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Équations aux dérivées partielles/*Partial Differential Equations*

Small energy solutions to the Ginzburg–Landau equation

Fabrice BETHUEL [a], Haïm BREZIS [a,*], Giandomenico ORLANDI [a,b]

[a] Laboratoire d'analyse numérique, Université Pierre-et-Marie-Curie (Paris-6), boîte courrier 187, 4, place Jussieu, 75252 Paris cedex 05, France
[b] Dip. Scientifico e Tecnologico, Università di Verona, strada le Grazie, 37134 Verona, Italy

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Abstract. We generalize to arbitrary dimensions a result obtained by F.H. Lin and T. Rivière [4,5] in dimension three for solutions to the Ginzburg–Landau equation, as well as, in arbitrary dimensions, in the case of minimizing solutions. © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Solutions d'énergie petite de l'équation de Ginzburg–Landau

Résumé. *Nous généralisons en dimension quelconque un résultat de F.H. Lin et T. Rivière* [4,5] *démontré en dimension trois, pour des solutions de l'équation de Ginzburg–Landau, ainsi que pour les solutions minimisantes en dimension quelconque.* © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Soient N un entier supérieur à deux, et $\mathrm{B}(0,1)$ la boule unité de $\mathbb{R}^N$. On considère, pour un petit paramètre $0<\varepsilon<1$, les solutions u de l'équation de Ginzburg–Landau :

$$-\Delta u = \frac{1}{\varepsilon^2}\, u\big(1-|u|^2\big) \quad \text{dans } \mathrm{B}(0,1). \tag{1}$$

Notre résultat principal est le suivant :

THÉORÈME 1. – *Il existe une constante $C>0$ dépendant seulement de N telle que, si*

$$\int_{\mathrm{B}(0,1)} \frac{1}{2}\,|\nabla u|^2 + \frac{1}{4\varepsilon^2}\big(1-|u|^2\big)^2 \leqslant \eta \log\frac{1}{\varepsilon},$$

alors

$$\big|u(0)\big| \geqslant 1 - C\,\eta^{\frac{1}{4N(N+2)}}.$$

Ici η désigne un nombre positif arbitraire.

Note présentée par Haïm BREZIS.

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Le cas $N=2$ découle directement des estimations en [1]. Le cas $N=3$ et le cas minimisant pour N quelconque ont été prouvés par F.H. Lin et T. Rivière [5,6]. Notre approche reprend en partie la stratégie développée dans [5,6]. Elle s'appuie sur une identité de Pohozaev, une formule de monotonie pour l'équation (1) (*voir* [1,5–7]), et sur une décomposition de Hodge adéquate de la quantité $u \times \nabla u$ (comme dans [1]), qui représente, en gros, le gradient de la phase de u.

1. Introduction

Let N be an integer larger than two, and let $\mathrm{B}(0,1)$ be the unit ball of $\mathbb{R}^N$. For $0<\varepsilon<1$ a small parameter, we consider solutions u of the Ginzburg–Landau equation:

$$-\Delta u = \frac{1}{\varepsilon^2}\, u\big(1-|u|^2\big) \quad \text{in } \mathrm{B}(0,1). \tag{1}$$

Our main result is the following:

THEOREM 1. – *There exists a constant $C>0$ depending only on N such that if*

$$\int_{\mathrm{B}(0,1)} \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}\big(1-|u|^2\big)^2 \leqslant \eta \log\frac{1}{\varepsilon}, \tag{2}$$

where η is any arbitrary positive constant, then

$$\big|u(0)\big| \geqslant 1 - C\,\eta^{\frac{1}{4N(N+2)}}.$$

The case $N=2$ can be easily derived from estimates in [1]. The case $N=3$ and the case of minimizing solutions (N arbitrary) were derived by F.H. Lin and T. Rivière [5,6].

By scaling, it follows from Theorem 1 that if if u is a solution of (1) on the ball $\mathrm{B}(0,R)$, and if

$$R^{2-N}\int_{\mathrm{B}(0,R)} \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}\big(1-|u|^2\big)^2 \ \leqslant \eta\log\frac{R}{\varepsilon},$$

then, as before, $|u(0)| \geqslant 1 - C\,\eta^{\frac{1}{4N(N+2)}}$. It suffices to make the changes of variable $x \to Rx$ and of parameter $\varepsilon \to R\varepsilon$.

Our proof borrows many ideas and the general outline from [5,6]. We will sketch the main ideas involved. If not otherwise stated, C will denote any arbitrary absolute constant, depending only on N. We will denote $\mathrm{B}_r(x_0)$ the ball of radius r centered at the point x_0. In case $x_0=0$, we simply write B_r. We will also assume $\eta<1/4$. In Section 2 we recall a few preliminary estimates, already derived in [1–3,5–7].

