{K}$, such that

$$\mu = d^*\Phi \qquad \text{in } \mathbb{R}^N,$$

$$d\Phi = 0 \qquad \text{in } \mathbb{R}^N,$$

and

$$|\Phi(x)|\,|x|^{N-1} \to 0 \qquad \text{as } |x| \to +\infty.$$

Proof. It suffices to check that H, Φ given by Proposition A.4 are such that $H = 0$: this is a consequence of the fact that $\Delta H = 0$.

For (A.40), note that, integrating (A.39), we obtain

$$\int_{\mathbb{R}^N} \mu = 0. \tag{A.41}$$

On the other hand, by (A.36),

$$\begin{aligned}\Phi = d(G * \mu) &= d\left(\sum_I (G * \mu_I)\, dx_I\right)\\ &= \sum_{I,j} \frac{\partial G}{\partial x_j} * \mu_I\, dx_j \wedge dx_I\,.\end{aligned}$$

The conclusion (A.40) then follows from the fact that $|\frac{\partial G}{\partial x_j}|\,|x|^{N-1}$ remains bounded as $|x| \to +\infty$, combined with (A.41).

A.10. *Hodge–de Rham Decomposition on* $\mathbb{R}^N_+$

The previous results can be adapted with minor changes to the case $\Omega = \mathbb{R}^N_+ \equiv \mathbb{R}^{N-1} \times (0, +\infty)$. In Section III.2 ($\eta$-ellipticity at the boundary), we use the following

Proposition A.8. *Let $\mu \in \mathscr{D}'(\Lambda^l \mathbb{R}^N_+)$ be a bounded measure on $\mathbb{R}^N_+$ with compact support $\mathscr{K}$ in $\mathbb{R}^N_+$. There exists a unique $H \in L^p_{loc}(\Lambda^{l-1} \mathbb{R}^N_+)$, a unique $\Phi \in L^p_{loc}(\Lambda^{l+1} \mathbb{R}^N_+)$, with $|\nabla H| \in L^p(\mathbb{R}^N_+)$, $|\nabla \Phi| \in L^p(\mathbb{R}^N_+)$, for any $1 \leqslant p < \frac{N}{N-1}$, such that*

$$\begin{cases} \mu = dH + d^*\Phi & \text{in } \mathbb{R}^N_+ \\ d^*H = 0, \quad d\Phi = 0 & \text{in } \mathbb{R}^N_+ \\ H_\top = 0, \quad \Phi_\top = 0 & \text{on } \partial\mathbb{R}^N_+ = \mathbb{R}^{N-1} \times \{0\}. \end{cases}$$

Moreover, $H \in C^\infty(\mathbb{R}^N_+ \setminus \mathscr{K})$, $\Phi \in C^\infty(\mathbb{R}^N_+ \setminus \mathscr{K})$, and

$$|H(x)|\,|x|^{N-1} \leqslant C, \qquad |\Phi(x)|\,|x|^{N-1} \leqslant C, \qquad \text{if} \quad |x| > R,$$

where R is such that $\mathscr{K} \subset B_R$ and C is some constant depending on μ.

If $d^*\mu = 0$ then $H = 0$, i.e., $\mu = d^*\Phi$.

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REFERENCES

[Alberti-Baldo-Orlandi] G. Alberti, S. Baldo, and G. Orlandi, Variational convergence for functionals of Ginzburg–Landau type, preprint 2001.

[Allard] W. Allard, On the first variation of a varifold, *Ann. of Math.* **95** (1972), 417–491.

[Almeida] L. Almeida, Topological sectors for Ginzburg–Landau energies, *Rev. Mat. Iberoamericana* **15** (1999), 487–545.

[Almeida-Bethuel 1] L. Almeida and F. Bethuel, Multiplicity results for the Ginzburg–Landau equation in presence of symmetries, *Houston J. Math.* **23** (1997), 733–764.

