Homotopy classes in Sobolev spaces

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Let $M$ and $N$ be two compact Riemannian manifolds. One of the central questions in Topology is the study of homotopy classes of $C^0(M, N)$, i.e. the connected (or equivalently path-connected) components of the metric space $C^0(M, N)$. In other words, homotopy classes are the equivalence classes corresponding to the equivalence relation: $f \sim g$ if there exists a path $h(t) \in C^0([0, 1]; C^0(M, N))$ connecting $f$ to $g$, i.e. $h(0) = f$ and $h(1) = g$. For example, when $M = N = S^n$, it is well-known that $f \sim g$ if and only if $\deg(f) = \deg(g)$.

Our aim is to present some new results from the paper H. Brezis – Y. Li [1] concerning the study of homotopy classes, when the space $C^0(M, N)$ is replaced by the scale of Sobolev spaces $W^{1,p}(M, N)$.

Throughout this note $M$ and $N$ are smooth, connected, compact, oriented Riemannian manifolds. We will always assume that $\partial N = \phi$ but $M$ may or may not have a boundary; in particular the case where $M$ is a domain in $\mathbb{R}^N$ is of interest. The reader is encouraged to keep in mind elementary manifolds, such as spheres, balls and their products; the results are already of interest for such a simple situation.

Let us start with a simple observation about the scale $C^k(M, N)$ equipped with the metric
\[ d(f, g) = \sum_{|\alpha| \leq k} \sup_{x \in M} d(D^\alpha f(x), D^\alpha g(x)). \]

In principle, for each $k$, one may introduce a new equivalence relation:
\[ f \sim g \text{ in } C^k \iff f, g \text{ belong to the same path-connected component of } C^k \]
i.e. there exists a homotopy $h(t) \in C([0, 1], C^k(M, N))$ such that $h(0) = f$ and $h(1) = g$.

In fact such a notion has no interest because of the following "standard"

**Lemma 1.** Let $f, g \in C^k(M, N)$; then $f \sim g$ in $C^k$ if and only if $f \sim g$ in $C^0$.

The proof consists of smoothing the given homotopy $h \in C([0, 1]; C^0(M, N))$. For this purpose we may assume that $N \subset \mathbb{R}^K$ is an isometric embedding. Then $\rho_\varepsilon \ast h$ does not take its values into $N$, but $\rho_\varepsilon \ast h$ is uniformly close to $h$ as $\varepsilon \to 0$ (because $h$ is continuous). And then one may project $\rho_\varepsilon \ast h$ back onto $N$ for $\varepsilon$ small.

As a consequence of Lemma 1 we see that the components of $C^k(M, N)$ shrink as $k$ increase, but they "do not change their shape". By contrast, we will see that the situation is totally different in the scale of Sobolev spaces $W^{1,p}$. 

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Let $1 \leq p < \infty$ be a real number. Let $M$ and $N$ be as above and $N \subset \mathbb{R}^K$. The spaces $W^{1,p}(M, \mathbb{R})$ and $W^{1,p}(M, \mathbb{R}^K)$ are defined as usual and equipped with the standard norm $\|f\|_{W^{1,p}}$. Set

$$W^{1,p}(M, N) = \{ f \in W^{1,p}(M, \mathbb{R}^K); f(x) \in N \text{ a.e.} \},$$

equipped with the distance

$$d(f, g) = \|f - g\|_{W^{1,p}(M, \mathbb{R}^K)}.$$

As a metric space, $W^{1,p}(M, N)$ admits connected components and also path-connected components. We do not know whether they are the same. This boils down to

**Open problem 1.** Given $f \in W^{1,p}(M, N)$, can one find $\varepsilon > 0$ (depending on $f$) such that

$$d(g, f) < \varepsilon \quad \Rightarrow \quad g \sim f \text{ in } W^{1,p}?$$

Here we say that $g \sim f$ in $W^{1,p}$ if there is a path $h(t) \in C([0,1]; W^{1,p}(M, N))$ such that $h(0) = f$ and $h(1) = g$.

A simple but useful example is the case $M = N = S^2$ who’s study was initiated in Brezis–Coron [1].

