On the structure of the Sobolev space $H^{1/2}$
with values into the circle

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(Reçu et accepté le 14 mai 2000)

Abstract.
We are concerned with properties of $H^{1/2} = 2$ ($\Omega$, $S^1$) where $\Omega$ is the boundary of a domain in $\mathbb{R}^3$. To every $u \in H^{1/2} = 2$ ($\Omega$, $S^1$) we associate a distribution $T(u)$ which, in some sense, describes the location and the topological degree of singularities of $u$. The closure $Y$ of $C^1$ ($\Omega$, $S^1$) in $H^{1/2}$ coincides with the $u$'s such that $T(u) = 0$. Moreover, every $u \in Y$ admits a unique (mod. $2\pi$) lifting in $H^{1/2} + W^{1,1}$. We also discuss an application to the 3-d Ginzburg–Landau problem.

On the structure of the Sobolev space $H^{1/2}$ à valeurs dans le cercle

Résumé. On s’intéresse aux propriétés des fonctions de $H^{1/2} = 2$ ($\Omega$, $S^1$) où $\Omega$ est le bord d’un domaine de $\mathbb{R}^3$. A tout $u \in H^{1/2} = 2$ ($\Omega$, $S^1$) on associe une distribution $T(u)$ qui décrit l’emplacement et le degré topologique des singularités de $u$. La fermeture $Y$ de $C^\infty$ ($\Omega$, $S^1$) dans $H^{1/2}$ coïncide avec l’ensemble des $u$ tels que $T(u) = 0$. De plus, tout $u \in Y$ s’écrit de manière unique (mod. $2\pi$) sous la forme $u = e^{i\alpha}$ avec $\varphi \in H^{1/2} + W^{1,1}$. On présente aussi une application au problème de Ginzburg–Landau en 3-d. © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Soit $G \subset \mathbb{R}^3$ un domaine borné régulier tel que $\Omega = \partial G$ soit simplement connexe. On s’intéresse aux propriétés des fonctions de $H^{1/2} = 2$ ($\Omega$, $S^1$). Par analogie avec les résultats de [8] et [3], on associe à tout $u \in H^{1/2} = 2$ ($\Omega$, $S^1$) une distribution $T(u)$ qui agit sur $C^1$ ($\Omega$). Lorsque $u$ admet seulement un nombre fini de singularités $(a_j)$ de degré $(d_j)$, on a $T(u) = 2\pi \sum_j d_j \delta_{a_j}$. On sait alors que $\sup \{\langle T(u), \varphi \rangle; \|\varphi\|_{\text{Lip}} \leq 1\}$ est la longueur de la connexion minimale (au sens de [8]) associée aux singularités de $u$. On montre que $u \in Y$
si et seulement si $T(u) = 0$ (ceci est l’analogue $H^{1/2}$ d’un résultat de Bethuel [1] concernant les fonctions de $H^1(B^3; S^2)$). On prouve que toute fonction $u \in Y$ s’écrit (de manière unique mod. $2\pi$) sous la forme

$u = e^{i\varphi}$ avec $\varphi \in H^{1/2}(\Omega; \mathbb{R}) + W^{1,1}(\Omega; \mathbb{R})$. De plus, on a l’estimation $\|\varphi\|_{H^{1/2}, W^{1,1}} \leq C(1 + \|u\|_{H^{1/2}}^2)$. La preuve de cette estimeé utilise la théorie des paraproducts au sens de J.-M. Bony et Y. Meyer.

Enfin, on considère l’énergie de Ginzburg–Landau $E$, définie par (11), où $g_0$ est une approximation de $g$ au sens de (10). On suppose que $g \in Y$ et on écrit $g = e^{i\varphi_0}$ avec $\varphi_0 \in H^{1/2}(\Omega; \mathbb{R}) + W^{1,1}(\Omega; \mathbb{R})$. Alors les minimiseurs $u_\varepsilon$ de (11) convergent vers $u_* = e^{i\varphi}$, où $\varphi$ est la solution de (12).

