Homotopy classes in Sobolev spaces

Haïm Brezis

Proceedings of the Conference held in Taiwan
on the occasion of the 75th birthday of Louis Nirenberg

Let M and N be two compact Riemannian manifolds. One of the central questions in Topology is the study of homotopy classes of $C^0(M,N)$, i.e. the connected (or equivalently path-connected) components of the metric space $C^0(M,N)$. In other words, homotopy classes are the equivalence classes corresponding to the equivalence relation: $f \sim g$ if there exists a path $h(t) \in C^0([0,1]; C^0(M,N))$ connecting f to g, i.e. $h(0) = f$ and $h(1) = g$. For example, when $M = N = S^n$, it is well-known that $f \sim g$ if and only if deg $(f) =$ deg(g).

Our aim is to present some new results from the paper H. Brezis – Y. Li [1] concerning the study of homotopy classes, when the space $C^0(M,N)$ is replaced by the scale of Sobolev spaces $W^{1,p}(M,N)$.

Throughout this note M and N are smooth, connected, compact, oriented Riemannian manifolds. We will always assume that $\partial N = \phi$ but M may or may not have a boundary; in particular the case where M is a domain in $\mathbb{R}^N$ is of interest. The reader is encouraged to keep in mind elementary manifolds, such as spheres, balls and their products; the results are already of interest for such a simple situation.

Let us start with a simple observation about the scale $C^k(M,N)$ equipped with the metric

$$d(f,g) = \sum_{|\alpha| \le k} \sup_{x \in M} d(D^\alpha f(x), D^\alpha g(x)).$$

In principle, for each k, one may introduce a new equivalence relation:

$$f \sim g \text{ in } C^k \iff f, g \text{ belong to the same path-connected component of } C^k$$

i.e. there exists a homotopy $h(t) \in C([0,1], C^k(M,N))$ such that $h(0) = f$ and $h(1) = g$.

In fact such a notion has no interest because of the following "standard"

Lemma 1. *Let $f, g \in C^k(M,N)$; then $f \sim g$ in C^k if and only if $f \sim g$ in C^0.*

The proof consists of smoothing the given homotopy $h \in C([0,1]; C^0(M,N))$. For this purpose we may assume that $N \subset \mathbb{R}^K$ is an isometric embedding. Then $\rho_\varepsilon \star h$ does not take its values into N, but $\rho_\varepsilon \star h$ is uniformly close to h as $\varepsilon \to 0$ (because h is continuous). And then one may project $\rho_\varepsilon \star h$ back onto N for ε small.

As a consequence of Lemma 1 we see that the components of $C^k(M,N)$ shrink as k increase, but they "do not change their shape". By contrast, we will see that the situation is *totally different* in the scale of Sobolev spaces $W^{1,p}$.

Let $1 \le p < \infty$ be a real number. Let M and N be as above and $N \subset \mathbb{R}^K$. The spaces $W^{1,p}(M,\mathbb{R})$ and $W^{1,p}(M,\mathbb{R}^K)$ are defined as usual and equipped with the standard norm $\|f\|_{W^{1,p}}$. Set

$$W^{1,p}(M,N) = \{f \in W^{1,p}(M,\mathbb{R}^K); f(x) \in N \text{ a.e.}\},$$

equipped with the distance

$$d(f,g) = \|f-g\|_{W^{1,p}(M,\mathbb{R}^K)}.$$

As a metric space, $W^{1,p}(M,N)$ admits connected components and also path-connected components. We do not know whether they are the same. This boils down to

Open problem 1. Given $f \in W^{1,p}(M,N)$, can one find $\varepsilon > 0$ (depending on f) such that

$$d(g,f) < \varepsilon \Longrightarrow g \sim f \text{ in } W^{1,p}?$$

Here we say that $g \sim f$ in $W^{1,p}$ if there is a path $h(t) \in C([0,1]; W^{1,p}(M,N))$ such that $h(0) = f$ and $h(1) = g$.

