SOME SIMPLE NONLINEAR PDE'S WITHOUT SOLUTIONS

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0. Introduction.

The original motivation of this work is the following. Consider the simple problem

$$\begin{cases} -\Delta u = a(x)u^2 + f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{0.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N, N \geq 3$. If $a(x) \in L^p(\Omega)$ and $p > N/2$, then for any $f \in L^p(\Omega)$ with $\|f\|_p$ small, problem (0.1) has a unique small solution u in $W^{2,p}(\Omega)$. This is an easy consequence of the Inverse Function Theorem applied to $F(u) = -\Delta u - a(x)u^2$ which maps $X = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ into $Y = L^p(\Omega)$ (recall that $W^{2,p}(\Omega) \subset L^\infty(\Omega)$ by the Sobolev imbedding theorem), since its differential at $0, DF(0) = -\Delta$ is bijective.

As a special case suppose $a(x) = |x|^{-\alpha}$ in a domain Ω containing 0, with $0 < \alpha < 2$. Then for any small constant c the problem

$$\begin{cases} -\Delta u = \dfrac{u^2}{|x|^\alpha} + c & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{0.2}$$

has a unique small solution.

The case $\alpha = 2$ is interesting since $a(x) = |x|^{-2}$ does **not** belong to $L^p(\Omega)$ for $p > N/2$. One may then wonder what happens to the problem

$$\begin{cases} -\Delta u = \dfrac{u^2}{|x|^2} + c & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{0.3}$$

On the one hand, a **formal** computation suggests that since the linearized operator at 0 is $-\Delta$, which is bijective, problem (0.3) has a solution for small c. On the other hand, the F above does **not** map $X = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ into $Y = L^p(\Omega)$ for any $1 < p < \infty$. One may then try to construct other function spaces, for example weighted spaces, where the Inverse Function Theorem might apply. This is doomed to fail. In fact, our main results show that for any constant $c > 0$ (no matter how small) problem (0.3) has **no** solution, even in a weak sense. When $c < 0$, problem (0.3) does, however, have a solution (see Remark 1.4).

In Sections 1 and 4 we propose several notions of weak solutions and establish nonexistence. A basic ingredient in Section 1 is the following:

Theorem 0.1. *Assume* $0 \in \Omega$. *If* $u \in L^1_{\mathrm{loc}}(\Omega)$, $u \geq 0$ *a.e. with* $\dfrac{u^2}{|x|^2} \in L^1_{\mathrm{loc}}(\Omega)$ *is such that*

$$-\Delta u \geq \frac{u^2}{|x|^2} \qquad \text{in } \mathcal{D}'(\Omega) \tag{0.4}$$

then $u \equiv 0$.

The proof of Theorem 0.1 uses an adaptation of a method introduced in [4]. In Section 4 we prove a stronger result, namely:

Theorem 0.2. *Assume* $0 \in \Omega$. *If* $u \in L^2_{\mathrm{loc}}(\Omega \backslash \{0\})$, $u \geq 0$ *a.e. is such that*

$$-|x|^2 \Delta u \geq u^2 \quad \text{in } \mathcal{D}'(\Omega\backslash\{0\}) \tag{0.5}$$

then $u \equiv 0$.

Theorem 0.2 is proved using appropriate powers of testing functions—an idea due to Baras-Pierre [2]. As a consequence we obtain the **nonexistence of local solutions** (i.e., in any neighborhood of 0, without prescribing any boundary condition) for a very simple nonlinear equation:

Theorem 0.3. *Assume* $0 \in \Omega$ *and* $c > 0$. *There is no function* $u \in L^2_{\mathrm{loc}}(\Omega\backslash\{0\})$ *satisfying*

$$-|x|^2 \Delta u = u^2 + c \quad \text{in } \mathcal{D}'(\Omega\backslash\{0\}). \tag{0.6}$$

In Section 3 we examine what happens to a natural approximation procedure of (0.3). Consider for example the equation

$$\begin{cases} -\Delta u = \dfrac{\min\{u^2, n\}}{|x|^2 + (1/n)} + c & \text{in } \Omega, \quad c > 0, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{0.7}$$

For any n there is a minimal solution u_n. We prove that $u_n(x) \to +\infty$ **for every** $x \in \Omega$ as $n \to \infty$, i.e., there is **complete blow-up** in the sense of Baras-Cohen [1]. Again, this rules out any reasonable notion of weak solution for (0.3).

In Section 2 we extend the previous results to more general problems such as

$$-\Delta u = a(x)g(u) + b(x)$$

assuming only that $g \geq 0$ on $\mathbb{R}$, g is nondecreasing on $[0,\infty)$ and

$$\int^{\infty} \frac{ds}{g(s)} < \infty,$$

with $a \in L^1_{\text{loc}}(\Omega)$, $a \geq 0$ and

$$\int \frac{a(x)}{|x|^{N-2}} = \infty.$$

The original motivation of our research came from observations made in [6] and [4].

For any $N \geq 3$ the problem

$$\begin{cases} -\Delta u = 2(N-2)e^u & \text{in } B_1 = \{x \in \mathbb{R}^N; |x| < 1\} \\ u = 0 & \text{on } \partial B_1 \end{cases}$$

admits the weak solution $\overline{u}(x) = \log(1/|x|^2)$. It was observed in [6] that when $N \geq 11$ the linearized operator at $\overline{u}$ namely

$$\begin{aligned} Lv &= -\Delta v - 2(N-2)e^{\overline{u}}v \\ &= -\Delta v - \frac{2(N-2)}{|x|^2}v \end{aligned}$$

is **coercive** and thus formally bijective; this is a simple consequence of Hardy's inequality:

$$\int |\nabla v|^2 \geq \frac{(N-2)^2}{4} \int \frac{v^2}{|x|^2} \qquad \forall v \in H^1_0(B_1)$$

(note that $\dfrac{(N-2)^2}{4} > 2(N-2)$ when $N \geq 11$). On the other hand the results of [4] show that when $N \geq 10$ the perturbed problem

$$\begin{cases} -\Delta u = 2(N-2)e^u + c & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1 \end{cases}$$

has **no** solution even in a weak sense and no matter how small c is, provided $c > 0$.

This strange "failure" of the Inverse Function Theorem is only apparent. As was pointed out in [6] this just means that there is no functional setting in which it can be correctly applied. We have tried here, in the spirit of Open Problem 6 in [6], to analyze the same phenomenon for simple examples in low dimensions.

After our investigation was completed we learned about an interesting work of N. J. Kalton and I. E. Verbitsky [8] (which was carried out independently of ours). Consider for example the problem

$$\begin{cases} -\Delta u = a(x)u^2 + c & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{0.8}$$

with $a \in L^1_{\mathrm{loc}}(\Omega)$, $a \geq 0$ and c a positive constant.

Their result says that if (0.8) has a weak solution, then necessarily

$$G(a\delta^2) \leq C\delta \quad \text{in } \Omega \tag{0.9}$$

for some constant C, where $G = (-\Delta)^{-1}$ (with zero boundary condition) and $\delta(x) = \mathrm{dist}(x, \partial\Omega)$. In particular, if $0 \in \Omega$ and $a(x) = 1/|x|^2$, then $G(a\delta^2) \simeq |\log|x||$ as $x \to 0$ and thus (0.9) fails; hence (0.8) has no weak solution.

We present in Section 5 a very simple proof of the main result of [8] using a variant of the method developed in Section 1.

Finally, in Section 6 we present a parabolic analogue of Theorem 0.2. It extends, in particular, a result of I. Peral and J. L. Vázquez [12]. Namely, the problem

$$\begin{cases} u_t - \Delta u = 2(N-2)e^u & \text{in } B_1 \times (0,T) \\ \qquad u = 0 & \text{on } \partial B_1 \times (0,T) \\ \quad u(x,0) = u_0 \end{cases}$$

with $u_0 \geq \overline{u} = \log(1/|x|^2)$, $u_0 \not\equiv \overline{u}$, has no solution $u \geq \overline{u}$ even for small time: **instantaneous** and **complete blow-up** occurs.

The plan of the paper is the following:

(1) Proof of Theorem 0.1
(2) General nonlinearities
(3) Complete blow-up
(4) Very weak solutions. Proofs of Theorems 0.2 and 0.3
(5) Connection with a result of Kalton-Verbitsky
(6) Evolution equations

Notation. Throughout this paper, Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 1$, such that $0 \in \Omega$. We write

$$\delta(x) = \mathrm{dist}(x, \partial\Omega)$$

and $L^1_\delta(\Omega) = L^1(\Omega, \delta(x)dx)$. We denote by $C_0^\infty(\Omega)$ the space of C^∞ functions with compact support in Ω, and by $\mathcal{D}'(\Omega)$ the space of distributions in Ω. By C we denote a positive constant which may be different in each inequality.

1. Proof of Theorem 0.1.

In this section we prove Theorem 0.1 and its consequences. We first introduce some terminology about weak solutions.

Definition 1.1. Let $h(x,u)$ be a Caratheodory function in $\Omega \times \mathbb{R}$, that is, $h(x,u)$ is measurable in x and continuous in u for a.e. x.

(a) We say that

$$-\Delta u = h(x,u) \qquad \text{in } \mathcal{D}'(\Omega)$$

if $u \in L^1_{\mathrm{loc}}(\Omega)$, $h(x,u) \in L^1_{\mathrm{loc}}(\Omega)$ and $-\int u\Delta\varphi = \int h(x,u)\varphi$ for any $\varphi \in C_0^\infty(\Omega)$.

(b) We say that

$$-\Delta u \geq h(x,u) \qquad \text{in } \mathcal{D}'(\Omega)$$

if $u \in L^1_{\mathrm{loc}}(\Omega), h(x,u) \in L^1_{\mathrm{loc}}(\Omega)$ and $-\int u\Delta\varphi \geq \int h(x,u)\varphi$ for any $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 0$.

(c) We say that u is a weak solution of

$$\begin{cases} -\Delta u = h(x,u) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases}$$

if $u \in L^1(\Omega)$, $h(x,u) \in L^1_\delta(\Omega)$ and $-\int_\Omega u\Delta\zeta = \int_\Omega h(x,u)\zeta$ for any $\zeta \in C^2(\overline{\Omega})$ with $\zeta = 0$ on $\partial\Omega$.

The following is the main result of this section; it is Theorem 0.1 of the Introduction.

Theorem 1.2. *Let $N \geq 1$ and $u \in L^1_{\mathrm{loc}}(\Omega)$ satisfy $u \geq 0$ a.e. in Ω, $\dfrac{u^2}{|x|^2} \in L^1_{\mathrm{loc}}(\Omega)$ and*

$$-\Delta u \geq \frac{u^2}{|x|^2} \quad \text{in } \mathcal{D}'(\Omega). \tag{1.1}$$

Then $u \equiv 0$.

This theorem easily implies two nonexistence results. The first one deals with the following boundary value problem.