[Almeida-Bethuel 2] L. Almeida and F. Bethuel, Topological methods for the Ginzburg–Landau equation, *J. Math. Pures Appl.* **11** (1998), 1–49.

[Ambrosio-Soner] L. Ambrosio and H. M. Soner, A measure theoretic approach to higher codimension mean curvature flow, *Ann. Sc. Norm. Sup. Pisa, Cl. Sci. (4)* **25** (1997), 27–49.

[Baldo-Orlandi] S. Baldo and G. Orlandi, A note on the Hodge theory for functionals with linear growth, *Manuscripta Math.* **97** (1998), 453–467.

[Bethuel] F. Bethuel, Variational methods for Ginzburg–Landau equations, *in* "Calculus of Variations and Geometric Evolution Problems" (S. Hildebrandt and M. Struwe, Eds.), Springer-Verlag, Berlin/New York, 1999.

[Bethuel-Brezis-Hélein 1] F. Bethuel, H. Brezis, and F. Hélein, Asymptotics for the minimizers of a Ginzburg–Landau functional, *Calc. Var.* and *Partial Differential Equations* **1** (1993), 123–148.

[Bethuel-Brezis-Hélein 2] F. Bethuel, H. Brezis, and F. Hélein, "Ginzburg–Landau vortices," Birkhäuser, Basel, 1994.

[Bethuel-Brezis-Orlandi] F. Bethuel, H. Brezis, and G. Orlandi, Small energy solutions to the Ginzburg–Landau equation, *C.R. Acad. Sci. Paris (I)* **331** (2000), 763–770.

[Bethuel-Rivière] F. Bethuel and T. Rivière, A minimization problem related to superconductivity, *Ann. Inst. H. Poincaré, Anal. Non Linéaire* (1995), 243–303.

[Bethuel-Saut] F. Bethuel and J. C. Saut, Travelling waves for the Gross–Pitaevskii equation, *Ann. Inst. H. Poincaré, Phys. Théor.* **70**, (1999).

[Bourgain-Brezis-Mironescu 1] J. Bourgain, H. Brezis, and P. Mironescu, Lifting in Sobolev spaces, *J. d'Analyse Math.* **80** (2000), 37–86.

[Bourgain-Brezis-Mironescu 2] J. Bourgain, H. Brezis, and P. Mironescu, On the structure of the Sobolev space $H^{1/2}$ with values into the circle, *C.R. Acad. Sci. Paris (I)* **331** (2000), 119–124, and detailed paper in preparation.

[Brezis] H. Brezis (with the assistance of P. Mironescu), Lectures on the Ginzburg–Landau equations, in preparation.

[Brezis-Coron-Lieb] H. Brezis, J. M. Coron, and E. Lieb, Harmonic maps with defects, *Comm. Math. Phys.* **107** (1986), 649–705.

[Brezis-Mironescu] H. Brezis and P. Mironescu, Sur une conjecture de E. De Giorgi relative à l'énergie de Ginzburg–Landau, *C.R. Acad. Sci. Paris (I)* **319** (1994), 167–170.

[Brezis-Nirenberg] H. Brezis and L. Nirenberg, Degree theory and BMO I. Compact manifolds without boundaries, *Selecta Math.* **1** (1995), 197–263.

[Chen-Lin] Y. Chen and F. H. Lin, Evolution of harmonic maps with Dirichlet boundary conditions, *Comm. Anal. Geom.* **1** (1993), 327–346.

[Chen-Struwe] Y. Chen and M. Struwe, Existence and partial regularity results for the heat flow for harmonic maps, *Math. Z.* **201** (1989), 83–103.

[Del Pino-Felmer] M. Del Pino and P. Felmer, On the basic concentration estimate for the Ginzburg–Landau equation, *Differential Integral Equations* **11** (1998), 771–779.

[Evans-Gariepy] L. Evans and F. Gariepy, "Measure Theory and Fine Properties of Functions," CAS Press, New York, 1992.