A simple but useful example is the case $M = N = S^2$ who's study was initiated in Brezis–Coron [1].

Let us examine the space $W^{1,p}(S^2,S^2)$, $1 \le p < \infty$, from the point of view of its components. One has to consider 3 different cases.

a) *Case* $p > 2$. It is not difficult to prove, following the same idea as in Lemma 1, and using the Sobolev imbedding,

Lemma 2. *Let $f, g \in W^{1,p}(S^2,S^2)$; then $f \sim g$ in $W^{1,p}(S^2,S^2)$ if and only if $f \sim g$ in C^0. In particular, the homotopy classes of $W^{1,p}(S^2,S^2)$ can be classified using their standard degree.*

b) *Case* $p = 2$. This is a very interesting case because it is a *limiting case* for the Sobolev imbedding. And $W^{1,2}$ is *not* contained in C^0, so that the standard notion of degree is not well-defined. One may nevertheless still define a degree. This was first done in Brezis–Coron [1] using the following strategy:

Step 1. If $f \in C^1(S^2,S^2)$ a well-know integral *formula* for computing the degree (going back to Kronecker) asserts that

$$deg(f) = \frac{1}{4\pi}\int_{S^2} \det(\nabla f) \tag{1}$$

When $f \in W^{1,2}(S^2,S^2)$ the integral in (1) still makes sense because $\det(\nabla f)$ is a 2×2 determinant and $\nabla f \in L^2$. Unfortunately it is not clear that the right-hand side in (1) is an integer. For this purpose one relies on

Step 2. This is an important observation due to Schoen–Uhlenbeck [1]. If $f \in W^{1,2}(S^2,S^2)$, then $\rho_\varepsilon \star f$ does *not* take its values into S^2. And $\rho_\varepsilon \star f$ does *not* converge uniformly to f (as $\varepsilon \to 0$) (otherwise f would be continuous). However one can prove that $|(\rho_\varepsilon \star f)(x)| \to 1$

uniformly in x. This is a consequence of the fact that $f \in VMO$ (see Brezis–Nirenberg [1]) which in turn follows from of Poincaré's inequality

$$\int_B \left| f - \fint_B f \right| \leq |B|^{1/2} \int_B |\nabla f|.$$

As a result one may consider

$$f_\varepsilon(x) = P\left((\rho_\varepsilon \star f)(x)\right)$$

where P is the projection on S^2, which is well-defined near S^2. Then $f_\varepsilon \in C^\infty(S^2, S^2)$ and $f_\varepsilon \to f$ in $W^{1,2}$.

Since $deg(f_\varepsilon) \in \mathbb{Z}$ and

$$deg(f_\varepsilon) \to \frac{1}{4\pi} \int_{S^2} \det(\nabla f) \text{ as } \varepsilon \to 0$$

we conclude that

$$\frac{1}{4\pi} \int_{S^2} \det(\nabla f) \in \mathbb{Z}.$$

We now *set*, for every $f \in W^{1,2}$

$$deg(f) = \frac{1}{4\pi} \int_{S^2} \det(\nabla f).$$

It was later observed in Brezis–Nirenberg [1] that $deg(f)$ can still be defined for every $f \in W^{1,2}$ *without* using formula (1), only Step 2. Indeed, if $f \in W^{1,2}(S^2, S^2)$ consider f_ε as above (for $\varepsilon < \varepsilon_0$). Then $deg(f_\varepsilon)$ is independent of ε, for $\varepsilon \in (0, \varepsilon_0)$, because one may use ε itself as homotopy parameter (or connect f_{ε_1} and f_{ε_2} via the homotopy $h(t) = f_{t_{\varepsilon_1 + (1-t)\varepsilon_2}}$). We take *as definition* of $deg\, f$, the integer $deg(f_\varepsilon)$, (for $0 < \varepsilon < \varepsilon_0$).