Corollary 1.3. *Let $N \geq 1$ and $f \in L^1_\delta(\Omega)$ satisfy $f \geq 0$ a.e. and $f \not\equiv 0$. Then there is no weak solution of*

$$\begin{cases} -\Delta u = \dfrac{u^2}{|x|^2} + f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

in the sense of Definition 1.1.

Remark 1.4. When $f \equiv 0$, problem (1.2) has $u \equiv 0$ as the only weak solution; this follows immediately from Theorem 1.2. When $N \geq 3$ and $f \in L^1(\Omega)$ satisfies $f \leq 0$ and $f \not\equiv 0$, (1.2) has a weak solution $u \leq 0$, which is the unique solution among nonpositive functions. This fact is a consequence of a more general result of Gallouët and Morel [7], which extends work of Brezis and Strauss [5].

As a consequence of Theorem 1.2, we may write down a simple PDE without local solutions, i.e., no solution exists in any neighborhood of 0. Here, we do not impose any boundary condition.

Corollary 1.5. *Let $N \geq 3$ and $c > 0$ be any positive constant. Then there is no function u such that $\dfrac{u^2}{|x|^2} \in L^1_{\mathrm{loc}}(\Omega)$ and*

$$-\Delta u = \frac{u^2 + c}{|x|^2} \quad \text{in } \mathcal{D}'(\Omega). \tag{1.3}$$

Remark 1.6. In contrast with (1.3), the equation

$$-\Delta u = \frac{u^2}{|x|^2} + c \tag{1.4}$$

has a weak solution in some neighborhood of 0, if $N \geq 3$ and $c > 0$ is a constant. This solution is nonpositive and can be obtained from the results of [7] as follows.

We introduce the new unknown $v = -u - c|x|^2$ so that (1.4) becomes

$$-\Delta v + \frac{v^2}{|x|^2} + 2cv = c(2N-1) - c^2|x|^2 \equiv g(x). \tag{1.5}$$

We solve (1.5) on B_R with the boundary condition $v = 0$ on ∂B_R. Note that $g \geq 0$ on B_R provided R is sufficiently small ($R^2 \leq (2N-1)/c$). The results of [7] give a unique solution $v \geq 0$.

In Section 4 we will prove stronger nonexistence results for problems (1.2) and (1.3).

The proof of Theorem 1.2 is based on the following variant of Kato's inequality [9].

Lemma 1.7. *Let $u \in L^1_{\rm loc}(\Omega)$ and $f \in L^1_{\rm loc}(\Omega)$ satisfy*

$$-\Delta u \geq f \qquad \text{in } \mathcal{D}'(\Omega).$$

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a C^1, concave function such that

$$0 \leq \phi' \leq C \qquad \text{in } \mathbb{R}$$

for some constant C. Then $\phi(u) \in L^1_{\rm loc}(\Omega)$ and

$$-\Delta \phi(u) \geq \phi'(u) f \qquad \text{in } \mathcal{D}'(\Omega).$$

The proof is standard, smoothing u and φ by convolution; see also Lemma 2 in [4].

The proof of Theorem 1.2 is a variant of a method introduced in [4]. Consider the function $\phi(s) = \dfrac{1}{\varepsilon} - \dfrac{1}{s}$. It is nonnegative, bounded, increasing and concave in the interval $(\varepsilon, \infty), \varepsilon > 0$. Note that if $u \geq \varepsilon$ satisfies

$$-\Delta u \geq \frac{u^2}{|x|^2},$$

then

$$-\Delta \phi(u) \geq \phi'(u) \frac{u^2}{|x|^2} = \frac{1}{|x|^2}.$$

This will lead to a contradiction with the fact that $\phi(u)$ is bounded. The details go as follows.

Proof of Theorem 1.2. Suppose that u is as in the theorem, and that $u \not\equiv 0$. Since $u \geq 0, u \not\equiv 0$, $-\Delta u \geq 0$ in $\mathcal{D}'(\Omega)$ and Ω is connected, we have that

$$u \geq \varepsilon \qquad \text{a.e. in } B_\eta,$$

for some $\varepsilon > 0$ and $B_\eta = B_\eta(0)$ with closure in Ω. If $N \leq 2$, this is already a contradiction with $u^2/|x|^2 \in L^1_{\rm loc}(\Omega)$. Suppose $N \geq 3$. Let

$$\phi(s) = \frac{1}{\varepsilon} - \frac{1}{s}, \quad \text{for } s \geq \varepsilon,$$

and extend it by $\phi(s) = \dfrac{1}{\varepsilon^2}(s - \varepsilon)$ for $s \leq \varepsilon$. Note that $\phi : \mathbb{R} \to \mathbb{R}$ is C^1, concave and $0 \leq \phi' \leq \dfrac{1}{\varepsilon^2}$, so that ϕ satisfies all the conditions of Lemma 1.7. Recall that $u \geq \varepsilon$ in B_η, and consider

$$v = \phi(u) = \frac{1}{\varepsilon} - \frac{1}{u} \quad \text{in } B_\eta.$$

It satisfies $0 \le v \le \dfrac{1}{\varepsilon}$ in B_η and, by Lemma 1.7,

$$-\Delta v \ge \phi'(u)\frac{u^2}{|x|^2} = \frac{1}{|x|^2} \quad \text{in } \mathcal{D}'(B_\eta).$$

Hence $v - \dfrac{1}{N-2}\log\dfrac{1}{|x|} \in L^1(B_\eta)$ and

$$-\Delta\left(v - \frac{1}{N-2}\log\frac{1}{|x|}\right) \ge 0 \quad \text{in } \mathcal{D}'(B_\eta),$$

which implies

$$v - \frac{1}{N-2}\log\frac{1}{|x|} \ge -C \quad \text{in } B_{\eta/2}$$

for some constant $C > 0$. In particular, $v(x) \to +\infty$ as $x \to 0$, which is a contradiction with the fact that $v \le \dfrac{1}{\varepsilon}$. □

Remark 1.8. In the previous proof we could have used (in the spirit of [4]) the function

$$w = \frac{u}{\varepsilon u + 1}$$

for any $\varepsilon > 0$, instead of $v = \dfrac{1}{\varepsilon} - \dfrac{1}{u}$. Note that

$$\psi(s) = \frac{s}{\varepsilon s + 1}$$

satisfies $\psi'(s) = (\varepsilon s + 1)^{-2}$ and hence

$$\psi'(s)s^2 = \psi(s)^2.$$

Moreover, ψ is bounded in $[0,\infty)$ and satisfies all the conditions of Lemma 1.7 in $[0,\infty)$. In particular, $-\Delta u \ge \dfrac{u^2}{|x|^2}$, $u \ge 0$, implies

$$-\Delta w \ge \psi'(u)\frac{u^2}{|x|^2} = \frac{w^2}{|x|^2},$$

and we can conclude as before, since $\dfrac{w^2}{|x|^2} \ge \dfrac{\nu}{|x|^2}$, for some constant $\nu > 0$, in a subdomain of Ω.

Analogous versions of the functions ϕ and ψ will appear, in Sections 2 and 5, when the nonlinearity u^2 is replaced by more general nonlinearities $g(u)$.

Finally, we use Theorem 1.2 to prove the nonexistence results of this section.

Proof of Corollary 1.3. Suppose that u is a weak solution of (1.2). Since $\int_\Omega u(-\Delta\zeta) \geq 0$ for any $\zeta \in C^2(\overline{\Omega})$ with $\zeta \geq 0$ in Ω and $\zeta = 0$ on $\partial\Omega$, we easily deduce $u \geq 0$ in Ω. Moreover, $-\Delta u \geq \dfrac{u^2}{|x|^2}$ in $\mathcal{D}'(\Omega)$. We obtain, by Theorem 1.2, $u \equiv 0$. This is a contradiction with (1.2), since $f \not\equiv 0$. □

Proof of Corollary 1.5. Suppose that u is a solution of (1.3) in $\mathcal{D}'(\Omega)$. Then $-\Delta u \geq \dfrac{c}{|x|^2}$ in $\mathcal{D}'(B_\eta)$, for some ball $B_\eta = B_\eta(0)$ with closure in Ω. As in the proof of Theorem 1.2, we deduce that

$$u - \frac{c}{N-2}\log\frac{1}{|x|} \geq -C \qquad \text{in } B_{\eta/2},$$

for some constant C. In particular, $u \geq 0$ in B_ν for some small $\nu > 0$.

We therefore have

$$u \geq 0 \quad \text{in } B_\nu$$
$$-\Delta u \geq \frac{u^2}{|x|^2} \quad \text{in } \mathcal{D}'(B_\nu).$$

By Theorem 1.2, $u \equiv 0$ in B_ν, which is a contradiction with equation (1.3). □

2. General nonlinearities.

In this section we extend the previous nonexistence results to more general problems of the form

$$-\Delta u = a(x)g(u) + b(x).$$

We assume (here and throughout the rest of this section) that $g : \mathbb{R} \longrightarrow [0, \infty)$ is continuous on $\mathbb{R}$, nondecreasing on $[0, \infty)$, $g(s) > 0$ if $s > 0$, and

$$\int_1^\infty \frac{ds}{g(s)} < \infty. \tag{2.1}$$

Power functions $g(u) = u^p$, with $p > 1$ are examples of such nonlinearities. We suppose that $N \geq 3$.

For the potential, $a(x)$ we assume in this section that $a \in L^1_{\text{loc}}(\Omega)$, $a \geq 0$ in Ω, and

$$\int_{B_\eta(0)} \frac{a(x)}{|x|^{N-2}} = \infty \tag{2.2}$$

for some $\eta > 0$ small enough (or, equivalently, for any $\eta > 0$ small). We then have the following extensions of the results of Section 1.

Theorem 2.1. *Assume $N \geq 3$, (2.1) and (2.2).*

(a) *Let $u \geq 0$ a.e. in Ω satisfy*

$$-\Delta u \geq a(x)g(u) \quad \text{in } \mathcal{D}'(\Omega).$$

Then $u \equiv 0$.

(b) *Let $f \in L^1_\delta(\Omega)$ satisfy $f \geq 0$ a.e. and $f \not\equiv 0$. Then there is no weak solution of*

$$\begin{cases} -\Delta u = a(x)g(u) + f(x) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases}$$

(c) *If $b(x)$ satisfies the same conditions as $a(x)$, then there is no weak solution of*

$$-\Delta u = a(x)g(u) + b(x) \quad \text{in } \mathcal{D}'(\Omega).$$

This theorem is proved with the same method as in the previous section. We only need to adapt two points. First, $\phi(s) = \dfrac{1}{\varepsilon} - \dfrac{1}{s}$ has to be replaced by a solution of

$$\phi'(s) = \frac{1}{g(s)} \quad \text{if } s \geq \varepsilon,$$

where $\varepsilon > 0$ is a constant.