2. Some preliminary estimates

LEMMA 1 ([1,2]). – *Assume u verifies* (1). *Then*

$$u \in \mathbf{C}^\infty\big(\mathrm{B}(0,1)\big), \quad |u| \leqslant C, \quad |\nabla u| \leqslant \frac{C}{\varepsilon} \quad \textit{on } \mathrm{B}\left(0,\frac{1}{2}\right). \tag{3}$$

LEMMA 2 (Pohozaev's identity). – *Let* $x_0 \in \mathrm{B}(0,1)$, $r > 0$ *such that* $\mathrm{B}(x_0,r) \subset \mathrm{B}(0,1)$. *Define* $(N-2)F_\varepsilon(x_0,r) \equiv \frac{N-2}{2}\int_{\mathrm{B}_r(x_0)} |\nabla u|^2 + \frac{N}{4\varepsilon^2}\int_{\mathrm{B}_r(x_0)} (1-|u|^2)^2$. *One then has*

$$(N-2)F_\varepsilon(x_0,r) = r\int_{\partial \mathrm{B}_r(x_0)} \frac{|\nabla_\tau u|^2}{2} - \frac{1}{2}\left|\frac{\partial u}{\partial n}\right|^2 + \frac{1}{4\varepsilon^2}\left(1-|u|^2\right)^2. \tag{4}$$

A straightforward consequence of Lemma 2 is the following (*see* [7])

LEMMA 3 (Monotonicity formula). – *With the notations of Lemma* 2, *it holds*

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{2-N}F_\varepsilon(x_0,r)\right) = r^{2-N}\int_{\partial \mathrm{B}_r(x_0)} |\partial_n u|^2 + \left(2(N-2)\varepsilon^2\right)^{-1}\left(1-|u|^2\right)^2. \tag{5}$$

Integrating (5) we obtain the crucial estimate

$$\int_{\mathrm{B}_r(x_0)} \rho^{2-N}\left(|\partial_\rho u|^2 + \left(2(N-2)\varepsilon^2\right)^{-1}\left(1-|u|^2\right)^2\right) \leqslant r^{2-N}F_\varepsilon(x_0,r), \tag{6}$$

where $\rho = |x-x_0|$ (note that ρ^{2-N} is, up to a multiplicative constant, the Green function for the Laplacian: this remark turns out to be fundamental in our analysis).

COROLLARY 1. – *Let* $0 < \delta < 1/16$ *and assume* $\varepsilon \leqslant \delta^2/4$. *If* u *verifies* (1) *and* (2), *then there exists some* $r_0 \in [\varepsilon^{1/2}, \delta/2]$, *depending on* u, *such that, for some absolute constant* C *and* $U_0 = \mathrm{B}_{\delta^{-1}r_0} \setminus \mathrm{B}_{\delta^2 r_0}$,

$$\int_{U_0} \rho^{2-N}\left(|\partial_\rho u|^2 + \left(2(N-2)\varepsilon^2\right)^{-1}\left(1-|u|^2\right)^2\right) \leqslant C\eta|\log\delta|. \tag{7}$$

Proof. – (*See* [2,7]): let $3k+1$ be the largest integer smaller than $|\log(2\varepsilon^{1/2})|\,|\log\delta|^{-1}$, so that $k \geqslant C\,|\log\varepsilon|\,|\log\delta|^{-1}$. Let $V_j = \mathrm{B}_{\delta^{-3j-3}\varepsilon^{1/2}} \setminus \mathrm{B}_{\delta^{-3j}\varepsilon^{1/2}}$. Since the V_j are disjoint, by (6) and (12)

$$\sum_{j=0}^{k}\int_{V_j} \rho^{2-N}\left(|\partial_\rho u|^2 + \left(2(N-2)\varepsilon^2\right)^{-1}\left(1-|u|^2\right)^2\right) \leqslant F_\varepsilon(0,1) \leqslant 3\eta|\log\varepsilon|.$$

Hence, for some j, the integral $\int_{V_j} \dots$ is smaller than $\eta\,|\log\varepsilon|/(k+1)$, and the result (7) follows with $r_0 = \delta^{-3j+2}\varepsilon^{1/2}$.