Proposition 1. *Assume $f, g \in W^{1,2}(S^2, S^2)$. Then $f \sim g$ in $W^{1,2}$ if and only if $deg(f) = deg(g)$. Consequently, $W^{1,2}(S^2, S^2)$ still admits infinitely many homotopy classes and they are classified using degree.*

c) *Case* $p < 2$. This case was not considered in Brezis–Coron [1]. But in Brezis–Nirenberg [1] we observed that no degree theory (in some reasonable sense) could be defined. Indeed the identity map can be homotopied to a constant map! This is done as follows. Fix any point $a \in \mathbb{R}^3$ with $|a| = 2$. Consider the path

$$h(t)(x) = \frac{x - ta}{|x - ta|}, \quad t \in [0, 1],\ x \in S^2.$$

Clearly h is smooth for $t \in [0, 1/2) \cup [1/2, 1]$ and $x \in S^2$. The only difficulty occurs at $t = 1/2$ because $a/2 \in S^2$. However it is not difficult to check that $h(1/2) \in W^{1,p}(S^2, S^2)$ for every $p < 2$ and moreover

$$h(t) \in C\big([0, 1]; W^{1,p}(S^2, S^2)\big)$$

for every $p < 2$. Finally it is clear that $h(1)$ has degree zero, and thus it can be homotopied to a constant in the C^0 sense (and thus in the C^1 sense by Lemma 1, and hence in the $W^{1,p}$ sense). Putting all this together we see that $h(0) = Id$ is homotopic to a constant in every $W^{1,p}$, $p < 2$.

We went one step further in Brezis–Li [1] and proved

Theorem 1. *The space $W^{1,p}(S^2, S^2)$ is path-connected for every $p < 2$.*

We now have a full picture for $W^{1,p}(S^2, S^2)$. When $p < 2$, $W^{1,p}(S^2, S^2)$ consists of one piece. At $p = 2$, $W^{1,2}(S^2, S^2)$ splits into infinitely many pieces. As p increases from 2 to ∞, these pieces "*shrink*" but *do not change their shape*.

Exactly the same type of conclusion holds for $W^{1,p}(S^n, S^n)$. When $p < n$, $W^{1,p}(S^n, S^n)$ is path-connected. At $p = n$, a degree theory is well-defined, and thus $W^{1,n}(S^n, S^n)$ splits into infinitely many pieces. As p increases from n to ∞, these pieces shrink without changing their shape.

At this stage, one would be inclined to believe that this is a general phenomenon. When $p < \dim M$, $W^{1,p}(M, N)$ is path-connected. As p increases from $\dim M$ to ∞, $W^{1,p}(M, N)$ admits path-connected component similar to the ones of $C^0(M, N)$ and they shrink without changing their shape. The second assertion (for $p \geq \dim M$) is indeed true. However the first assertion (for $p < \dim M$) is totally wrong. This was first pointed out in a very interesting paper of Rubinstein–Sternberg [1]. Namely, some "topology" still survives for $W^{1,p}(M, N)$ even when $p < \dim M$. In fact $W^{1,p}(M, N)$, may have a very rich structure from the point of view of homotopy classes when $p < \dim M$. I will present later some remark examples.

Theorem 2. (Rubinstein–Sternberg [1]). *Let $M = \Omega = S^1 \times D$ where D is the unit disc in $\mathbb{R}^2$, i.e. Ω is a solid torus in $\mathbb{R}^3$. Let $N = S^1$. Then any $f \in W^{1,2}(M, N)$ admits a well-defined degree (stable under $W^{1,2}$ convergence).*

More precisely write

$$f(x, \lambda) : S^1 \times D \to S^1$$

then

$$\varphi(\lambda) = deg\, f(\cdot, \lambda),$$

which is well-defined for a.e. $\lambda \in D$, (since $f(\cdot, \lambda) \in W^{1,2}(S^1, S^2)$ for a.e. $\lambda \in D$) is in fact a constant (a.e).