We therefore define

$$\phi(s) = \int_\varepsilon^s \frac{dt}{g(t)} \qquad \text{if } s \geq \varepsilon,$$

which satisfies $0 \leq \phi \leq \int_\varepsilon^\infty \frac{dt}{g(t)} < \infty$ in $[\varepsilon, \infty)$ (by assumption (2.1)), $\phi(\varepsilon) = 0$, $\phi'(\varepsilon) = \frac{1}{g(\varepsilon)}$, ϕ is C^1, concave (since $\phi'(s) = \frac{1}{g(s)}$ is nonincreasing) and $0 \leq \phi' \leq \frac{1}{g(\varepsilon)}$ in $[\varepsilon, \infty)$. Extending ϕ by $\phi(s) = \frac{1}{g(\varepsilon)}(s - \varepsilon)$ for $s \leq \varepsilon$, we obtain a function ϕ on all of $\mathbb{R}$, satisfying the conditions of Lemma 1.7 and with ϕ bounded from above.

To complete the proof of Theorem 2.1, we only need to consider a solution $w \in L^1_{\text{loc}}(B_\eta)$, where $B_\eta = B_\eta(0)$ with closure in Ω, of

$$-\Delta w = a(x) \qquad \text{in } \mathcal{D}'(B_\eta)$$

(a solution always exists since $a \in L^1(B_\eta)$) and show that

$$\operatorname*{ess\,inf}_{B_{1/n}(0)} w \longrightarrow +\infty \quad \text{as } n \to +\infty. \tag{2.3}$$

For this purpose, we consider the convolution in B_η, $\widetilde{w} = a \star \frac{C_N}{|x|^{N-2}}$, where C_N is chosen such that $-\Delta(C_N|x|^{2-N}) = \delta_0$. Then $w - \widetilde{w}$ is harmonic in B_η and hence bounded in $B_{\eta/2}$. In particular, it suffices to show (2.3) for $\widetilde{w}$. But this is true since, for $|x| \leq \frac{1}{n}$,

$$\begin{aligned}\widetilde{w}(x) &= \int_{B_\eta} \frac{C_N a(y)}{|y-x|^{N-2}} dy \\ &\geq C \int_{B_\eta} \frac{a(y)}{|y|^{N-2} + \frac{1}{n^{N-2}}} dy \quad \to +\infty\end{aligned}$$

as $n \to \infty$, by (2.2).

3. Complete blow-up.

In Corollary 1.3 and Theorem 2.1 we have proved the nonexistence of weak solutions of some problems of the form

$$\begin{cases} -\Delta u = a(x)g(u) + f(x) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

In this section we prove that, under the same assumptions on $a(x), g(u)$ and $f(x)$ made in Section 2, approximate solutions of (3.1) blow up everywhere in Ω, that is, there is complete blow up. More precisely, we have the following.

Let $g(u)$ and $a(x)$ be as in Section 2. Suppose that $f \in L^1_\delta(\Omega)$, $f \geq 0$ a.e. and $f \not\equiv 0$. Let (g_n) be a sequence of nonnegative, bounded, nondecreasing and continuous functions in $[0, \infty)$ such that $g_n(u)$ increases pointwise to $g(u)$. Let a_n and f_n be two sequences of nonnegative bounded functions in Ω, increasing pointwise to a and f, respectively.

Theorem 3.1. *Under the above assumptions, let $\underline{u}_n$ be the minimal nonnegative solution of the approximate problem*

$$\begin{cases} -\Delta u = a_n(x)g_n(u) + f_n(x) & \textit{in } \Omega \\ \quad u = 0 & \textit{on } \partial\Omega. \end{cases} \tag{3.2$_n$}$$

Then, as $n \to +\infty$,

$$\frac{\underline{u}_n(x)}{\delta(x)} \to +\infty \quad \textit{uniformly in } \Omega.$$

In the proof of Theorem 3.1 we use two ingredients. First, the nonexistence result of Theorem 2.1(b) (see also Corollary 1.3) and, second, the following estimate for the linear Laplace equation. It asserts that, for some positive constant c,

$$G(x,y) \geq c\delta(x)\delta(y) \quad \text{in } \overline{\Omega} \times \overline{\Omega}, \tag{3.3}$$

where G is the Green's function of the Laplacian in Ω with zero Dirichlet condition. In an equivalent way, we can state this lower bound on G as follows.

Lemma 3.2. *Suppose that $h \geq 0$ belongs to $L^\infty(\Omega)$. Let v be the solution of*

$$\begin{cases} -\Delta v = h & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Then

$$\frac{v(x)}{\delta(x)} \geq c \int_\Omega h\delta \qquad \forall x \in \Omega, \tag{3.4}$$

where $c > 0$ is a constant depending only on Ω.

Estimate (3.3) was proved by Morel and Oswald [11] (in unpublished work), and by Zhao [13] (in a stronger form). For the convenience of the reader we give a simple proof of (3.3).

Proof of Lemma 3.2. We proceed in two steps.

Step 1. For any compact set $K \subset \Omega$, we first show

$$v(x) \geq c \int_\Omega h\delta \qquad \forall x \in K, \tag{3.5}$$

where c is a positive constant depending only on K and Ω. To prove (3.5), let $\rho = \operatorname{dist}(K, \partial\Omega)/2$, and take m balls of radius ρ such that

$$K \subset B_\rho(x_1) \cup \ldots \cup B_\rho(x_m) \subset \Omega.$$

Let $\zeta_1, \ldots, \zeta_m$ be the solutions of

$$\begin{cases} -\Delta \zeta_i = \chi_{B_\rho(\xi_i)} & \text{in } \Omega \\ \zeta_i = 0 & \text{on } \partial\Omega, \end{cases}$$

where χ_A denotes the characteristic function of A. The Hopf boundary lemma implies that there is a constant $c > 0$ such that

$$\zeta_i(x) \geq c\delta(x) \qquad \forall x \in \Omega \qquad \forall 1 \leq i \leq m.$$

Here and in the rest of the proof, c denotes various constants depending only on K and Ω. Let now $x \in K$, and take a ball $B_\rho(x_i)$ containing x. Then $B_\rho(x_i) \subset B_{2\rho}(x) \subset \Omega$, and since $-\Delta v \geq 0$ in Ω, we conclude

$$\begin{aligned} v(x) &\geq \fint_{B_{2\rho}(x)} v = c \int_{B_{2\rho}(x)} v \geq c \int_{B_\rho(x_i)} v \\ &= c \int_\Omega v(-\Delta \zeta_i) = c \int_\Omega h\zeta_i \\ &\geq c \int_\Omega h\delta. \end{aligned}$$

Step 2. Fix a smooth compact set $K \subset \Omega$. By (3.5), $v \geq c \int_\Omega h\delta$ in K, so that it suffices to prove (3.4) for $x \in \Omega \backslash K$.

Let w be the solution of

$$\begin{cases} -\Delta w = 0 & \text{in } \Omega\backslash K \\ w = 0 & \text{on } \partial\Omega \\ w = 1 & \text{on } \partial K. \end{cases}$$

The Hopf boundary lemma gives again

$$w(x) \geq c\delta(x) \qquad \forall x \in \Omega\backslash K.$$

Since v is superharmonic and $v \geq c \int_\Omega h\delta$ on ∂K, the maximum principle implies

$$\begin{aligned} v(x) &\geq c \left(\int_\Omega h\delta \right) w(x) \\ &\geq c \left(\int_\Omega h\delta \right) \delta(x) \qquad \forall x \in \Omega\backslash K. \end{aligned}$$

This completes the proof. □

Proof of Theorem 3.1. Consider the approximate problem

$$(3.6)_n \qquad \begin{cases} -\Delta u = a_n(x) g_n(u) + f_n(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $0 \leq a_n(x)g_n(s) + f_n(x) \leq C_n$ in $\Omega \times [0, \infty)$ for some constant C_n, we have that $C_n z$ is a supersolution of $(3.6)_n$ where

$$(3.7) \qquad \begin{cases} -\Delta z = 1 & \text{in } \Omega \\ z = 0 & \text{on } \partial\Omega. \end{cases}$$

On the other hand, 0 is a subsolution of $(3.6)_n$. We therefore obtain a minimal solution $\underline{u}_n$ of $(3.6)_n$ by monotone iteration:

$$\begin{cases} -\Delta u_{m+1} = a_n(x) g_n(u_m) + f_n(x) & \text{in } \Omega \\ u_{m+1} = 0 & \text{on } \partial\Omega, \end{cases}$$

starting with $u_0 \equiv 0$. In particular, since $a_n(x)g_n(s) + f_n(x)$ increases with n, $\underline{u}_{n+1}$ is a supersolution of $(3.6)_n$, and hence

$$\underline{u}_n \leq \underline{u}_{n+1}.$$

We claim that

$$\int_\Omega a_n(x)g_n(\underline{u}_n)\delta \nearrow +\infty \quad \text{as } n \nearrow +\infty.$$

Lemma 3.2 then gives

$$\frac{\underline{u}_n(x)}{\delta(x)} \nearrow +\infty \quad \text{uniformly in } \Omega$$

as $n \to +\infty$; this proves Theorem 3.1. Thus, it only remains to show the claim. Suppose not, that

$$\int_\Omega a_n(x)g_n(\underline{u}_n)\delta \leq C \quad \forall n. \tag{3.8}$$

Then, multiplying $(3.6)_n$ by the solution z of (3.7), we see that

$$\int_\Omega \underline{u}_n \leq C \quad \forall n$$

(we have used that $0 \leq f_n \leq f \in L^1_\delta(\Omega)$). Hence, $\underline{u}_n \nearrow u$ in $L^1(\Omega)$, for some u, by monotone convergence.

Since g_n is a nondecreasing function, $a_n g_n(\underline{u}_n) + f_n$ increases to $ag(u) + f$ a.e. in Ω; (3.8) also gives

$$a_n g_n(\underline{u}_n) + f_n \nearrow ag(u) + f \quad \text{in } L^1_\delta(\Omega),$$

again by monotone convergence. We can now pass to the limit in the weak formulation of $(3.6)_n$ (recall Definition 1.1(c)), and obtain that u is a weak solution of

$$\begin{cases} -\Delta u = a(x)g(u) + f(x) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases}$$

This is impossible by Theorem 2.1(b). □

4. Very weak solutions. Proofs of Theorems 0.2 and 0.3.

In this section we return to the study of equation

$$-\Delta u = \frac{u^2}{|x|^2} + f(x)$$

and its corresponding boundary value problem. We prove stronger versions of the nonexistence results of Section 1 by considering a more general notion of solutions, which we call very weak solutions. More precisely, we prove the following results.

Theorem 4.1. *Let $N \geq 2$ and $u \in L^2_{\mathrm{loc}}(\Omega\backslash\{0\})$ satisfy $u \geq 0$ a.e. in Ω and*

$$-|x|^2 \Delta u \geq u^2 \qquad \text{in } \mathcal{D}'(\Omega\backslash\{0\}), \tag{4.1}$$

in the sense that $-\int u\Delta(|x|^2\varphi) \geq \int u^2\varphi$ for any $\varphi \geq 0$, $\varphi \in C_0^\infty(\Omega\backslash\{0\})$. Then $u \equiv 0$.