COROLLARY 2. – *There exists* $r_1 \in [\delta^2 r_0, 2\delta^2 r_0]$ *and* $r_2 \in [(4\delta)^{-1}r_0, (2\delta)^{-1}r_0]$ *such that,*

$$\int_{\partial \mathrm{B}_{r_i}} |\partial_\rho u|^2 + \left(2(N-2)\varepsilon^2\right)^{-1}\left(1-|u|^2\right)^2 \leqslant C\eta|\log\delta|r_i^{N-3}, \qquad i=1,2, \tag{8}$$

$$r_i^{3-N}\int_{\partial \mathrm{B}_{r_i}} |\nabla_\tau u|^2 \leqslant C\left(\eta|\log\delta| + \left(\delta^{-1}r_0\right)^{2-N}F_\varepsilon\left(0,\delta^{-1}r_0\right)\right), \quad i=1,2. \tag{9}$$

Proof. – A simple mean-value argument and (7) yield (8). For (9) use (4) and (5), so that

$$\begin{aligned} r_i^{3-N}\int_{\partial \mathrm{B}_{r_i}} |\nabla_\tau u|^2 &\leqslant r_i^{2-N}(N-2)F_\varepsilon(0,r_i) + \frac{1}{2}r_i^{3-N}\int_{\partial \mathrm{B}_{r_i}} |\partial_\rho u|^2 \\ &\leqslant (N-2)\frac{F_\varepsilon(0,\delta^{-1}r_i)}{(\delta^{-1}r_i)^{N-2}} + C\,\eta\,|\log\delta|. \end{aligned}$$

The main ingredient in the proof of Theorem 1 is:

PROPOSITION 1. – *Choose* $\delta=\eta^{1/8N}$, *and let* $W_0=\mathrm{B}_{2r_0}\setminus\mathrm{B}_{r_0}\subset U_0$. *Then we have*

$$\int_{W_0}\frac{|\nabla u|^2}{2}+\frac{1}{4\varepsilon^2}\left(1-|u|^2\right)^2\leqslant C\,\eta^{1/8}\left[F_\varepsilon\left(0,\delta^{-1}r_0\right)+r_0^{N-2}\right]. \tag{10}$$

3. Proof of Proposition 1

We use the same Hodge type decomposition as in [1]: remark that

$$4|u|^2|\nabla u|^2=4|u\times\nabla u|^2+\left|\nabla|u|^2\right|^2. \tag{11}$$

Estimate of $|u\times\nabla u|$. – The starting point, as in [1], is the equation $\operatorname{div}(u\times\nabla u)=0$, obtained through exterior multiplication by u of equation (1). Using the formalism of differential forms we write the previous relation as

$$\mathrm{d}^*(u\times\mathrm{d}u)=0\quad\text{in }\Omega,\quad\text{where }\mathrm{d}u=\sum\partial_i u\,\mathrm{d}x_i,\quad\mathrm{d}^*=\pm\star\mathrm{d}\star, \tag{12}$$

and $\star$ denotes the Hodge star operator. We introduce the auxiliary Neumann problem:

$$\Delta\xi=0\quad\text{on }U_1\equiv\mathrm{B}_{r_2}\setminus\mathrm{B}_{r_1},\quad\partial_n\xi=u\times\partial_n u\quad\text{on }\partial U_1.$$

The solution ξ clearly exists, since $\int_{\partial U_1}u\times\partial_n u=0$ by (12). Moreover, combining (8) and standard elliptic estimates, We obtain

$$r_0^{2-N}\int_{U_1}|\nabla\xi|^2\leqslant C\,\eta|\log\delta|. \tag{13}$$

By construction we have now $\mathrm{d}^*((u\times\mathrm{d}u-\mathrm{d}\xi)\mathbf{1}_{U_1})=0$ on $\mathbb{R}^N$, where $\mathbf{1}_{U_1}$ denotes the characteristic function of U_1. By the classical Hodge theory (*see* [4]), there exists some 2-form φ on $\mathbb{R}^N$ such that (here $1<p<\infty$)

$$\mathrm{d}^*\varphi=(u\times\mathrm{d}u-\mathrm{d}\xi)\mathbf{1}_{U_1},\quad\mathrm{d}\varphi=0,\quad\varphi\in\mathrm{H}^1,\quad\|\nabla\varphi\|_p\leqslant C_p\left(\|\nabla u\|_{\mathrm{L}^p(U_1)}+\|\nabla\xi\|_{\mathrm{L}^p(U_1)}\right). \tag{14}$$

In view of the identity $u\times\mathrm{d}u=\mathrm{d}^*\varphi+\mathrm{d}\xi$, and since we already have derived a good bound (13) for $\nabla\xi$, we observe, that it in order to bound $|u\times\mathrm{d}u|$, it suffices to bound $|\nabla\varphi|$. We therefore look for an elliptic equation for φ. For that purpose, let $0<\beta<1/2$ to be determined later, and let $f:\mathbb{R}\to[1,1/(1-\beta)]$ be a smooth function such that $f(t)=1/t$ if $t\geqslant 1-\beta$, $f(t)=1$ if $t\leqslant 1-2\beta$, and $|f'|\leqslant 4$.