Remark 1. Is is quite surprizing that a degree may still be defined even though $p = 2 < 3 = \dim M$. I should point out however that the same conclusion *fails* for $p \in [1, 2)$. Note that when $p \in [1, 2)$ $\varphi(\lambda)$ is still well-defined for a.e. $\lambda \in D$. However φ is *not* constant in general.

Remark 2. The conclusion that φ is a constant may be related to a variety of results which have emerged in recent years in the work of Bourgain–Brezis–Mironescu [1] [2] (see also Brezis [1]) about conditions implying that a given measurable function is constant. Here is such a typical result.

Theorem 3. *Let $G \subset \mathbb{R}^N$ be a connected open set and let $\varphi : G \to \mathbb{Z}$ be a measurable function. Assume that*

$$\int_G \int_G \frac{|\varphi(\lambda) - \varphi(\mu)|^p}{|\lambda - \mu|^{N+1}} d\lambda \, d\mu < \infty$$

(any $p \geq 1$). Then φ is a constant (a.e.).

Here is an extension of Theorem 2,

Theorem 4. (Brezis–Li–Mironescu–Nirenberg [1]). *Let $M = \Omega = S^n \times \Lambda$, where $n \geq 1$, and $\Lambda \subset \mathbb{R}^k$ is any open connected set with $k \geq 1$.*

If $p \geq n+1$, any $f \in W^{1,p}(M, N)$ has a well-defined degree.

More precisely, write

$$f(x, \lambda) : S^n \times \Lambda \to S^n,$$

then

$$\varphi(\lambda) = deg\, f(\cdot, \lambda),$$

which is well-defined for a.e. $\lambda \in \Lambda$ (since $p > n$), is in fact a constant.

Remark 3. Theorem 4 shows again that some topology still "persists" much below the critical Sobolev exponent $p = \dim M$. The condition $p \geq n+1$ is usually *much weaker* than the condition $p \geq \dim M = n + k$, especially when k is large.

Going back to the study of the path-connected comments for $W^{1,p}(M, N)$, here is a complement to Theorem 4, which gives a *complete classification* of $W^{1,p}$ when the parameter space Λ is a ball.

Theorem 5. (Brezis–Li [1]). *Let $M = \Omega = S^n \times \Lambda$ where $n \geq 1$ and Λ is the unit ball in $\mathbb{R}^k$ (any $k \geq 1$). Let $f, g \in W^{1,p}(M, N)$. If $p \geq n+1$, then $f \sim g$ in $W^{1,p}$ if and only if $deg(f) = deg(g)$ (where deg is meant in the sense of Theorem 4). If $p < n+1$, then $f \sim g$ in $W^{1,p}$, i.e. $W^{1,p}$ is path-connected.*

At this stage we decide with Yanyan Li to initiate a general investigation of the homotopy classes of $W^{1,p}(M, N)$ for a general pair of manifolds M and N, and for a general $p \geq 1$.

Here is a first (somewhat suprizing) result

Theorem 6. (Brezis–Li [1]) *For any pair M and N with $\dim M \geq 2$ and for any $p \in [1, 2)$, $W^{1,p}(M, N)$ is path-connected.*

We also introduced in Brezis–Li [1] a concept which plays a very important role in describing the possible changes in homotopy classes for $W^{1,p}$ as p varies. Here it is.

Let $p > 1$. Let $0 < \varepsilon < p - 1$. Clearly

$$W^{1,p+\varepsilon}(M, N) \subset W^{1,p-\varepsilon}(M, N).$$

Given $f \in W^{1,p}(M, N)$, we denote by $[f]_p$ its homotopy class in $W^{1,p}$. We denote by $W^{1,p}/_{\underset{p}{\sim}}$ the quotient of $W^{1,p}$ by the equivalence relation $f \underset{p}{\sim} g$ (meaning $f \sim g$ in $W^{1,p}$).