Note that now we are testing (4.1) only against functions with compact support in Ω and vanishing in a neighborhood of 0. As a consequence of the previous theorem, we will prove the following stronger version of Corollary 1.3.

Corollary 4.2. *Let $N \geq 2$ and $f \in L^1_{\mathrm{loc}}(\Omega\backslash\{0\})$, f integrable near $\partial\Omega$, $f \geq 0$ a.e. in Ω, $f \not\equiv 0$. Then there is no very weak solution of*

$$\begin{cases} -\Delta u = \dfrac{u^2}{|x|^2} + f(x) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

in the sense that $u \in L^2_{\mathrm{loc}}(\Omega\backslash\{0\})$, u and $u^2\delta$ are integrable near $\partial\Omega$ and

$$-\int_\Omega u\Delta\left(|x|^2\zeta\right) = \int_\Omega u^2\zeta + \int_\Omega f|x|^2\zeta$$

for any $\zeta \in C^2(\overline{\Omega})$, $\zeta = 0$ on $\partial\Omega$ and $\zeta \equiv 0$ in a neighborhood of 0.

Remark 4.3. Theorem 4.1 does not hold in dimension $N = 1$; a direct computation shows that $u(x) = |x|^\alpha$, for any $0 < \alpha < 1$, satisfies (4.1) when Ω is a small interval containing 0. Corollary 4.2 is also false in dimension $N = 1$; in fact, for any $p > 1$ and $f \in L^p(-1, 1)$, small in L^p, there exists a very weak solution u of (4.2) in $\Omega = (-1, 1)$ which is continuous in $[-1, 1]$ and satisfies $u(0) = 0$. This solution u is defined in $(0, 1)$ to be the solution of

$$\begin{cases} -u'' = \dfrac{u^2}{|x|^2} + f(x) & \text{in } (0,1) \\ u(0) = u(1) = 0, \end{cases} \tag{4.3}$$

and similarly in $(-1, 0)$; we obtain in this way a very weak solution of (4.2)

Note that if $\|f\|_{L^p(0,1)}$ is small then there is a solution $u \in C^2(0, 1) \cap C^1([0, 1])$ of (4.3). It can be obtained through the Inverse Function Theorem applied to the operator $-u'' - \dfrac{u^2}{|x|^2}$, which maps $X = W^{2,p}(0, 1) \cap W_0^{1,p}(0, 1)$ into $L^p(0, 1)$; note that $u \in X$ implies $\dfrac{u}{|x|} \in L^\infty(0, 1)$ and $u \in C^1([0, 1])$.

We also extend the local nonexistence result of Corollary 1.5, now for all $N \geq 1$.

Corollary 4.4. *Let $N \geq 1$ and $c > 0$ be any positive constant. Then there is no very weak solution of*

$$-|x|^2 \Delta u = u^2 + c \quad \text{in } \mathcal{D}'(\Omega\backslash\{0\}), \tag{4.4}$$

in the sense that $u \in L^2_{\mathrm{loc}}(\Omega\backslash\{0\})$, and $-\int u\Delta(|x|^2\varphi) = \int(u^2+c)\varphi$ for any function $\varphi \in C_0^\infty(\Omega\backslash\{0\})$.

The proofs of the results of this section consist of using appropriate powers of testing functions; this is an idea due to Baras and Pierre [2] and employed in [2] for the study of removable singularities of solutions of semilinear equations. In fact, the first step in our proof of Theorem 4.1 will be to use equation (4.1) to show that $\frac{u}{|x|} \in L^2_{\mathrm{loc}}(\Omega)$; this can be interpreted as a "removable singularity" result. The second step of the proof is to show that $-\Delta u \geq \frac{u^2}{|x|^2}$ is satisfied in $\mathcal{D}'(\Omega)$. We may then conclude that $u \equiv 0$ with the help of Theorem 1.2. We present here an alternative proof of Theorem 1.2 based on multiplication by a sequence of appropriate testing functions.

Proof of Theorem 4.1.

Step 1. We prove that $\frac{u}{|x|} \in L^2_{\mathrm{loc}}(\Omega)$. For this purpose, let $\zeta_n \in C_0^\infty(\Omega\backslash\{0\})$ be such that $0 \leq \zeta_n \leq 1$,

$$\zeta_n = \begin{cases} 0 & \text{if } |x| \leq \dfrac{1}{n} \\ 1 & \text{if } |x| \geq \dfrac{2}{n}, x \in \omega \\ \zeta & \text{if } x \in \Omega\backslash\omega, \end{cases}$$

where ω is open, $0 \in \omega$, $\overline{\omega} \subset \Omega$ and ζ is a fixed "tail" for all ζ_n. We take ζ_n such that if $\frac{1}{n} < |x| < \frac{2}{n}$ then

$$\begin{aligned} \nabla\zeta_n^4 &= 4\zeta_n^3\nabla\zeta_n \\ \Delta\zeta_n^4 &= 4\zeta_n^3\Delta\zeta_n + 12\zeta_n^2|\nabla\zeta_n|^2 \end{aligned}$$

and

$$|\Delta\zeta_n^4| \leq C\zeta_n^2 n^2,$$

for some constant C independent of n. Multiplying (4.1) by $\frac{\zeta_n^4}{|x|^2}$ yields

$$\begin{aligned} \int \frac{u^2}{|x|^2}\zeta_n^4 &\leq -\int u\Delta\zeta_n^4 \\ &\leq C \int\limits_{\{\frac{1}{n}<|x|<\frac{2}{n}\}} \frac{u}{|x|}\zeta_n^2 + C. \end{aligned}$$

Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\int \frac{u^2}{|x|^2}\zeta_n^4 &\leq Cn \int_{\{\frac{1}{n}<|x|<\frac{2}{n}\}} \frac{u}{|x|}\zeta_n^2 + C\\ &\leq \frac{C}{n^{\frac{N}{2}-1}}\left(\int \frac{u^2}{|x|^2}\zeta_n^4\right)^{1/2} + C.\end{aligned}$$

Since $N \geq 2$ we deduce that

$$\int \frac{u^2}{|x|^2}\zeta_n^4 \leq C,$$

and hence $\dfrac{u}{|x|} \in L^2_{\mathrm{loc}}(\Omega)$.

For later purposes, let us retain two more consequences of the previous proof. First, we did not use $u \geq 0$. Second, if carried out for $N = 1$, the proof gives

$$\int_{2/n}^{3/n} \frac{u^2}{|x|^2} \leq Cn + C, \quad (N = 1). \tag{4.5}$$

Step 2. We show that

$$-\Delta u \geq \frac{u^2}{|x|^2} \qquad \text{in } \mathcal{D}'(\Omega). \tag{4.6}$$

Indeed, let $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$, and $\eta_n(x) = \eta_1(n|x|)$ be such that $0 \leq \eta_1 \leq 1$ and

$$\eta_1(x) = \begin{cases} 0 & \text{if } |x| \leq 1\\ 1 & \text{if } |x| \geq 2.\end{cases}$$

Multiplying (4.1) by $\dfrac{\varphi}{|x|^2}\eta_n$ yields

$$\int \frac{u^2}{|x|^2}\varphi\eta_n \leq -\int u\Delta(\varphi\eta_n).$$

If we show that, as $n \to \infty$,

$$\int u|\nabla\varphi|\,|\nabla\eta_n| \to 0 \tag{4.7}$$

and

$$\int u\varphi|\Delta\eta_n| \to 0, \tag{4.8}$$

then we obtain $-\int u\Delta(\varphi\eta_n) \to -\int u\Delta\varphi$ and hence the statement of Step 2. To prove (4.7) and (4.8), we use that $\dfrac{u}{|x|} \in L^2_{\mathrm{loc}}(\Omega)$—which we established in Step 1. Hence

$$\begin{aligned}\int u|\nabla\varphi|\,|\nabla\eta_n| &\le Cn \int\limits_{\{\frac{1}{n}<|x|<\frac{2}{n}\}} u \\ &\le C \int\limits_{\{\frac{1}{n}<|x|<\frac{2}{n}\}} \frac{u}{|x|} \to 0\end{aligned}$$

and

$$\begin{aligned}\int u\varphi|\Delta\eta_n| &\le Cn^2 \int_{\{\frac{1}{n}<|x|<\frac{2}{n}\}} u \le Cn \int_{\{\frac{1}{n}<|x|<\frac{2}{n}\}} \frac{u}{|x|} \\ &\le C\frac{1}{n^{\frac{N}{2}-1}} \left(\int_{\{\frac{1}{n}<|x|<\frac{2}{n}\}} \frac{u^2}{|x|^2} \right)^{1/2} \to 0.\end{aligned}$$

Note that, as in Step 1, we have not used $u \ge 0$.

Step 3. We show that $u \equiv 0$ (it is only here where we use $u \ge 0$). Let us suppose that $u \not\equiv 0$. Then, since $u \ge 0$, $-\Delta u \ge 0$ in $\mathcal{D}'(\Omega)$ and Ω is connected, we have that

$$u \ge \varepsilon \quad \text{a.e. in } B_\eta,$$

for some $\varepsilon > 0$ and $B_\eta = B_\eta(0)$ with closure in Ω. When $N = 2$ this is impossible since $\dfrac{u}{|x|} \ge \dfrac{\varepsilon}{|x|}$ near 0 and hence $\dfrac{u}{|x|} \notin L^2_{\mathrm{loc}}(\Omega)$—a contradiction with Step 1.

When $N \ge 3$, we use that

$$-\Delta u \ge \frac{\varepsilon^2}{|x|^2} = -\Delta\left(\frac{\varepsilon^2}{N-2}\log\frac{1}{|x|}\right) \quad \text{in } \mathcal{D}'(B_\eta)$$

and we conclude (as in the proof of Theorem 1.2) that

$$u \ge \frac{\varepsilon^2}{N-2}\,|\log\,|x|| - C \quad \text{near } 0 \tag{4.9}$$

for some $C > 0$.

Let us now choose a sequence of functions $\chi_n(x) = \chi_1(n|x|)$ such that $0 \leq \chi_n \leq 1$ and

$$\chi_n(x) = \begin{cases} 1 & \text{if } |x| \leq 1/n \\ 0 & \text{if } |x| \geq 2/n. \end{cases}$$

Mutiplying (4.6) by χ_n^4 yields

$$\begin{aligned} \int \frac{u^2}{|x|^2}\chi_n^4 &\leq \int u|\Delta\chi_n^4| \leq n^2 \int\limits_{\{\frac{1}{n}<|x|<\frac{2}{n}\}} u\chi_n^2 \\ &\leq 2n \int \frac{u}{|x|}\chi_n^2 \leq \frac{C}{n^{\frac{N}{2}-1}} \left(\int \frac{u^2}{|x|^2}\chi_n^4\right)^{1/2}, \end{aligned}$$

and therefore

$$\int \frac{u^2}{|x|^2}\chi_n^4 \leq \frac{C}{n^{N-2}}.$$

But, using (4.9), we have

$$\int \frac{u^2}{|x|^2}\chi_n^4 \geq c \int\limits_{\{\frac{1}{2n}<|x|<\frac{1}{n}\}} n^2|\log n|^2 \sim \frac{|\log n|^2}{n^{N-2}}$$

which contradicts the previous statement. □

Finally we give the proofs of Corollaries 4.2 and 4.3.