Let us examine the space $W^{1,p}(S^2, S^2)$, $1 \leq p < \infty$, from the point of view of its components. One has to consider 3 different cases.

a) **Case $p > 2$.** It is not difficult to prove, following the same idea as in Lemma 1, and using the Sobolev imbedding,

**Lemma 2.** Let $f, g \in W^{1,p}(S^2, S^2)$; then $f \sim g$ in $W^{1,p}(S^2, S^2)$ if and only if $f \sim g$ in $C^0$. In particular, the homotopy classes of $W^{1,p}(S^2, S^2)$ can be classified using their standard degree.

b) **Case $p = 2$.** This is a very interesting case because it is a limiting case for the Sobolev imbedding. And $W^{1,2}$ is not contained in $C^0$, so that the standard notion of degree is not well-defined. One may nevertheless still define a degree. This was first done in Brezis–Coron [1] using the following strategy:

**Step 1.** If $f \in C^1(S^2, S^2)$ a well-know integral formula for computing the degree (going back to Kronecker) asserts that

$$\deg(f) = \frac{1}{4\pi} \int_{S^2} \det(\nabla f).$$

When $f \in W^{1,2}(S^2, S^2)$ the integral in (1) still makes sense because $\det(\nabla f)$ is a $2 \times 2$ determinant and $\nabla f \in L^2$. Unfortunately it is not clear that the right-hand side in (1) is an integer. For this purpose one relies on

**Step 2.** This is an important observation due to Schoen–Uhlenbeck [1]. If $f \in W^{1,2}(S^2, S^2)$, then $\rho_\varepsilon \ast f$ does not take its values into $S^2$. And $\rho_\varepsilon \ast f$ does not converge uniformly to $f$ (as $\varepsilon \to 0$) (otherwise $f$ would be continuous). However one can prove that $|(\rho_\varepsilon \ast f)(x)| \to 1$
uniformly in $x$. This is a consequence of the fact that $f \in VMO$ (see Brezis–Nirenberg [1]) which in turn follows from of Poincaré’s inequality

$$\int_B |f - \frac{1}{|B|^{1/2}} \int_B f| \leq \int_B |\nabla f|.$$ 

As a result one may consider

$$f_\varepsilon(x) = \frac{1}{4\pi} \int_{S^2} \det(\nabla f)$$

we conclude that

$$\int_{S^2} \det(\nabla f) \in \mathbb{Z}.$$ 

We now set, for every $f \in W^{1,2}$

$$deg(f) = \frac{1}{4\pi} \int_{S^2} \det(\nabla f).$$

It was later observed in Brezis–Nirenberg [1] that $deg(f)$ can still be defined for every $f \in W^{1,2}$ without using formula (1), only Step 2. Indeed, if $f \in W^{1,2}(S^2, S^2)$ consider $f_\varepsilon$ as above (for $\varepsilon < \varepsilon_0$). Then $deg(f_\varepsilon)$ is independent of $\varepsilon$, for $\varepsilon \in (0, \varepsilon_0)$, because one may use $\varepsilon$ itself as homotopy parameter (or connect $f_{\varepsilon_1}$ and $f_{\varepsilon_2}$ via the homotopy $h(t) = f_{\varepsilon_1 + (1-t)\varepsilon_2}$). We take as definition of $deg f$, the integer $deg(f_\varepsilon)$, (for $0 < \varepsilon < \varepsilon_0$).

**Proposition 1.** Assume $f, g \in W^{1,2}(S^2, S^2)$. Then $f \sim g$ in $W^{1,2}$ if and only if $deg(f) = deg(g)$. Consequently, $W^{1,2}(S^2, S^2)$ still admits infinitely many homotopy classes and they are classified using degree.

c) Case $p < 2$. This case was not considered in Brezis–Coron [1]. But in Brezis–Nirenberg [1] we observed that no degree theory (in some reasonable sense) could be defined. Indeed the identity map can be homotopied to a constant map! This is done as follows. Fix any point $a \in \mathbb{R}^3$ with $|a| = 2$. Consider the path

$$h(t)(x) = \frac{x - \frac{t}{|x|}a}{|x - ta|}, \quad t \in [0, 1], \quad x \in S^2.$$ 

Clearly $h$ is smooth for $t \in [0, 1/2] \cup [1/2, 1]$ and $x \in S^2$. The only difficulty occurs at $t = 1/2$ because $a/2 \in S^2$. However it is not difficult to check that $h(1/2) \in W^{1,p}(S^2, S^2)$ for every $p < 2$ and moreover

$$h(t) \in C([0, 1]; W^{1,p}(S^2, S^2))$$
for every \( p < 2 \). Finally it is clear that \( h(1) \) has degree zero, and thus it can be homotoped to a constant in the \( C^0 \) sense (and thus in the \( C^1 \) sense by Lemma 1, and hence in the \( W^{1,p} \) sense). Putting all this together we see that \( h(0) = Id \) is homotopic to a constant in every \( W^{1,p} \), \( p < 2 \).