Define on $\mathbb{R}^N$ the function α such that $\alpha(x)=f(|u(x)|)^2$ on U_1, $\alpha\equiv 1$ otherwise, so that $0\leqslant\alpha-1\leqslant 4\beta$ on $\mathbb{R}^N$. Note that $f(|u|)^2u\times\mathrm{d}u=f(|u|)u\times\mathrm{d}(f(|u|)u)$, hence (14) gives

$$\begin{aligned}\mathrm{d}(\alpha\mathrm{d}^*\varphi)=\omega_1+\omega_2+\omega_3\equiv&\sum_{i<j}\mathbf{1}_{U_1}\left(f\left(|u|\right)u\right)_{x_i}\times\left(f\left(|u|\right)u\right)_{x_j}\mathrm{d}x_i\wedge\mathrm{d}x_j\\&+\left(\sigma_{\partial\mathrm{B}_{r_2}}-\sigma_{\partial\mathrm{B}_{r_1}}\right)f\left(|u|\right)u\times\mathrm{d}u\wedge\mathrm{d}\rho-\mathrm{d}(\mathbf{1}_{U_1}\alpha\mathrm{d}\xi),\end{aligned}$$

where $\sigma_{\partial\mathrm{B}_{r_i}}$ stands for the surface measure on $\partial\mathrm{B}_{r_i}$, $i=1,2$. We claim that

$$|\omega_1|\leqslant C\beta^{-2}\varepsilon^{-2}\left(1-|u|^2\right)^2. \tag{15}$$

Indeed, if $|u|\geqslant 1-\beta$ then $(f(|u|)u)_{x_i}\times(f(|u|)u)_{x_j}=\frac{u}{|u|}_{x_i}\times\frac{u}{|u|}_{x_j}=0$, otherwise we have $\beta^2\leqslant(1-|u|^2)^2$, and by (3) $|\omega_1|\leqslant C/\varepsilon^2$. Hence $|\omega_1|\leqslant C\,\varepsilon^{-2}\beta^{-2}\beta^2\leqslant C\,\beta^{-2}\varepsilon^{-2}(1-|u|^2)^2$.

Since $\mathrm{d}\varphi = 0$, $\Delta\varphi = \mathrm{dd}^*\varphi = \mathrm{d}(\alpha \mathrm{d}^*\varphi) + \mathrm{d}((1-\alpha)\mathrm{d}^*\varphi)$, we are led to write $\varphi = \sum_{i=1}^4 \varphi_i$, where $\varphi_i = G * \omega_i$ for $i = 1, 2, 3$ and $G = c_N |x|^{2-N}$ is the fundamental solution of the Laplacian in $\mathbb{R}^N$, so that $\Delta\varphi_i = \omega_i$ for $i = 1, 2, 3, 4$. The form φ_4 represents the remainder, i.e., $\varphi_4 = \varphi - \sum_{i=1}^3 \varphi_i$. Next, we will estimate each $\|\nabla\varphi_i\|_{\mathrm{L}^2(W_0)}$ separately for $i = 1, 2, 3$. While the estimate for φ_2, φ_3 and φ_4 are more or less standard, the estimate for φ_1 is the central observation of the paper.

Estimate for φ_2. – We have (*see* Appendix)

$$\|\nabla\varphi_2\|^2_{\mathrm{L}^2(\mathbb{R}^N)} \leqslant C \sum_{i=1}^2 r_i \int_{\partial \mathrm{B}_{r_i}} |\nabla_\tau u|^2 \leqslant C \sum_{i=1}^2 \big(r_i^{N-2}\eta |\log\delta| + F_\varepsilon\big(x_0, \delta^{-1} r_0\big)\big), \tag{16}$$

$$\int_{W_0} |\nabla\varphi_2|^2 \leqslant C\big(\delta^2 r_0^{N-2}\eta\,|\log\delta| + \delta^N F_\varepsilon\big(x_0, \delta^{-1} r_0\big)\big). \tag{17}$$