Proof of Corollary 4.2. Recall that Steps 1 and 2 of the previous proof hold for any u satisfying (4.1)—without the assumption $u \geq 0$. Therefore, since $f \geq 0$, Step 1 of the proof of Theorem 4.1 gives

$$\frac{u}{|x|} \in L^2_{\text{loc}}(\Omega).$$

Moreover, proceeding as in Step 2 of the same proof, we see that

$$-\int_\Omega u\Delta\zeta = \int_\Omega \frac{u^2}{|x|^2}\zeta + \int_\Omega f\zeta$$

for any $\zeta \in C^2(\overline{\Omega})$, $\zeta = 0$ on $\partial\Omega$. In particular, $-\int u\Delta\zeta \geq 0$ if, in addition, $\zeta \geq 0$ in Ω. We conclude that $u \geq 0$ in Ω. Theorem 4.1 implies that $u \equiv 0$, which contradicts (4.2) and $f \not\equiv 0$. □

Proof of Corollary 4.4. Suppose that u satisfies

$$-|x|^2\Delta u = u^2 + c \qquad \text{in } \mathcal{D}'(\Omega\backslash\{0\}).$$

The proof of Step 1 in Theorem 4.1 gives

$$\frac{u^2}{|x|^2} + \frac{c}{|x|^2} \in L^1_{\text{loc}}(\Omega) \quad \text{when } N \geq 2.$$

This is impossible when $N = 2$, since $\dfrac{1}{|x|^2}$ is not integrable near 0.

When $N \geq 3$ we get (as in the proof of Theorem 4.1, Step 2) that

$$-\Delta u = \frac{u^2}{|x|^2} + \frac{c}{|x|^2} \quad \text{in } \mathcal{D}'(\Omega).$$

We now proceed as in the proof of Corollary 1.5 (i.e., we compare u with $\log \frac{1}{|x|}$). We obtain that $u \geq 0$ in a neighborhood of 0 and hence, by Theorem 4.1, $u \equiv 0$. This is a contradiction with equation (4.4).

We finally treat the case $N = 1$. We would have $u \in L^2_{\text{loc}}(\Omega\backslash\{0\})$ and

$$-x^2 u'' = u^2 + c \quad \text{in } \mathcal{D}'(\Omega\backslash\{0\}).$$

In particular, u belongs to $C^2(0, a)$ for some $a > 0$. Integrating the inequality $-u'' \geq \dfrac{c}{x^2}$ in $(s, \dfrac{a}{2})$, we obtain

$$u'(s) \geq \frac{c}{s} - C.$$

for some constant C. Integrating again yields

$$-u(s) \geq -c \log s - C.$$

Thus

$$|u(s)| \geq c|\log s| - C, \qquad \text{near } 0.$$

On the other hand, we recall (4.5):

$$\int_{2/n}^{3/n} \frac{u^2}{s^2} ds \leq Cn + C,$$

which was proved in Step 1 of the proof of Theorem 4.1. Using $|u(s)| \geq c|\log s| - C$, this inequality yields

$$\begin{aligned} \frac{c}{6}|\log n|^2 n = c|\log n|^2 \int_{2/n}^{3/n} \frac{ds}{s^2} \\ \leq C \int_{2/n}^{3/n} \frac{u^2}{s^2} ds \leq Cn + C, \end{aligned}$$

which is a contradiction. $\square$

5. Connection with a result of Kalton-Verbitsky.

In this section we consider the problem (for $u \geq 0$)

$$\begin{cases} -\Delta u = a(x)u^p + f(x) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where $p > 1$, $a \geq 0$ and $f \geq 0$ in Ω. Recently, Kalton and Verbitsky [8] have found an interesting necessary condition for the existence of a weak solution of (5.1). Their result states that if (5.1) has a weak solution, then necessarily

$$G(aG(f)^p) \leq CG(f) \quad \text{in } \Omega \tag{5.2}$$

for some constant C, where $G = (-\Delta)^{-1}$ (with zero Dirichlet boundary condition). In [8] the authors also prove (5.2) for more general second-order elliptic operators.

In this section we give a simple proof of the necessary condition (5.2) (for the Laplacian) using a refinement of the method that we have developed in Section 1. Our proof gives (5.2) with constant $C = \frac{1}{p-1}$. Next, we replace $f(x)$ by $\lambda f(x)$ in (5.1) (where $\lambda > 0$ is a parameter) and we study the problem of existence of solution depending on the value of λ.

As pointed out in the Introduction, condition (5.2) easily implies some of our nonexistence results. For instance, it gives the result of Theorem 2.1(b) when $g(u) = u^p$, for some $p > 1$, and $f \in L^\infty$, since in this case $G(f) \sim \delta$.

We recall that there is another necessary condition—due to Baras and Pierre [3]—for the existence of a weak solution of (5.1). Its proof consists of multiplying (5.1) by test functions and using Young's inequality.

Throughout this section we assume that

$$a \in L^1_{\text{loc}}(\Omega), \quad a \geq 0 \quad \text{a.e., } a \not\equiv 0. \tag{5.3}$$

and

$$f \in L^1_\delta(\Omega), \quad f \geq 0 \quad \text{a.e., } f \not\equiv 0. \tag{5.4}$$

A function $u \in L^1(\Omega)$, $u \geq 0$ a.e. is a weak solution of (5.1) if $au^p \in L^1_\delta(\Omega)$ and (5.1) is satisfied in the sense of Definition 1.1.(c). Finally, for $h \in L^1_\delta(\Omega)$ we denote by $G(h)$ the unique function in $L^1(\Omega)$ satisfying

$$\begin{cases} -\Delta(G(h)) = h & \text{in } \Omega \\ \quad G(h) = 0 & \text{on } \partial\Omega \end{cases}$$

again in the sense of Definition 1.1.(c)—see e.g. Lemma 1 of [4] for such result about the linear Laplace equation. We now give a necessary condition and a sufficient condition for the existence of a weak solution of (5.1).

Theorem 5.1. *Assume* (5.3) *and* (5.4).

(a) *If*

$$\begin{cases} -\Delta u = a(x)u^p + f(x) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{5.5}$$

has a weak solution, then $aG(f)^p \in L^1_\delta(\Omega)$ *and*

$$\frac{G(aG(f)^p)}{G(f)} \le \frac{1}{p-1} \qquad \text{in } \Omega.$$

(b) *If* $aG(f)^p \in L^1_\delta(\Omega)$ *and*

$$\frac{G(aG(f)^p)}{G(f)} \le \left(\frac{p-1}{p}\right)^p \frac{1}{p-1} \qquad \text{in } \Omega,$$

then (5.5) *has a weak solution* u *satisfying* $u \le CG(f)$ *in* Ω *for some constant* C.

The second result of this section is the following.

Theorem 5.2. *Assume* (5.3), (5.4) *and*

$$\frac{G(aG(f)^p)}{G(f)} \in L^\infty(\Omega).$$

For λ *a positive parameter, consider the problem*

$$\begin{cases} -\Delta u = a(x)u^p + \lambda f(x) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.6$_\lambda$}$$

Then there exists $\lambda^\star \in (0,\infty)$ *such that*

(i) *if* $0 < \lambda < \lambda^\star$, *then* $(5.6)_\lambda$ *has a weak solution* u_λ *satisfying*

$$\lambda \le \frac{u_\lambda}{G(f)} \le C(\lambda),$$

for some constant $C(\lambda)$ *depending on* λ.

(ii) *if* $\lambda = \lambda^\star$, *then* $(5.6)_\lambda$ *has a weak solution.*

(iii) *if* $\lambda > \lambda^\star$, *then* $(5.6)_\lambda$ *has no weak solution.*

Moreover,

$$\left(\frac{p-1}{p}\right)^p \frac{1}{p-1} \le (\lambda^\star)^{p-1} \left\| \frac{G(aG(f)^p)}{G(f)} \right\|_{L^\infty(\Omega)} \le \frac{1}{p-1}. \tag{5.7}$$

The main ingredient in the proof of the above theorems is the following.

Lemma 5.3. *Suppose u and v are C^2 functions in Ω, and that $v > 0$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a C^2, concave function. Then*

$$-\Delta \left[v\phi\left(\frac{u}{v}\right)\right] \geq \phi'\left(\frac{u}{v}\right)(-\Delta u) + \left[\phi\left(\frac{u}{v}\right) - \frac{u}{v}\phi'\left(\frac{u}{v}\right)\right](-\Delta v).$$

If, in addition, $-\Delta v \geq 0$ in Ω, then

$$-\Delta \left[v\phi\left(\frac{u}{v}\right)\right] \geq \phi'\left(\frac{u}{v}\right)[-\Delta u + \Delta v] + \phi(1)(-\Delta v). \tag{5.8}$$

Proof. We simply compute and use that $\phi'' \leq 0$ and $v > 0$. We have (using the notation $\partial_i v = v_i$)

$$\begin{aligned}
-\Delta \left[v\phi\left(\frac{u}{v}\right)\right] &= -\sum_{i=1}^{N}\left[\phi'\left(\frac{u}{v}\right) v \left(\frac{u}{v}\right)_i + \phi\left(\frac{u}{v}\right) v_i\right]_i \\
&= -\phi''\left(\frac{u}{v}\right) v \left|\nabla\left(\frac{u}{v}\right)\right|^2 - \sum_{i=1}^{N}\left\{\phi'\left(\frac{u}{v}\right)\left[v\left(\frac{u}{v}\right)_i\right]_i + \left[\phi\left(\frac{u}{v}\right)v_i\right]_i\right\}
\end{aligned}$$

$$\begin{aligned}
&\geq -\sum_{i=1}^{N}\left\{\phi'\left(\frac{u}{v}\right)\left[u_i - \frac{u}{v}v_i\right]_i + \phi\left(\frac{u}{v}\right)v_{ii} + \phi'\left(\frac{u}{v}\right)\left(\frac{u}{v}\right)_i v_i\right\} \\
&= \phi'\left(\frac{u}{v}\right)(-\Delta u) + \phi'\left(\frac{u}{v}\right)\nabla\left(\frac{u}{v}\right)\nabla v - \frac{u}{v}\phi'\left(\frac{u}{v}\right)(-\Delta v) \\
&+ \phi\left(\frac{u}{v}\right)(-\Delta v) - \phi'\left(\frac{u}{v}\right)\nabla\left(\frac{u}{v}\right)\nabla v \\
&= \phi'\left(\frac{u}{v}\right)(-\Delta u) + \left[\phi\left(\frac{u}{v}\right) - \frac{u}{v}\phi'\left(\frac{u}{v}\right)\right](-\Delta v),
\end{aligned}$$

which is the first inequality of the lemma. From this, we easily deduce the second inequality, as follows. Since ϕ is concave, we have

$$\phi(s) + (1-s)\phi'(s) \geq \phi(1) \qquad \forall s \in \mathbb{R}.$$

Thus

$$\phi\left(\frac{u}{v}\right) - \frac{u}{v}\phi'\left(\frac{u}{v}\right) \geq -\phi'\left(\frac{u}{v}\right) + \phi(1);$$

multiplying this inequality by $-\Delta v$ (which is nonnegative by assumption), we easily deduce (5.8). □

To use the previous lemma, we will need (5.8) in its weak version for L^1 functions.