[Giaquinta] M. Giaquinta, "Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems," Princeton Univ. Press, Princeton, NJ, 1983.

[Giaquinta-Modica-Souček] M. Giaquinta, G. Modica, and J. Souček, "Cartesian Currents and the Calculus of Variations," Springer-Verlag, Berlin/New York, 1998.

[Gilbarg-Trudinger] D. Gilbarg and N. Trudinger, "Elliptic Partial Differential Equations of Second Order," Springer-Verlag, Berlin/New York, 1983.

[Hélein] F. Hélein, "Applications harmoniques, lois de conservations et repères mobiles," Diderot, Paris, 1996.

[Iwaniec-Scott-Stroffolini] T. Iwaniec, C. Scott, and B. Stroffolini, Nonlinear Hodge theory on manifolds with boundary, *Ann. Mat. Pura Appl. (4)* **157** (1999), 37–115.

[Jerrard-Soner] R. Jerrard, and H. M. Soner, The Jacobian and the Ginzburg–Landau energy, preprint, 1999.

[Lin] F. H. Lin, A remark on the map $x/|x|$, *C.R. Acad. Sci. Paris (I)* **305** (1987), 529–531.

[Lin-Rivière 1] F. H. Lin and T. Rivière, Complex Ginzburg–Landau equation in high dimensions and codimension 2 area minimizing currents, *J. Eur. Math. Soc.* **1** (1999), 237–311. [erratum, *J. Eur. Math. Soc.* **1** (1999)]

[Lin-Rivière 2] F. H. Lin and T. Rivière, A quantization property for static Ginzburg–Landau vortices, preprint, 2000.

[Mironescu] P. Mironescu, Les minimiseurs locaux pour l'équation de Ginzburg–Landau sont à symétrie radiale, *C.R. Acad. Sci. Paris (I)* **323** (1996), 593–598.

[Ochvinnikov-Sigal] Y. Ochvinnikov and I. Sigal, paper in preparation.

[Rivière 1] T. Rivière, Line vortices in the $U(1)$ Higgs model, *COCV* **1** (1996), 77–167.

[Rivière 2] T. Rivière, Asymptotic analysis for the Ginzburg–Landau equations, *Boll. UMI* **8** (1998), 537–575.

[Sandier] E. Sandier, Ginzburg–Landau minimizers from $\mathbb{R}^{N+1}$ to $\mathbb{R}^N$ and minimal connections, preprint, 1999.

[Sandier-Serfaty] E. Sandier and S. Serfaty, On the energy of type-II superconductors in the mixed phase, *Rev. Math. Phys.* **12** (2000), 1219–1257.

[Schoen-Uhlenbeck 1] R. Schoen and K. Uhlenbeck, A regularity theory of harmonic maps, *J. Differential Geom.* **17** (1982), 307–335.

[Schoen-Uhlenbeck 2] R. Schoen and K. Uhlenbeck, Boundary regularity and the Dirichlet problem for harmonic maps, *J. Differential Geom.* **18** (1983), 253–268.

[Serfaty] S. Serfaty, Local minimizers for the Ginzburg–Landau energy near critical magnetic fields, I and II, *Comm. Contemp. Math.* **1** (1999), 213–254 and 295–333.

[Simon] L. Simon, Lectures on geometric measure theory, *in* "Proceedings of the Centre for Math. Analysis, Australian Nat. Univ.," Canberra, 1983.

[Struwe] M. Struwe, On the asymptotic behavior of the Ginzburg–Landau model in 2 dimensions, *J. Differential Equations* **7** (1994), 1613–1624. [Erratum, **8** (1995) 224]

[Zhou-Zhou] F. Zhou and Q. Zhou, A remark on the multiplicity of solutions for the Ginzburg–Landau equation, *Ann. Inst. H. Poincaré, Analyse Non Linéaire* **16** (1999), 255–267.