We went one step further in Brezis–Li [1] and proved

**Theorem 1.** The space \( W^{1,p}(S^2, S^2) \) is path-connected for every \( p < 2 \).

We now have a full picture for \( W^{1,p}(S^2, S^2) \). When \( p < 2 \), \( W^{1,p}(S^2, S^2) \) consists of one piece. At \( p = 2 \), \( W^{1,2}(S^2, S^2) \) splits into infinitely many pieces. As \( p \) increases from 2 to \( \infty \), these pieces “shrink” but do not change their shape.

Exactly the same type of conclusion holds for \( W^{1,p}(S^n, S^n) \). When \( p < n \), \( W^{1,p}(S^n, S^n) \) is path-connected. At \( p = n \), a degree theory is well-defined, and thus \( W^{1,n}(S^n, S^n) \) splits into infinitely many pieces. As \( p \) increases from \( n \) to \( \infty \), these pieces shrink without changing their shape.

At this stage, one would be inclined to believe that this is a general phenomenon. When \( p < \dim M \), \( W^{1,p}(M, N) \) is path-connected. As \( p \) increases from \( \dim M \) to \( \infty \), \( W^{1,p}(M, N) \) admits path-connected component similar to the ones of \( C^0(M, N) \) and they shrink without changing their shape. The second assertion (for \( p \geq \dim M \)) is indeed true. However the first assertion (for \( p < \dim M \)) is totally wrong. This was first pointed out in a very interesting paper of Rubinstein–Sternberg [1]. Namely, some “topology” still survives for \( W^{1,p}(M, N) \) even when \( p < \dim M \). In fact \( W^{1,p}(M, N) \), may have a very rich structure from the point of view of homotopy classes when \( p < \dim M \). I will present later some remark examples.

**Theorem 2.** (Rubinstein–Sternberg [1]). Let \( M = \Omega = S^1 \times D \) where \( D \) is the unit disc in \( \mathbb{R}^2 \), i.e. \( \Omega \) is a solid torus in \( \mathbb{R}^3 \). Let \( N = S^1 \). Then any \( f \in W^{1,2}(M, N) \) admits a well-defined degree (stable under \( W^{1,2} \) convergence).

More precisely write

\[
f(x, \lambda) : S^1 \times D \to S^1
\]

then

\[
\varphi(\lambda) = \deg f(\cdot, \lambda),
\]

which is well-defined for a.e. \( \lambda \in D \), (since \( f(\cdot, \lambda) \in W^{1,2}(S^1, S^2) \) for a.e. \( \lambda \in D \)) is in fact a constant (a.e).

**Remark 1.** Is is quite surprising that a degree may still be defined even though \( p = 2 < 3 = \dim M \). I should point out however that the same conclusion fails for \( p \in [1, 2) \). Note that when \( p \in [1, 2) \) \( \varphi(\lambda) \) is still well-defined for a.e. \( \lambda \in D \). However \( \varphi \) is not constant in general.

**Remark 2.** The conclusion that \( \varphi \) is a constant may be related to a variety of results which have emerged in recent years in the work of Bourgain–Brezis–Mironescu [1] [2] (see also Brezis [1]) about conditions implying that a given measurable function is constant. Here is such a typical result.
**Theorem 3.** Let $G \subset \mathbb{R}^N$ be a connected open set and let $\varphi : G \to \mathbb{Z}$ be a measurable function. Assume that
\[
\int_G \int_G \frac{|\varphi(\lambda) - \varphi(\mu)|^p}{|\lambda - \mu|^{N+1}} d\lambda d\mu < \infty
\]
(any $p \geq 1$). Then $\varphi$ is a constant (a.e.).

Here is an extension of Theorem 2,

**Theorem 4.** (Brezis–Li–Mironescu–Nirenberg [1]). Let $M = \Omega = S^n \times \Lambda$, where $n \geq 1$, and $\Lambda \subset \mathbb{R}^k$ is any open connected set with $k \geq 1$.