Lemma 5.4. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a C^1, concave function with ϕ' bounded. Let h and k belong to $L^1_\delta(\Omega)$, with $k \geq 0$, $k \not\equiv 0$, and let u and v be the $L^1(\Omega)$ solutions of*

$$\begin{cases} -\Delta u = h & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} -\Delta v = k & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Then

$$-\Delta \left[v\phi\left(\frac{u}{v}\right)\right] \geq \phi'\left(\frac{u}{v}\right)(h-k) + \phi(1)k, \tag{5.9}$$

in the sense that $v\phi\left(\frac{u}{v}\right) \in L^1(\Omega)$, $\phi'\left(\frac{u}{v}\right)(h-k) + \phi(1)k \in L^1_\delta(\Omega)$ and

$$-\int_\Omega v\phi\left(\frac{u}{v}\right)\Delta\zeta \geq \int_\Omega \left\{\phi'\left(\frac{u}{v}\right)(h-k) + \phi(1)k\right\}\zeta$$

for all $\zeta \in C^2(\overline{\Omega})$, $\zeta \geq 0$, with $\zeta = 0$ on $\partial\Omega$.

Proof. We first point out that (5.8) holds when ϕ is C^1 and concave—not necessarily C^2. This follows immediately from Lemma 5.3 convoluting ϕ with mollifiers.

We approximate h and k in $L^1_\delta(\Omega)$ by sequences (h_n) and (k_n), respectively, of $C^\infty_0(\Omega)$ functions and with $k_n \geq 0$, $k_n \not\equiv 0$. Let u_n, v_n be the solutions of

$$\begin{cases} -\Delta u_n = h_n & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} -\Delta v_n = k_n & \text{in } \Omega \\ v_n = 0 & \text{on } \partial\Omega. \end{cases}$$

It follows that $u_n \to u$ and $v_n \to v$ in $L^1(\Omega)$ (for this, substract the equations for u_n and u, multiply by $G(1)$ and integrate). Moreover, $u_n, v_n \in C^2(\Omega)$ and $v_n > 0, -\Delta v_n \geq 0$ in Ω.

By (5.8) we have

$$-\Delta \left[v_n\phi\left(\frac{u_n}{v_n}\right)\right] \geq \phi'\left(\frac{u_n}{v_n}\right)(h_n-k_n) + \phi(1)k_n.$$

Moreover, using that ϕ' is bounded, we see that

$$\begin{aligned}\left|v_n\phi\left(\frac{u_n}{v_n}\right)\right| &= \left|v_n\left(\phi\left(\frac{u_n}{v_n}\right)-\phi(0)\right)+\phi(0)v_n\right| \\ &\leq C(|u_n|+|v_n|)\end{aligned}\tag{5.10}$$

for some constant C. Hence, $v_n\phi\left(\frac{u_n}{v_n}\right)$ vanishes on $\partial\Omega$ and thus

$$-\int_\Omega v_n\phi\left(\frac{u_n}{v_n}\right)\Delta\zeta \geq \int_\Omega \left\{\phi'\left(\frac{u_n}{v_n}\right)(h_n-k_n)+\phi(1)k_n\right\}\zeta\tag{5.11}$$

for all $\zeta\in C^2(\overline{\Omega})$, $\zeta\geq 0$ in Ω and $\zeta=0$ on $\partial\Omega$.

Note that $v>0$ a.e. in Ω, so that $v\phi\left(\frac{u}{v}\right)$ is well defined a.e. Moreover, $v_n\phi\left(\frac{u_n}{v_n}\right)$ converges a.e. to $v\phi\left(\frac{u}{v}\right)$—up to a subsequence. Since u_n and v_n converge in $L^1(\Omega)$, they are dominated (also up to a subsequence) by an $L^1(\Omega)$ function. Thus, by (5.10), $|v_n\phi\left(\frac{u_n}{v_n}\right)|$ is also dominated by an L^1 function (for a subsequence). We conclude that

$$v_n\phi\left(\frac{u_n}{v_n}\right)\longrightarrow v\phi\left(\frac{u}{v}\right)\qquad\text{in } L^1(\Omega).$$

Passing to the limit in (5.11), we finally obtain (5.9) and the lemma. □

We write explicitly the concave functions ϕ that we use in this section. For Theorem 5.1 we will use

$$\phi(s)=\int_1^s\frac{dt}{t^p}=\frac{1}{p-1}\left(1-\frac{1}{s^{p-1}}\right)\qquad\text{for } s\geq 1.\tag{5.12}$$

It satisfies

$$\phi'(s)s^p=1\qquad\text{for } s\geq 1,$$

and hence ϕ is concave and ϕ' is bounded in $[1,\infty)$. Moreover,

$$\phi(1)=0\qquad\text{and}\quad 0\leq\phi(s)\leq\frac{1}{p-1}\ \text{ for }\ s\geq 1.$$

Since $\phi'(1)=1$, we can extend ϕ by $\phi(s)=s-1$ in $(-\infty,1]$ obtaining a function ϕ that satisfies the conditions of Lemma 5.4.

Note that the functions ϕ above are analogous versions of the ones that we employed in Sections 1 and 2, in the sense that they all satisfy $\phi'(s)g(s) = 1$ (in an appropriate interval) where g is the nonlinearity.

In Theorem 5.2 we will be led to take ϕ (that we denote now by ψ) satisfying another differential equation, namely: $\psi'(s)s^p = \psi(s)^p$. Precisely, we will take

$$\psi(s) = \frac{s}{(\varepsilon s^{p-1}+1)^{\frac{1}{p-1}}} \quad \text{for } s \geq 0. \tag{5.13}$$

It satisfies

$$\psi'(s) = \frac{1}{(\varepsilon s^{p-1}+1)^{\frac{p}{p-1}}} \qquad \text{for } s \geq 0.$$

and hence

$$\psi'(s)s^p = \psi(s)^p \qquad \text{for } s \geq 0.$$

Note that ψ is concave and ψ' is bounded in $[0,\infty)$;

$$\psi(1) = \left(\frac{1}{1+\varepsilon}\right)^{\frac{1}{p-1}} \quad \text{and} \quad 0 \leq \psi(s) \leq \left(\frac{1}{\varepsilon}\right)^{\frac{1}{p-1}} \quad \text{for } \ s \geq 0.$$

Extending ψ by $\psi(s) = s$ in $(-\infty, 0]$, we obtain a function ψ which satisfies the conditions (for ϕ) of Lemma 5.4. Note that, for $p = 2$, ψ was already considered in Remark 1.8.

Proof of Theorem 5.1.

Part (a). Let $u \geq 0$ be a weak solution of

$$\begin{cases} -\Delta u = a(x)u^p + f(x) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases}$$

We consider

$$v = G(f);$$

note that, since $a(x)u^p \geq 0$,

$$\frac{u}{v} = \frac{u}{G(f)} \geq 1 \quad \text{in } \Omega$$

by the weak maximum principle (which is an easy consequence of the weak formulation of Definition 1.1.(c) used here).

We take ϕ, defined by (5.12), and apply Lemma 5.4 to obtain (we use $u/v \geq 1$ and the properties of $\phi(s)$ for $s \geq 1$)

$$\begin{aligned} -\Delta\left[v\phi\left(\frac{u}{v}\right)\right] &\geq \phi'\left(\frac{u}{v}\right)(au^p + f - f) + \phi(1)f \\ &= \phi'\left(\frac{u}{v}\right)au^p \\ &= av^p = aG(f)^p \end{aligned}$$

in the weak sense of the lemma. In particular $aG(f)^p \in L^1_\delta(\Omega)$, and (by the weak maximum principle)

$$\begin{aligned} G(aG(f)^p) &\leq v\phi\left(\frac{u}{v}\right) \\ &\leq \frac{1}{p-1}v = \frac{1}{p-1}G(f). \end{aligned}$$

Part (a) is now proved.

Part (b). We assume that $aG(f)^p \in L^1_\delta(\Omega)$ and

$$G(aG(f)^p) \leq \left(\frac{p-1}{p}\right)^p \frac{1}{p-1}G(f).$$

It follows that the $L^1(\Omega)$ function

$$\tilde{u} := \left(\frac{p}{p-1}\right)^p G(aG(f)^p) + G(f)$$

satisfies

$$\tilde{u} \leq \frac{p}{p-1}G(f).$$

Therefore

$$\begin{aligned} -\Delta\tilde{u} &= \left(\frac{p}{p-1}\right)^p aG(f)^p + f \\ &\geq a\tilde{u}^p + f \end{aligned}$$

in the weak sense. That is, $\tilde{u}$ is a weak supersolution of (5.5). On the other hand, 0 is a subsolution of the problem. It is then easy to obtain a weak solution u of (5.5) by monotone iteration (see e.g. Lemma 3 of [4]). Moreover

$$G(f) \leq \tilde{u} \leq \frac{p}{p-1}G(f).$$

This completes the proof of Theorem 5.1. $\square$

Proof of Theorem 5.2. We assume that

$$\frac{G(aG(f)^p)}{G(f)} \in L^\infty(\Omega);$$

moreover, $a \not\equiv 0$ and $f \not\equiv 0$ and hence

$$0 < M_\infty := \left\| \frac{G(aG(f)^p)}{G(f)} \right\|_{L^\infty(\Omega)} < \infty.$$

Theorem 5.1 (applied with f replaced by λf) implies that if $(5.6)_\lambda$ has a weak solution, then

$$\lambda^{p-1} M_\infty \leq \frac{1}{p-1}.$$

The theorem also gives that if

$$\lambda^{p-1} M_\infty \leq \left(\frac{p-1}{p}\right)^p \frac{1}{p-1}$$

then $(5.6)_\lambda$ has a weak solution. Hence, defining

$$\lambda^\star = \sup\{\lambda > 0; (5.6)_\lambda \text{ has a weak solution}\},$$

we have $0 < \lambda^\star < \infty$, and also estimate (5.7) of Theorem 5.2. Note that part (iii) of the theorem is obvious.