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Let $G \subset \mathbb{R}^3$ be a smooth bounded domain with $\Omega = \partial G$ simply connected. We are concerned with the properties of the space

$$H^{1/2}(\Omega; S^1) = \{ u \in H^{1/2}(\Omega; \mathbb{R}^2); |u| = 1 \text{ a.e. on } \Omega \}.$$

Recall (see [5]) that there are functions in $H^{1/2}(\Omega; S^1)$ which cannot be written in the form $u = e^{i\varphi}$ with $\varphi \in H^{1/2}(\Omega; \mathbb{R})$. For example, we may assume that locally, near a point on $\Omega$, say 0, $\Omega$ is a disc $B_1$; then take

$$u(x, y) = (x, y)/(x^2 + y^2)^{1/2} \quad \text{on } B_1. \quad (1)$$

Recall also (see [12]) that there are functions in $H^{1/2}(\Omega; S^1)$ which cannot be approximated in the $H^{1/2}$-norm by functions in $C^\infty(\Omega, S^1)$. Consider, for example, again a function $u$ which is the same as in (1) near 0.

It is therefore natural to introduce the classes

$$X = \{ u \in H^{1/2}(\Omega; S^1); \ u = e^{i\varphi} \text{ for some } \varphi \in H^{1/2}(\Omega; \mathbb{R}) \}$$

and

$$Y = C^\infty(\Omega; S^1)^{H^{1/2}}.$$

Clearly, we have

$$X \subset Y \subset H^{1/2}(\Omega; S^1).$$

Moreover, these inclusions are strict. Any function $u \in H^{1/2}(\Omega; S^1)$ which satisfies (1) does not belong to $Y$. On the other hand the function

$$u(x, y) = \begin{cases} e^{2\pi i r^\alpha} & \text{on } B_1, \\ 1 & \text{on } \Omega \setminus B_1, \end{cases} \quad (2)$$

with $r = (x^2 + y^2)^{1/2}$ and $1/2 \leq \alpha < 1$, belongs to $Y$, but not to $X$ (see [5]).

To every function $u \in H^{1/2}(\Omega; \mathbb{R}^2)$ we associate a distribution $T = T(u) \in \mathcal{D}'(\Omega; \mathbb{R})$. When $u \in H^{1/2}(\Omega; S^1)$ the distribution $T(u)$ describes the location and the topological degree of its singularities.

Given $u \in H^{1/2}(\Omega; \mathbb{R}^2)$ and $\varphi \in C^1(\Omega; \mathbb{R})$ consider any $U \in H^1(G; \mathbb{R}^2)$ and any $\Phi \in C^1(G; \mathbb{R}^2)$ such that

$$U|_{\Omega} = u \quad \text{and} \quad \Phi|_{\Omega} = \varphi.$$

Set

$$H = 2(U_y \wedge U_z, U_z \wedge U_x, U_x \wedge U_y);$$

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this $H$ is independent of the choice of direct orthonormal bases in $\mathbb{R}^3$ (to compute derivatives) and in $\mathbb{R}^2$ (to compute $\wedge$-products). Next, consider

$$
\int_G H \cdot \nabla \Phi.
$$

(3)

It is not difficult to show (see [6]) that (3) is independent of the choice of $U$ and $\Phi$; it depends only on $u$ and $\varphi$. We may thus define the distribution $^1 T(u) \in D'(\Omega; \mathbb{R})$ by:

$$
\langle T(u), \varphi \rangle = \int_G H \cdot \nabla \Phi.
$$

If there is no ambiguity we will simply write $T$ instead of $T(u)$.

When $u$ has a little more regularity we may also express $T$ in a simpler form.

**Lemma 1.** – If $u \in H^{1/2}(\Omega; \mathbb{R}^2) \cap W^{1,1}(\Omega; \mathbb{R}^2) \cap L^\infty(\Omega; \mathbb{R}^2)$, then

$$
\langle T(u), \varphi \rangle = \int (u \wedge u_x)\varphi_y - (u \wedge u_y)\varphi_x, \quad \forall \varphi \in C^1(\Omega; \mathbb{R}),
$$

for any choice of local orthonormal coordinates $(x, y)$ on $\Omega$ such that $(x, y, n)$ is direct, where $n$ is the outward normal to $G$.

By analogy with the results of [8] and [3] we introduce, for every $u \in H^{1/2}(\Omega; \mathbb{R}^2)$, the number

$$
L(u) = \frac{1}{2\pi} \sup_{\varphi \in C^1(\Omega; \mathbb{R})} \sup_{\|\varphi\|_{L^p} \leq 1} \langle T(u), \varphi \rangle,
$$

where $\|\varphi\|_{L^p}$ refers to a given metric on $\Omega$. There are three (equivalent) metrics on $\Omega$ which are of interest:

$$
d_{_{\mathbb{R}^3}}(x, y) = |x - y|,
$$

$$
d_G(x, y) = \text{the geodesic distance in } \Omega \text{, and}
$$

$$
d_{_{\Omega}}(x, y) = \text{the geodesic distance in } \Omega.
$$

(4)