If $p \geq n + 1$, any $f \in W^{1,p}(M,N)$ has a well-defined degree. More precisely, write
\[
f(x,\lambda) : S^n \times \Lambda \to S^n,
\]
then
\[
\varphi(\lambda) = \deg f(\cdot,\lambda),
\]
which is well-defined for a.e. $\lambda \in \Lambda$ (since $p > n$), is in fact a constant.

**Remark 3.** Theorem 4 shows again that some topology still “persists” much below the critical Sobolev exponent $p = \dim M$. The condition $p \geq n + 1$ is usually much weaker than the condition $p \geq \dim M = n + k$, especially when $k$ is large.

Going back to the study of the path-connected comments for $W^{1,p}(M,N)$, here is a complement to Theorem 4, which gives a complete classification of $W^{1,p}$ when the parameter space $\Lambda$ is a ball.

**Theorem 5.** (Brezis–Li [1]). Let $M = \Omega = S^n \times \Lambda$ where $n \geq 1$ and $\Lambda$ is the unit ball in $\mathbb{R}^k$ (any $k \geq 1$). Let $f, g \in W^{1,p}(M,N)$. If $p \geq n + 1$, then $f \sim g$ in $W^{1,p}$ if and only if $\deg(f) = \deg(g)$ (where $\deg$ is meant in the sense of Theorem 4). If $p < n + 1$, then $f \sim g$ in $W^{1,p}$, i.e. $W^{1,p}$ is path-connected.

At this stage we decide with Yanyan Li to initiate a general investigation of the homotopy classes of $W^{1,p}(M,N)$ for a general pair of manifolds $M$ and $N$, and for a general $p \geq 1$.

Here is a first (somewhat surprising) result

**Theorem 6.** (Brezis–Li [1]) For any pair $M$ and $N$ with $\dim M \geq 2$ and for any $p \in [1,2)$, $W^{1,p}(M,N)$ is path-connected.

We also introduced in Brezis–Li [1] a concept which plays a very important role in describing the possible changes in homotopy classes for $W^{1,p}$ as $p$ varies. Here it is.

Let $p > 1$. Let $0 < \varepsilon < p - 1$. Clearly
\[
W^{1,p+\varepsilon}(M,N) \subset W^{1,p-\varepsilon}(M,N).
\]
Given $f \in W^{1,p}(M,N)$, we denote by $[f]_p$ its homotopy class in $W^{1,p}$. We denote by $W^{1,p}/_p$ the quotient of $W^{1,p}$ by the equivalence relation $f \sim g$ (meaning $f \sim g$ in $W^{1,p}$).
Obviously, if \( f, g \in W^{1,p+\varepsilon} \) and \( f \sim g \) in \( W^{1,p+\varepsilon} \), then \( f \sim g \) in \( W^{1,p-\varepsilon} \). As a result we have a canonical map

\[
i_{p+\varepsilon,p-\varepsilon} : W^{1,p+\varepsilon} \rightarrow W^{1,p-\varepsilon}.
\]

**Definition.** We say that a change of topology occurs at \( p \) if for every \( \varepsilon \in (0, p-1) \), the map \( i_{p+\varepsilon,p-\varepsilon} \) is not bijective. We denote by \( CT(M,N) \) the set of \( p \)'s for which a change of topology occurs at \( p \).

If a change of topology occurs at \( p \), two things may happen, \( i_{p+\varepsilon,p-\varepsilon} \) is not injective or \( i_{p+\varepsilon,p-\varepsilon} \) is not surjective (or both!). Roughly speaking not injective means that there are 2 maps \( f, g \in W^{1,p+\varepsilon} \) such that \( f \) and \( g \) are not homotopic in \( W^{1,p+\varepsilon} \) while \( f \sim g \) in \( W^{1,p-\varepsilon} \).

In other words, one component of \( W^{1,q}(M,N) \) splits into two distinct components as \( q \) increases from \( p-\varepsilon \) to \( p+\varepsilon \). Another view point is to say that two distinct components of \( W^{1,q} \) have coalesced as \( q \) decreases from \( p+\varepsilon \) to \( p-\varepsilon \). This is a common situation, which we have already encountered above. For example a change of topology for \( W^{1,p}(S^2,S^2) \) occurs at \( p = 2 \), because \( i_{2+\varepsilon,2-\varepsilon} \) is not injective: any two maps \( f, g \in W^{1,2+\varepsilon} \) are homotopic in \( W^{1,2-\varepsilon} \), while they need not be in \( W^{1,2+\varepsilon} \) (unless their degree is the same).