To prove part (i), we have to show that if $0 < \lambda < \mu$ and $(5.6)_\mu$ has a weak solution u then $(5.6)_\lambda$ has a weak solution u_λ satisfying

$$\lambda \leq \frac{u_\lambda}{G(f)} \leq C(\lambda). \tag{5.14}$$

For this purpose, we consider

$$v = G(\mu f) = \mu G(f),$$

and the function ψ defined by (5.13) with $\varepsilon > 0$ chosen small enough such that

$$\lambda \leq \left(\frac{1}{1+\varepsilon}\right)^{\frac{1}{p-1}} \mu = \psi(1)\mu.$$

We apply Lemma 5.4 (with ϕ replaced by ψ) to obtain

$$\begin{aligned} -\Delta \left[v\psi\left(\frac{u}{v}\right)\right] &\geq \psi'\left(\frac{u}{v}\right)(au^p + \mu f - \mu f) + \psi(1)\mu f \\ &= av^p \psi\left(\frac{u}{v}\right)^p + \psi(1)\mu f \\ &\geq a\left[v\psi\left(\frac{u}{v}\right)\right]^p + \lambda f \end{aligned}$$

in the weak sense of the lemma. Hence $v\psi\left(\frac{u}{v}\right)$ is a weak supersolution of $(5.6)_\lambda$. Again by monotone iteration we deduce that $(5.6)_\lambda$ has a weak solution u_λ such that

$$u_\lambda \leq v\psi\left(\frac{u}{v}\right) \leq Cv = C\mu G(f).$$

This, together with the immediate bound $u_\lambda \geq G(\lambda f)$, gives (5.14) and proves (i).

It remains to show part (ii). For $\lambda < \lambda^\star$ we can take u_λ to be the minimal solution of $(5.6)_\lambda$, i.e., the solution obtained by monotone iteration starting from the function 0. In this manner, $u_\lambda \leq u_\mu$ if $0 < \lambda < \mu < \lambda^\star$. Hence, in order to obtain a weak solution of $(5.6)_{\lambda^\star}$, it suffices to show

$$\int_\Omega \left[a(x)u_\lambda^p + \lambda f(x)\right]\delta \leq C \tag{5.15}$$

for some constant C independent of λ.

To prove (5.15) we proceed as follows. Since $a \geq 0$ and $a \not\equiv 0$, there exists a constant $M \in (0, \infty)$ such that

$$a_M := \chi_{\{a \leq M\}} a$$

satisfies $0 \leq a_M \leq M$ and $a_M \not\equiv 0$ (here $\chi_{\{a\leq M\}}$ denotes the characteristic function of $\{a \leq M\}$).

It is well-known that the problem

$$\begin{cases} -\Delta w = a_M^{1/p} w^{1/p} & \text{in } \Omega \\ \quad w > 0 & \text{in } \Omega \\ \quad w = 0 & \text{on } \partial\Omega \end{cases} \tag{5.16}$$

has a unique solution $w \in W^{2,r}(\Omega)$ (for any $1 < r < \infty$) with $w \not\equiv 0$. This solution can be obtained for example by minimizing in $H_0^1(\Omega)$ the energy associated to (5.16):

$$E(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \frac{p}{p+1}\int_\Omega a_M^{1/p}(v^+)^{(p+1)/p},$$

which is a coercive functional, bounded from below, in $H_0^1(\Omega)$ (note that $a_M^{1/p} \in L^\infty(\Omega)$ and $1 < \dfrac{p+1}{p} < 2$). Note that $E(t\varphi_1) < 0$ for t small, if φ_1 denotes the first eigenfunction of $-\Delta$. In particular, the minimizer w of E satisfies $E(w) < 0$, and thus $w \not\equiv 0$. The strong maximum principle then gives

$$w \geq C\delta \tag{5.17}$$

for some positive constant C.

Since $\Delta w \in L^\infty(\Omega)$ we can multiply $(5.6)_\lambda$ by w and integrate by parts. We also use Young's inequality—in the spirit of the ideas of Baras and Pierre [3]. We obtain

$$\begin{aligned}\int_\Omega a u_\lambda^p w + \int_\Omega \lambda f w &= \int_\Omega u_\lambda(-\Delta w)\\ &= \int_\Omega a_M^{1/p} u_\lambda w^{1/p} \le \int_\Omega a^{1/p} u_\lambda w^{1/p}\\ &\le \frac{1}{p}\int_\Omega a u_\lambda^p w + \frac{p-1}{p}\int_\Omega 1,\end{aligned}$$

and therefore

$$\int_\Omega a u_\lambda^p w + \int_\Omega \lambda f w \le C$$

for some constant independent of λ. Using (5.17), we conclude (5.15), and hence the proof of part (ii). $\square$

Remark 5.5. An analogous version of Theorem 5.2 also holds for the problem

$$\begin{cases} -\Delta u = \lambda(a(x)u^p + f(x)) & \text{in } \Omega\\ \quad u = 0 & \text{on } \partial\Omega. \end{cases}$$

This follows from Theorem 5.2 rescaling the solution u, i.e., considering the problem for αu, for appropriate α.

Remark 5.6. In Theorem 5.2 we have shown that, for $\lambda < \lambda^\star$, $(5.6)_\lambda$ has a weak solution u_λ satisfying

$$\frac{u_\lambda}{G(f)} \in L^\infty(\Omega). \tag{5.18}$$

We point out that this property may not be true for the minimal solution obtained for $\lambda = \lambda^\star$. To give an example of this, consider first the problem

$$(5.19)_\lambda \qquad \begin{cases} -\Delta v = \lambda(v+1)^p & \text{in } B_1 = \{|x|<1\} \subset \mathbb{R}^N\\ \quad v = 0 & \text{on } \partial B_1. \end{cases}$$

For N and p sufficiently large, $(5.19)_\lambda$ has

$$\overline{v}(x) = |x|^{-\frac{2}{p-1}} - 1$$

as weak solution for a certain parameter $\overline{\lambda} > 0$; moreover $\overline{v}$ is the pointwise, increasing limit (as $\lambda \nearrow \overline{\lambda}$) of the classical minimal solutions v_λ of $(5.19)_\lambda$ for $\lambda < \overline{\lambda}$ (see e.g. [6]). Let us define

$$f = (\overline{v} + 1)^p - \overline{v}^p,$$

so that the problem

$$(5.20)_\lambda \qquad \begin{cases} -\Delta u = \overline{\lambda} u^p + \lambda f(x) & \text{in } B_1 \\ \quad u = 0 & \text{on } \partial B_1 \end{cases}$$

has $u = \overline{v}$ as weak solution for $\lambda = \overline{\lambda}$. Note that $(5.20)_\lambda$ is of the form $(5.6)_\lambda$ considered in Theorem 5.2. We claim that $\overline{v}$ is the minimal weak solution of $(5.20)_{\overline{\lambda}}$. This is shown as follows: for $\lambda < \lambda$, let u_λ be the minimal weak solution of $(5.20)_\lambda$. Since $u_\lambda \le \overline{v}$, we have $(u_\lambda + 1)^p - u_\lambda^p \le f$, and hence u_λ is a weak supersolution of $(5.19)_\lambda$. Thus $u_\lambda \ge v_\lambda$, and since $v_\lambda \nearrow \overline{v}$ as $\lambda \nearrow \overline{\lambda}$, we conclude that $\overline{v}$ is the minimal weak solution of $(5.20)_{\overline{\lambda}}$.

Finally, we have that

$$(5.21) \qquad \frac{\overline{v}}{G(f)} \notin L^\infty(B_1),$$

since $\overline{v} \sim |x|^{-\frac{2}{p-1}}$ near 0 and $G(f) \le C \log \frac{1}{|x|}$ (note that $f \le p(\overline{v}+1)^{p-1} = p|x|^{-2}$ in B_1). This proves that (5.18) does not hold for $\lambda = \overline{\lambda}$ and, in particular, that $\overline{\lambda}$ is the extremal parameter $\lambda^\star$ for problem $(5.20)_\lambda$.

6. Evolution equations.

In section 4 we proved that $u \equiv 0$ is the only nonnegative very weak supersolution of

$$-\Delta u \ge \frac{u^2}{|x|^2} \qquad \text{in } \mathcal{D}'(\Omega \backslash \{0\}).$$

Here we prove an analogous result for the equation

$$u_t - \Delta u \ge \frac{u^2}{|x|^2} \qquad \text{in } \mathcal{D}'((\Omega \backslash \{0\}) \times (0, T)).$$

More precisely, we have the following parabolic analogue of Theorem 4.1.

Theorem 6.1. *Let $N \geq 2$, $T > 0$ and $u \in L^2_{\text{loc}}((\Omega\backslash\{0\}) \times (0,T))$ satisfy $u \geq 0$ a.e. in $\Omega \times (0,T)$ and*

$$|x|^2(u_t - \Delta u) \geq u^2 \qquad \text{in } \mathcal{D}'((\Omega\backslash\{0\}) \times (0,T)), \tag{6.1}$$

in the sense that

$$-\iint u\{(|x|^2\varphi)_t + \Delta(|x|^2\varphi)\} \geq \iint u^2\varphi$$

for any $\varphi \geq 0$, $\varphi \in C_0^\infty((\Omega\backslash\{0\}) \times (0,T))$.

Then $u \equiv 0$.

As an immediate consequence of the theorem we obtain an extension of a result of Peral and Vázquez (Theorem 7.1 of [12]). Here $\Omega = B_1$, the unit ball of $\mathbb{R}^N$, and $N \geq 3$. We consider the function

$$\overline{u}(x) = \log \frac{1}{|x|^2},$$

which is a weak solution of

$$\begin{cases} -\Delta\overline{u} = 2(N-2)e^{\overline{u}} & \text{in } B_1 \\ \quad \overline{u} = 0 & \text{on } \partial B_1. \end{cases}$$

Corollary 6.2. *Let $N \geq 3, T > 0$ and $u \in L^1_{\text{loc}}((B_1\backslash\{0\}) \times (0,T))$ be such that $e^u \in L^1_{\text{loc}}((B_1\backslash\{0\}) \times (0,T))$,*

$$u(x,t) \geq \overline{u}(x) \qquad \text{a.e. in } B_1 \times (0,T),$$

and

$$u_t - \Delta u = 2(N-2)e^u \quad \text{in } \mathcal{D}'((B_1\backslash\{0\}) \times (0,T)) \tag{6.2}$$

in the sense that

$$-\iint u(\varphi_t + \Delta\varphi) = 2(N-2)\iint e^u\varphi$$

for any $\varphi \in C_0^\infty((B_1\backslash\{0\}) \times (0,T))$.

Then $u(x,t) \equiv \overline{u}(x)$.

Note again that we only assume equation (6.2) to be satisfied in the distributional sense and away from $\{x = 0\} \times (0,T)$. In particular, given any $u_0(x) \geq \overline{u}(x)$ a.e. in $B_1, u_0 \not\equiv \overline{u}$, and any $T > 0$, there is no weak solution u, with

$$u(x,t) \geq \overline{u}(x) \quad \text{in } B_1 \times (0,T),$$

of the problem

$$\begin{cases} u_t - \Delta u = 2(N-2)e^u & \text{in } B_1 \times (0,T) \\ u = 0 & \text{on } \partial B_1 \times (0,T) \\ u(x,0) = u_0 & \text{on } B_1 \end{cases} \tag{6.3}$$

(as stated in [12]).