It is easy to see that

$$
|L(u)| \leq C\|u\|^2_{H^{1/2}}, \quad \forall u \in H^{1/2}(\Omega; \mathbb{R}^2)
$$

(5)

and

$$
|L(u) - L(v)| \leq C\|u - v\|_{H^{1/2}}\left(\|u\|_{H^{1/2}} + \|v\|_{H^{1/2}}\right), \quad \forall u, v \in H^{1/2}(\Omega; \mathbb{R}^2).
$$

(6)

When $u$ takes its values in $S^1$ and has only a finite number of singularities there are very simple expressions for $T(u)$ and $L(u)$:

**Lemma 2.** – If $u \in H^{1/2}(\Omega; S^1) \cap H^1_{\text{loc}}(\Omega \setminus \bigcup_{j=1}^k \{a_j\}; S^1)$, then

$$
T(u) = 2\pi \sum_{j=1}^k d_j \delta_{a_j},
$$

where $d_j = \text{deg}(u, a_j)$ and $L(u)$ is the length of the minimal connection associated to the configuration $(a_j, d_j)$ and to the specific metric on $\Omega$ (in the sense of [8]; see also [13]).
Remark 1. – Here $\deg(u; a_j)$ denotes the topological degree of $u$ restricted to any small circle around $a_j$, positively oriented with respect to the outward normal. It is well defined using the degree theory for maps in $H^{1/2}(S^1; S^1)$ (see [7] and [10]).

We will also make use of a density result of T. Rivière which is the $H^{1/2}$ analogue of a result of Bethuel and Zheng [4] concerning $H^1$ maps from $B^3$ to $S^2$ (see also a related result of Bethuel [2] in fractional Sobolev spaces). Let $\mathcal{R}$ denote the class of maps in $H^{1/2}(\Omega; S^1)$ which are $C^\infty$ on $\Omega$ except at a finite number of points.

**Lemma 3** (T. Rivière [19]). – The class $\mathcal{R}$ is dense in $H^{1/2}(\Omega; S^1)$.

Still some further elementary facts about $T$ and $L$:

**Lemma 4.** – For every $u, v \in H^{1/2}(\Omega; S^1)$ we have:

$$T(uv) = T(u) + T(v),$$
$$L(uv) \leq C ||u - v||_{H^{1/2}} (||u||_{H^{1/2}} + ||v||_{H^{1/2}}),$$
$$L(uv) \leq L(u) + L(v).$$

Here, we have identified $\mathbb{R}^2$ with $\mathbb{C}$ and $uv$ denotes complex multiplication.

Using Lemmas 3 and 4 we may extend the representation formula of Lemma 2 to general functions in $H^{1/2}(\Omega; S^1)$:

**Theorem 1.** – Given any $u \in H^{1/2}(\Omega; S^1)$ there are two sequences of points $(P_i)$ and $(N_i)$ in $\Omega$ such that

$$\sum_i |P_i - N_i| < \infty,$$
$$\langle T(u), \zeta \rangle = 2\pi \sum_i \left( \zeta(P_i) - \zeta(N_i) \right).$$

In addition, for any metric $d$ in (4)

$$L(u) = \text{Inf} \sum_i d(P_i, N_i),$$

where the infimum is taken over all possible sequences $(P_i), (N_i)$ satisfying (7), (8). In case the distribution $T$ is a measure (of finite total mass) then

$$T(u) = 2\pi \sum_{\text{finite}} d_j \delta_{a_j}$$

with $d_j \in \mathbb{Z}$ and $a_j \in \Omega$.

The last assertion in Theorem 1 is the $H^{1/2}$ analogue of a result of Jerrard and Soner [15,16] (see also Hang and Lin [14]) concerning maps in $W^{1,1}(\Omega; S^1)$. In Theorem 1, the last assertion can be derived from the first assertion via a direct abstract argument (see Smets [21]).

Maps in $Y$ can be characterized in terms of the distribution $T$:

**Theorem 2** (T. Rivière [19]). – Let $u \in H^{1/2}(\Omega; S^1)$, then $T(u) = 0$ if and only if $u \in Y$.

This result is the $H^{1/2}$-counterpart of a well-known result of Bethuel [1] characterizing the closure of smooth maps in $H^1(B^3; S^2)$ (see also Demengel [11]).