On the other hand, the fact that \( i_{p+\varepsilon,p-\varepsilon} \) is not surjective means that a new component of \( W^{1,q} \) appears (out of nowhere!) as \( q \) decreases from \( p+\varepsilon \) to \( p-\varepsilon \). This situation is more unusual and it does not occur in the example \( W^{1,p}(S^n,S^n) \). It means that some map \( f \in W^{1,p-\varepsilon} \) cannot be homotopied in \( W^{1,p-\varepsilon} \) to any \( g \in W^{1,p+\varepsilon} \). In particular, such a map \( f \) cannot be smooth and it cannot be homotopied in \( W^{1,p-\varepsilon} \) to any smooth map –so it must have a rather “solid” singularity!

We may now reformulate Theorem 1 and Theorem 5 using the above notion.

**Theorem 1’**. We have \( CT(S^2,S^2) = \{2\} \) or more generally \( CT(S^n,S^n) = \{n\} \) for any \( n \geq 2 \).

**Theorem 5’**. We have \( CT(S^n \times \Lambda,S^n) = \{n+1\} \), where \( \Lambda \) is the unit ball in \( \mathbb{R}^k \) (any \( k \geq 1 \)).

We also observed in Brezis–Li [1] that a change in topology may occurs at several values of \( p \) – not just one, as in the above examples. Here is such a situation, with a “cascade of mergings”

**Theorem 6**. We have \( CT(S^1 \times S^2,S^1 \times S^2) = \{2,3\} \).

This is somewhat natural because we have here (at least), two invariants: write \( f = (f_1(x,y), f_2(x,y)) \) and set \( d_1 = \deg f_1(\cdot,y) \) and \( d_2 = \deg f_2(x,\cdot) \). Note that \( d_1 \) is well-defined and independent of \( y \in S^2 \) when \( f \in W^{1,p} \) and \( p \geq 2 \) (by Theorem 4), while \( d_2 \) is well-defined and independent of \( x \in S^1 \) only when \( p \geq 3 \) (again by Theorem 4). Some new invariants appear, as \( p \) increases from 1 to \( \infty \), when crossing the values \( p = 2 \) and \( p = 3 \).

We also made in Brezis–Li [1] two conjectures:
Conjecture 1. $CT(M, N)$ consists only of integers, i.e. change of topology for $W^{1,p}$ occurs only when $p$ is an integer.

Conjecture 2. Any map $f \in W^{1,p}(M, N)$ can always be connected in $W^{1,p}$ to a smooth map (any $p$, any $M, N$).

They were both solved in a beautiful piece of work by Hang–Lin [1].

The answer to Conjecture 1 is positive:

**Theorem 7.** (Hang–Lin [1]) $CT(M, N)$ consists only of integers.

Theorem 7 is an immediate consequence of the following remarkable result which provides a “reduction” of the study of homotopy classes for $W^{1,p}$ to more classical concepts in Topology. Given $p \geq 2$, consider $M^{[p]-1}$ the $([p] - 1)$-skeleton of $M$.

**Theorem 8.** (Hang–Lin [1]) Let $f, g \in W^{1,p}(M, N)$. Then $f \sim g$ in $W^{1,p}$ if and only if $f|_{\Sigma} \sim g|_{\Sigma}$ in $C^0$ for a generic $\Sigma$ in $M^{[p]-1}$.

Note that $\dim \Sigma \leq [p]-1$, and for a generic $\Sigma$, $f|_{\Sigma} \in W^{1,p}(\Sigma)$, while $p > \dim \Sigma$ (since $[p] \geq \dim \Sigma + 1$). Thus for a generic $\Sigma$, $f|_{\Sigma} \in C^0$, by the Sobolev imbedding.

Concerning Conjecture 2, we had presented in Brezis–Li [1] several cases where Conjecture 2 is true: for example when $\dim M = 2$ (any $N$), or $\dim M = 3$ and $\partial M \neq \emptyset$ any $N$), or $N = S^1$ (any $M$). However Hang-Lin [1] found some situations where Conjecture 2 fails.

Here is such an example:

**Theorem 9.** (Hang–Lin [1]) Let $M = \mathbb{RP}^3$ and $N = \mathbb{RP}^2$. Then $CT(M, N) = \{2, 3\}$. Moreover there are maps $f \in W^{1,p}(M, N)$, with $p \in (2, 3)$, such that $f$ cannot be connected in $W^{1,p}$ to any smooth map.

References.


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