Estimate for φ_3. – Standard estimates lead to $\|\nabla\varphi_3\|_{\mathrm{L}^2(\mathbb{R}^N) \leqslant \|\alpha\|_{\mathrm{L}^\infty(U_1)} \|\nabla\xi\|_{\mathrm{L}^2(U_1)}}$, and by (13),

$$r_0^{2-N} \|\nabla\varphi_3\|^2_{\mathrm{L}^2(\mathbb{R}^N)} \leqslant C\,\eta |\log\delta|. \tag{18}$$

Estimate for φ_1. – It uses the monotonicity formula in a crucial way. We claim that

$$\|\nabla\varphi_1\|^2_{\mathrm{L}^2(\mathbb{R}^N)} \leqslant C\beta^{-4}\eta\,|\log\delta| \big[F_\varepsilon\big(0, \delta^{-1} r_0\big) + C\,\eta r_0^{N-2}\delta^{2-N} |\log\delta|\big]. \tag{19}$$

Indeed, by the maximum principle, $|\varphi_1|$ achieves its supremum on U_1. For $x \in U_1$,

$$|\varphi_1(x)| \leqslant C\,\beta^{-2} \int_{U_1} |x-y|^{2-N} \varepsilon^{-2} \big(1 - |u|^2\big)^2\,\mathrm{d}y \quad \text{(by definition and by (15))}.$$

Let $A_x = \mathrm{B}(x, (4\delta)^{-1} r_0) \cap U_1$, $D_x = U_1 \smallsetminus A_x$. The monotonicity formula (6) applied with $x_0 = x$ yields

$$\int_{A_x} |x-y|^{2-N} \varepsilon^{-2} \big(1 - |u|^2\big)^2\,\mathrm{d}y \leqslant \big((4\delta)^{-1} r_0\big)^{2-N} F_\varepsilon\big(x, (4\delta)^{-1} r_0\big) \leqslant C\big(\delta^{-1} r_0\big)^{2-N} F_\varepsilon\big(0, \delta^{-1} r_0\big),$$

while by Corollary 1 we have

$$\int_{D_x} |x-y|^{2-N} \varepsilon^{-2} \big(1 - |u|^2\big)^2\,\mathrm{d}y \leqslant C \int_{U_0} |y|^{2-N} \varepsilon^{-2} \big(1 - |u|^2\big)^2\,\mathrm{d}y \leqslant C\,\eta |\log\delta|.$$

It follows

$$\|\varphi_1\|_\infty \leqslant C\,\beta^{-2} \big(\big(\delta^{-1} r_0\big)^{2-N} \big(F_\varepsilon\big(0, \delta^{-1} r_0\big) + C\,\eta |\log\delta|\big).$$

On the other hand, combining Corollary 1 with (15), we deduce

$$\int_{\mathbb{R}^N} |\omega_1| \leqslant C\,\beta^{-2}\eta |\log\delta|.$$

Finally, we derive estimate (19) by multiplying the equation $\Delta\varphi_1 = \omega_1$ by φ_1, integrating by parts, and combining the two previous inequalities.

Estimate for φ_4. – By definition, $\varphi_4 = \varphi - \sum_{i=1}^3 \varphi_i$, so that $\Delta\varphi_4 = \Delta\varphi - \sum_{i=1}^3 \omega_i = \mathrm{d}((1-\alpha)\mathrm{d}^*\varphi)$, and one checks, using $0 \leqslant \alpha - 1 \leqslant 4\beta$, that

$$\|\nabla\varphi_4\|^2_{\mathrm{L}^2(\mathbb{R}^N)} \leqslant C\beta \sum_{i=1}^3 \|\nabla\varphi_i\|^2_{\mathrm{L}^2(\mathbb{R}^N)}. \tag{20}$$

Applying (20), together with (16), (18), (19) we find

$$\|\nabla\varphi_4\|^2_{L^2(\mathbb{R}^N)} \leqslant C\beta\big[\big(1+C\beta^{-4}\eta|\log\delta|\big)F_\varepsilon\big(0,\delta^{-1}r_0\big)+r_0^{N-2}\big(\beta^{-4}\eta^2\delta^{2-N}|\log\delta|^2+\eta\,|\log\delta|\big)\big]. \quad (21)$$

Finally, combining (21), (17), (18), (19) we obtain, choosing $\beta=\eta^{1/8}$, $\delta=\eta^{1/8N}$,

$$\int_{W_0}|\nabla\varphi|^2 \leqslant C\,\eta^{1/8}\big[F_\varepsilon\big(0,\delta^{-1}r_0\big)+r_0^{N-2}\big].$$

Since $|u\times \mathrm{d}u|\leqslant|\mathrm{d}^*\varphi|+|\mathrm{d}\xi|$, this yields likewise

$$\int_{W_0}\big|u\times \mathrm{d}u\big|^2 \leqslant C\,\eta^{1/8}\big[F_\varepsilon\big(0,\delta^{-1}r_0\big)+r_0^{N-2}\big]. \quad (22)$$