As was mentioned earlier, functions in $Y$ need not belong to $X$, i.e., they need not have a lifting in $H^{1/2}(\Omega; \mathbb{R})$. However we have:
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**Theorem 3.** For every $u \in Y$ there exists $\varphi \in H^{1/2}(\Omega; \mathbb{R}) + W^{1,1}(\Omega; \mathbb{R})$, which is unique (modulo $2\pi$) such that $u = e^{i\varphi}$. Conversely, if $u \in H^{1/2}(\Omega; S^1)$ can be written as $u = e^{i\varphi}$ with $\varphi \in H^{1/2} + W^{1,1}$ then $u \in Y$.

The existence is proved with the help of paraproducts (in the sense of J.-M. Bony and Y. Meyer) (see [6]). The heart of the matter is the estimate

$$
\|\varphi\|_{H^{1/2} + W^{1,1}} \leq C(1 + \|u\|_{H^{1/2}}^2),
$$

which holds for smooth $\varphi$. The uniqueness (modulo $2\pi$) of $\varphi$ is established as in [9]. Theorem 3 is still valid for domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$ (see [6]).

**Remark 2.** Using Theorem 3 and the basic estimate (9) one may prove that for every $u \in H^{1/2}(\Omega; S^1)$ there exists $\varphi \in H^{1/2}(\Omega; \mathbb{R}) + BV(\Omega; \mathbb{R})$ such that $u = e^{i\varphi}$. Of course this $\varphi$ is not unique, but there are some “distinguished” $\varphi$’s (see [6]).

The link between the Ginzburg–Landau energy and minimal connections has been first pointed out in the important work of T. Rivière [18] (see also [17] and [20]) in the case of boundary data with a finite number of singularities. We are concerned here with a general boundary condition $g$ in $H^{1/2}$.

Given $g \in H^{1/2}(\Omega; S^1)$ we may always approximate it by a sequence $g_\varepsilon \in C^\infty(\Omega; \mathbb{R}^2)$ such that

$$
\begin{cases}
\|g_\varepsilon - g\|_{L^2} \leq C\sqrt{\varepsilon}, & \|\nabla g_\varepsilon\|_{L^\infty} \leq C/\varepsilon, \\
g_\varepsilon \rightharpoonup g & \text{in } H^{1/2}.
\end{cases}
$$

Set

$$
E_\varepsilon = \min \left\{ \int_G |\nabla u|^2 + \frac{1}{4\varepsilon^2} \int_G ((|u|^2 - 1)^2); u \in H^1(G; \mathbb{R}^2) \text{ and } u = g_\varepsilon \text{ on } \Omega \right\}.
$$

**Theorem 4.** We have, as $\varepsilon \to 0$,

$$
E_\varepsilon = \pi L(g) \log(1/\varepsilon) + o(\log(1/\varepsilon)),
$$

where $L(g)$ corresponds to the metric $d_G$ on $\Omega$.

Finally, we study the convergence of minimizers $(u_\varepsilon)$ of (11). If $g \in X$ we may write $g = e^{i\varphi_0}$ with $\varphi_0 \in H^{1/2}(\Omega; \mathbb{R})$. A natural choice for $g_\varepsilon$ is $g_\varepsilon = e^{i\varphi_\varepsilon}$ where $\varphi_\varepsilon$ is an $\varepsilon$-regularization of $\varphi$ as in (10). In this case it is easy to prove that

$$
u_\varepsilon \rightharpoonup u_\ast = e^{i\varphi} \quad \text{in } H^1(G),
$$

where $\varphi$ is the solution of

$$
\Delta \varphi = 0 \quad \text{in } G, \quad \varphi = \varphi_0 \quad \text{on } \Omega.
$$

When $g \in Y$ we prove (see [6]):

**Theorem 5.** For every $g \in Y$ write (as in Theorem 3) $g = e^{i\varphi_0}$ with $\varphi_0 \in H^{1/2} + W^{1,1}$. Then (for any choice of $g_\varepsilon$ satisfying (10)) we have

$$
u_\varepsilon \rightharpoonup u_\ast = e^{i\varphi} \quad \text{in } W^{1,p}(G) \cap C^\infty(G), \quad \forall p < 3/2,
$$

where $\varphi$ is defined in (12).

**Remark 3.** We shall also present in [6] results concerning the convergence of $u_\varepsilon$ when $g \in H^{1/2}(\Omega; S^1)$ does not belong to $Y$. 

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The distribution $T(u)$ and the corresponding number $L(u)$ were originally introduced, for a general $u \in H^{1/2}$, by the authors in 1996 and these concepts were presented in various lectures.

References