Obviously, if $f, g \in W^{1,p+\varepsilon}$ and $f \sim g$ in $W^{1,p+\varepsilon}$, then $f \sim g$ in $W^{1,p-\varepsilon}$. As a result we have a canonical map

$$i_{p+\varepsilon,p-\varepsilon} : W^{1,p+\varepsilon}/_{\widetilde{p+\varepsilon}} \to W^{1,p-\varepsilon}/_{\widetilde{p-\varepsilon}}.$$

Definition. We say that a *change of topology* occurs at p if for every $\varepsilon \in (0, p-1)$, the map $i_{p+\varepsilon,p-\varepsilon}$ is *not* bijective. We denote by $CT(M, N)$ the set of p's for which a change of topology occurs at p.

If a change of topology occurs at p, two things may happen, $i_{p+\varepsilon,p-\varepsilon}$ is not injective or $i_{p+\varepsilon,p-\varepsilon}$ is not surjective (or both!). Roughly speaking *not injective* means that there are 2 maps $f, g \in W^{1,p+\varepsilon}$ such that f and g are *not* homotopic in $W^{1,p+\varepsilon}$ while $f \sim g$ in $W^{1,p-\varepsilon}$.

In other words, one component of $W^{1,q}(M, N)$ *splits* into two distinct components as q *increases* from $p-\varepsilon$ to $p+\varepsilon$. Another view point is to say that two distinct components of $W^{1,q}$ have *coalesced* as q decreases from $p+\varepsilon$ to $p-\varepsilon$. This is a common situation, which we have already encountered above. For example a change of topology for $W^{1,p}(S^2, S^2)$ occurs at $p = 2$, because $i_{2+\varepsilon,2-\varepsilon}$ is *not injective*: any two maps $f, g \in W^{1,2+\varepsilon}$ are homotopic in $W^{1,2-\varepsilon}$, while they need not be in $W^{1,2+\varepsilon}$ (unless their degree is the same).

On the other hand, the fact that $i_{p+\varepsilon,p-\varepsilon}$ is *not surjective* means that a new component of $W^{1,q}$ appears (out of nowhere!) as q decreases from $p + \varepsilon$ to $p - \varepsilon$. This situation is more unusual and it does not occur in the example $W^{1,p}(S^n, S^n)$. It means that some map $f \in W^{1,p-\varepsilon}$ *cannot be homotopied* in $W^{1,p-\varepsilon}$ to any $g \in W^{1,p+\varepsilon}$. In particular, such a map f cannot be smooth and it cannot be homotopied in $W^{1,p-\varepsilon}$ to any smooth map –so it must have a rather "solid" singularity!

We may now reformulate Theorem 1 and Theorem 5 using the above notion.

Theorem 1'. *We have $CT(S^2, S^2) = \{2\}$ or more generally $CT(S^n, S^n) = \{n\}$ for any $n \geq 2$.*

Theorem 5'. *We have $CT(S^n \times \Lambda, S^n) = \{n + 1\}$, where Λ is the unit ball in $\mathbb{R}^k$ (any $k \geq 1$).*

We also observed in Brezis–Li [1] that a change in topology may occurs at *several* values of p- not just one, as in the above examples. Here is such a situation, with a "cascade of mergings"

Theorem 6. *We have $CT(S^1 \times S^2, S^1 \times S^2) = \{2, 3\}$.*

This is somewhat natural because we have here (at least), two invariants: write $f = (f_1(x, y), f_2(x, y))$ and set $d_1 = deg\, f_1(\cdot, y)$ and $d_2 = deg\, f_2(x, \cdot)$. Note that d_1 is well-defined and independent of $y \in S^2$ when $f \in W^{1,p}$ and $p \geq 2$ (by Theorem 4), while d_2 is well-defined and independent of $x \in S^1$ only when $p \geq 3$ (again by Theorem 4). Some new invariants appear, as p increases from 1 to ∞, when crossing the values $p = 2$ and $p = 3$.

We also made in Brezis–Li [1] two conjectures:

Conjecture 1. $CT(M,N)$ consists only of integers, i.e. change of topology for $W^{1,p}$ occurs only when p is an integer.