Estimate of $|\nabla|u|^2|$. – The equation for $|u|^2$ reads $-\Delta(1-|u|^2)+\varepsilon^{-2}(1-|u|^2)|u|^2=|\nabla u|^2$. Multiplying by $(1-|u|^2)$ and integrating on U_1 yields

$$\int_{U_1}|\nabla|u|^2|^2+\varepsilon^{-2}\big(1-|u|^2\big)^2|u|^2=\int_{U_1}\big(1-|u|^2\big)|\nabla u|^2+\int_{\partial U_1}\big(1-|u|^2\big)\partial_n|u|^2.$$

By (3) we have $\big|\int_{\partial U_1}(1-|u|^2)\partial_n|u|^2\big|\leqslant C\int_{\partial U_1}|\partial_n u|$, and (here $U_\beta=\{||u|-1|\leqslant\beta\}$)

$$\int_{U_1}\big|1-|u|^2\big||\nabla u|^2\leqslant\int_{U_1\cap U_\beta}\beta|\nabla u|^2+\beta^{-1}\int_{U_1\setminus U_\beta}C\,\varepsilon^{-2}\big(1-|u|^2\big)^2, \quad (23)$$

so that, using (8), $\int_{U_1}|\nabla|u|^2|^2\leqslant\beta F_\varepsilon(0,\delta^{-1}r_0)+C\beta^{-1}\eta|\log\eta|\leqslant C\,\eta^{1/8}\big[F_\varepsilon(0,\delta^{-1}r_0)+r_0^{N-2}\big]$.

Combining this estimate with (11) and (22) we obtain $\int_{W_0}|u|^2|\nabla u|^2\leqslant C\,\eta^{1/8}\big[F_\varepsilon(0,\delta^{-1}r_0)+r_0^{N-2}\big]$. Finally, we complete the proof of Proposition 1 using (23) again.

4. Proof of Theorem 1

By the mean-value inequality applied to (10), there is some $r_3\in[r_0,2r_0]$ such that

$$r_3\int_{\partial B_{r_3}}|\nabla u|^2+\big(4\varepsilon^2\big)^{-1}\big(1-|u|^2\big)^2\leqslant C\,\eta^{1/8}\big[F_\varepsilon\big(0,\delta^{-1}r_0\big)+r_0^{N-2}\big].$$

By the Pohozaev's identity (4), this implies $F_\varepsilon(0,r_0)\leqslant F_\varepsilon(0,r_3)\leqslant C\,\eta^{1/8}\big[F_\varepsilon(0,\delta^{-1}r_0)+r_0^{N-2}\big]$, so that

$$\begin{aligned} r_0^{2-N}F_\varepsilon(0,r_0) &\leqslant C\,r_0^{2-N}\eta^{1/8}\big[F_\varepsilon\big(0,\delta^{-1}r_0\big)+r_0^{N-2}\big]\\ &\leqslant C\,\eta^{1/8}\delta^{2-N}\big[\big(\delta^{-1}r_0\big)^{2-N}F_\varepsilon\big(0,\delta^{-1}r_0\big)+1\big]\\ &\leqslant C\,\eta^{1/4N}\big[\big(\delta^{-1}r_0\big)^{2-N}F_\varepsilon\big(0,\delta^{-1}r_0\big)+1\big] \end{aligned} \quad (24)$$

(recall $\delta=\eta^{1/8N}$). On the other hand by monotonicity (5) and (7),

$$\big(\delta^{-1}r_0\big)^{2-N}F_\varepsilon\big(0,\delta^{-1}r_0\big)\leqslant r_0^{2-N}F_\varepsilon(0,r_0)+C\,\eta|\log\eta|.$$

Inserting this in (24) we obtain

$$\big(1-C\eta^{1/4N}\big)r_0^{2-N}F_\varepsilon(0,r_0)\leqslant C\eta^{1/4N}\big(1+\eta|\log\eta|\big).$$

Hence, if η_1 is such that $C\eta_1^{1/4N} = 1/2$, then for $\eta \leqslant \eta_1$ we conclude

$$r_0^{2-N} F_\varepsilon(0, r_0) \leqslant C\, \eta^{1/4N}. \tag{25}$$

Finally, we invoke the monotonicity (6) once more to assert, for $\eta \leqslant \eta_1$, by (25),

$$\varepsilon^{-N} \int_{\mathrm{B}_\varepsilon} \big(1 - |u|^2\big)^2 \leqslant \int_{\mathrm{B}_\varepsilon} \rho^{2-N} \varepsilon^{-2} \big(1 - |u|^2\big)^2 \leqslant \int_{\mathrm{B}_{r_0}} \rho^{2-N} \varepsilon^{-2} \big(1 - |u|^2\big)^2 \leqslant C\, \eta^{1/4N}.$$