A second consequence of Theorem 6.1 is the following nonexistence and complete blow-up result. For any $u_0 \geq 0, u_0 \not\equiv 0$ (say $u_0 \in C_0^\infty(\Omega)$) and for any $T > 0$, the problem

$$\begin{cases} u_t - \Delta u = \dfrac{u^2}{|x|^2} & \text{in } \Omega \times (0,T) \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0 & \text{on } \Omega \end{cases} \tag{6.4}$$

has no weak solution (by Theorem 6.1). Using similar ideas as in the elliptic case (see Section 3), we can prove that approximate solutions of (6.4) blow up completely. More precisely, let g_n be a sequence of nonnegative, nondecreasing and globally Lipschitz functions in $[0,\infty)$ such that $g_n(u)$ increases pointwise to u^2. Let a_n be a sequence of nonnegative bounded functions in Ω, increasing pointwise to $\dfrac{1}{|x|^2}$.

Theorem 6.3. *Under the above assumptions, let u_n be the solution of*

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - \Delta u_n = a_n(x) g_n(u_n) & \textit{in } \Omega \times (0,+\infty) \\ u_n = 0 & \textit{on } \partial\Omega \times (0,+\infty) \\ u_n(x,0) = u_0 & \textit{on } \Omega. \end{cases} \tag{6.5$_n$}$$

Then, for any $0 < \varepsilon < T$,

$$\frac{u_n(x,t)}{\delta(x)} \longrightarrow +\infty \quad \textit{uniformly in } x \in \Omega, t \in [\varepsilon, T] \tag{6.6}$$

as $n \to \infty$.

To prove Theorem 6.1 we adapt the method given in Section 4 for the elliptic case; it consists of using appropriate powers of testing functions.

Proof of Theorem 6.1. We proceed as in the proof of Theorem 4.1; we use the same notation as there. We also fix a cut-off function in time:

$$\psi \in C_0^\infty((0,T)), \quad 0 \leq \psi \leq 1,$$

with $\psi \equiv 1$ in a given compact sub-interval of $(0,T)$.

Step 1. We prove that $\dfrac{u}{|x|} \in L^2_{\text{loc}}(\Omega \times (0,T))$. For this purpose, we multiply (6.1) by $\dfrac{\zeta_n^4(x)\psi^2(t)}{|x|^2}$; it yields

$$\iint \frac{u^2}{|x|^2}\zeta_n^4\psi^2 \le -\iint u(\Delta\zeta_n^4)\psi^2 - \iint u\zeta_n^4(\psi^2)_t.$$

Let us denote by C different constants independent of n, but that may depend on Ω, T and the cut-off ψ. We have

$$-\iint u(\Delta\zeta_n^4)\psi^2 \le Cn \iint_{\{\frac{1}{n}<|x|<\frac{2}{n}\}\times(0,T)} \frac{u}{|x|}\zeta_n^2\psi + C,$$

and

$$-\iint u\zeta_n^4(\psi^2)_t \le C\iint \frac{u}{|x|}\zeta_n^2\psi.$$

Hence we can conclude, as in section 4, that

$$\iint \frac{u^2}{|x|^2}\zeta_n^4\psi^2 \le C.$$

Thus $\dfrac{u}{|x|} \in L^2_{\text{loc}}(\Omega \times (0,T))$.

Step 2. We show that

$$u_t - \Delta u \ge \frac{u^2}{|x|^2} \quad \text{in } \mathcal{D}'(\Omega \times (0,T)). \tag{6.7}$$

This is done exactly as in the proof of Theorem 4.1, where now $\varphi = \varphi(x,t)$ belongs to $C_0^\infty(\Omega \times (0,T))$.

Step 3. We finally prove $u \equiv 0$. We suppose $u \not\equiv 0$. Since $u \ge 0$, $u_t - \Delta u \ge 0$ in $\mathcal{D}'(\Omega \times (0,T))$ and Ω is connected, we have that

$$u \ge \varepsilon \quad \text{a.e. in } B_\eta \times (\tau, T),$$

for some $0 < \tau < T$, $\varepsilon > 0$ and $B_\eta = B_\eta(0)$ with closure in Ω. When $N = 2$ this is a contradiction with $\dfrac{u}{|x|} \in L^2_{\text{loc}}(\Omega \times (0,T))$.

When $N \geq 3$, (6.7) gives

$$u_t - \Delta u \geq \frac{\varepsilon^2}{|x|^2} = (\partial_t - \Delta)\left(\frac{\varepsilon^2}{N-2}\log\frac{1}{|x|}\right) \quad \text{in } \mathcal{D}'(B_\eta \times (\tau, T)).$$

We deduce that

$$u \geq \frac{\varepsilon^2}{N-2}\log\frac{1}{|x|} - C \quad \text{in } B_{\eta/2} \times (\tau', T) \tag{6.8}$$

for some $C > 0$ and $\tau < \tau' < T$.

Following the proof of the Theorem 4.1, we now multiply (6.7) by $\chi_n^4(x)\psi^2(t)$, with ψ a cut-off as in the beginning of this proof, and with $\psi \equiv 1$ in (τ', T') for some T' with $\tau' < T' < T$. We have

$$\iint \frac{u^2}{|x|^2}\chi_n^4\psi^2 \leq \iint u|\Delta\chi_n^4|\psi^2 + \iint u\chi_n^4|(\psi^2)_t|$$

(where we are integrating on $\{|x| < \frac{2}{n}\} \times (0, T)$, since it contains the support of $\chi_n^4\psi^2$). Hence

$$\iint \frac{u^2}{|x|^2}\chi_n^4\psi^2 \leq Cn\iint \frac{u}{|x|}\chi_n^2\psi,$$

where C is independent of n. From the Cauchy-Schwarz inequality, we deduce

$$\iint \frac{u^2}{|x|^2}\chi_n^4\psi^2 \leq \frac{C}{n^{N-2}}.$$

But using (6.8), we have

$$\iint \frac{u^2}{|x|^2}\chi_n^4\psi^2 \geq c\frac{|\log n|^2}{n^{N-2}},$$

which contradicts the previous statement. This proves the theorem. $\square$

Remark 6.4. Step 3 of the previous proof, (i.e., to show $u \equiv 0$ from $u \geq 0$ and (6.7)) could have been done using a parabolic analogue of the method of Section 1. That is, one considers

$$v = \frac{1}{\varepsilon} - \frac{1}{u}$$

in a subcylinder where $u \geq \varepsilon$. Then (with the aid of the parabolic Kato's inequality) v satisfies

$$v_t - \Delta v \geq \frac{1}{|x|^2},$$

which leads to contradiction since v is bounded.

Corollary 6.2 follows immediately from Theorem 6.1:

Proof of Corollary 6.2. Let u be as in the corollary. Consider $v(x,t) = u(x,t) - \overline{u}(x)$. It satisfies, from our assumptions,

$$v \geq 0 \quad \text{a.e. in } B_1 \times (0,T).$$

Moreover, in the distributional sense $\mathcal{D}'((B_1 \backslash \{0\}) \times (0,T))$,

$$\begin{aligned} v_t - \Delta v &= u_t - \Delta u + \Delta \overline{u} \\ &= 2(N-2)(e^u - e^{\overline{u}}) = 2(N-2)e^{\overline{u}}(e^v - 1) \\ &= \frac{2(N-2)}{|x|^2}(e^v - 1) \\ &\geq \frac{N-2}{|x|^2} v^2 \end{aligned}$$

since $v \geq 0$. Hence $(N-2)v \geq 0$ satisfies (6.1). By Theorem 6.1 we deduce $v \equiv 0$, that is $u \equiv \overline{u}$. □

We finally give the proof of the complete blow-up result.

Proof of Theorem 6.3. We proceed in three steps. Recall that $0 \leq u_n \leq u_{n+1}$, by the maximum principle.

Step 1. We prove that, for any $\tau > 0$,

$$\int_0^\tau \int_\Omega a_n g_n(u_n)\delta \to +\infty.$$

Suppose not, that $\int_0^\tau \int_\Omega a_n g_n(u_n)\delta \leq C$ for some $\tau > 0$. We multiply $(6.5)_n$ by the solution of

$$\begin{cases} -\Delta z = 1 & \text{in } \Omega \\ z = 0 & \text{on } \partial\Omega. \end{cases}$$

We obtain

$$\int_0^\tau \int_\Omega u_n + \int_\Omega u_n(x,\tau) z - \int_\Omega u_0 z = \int_0^\tau \int_\Omega a_n g_n(u_n) z \leq C$$

and, in particular

$$\int_0^\tau \int_\Omega u_n \leq C.$$

Hence u_n and $a_n g_n(u_n)\delta$ are bounded in $L^1(\Omega \times (0,\tau))$. By monotone convergence, we obtain that $u_n \to u$ in $L^1(\Omega \times (0,\tau))$ with u satisfying the assumptions of Theorem 6.1. By this theorem, $u \equiv 0$. Thus $u_1 \equiv 0$ and $u_0(x) = u_1(x,0) \equiv 0$, a contradiction.

Step 2. We show that

$$\int_\Omega u_n(x,t)\delta(x)dx \to +\infty \quad \text{uniformly in } t \in [\frac{\varepsilon}{2}, T].$$

Indeed, let us multiply $(6.5)_n$ by $e^{\lambda_1 t}\varphi_1$, where φ_1 is the first eigenfunction of $-\Delta$ in Ω with zero Dirichlet condition and λ_1 its corresponding eigenvalue. We then integrate in space and time, to obtain

$$e^{\lambda_1 \tau} \int_\Omega u_n(\cdot,\tau)\varphi_1 - \int_\Omega u_0\varphi_1 = \int_0^\tau \int_\Omega a_n g_n(u_n)\varphi_1 e^{\lambda_1 t}.$$

Hence, if $\tau \in [\frac{\varepsilon}{2}, T]$,

$$\int_\Omega u_n(x,\tau)\delta(x)dx \geq ce^{-\lambda_1 T} \int_0^{\varepsilon/2} \int_\Omega a_n g_n(u_n)\delta \to +\infty$$

by Step 1.

Step 3. We finally prove (6.6). For this purpose, we use a parabolic analogue of Lemma 3.2 due to Martel (see Lemma 2 of [10]); it asserts that

$$\frac{T(\tau)\cdot\varphi(x)}{\delta(x)} \geq c(\tau) \int_\Omega \varphi\delta \qquad \forall x \in \Omega$$

for any $\tau > 0$, where $c(\tau) > 0$ is a constant depending on τ, and where $T(\tau)$ is the heat semigroup at time τ. We apply this estimate with $\tau = \varepsilon/2$, and obtain

$$\frac{u_n(x,t)}{\delta(x)} \geq c\,(\varepsilon/2) \int_\Omega u_n(x, t - \frac{\varepsilon}{2})\delta(x)dx \to +\infty$$

uniformly in $t \in [\varepsilon, T]$ by Step 2, since $t - \frac{\varepsilon}{2} \geq \frac{\varepsilon}{2}$. □

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