Conjecture 2. Any map $f \in W^{1,p}(M,N)$ can always be connected in $W^{1,p}$ to a smooth map (any p, any M, N).

They were both solved in a beautiful piece of work by Hang–Lin [1].

The answer to Conjecture 1 is positive:

Theorem 7. (Hang–Lin [1]) *$CT(M,N)$ consists only of integers.*

Theorem 7 is an immediate consequence of the following remarkable result which provides a "reduction" of the study of homotopy classes for $W^{1,p}$ to more classical concepts in Topology. Given $p \geq 2$, consider $M^{[p]-1}$ the $([p]-1)$-skeleton of M.

Theorem 8. (Hang–Lin [1]) *Let $f, g \in W^{1,p}(M,N)$. Then $f \sim g$ in $W^{1,p}$ if and only if $f_{|\Sigma} \sim g_{|\Sigma}$ in C^0 for a generic Σ in $M^{[p]-1}$.*

Note that $\dim \Sigma \leq [p]-1$, and for a generic Σ, $f_{|\Sigma} \in W^{1,p}(\Sigma)$, while $p > \dim \Sigma$ (since $[p] \geq \dim \Sigma + 1$). Thus for a generic Σ, $f_{|\Sigma} \in C^0$, by the Sobolev imbedding.

Concerning Conjecture 2, we had presented in Brezis–Li [1] several cases where Conjecture 2 is true: for example when $\dim M = 2$ (any N), or $\dim M = 3$ and $\partial M \neq \phi$ any N), or $N = S^1$ (any M). However Hang-Lin [1] found some situations where Conjecture 2 fails.

Here is such an example:

Theorem 9. (Hang–Lin [1]) *Let $M = \mathbb{RP}^3$ and $N = \mathbb{RP}^2$. Then $CT(M,N) = \{2,3\}$. Moreover there are maps $f \in W^{1,p}(M,N)$, with $p \in (2,3)$, such that f cannot be connected in $W^{1,p}$ to any smooth map.*

References.

J. Bourgain, H. Brezis and P. Mironescu [1], Lifting in Sobolev spaces, J. Analyse Math. **80** (2000), p. 37-86.

[2], Another look at Sobolev spaces, in *Optimal Control and Partial Differential Equations* (J.L. Menaldi, E. Rofman et A. Sulem, eds) a volume in honour of A. Bensoussan's 60th birthday, IOS Press, 2001, p. 439–455.

H. Brezis [1], How to recognize constant functions. Connections with Sobolev spaces, Russian Math. Surveys (volume in honor of M. Vishik, to appear).

H. Brezis and J.-M. Coron [1], Large solutions for harmonic maps in two dimensions, Comm. Math. Phys. **92** (1983), p. 203–215.

H. Brezis and Y. Li [1], Topology and Sobolev spaces, J. Funct. Anal. **183** (2001), p. 321–369.

H. Brezis, Y.Li, P. Mironescu and L. Nirenberg [1], Degree and Sobolev spaces, Topological methods in Nonlinear Analysis **13** (1999), p. 181–190.

H. Brezis and L. Nirenberg [1], Degree theory and BMO, Part I : compact manifolds without boundaries, Selecta Math. **1** (1995), p. 197–263.

F.B. Hang and F.H. Lin [1], Topology of Sobolev mappings (to appear).

J. Rubinstein and P. Sternberg [1], Homotopy classification of minimizers of the Ginzburg-Landau energy and the existence of permanent currents, Comm. Math. Phys. **179** (1996), p. 257–263.

R. Schoen and K.Uhlenbeck [1], Boundary regularity and the Dirichlet problem for harmonic maps, J. Differential Geom. **18** (1983), p. 253–268.

Laboratoire Jacques-Louis Lions
Université Pierre et Marie Curie
Boîte courrier 187
4 place Jussieu
75252 Paris cedex 05

email: brezis@ann.jussieu.fr, brezis@ccr.jussieu.fr