We now complete the proof as in [1]. Set $k = |u(0)|$, by (3), we have $|u(x) - u(0)| \leqslant \frac{C}{\varepsilon}|x| \leqslant (1-k)/2$, provided $|x| \leqslant \varepsilon(1-k)/(2C) \equiv \gamma$. We distinguish two cases.
Case 1: $\gamma < \varepsilon$. Then $\int_{\mathrm{B}_\gamma} (1 - |u|^2)^2 \leqslant \int_{\mathrm{B}_\varepsilon} (1 - |u|^2)^2 \leqslant C\, \varepsilon^N \eta^{1/4N}$. On the other hand, on B_γ we have $(1 - |u|^2)^2 \geqslant (1 - |u|])^2 \geqslant (1-k)^2/4$, and therefore $\gamma^N (1-k)^2 \leqslant C\, \varepsilon^N \eta^{1/4N}$. Hence $(1-k)^{N+2} \leqslant C\, \eta^{1/4N}$, and the result follows.
Case 2: $\gamma \geqslant \varepsilon$. Then $(1 - |u|^2)^2 \geqslant (1-k)^2/4$ on B_ε, and one concludes as above.

Appendix: proof of estimates (16), (17) for φ_2

Recall that $G(x) = c_N |x|^{2-N}$ denotes the fundamental solution of the Laplacian in $\mathbb{R}^N$. For $r > 0$ let $\mathrm{B}_r \equiv \mathrm{B}_r(0) \subset \mathbb{R}^N$ and let $\sigma_{\partial \mathrm{B}_r}$ denote the surface measure on $\partial \mathrm{B}_r$.

LEMMA A1. – *Let $r > 0$ and $g \in \mathrm{L}^2(\partial \mathrm{B}_r)$. Set $\varphi = G * (\sigma_{\partial \mathrm{B}_r} g)$, i.e., for $x \in \mathbb{R}^N$,*

$$\varphi(x) = c_N \int_{\partial \mathrm{B}_r} |x - y|^{2-N} g(y)\, \mathrm{d}y.$$

Then $\nabla\varphi \in \mathrm{L}^2(\mathbb{R}^N)$ and $\|\nabla\varphi\|^2_{\mathrm{L}^2(\mathbb{R}^N)} \leqslant C\, r\, \|g\|^2_{\mathrm{L}^2(\partial \mathrm{B}_r)}$.

Proof. – By scaling it suffices to consider the case $r = 1$. Clearly, for $x \in \mathbb{R}^N \setminus \partial \mathrm{B}_1$,

$$|\nabla\varphi(x)| \leqslant C \int_{\partial \mathrm{B}_1} |x - y|^{1-N} \big|g(y)\big|\, \mathrm{d}y$$

so that for $x \in \mathbb{R}^N \setminus \partial \mathrm{B}_2$, we have $|\nabla\varphi(x)| \leqslant C|x|^{1-N}\, \|g\|_{\mathrm{L}^1(\partial \mathrm{B}_1)}$. Therefore,

$$\|\nabla\varphi\|^2_{\mathrm{L}^2(R^N \setminus \mathrm{B}_2)} \leqslant C \|g\|^2_{\mathrm{L}^2(\partial \mathrm{B}_1)}.$$

On B_2 we write $\varphi = \varphi_1 + \varphi_2$, where φ_1 solves $\Delta\varphi_1 = g\sigma_{\partial \mathrm{B}_1}$ in B_2, $\varphi_1 = 0$ on $\partial \mathrm{B}_2$, while φ_2 solves $\Delta\varphi_2 = 0$ in B_2, $\varphi_2 = \varphi$ on $\partial \mathrm{B}_2$. By the trace theorem we verify $\varphi_1 \in \mathrm{H}^1_0(\mathrm{B}_2)$ and

$$\|\nabla\varphi_1\|^2_{\mathrm{L}^2(\mathrm{B}_2)} \leqslant C \|g\|^2_{\mathrm{L}^2(\partial \mathrm{B}_1)}.$$

Similarly, since φ_2 is harmonic in B_2, one has (observe that $\|\varphi\|_{\mathrm{L}^\infty(\partial \mathrm{B}_2)} \leqslant C \|g\|_{\mathrm{L}^2(\partial \mathrm{B}_1)}$):

$$\|\nabla\varphi_2\|^2_{\mathrm{L}^2(\mathrm{B}_2)} \leqslant C \|\varphi\|^2_{\mathrm{H}^{1/2}(\partial \mathrm{B}_2)} \leqslant C \|g\|^2_{\mathrm{L}^2(\partial \mathrm{B}_1)}.$$

Combining the previous inequalities, the result follows easily.

Proof of (16). – Write $\varphi_2 = \varphi_{2,1} + \varphi_{2,2}$, where $\varphi_{2,i} = (-1)^i \sigma_{\partial \mathrm{B}_{r_i}} f(|u|) u \times \mathrm{d}u \wedge \mathrm{d}\rho$ for $i = 1, 2$. We apply previous lemma to obtain

$$\int_{\mathbb{R}^N} |\nabla\varphi_{2,1}|^2 \leqslant C\, r_i \int_{\partial \mathrm{B}_{r_i}} f\big(|u|\big)^2 |u|^2 |\mathrm{d}u|^2 \leqslant C\, r_i \int_{\partial \mathrm{B}_{r_i}} |\nabla u|^2,$$

and (16) follows.

Proof of (17). – We first bound $\|\nabla\varphi_2\|_{\mathrm{L}^\infty(W_0)}$. We estimate separately $\|\nabla\varphi_{2,i}\|_{\mathrm{L}^\infty(W_0)}$ for each $i=1,2$: remark that $\varphi_{2,i}$ is harmonic in $\mathbb{R}^N\setminus\partial\mathrm{B}_{r_i}$, therefore we may use the mean-value inequality for harmonic functions.

For $i=1$ and $x\in W_0$ we have $\mathrm{B}(x,r_0/2)\subset\mathbb{R}^N\setminus\partial\mathrm{B}_{r_1}$. Hence, using (9),

$$\begin{aligned}\left|\nabla\varphi_{2,1}(x)\right|^2 &\leqslant \left|\mathrm{B}(x,r_0/2)\right|^{-1}\|\nabla\varphi_{2,1}\|_{\mathrm{L}^2(\mathbb{R}^N)} \leqslant C\,r_0^{-N}r_1\int_{\partial\mathrm{B}_{r_1}}|\nabla u|^2\\ &\leqslant C\,r_0^{-N}\left(\delta^2 r_0\right)^{N-2}\left(\eta|\log\delta| + \left(\delta^{-1}r_0\right)^{2-N}F_\varepsilon\left(0,\delta^{-1}r_0\right)\right)\\ &\leqslant C\,r_0^{-N}\left(\delta^2 r_0^{N-2}\eta|\log\delta| + \delta^{3(N-2)}F_\varepsilon\left(0,\delta^{-1}r_0\right)\right).\end{aligned}$$

For $i=2$ and $x\in W_0$ we have $\mathrm{B}(x,\delta^{-1}r_0/8)\subset\mathbb{R}^N\setminus\partial\mathrm{B}_{r_2}$, so that by (9)

$$\begin{aligned}\left|\nabla\varphi_{2,2}(x)\right|^2 &\leqslant \left|\mathrm{B}\left(x,\delta^{-1}r_0/8\right)\right|^{-1}\|\nabla\varphi_{2,2}\|_{\mathrm{L}^2(\mathbb{R}^N)} \leqslant Cr_0^{-N}\delta^N r_2\int_{\partial\mathrm{B}_{r_2}}|\nabla u|^2\\ &\leqslant C\,r_0^{-N}\delta^N\left(\delta^{-1}r_0\right)^{N-2}\left(\eta|\log\delta| + \left(\delta^{-1}r_0\right)^{2-N}F_\varepsilon\left(0,\delta^{-1}r_0\right)\right)\\ &\leqslant C\,r_0^{-N}\left(\delta^2 r_0^{N-2}\eta|\log\delta| + \delta^N F_\varepsilon\left(0,\delta^{-1}r_0\right)\right).\end{aligned}$$

Finally, combining these estimates we obtain, for all $x\in W_0$,

$$\left|\nabla\varphi_2(x)\right|^2 \leqslant C\,r_0^{-N}\left(\delta^2 r_0^{N-2}\eta|\log\delta| + \left(\delta^N+\delta^{3(N-2)}\right)F_\varepsilon\left(0,\delta^{-1}r_0\right)\right).$$

Since $N\geqslant 3$, $3(N-2)\geqslant N$ and since $|W_0|\leqslant C\,r_0^N$, integrating the previous inequality on W_0 we obtain (17).

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