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A Lyapounov-Schmidt Procedure involving a nonlinear projection

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Dedicated in friendship to François Treves on his 65th birthday.

Section 1

The Lyapounov-Schmidt (L-S) procedure (see e.g. [N]) is a standard tool in bifurcation theory for solving equations of the form

$$F(x) = 0 \tag{1}$$

or

$$F(x, \lambda) = 0. \tag{1$'$}$$

In (1), F is, say, a smooth map from a neighbourhood of a point in a Banach space X into another Y. In (1)$'$ F is defined for (x, λ) in a neighbourhood in $X \times \Lambda$, with Λ a Banach space, the parameter space. In particular, the L-S procedure is useful in obtaining a family of solutions.

This paper is concerned with finding a local family of solutions of equation (1), and grew out of a recent result by V. Yudovich [Y1]. Throughout the paper, X and Y are Banach spaces, F is a smooth map from a neighbourhood U of the origin in X into Y, with

$$F(0) = 0.$$

We always assume

$$\left\{\begin{array}{l} X_2 = \ker F'(0) \quad \text{has a closed complementing} \\ \text{subspace } X_1, \text{ i.e., } X = X_1 \oplus X_2. \end{array}\right. \tag{2}$$

$$\left\{\begin{array}{l} Y_1 = \text{Range } F'(0) \text{ is closed and has a closed} \\ \text{complementing space } Y_2 \text{ in } Y. \end{array}\right. \tag{3}$$

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In [Y1], Yudovich treated (1) and introduced the notion of a cosymmetry map η of U into $Y^\star$, the dual space to Y. Its existence ensures that (1) has a *family* of solutions. Here is a result of [Y1]; the X and Y are asssumed to be Hilbert spaces but the result extends immediately to the following form.

THEOREM 1 (YUDOVICH). *Let X, Y, F be as above, i.e., (2) and (3) hold, and assume a stronger form of (3):*

$$(3)' \qquad Y_1 = \textit{Range } F'(0) \textit{ is closed and has codimension } 1.$$

Assume that there is a continuous map η of U into $Y^\star$, $\eta(0) \neq 0$ and

$$(4) \qquad \langle \eta(x), F(x) \rangle \equiv 0 \quad \textit{in } U.$$

A map with such properties is called a cosymmetry map. Then there is a smooth map u of a ball $B_r(0)$ in X_2 into X_1, with $u(0) = 0$, such that

$$F(u(x_2) + x_2) \equiv 0, \quad \forall x_2 \in B_r(0).$$

The proof is a simple application of L-S; for the convenience of the reader we include it:

PROOF. For x near the origin, by continuity of η,

$$0 = \langle \eta(x), F(x) \rangle = \langle \eta(0), F'(0)x \rangle + o(\|x\|).$$

Consequently $\eta(0)$ annihilates Range$F'(0) = Y_1$. We may decompose Y as $Y_1 \oplus Y_2$ where Y_2 is spanned by a unit vector y_2. Without loss of generality we may assume that

$$\langle \eta(0), y_2 \rangle = 1.$$

For x near the origin in X,

$$t(x) := \langle \eta(x), y_2 \rangle \quad \text{is close to } 1,$$

and

$$\bar{\eta}(x) := \eta(x) - t(x)\eta(0)$$

satisfies

$$\langle \bar{\eta}(x), y_2 \rangle = 0.$$

We now carry out the L-S procedure. Let P be the projection of Y onto Y_1 along Y_2. For x_2 near the origin in X_2, using the Implicit Function Theorem we solve the equation

$$(5) \qquad PF(x_1 + x_2) = 0$$

for $x_1 = u(x_2)$ in X_1, with $u(0) = 0$, u smooth. Then, for $x = x_2 + u(x_2)$, necessarily,

$$F(x) = \tau(x)y_2 \quad \text{with } \tau(0) = 0.$$

But

$$0 = \langle \eta(x), F(x) \rangle = t(x)\tau(x).$$

Consequently $\tau(x) \equiv 0$.

□

Theorem 1 extends to the case when, in place of $(3)'$, we have (see [Y4]):

$$(3)'' \qquad Y_1 = \text{Range } F'(0) \text{ is closed and has codimension } k < \infty$$

provided there are k continuous maps $\eta_1, \dots, \eta_k : U \to Y^\star$, which are linearly independent at the origin, each satisfying (4). In [Y1], Yudovich presents interesting applications of Theorem 1 including one to a problem on filtration convection. Papers [Y2], [Y3] contain further results while [Y4] contains extension (see there for more references).

The aim of this paper is to prove a more general result than Theorem 1, Theorem 2 below. The interest in this result is that in carrying out a L-S procedure, we use a **nonlinear** operator in place of a linear projection operator. However, in case Y is a Hilbert space, the standard L-S procedure applies, and the proof is then very simple.

THEOREM 2. *Let X, Y and F be as above and assume that $(3)''$ holds. Assume further that for some positive $\theta < 1$,*

$$dist(F(x), Y_1) \leq \theta ||F(x)|| \quad \forall x \in U. \tag{6}$$

Then for some $r > 0$, there is a unique smooth map u of $B_r^2 = \{x_2 \in X_2; ||x_2|| < r\}$ into X_1, with $u(0) = 0$, such that

$$F(x_2 + u(x_2)) = 0 \quad \forall x_2 \in B_r^2. \tag{7}$$

REMARK. Unfortunately, assumption (6) is not invariant under passage to an equivalent norm (even with another $\theta < 1$). But the conclusion of Theorem 2 is stable under such a change. It would be good to have a better condition than (6).

We shall actually prove the theorem under a weaker condition than $(3)''$, namely

$$(\tilde{3}) \quad \begin{cases} Y_1 = \text{Range } F'(0) \text{ is closed, and, for any } \varepsilon > 0, Y_1 \text{ admits a complementing} \\ \text{space } Y_2 \text{ and a Lipschitz continuous map } Q : Y_2 \to Y_1 \text{ satisfying} \\ ||y_2 - Q(y_2)|| \leq (1+\varepsilon) \text{ dist}(y_2, Y_1) \quad \forall y_2 \in Y_2. \end{cases}$$

Condition $(\tilde{3})$ automatically holds in case Y is a Hilbert space: Y_2 is then $Y_1^\perp$, and Q is the orthogonal projection onto Y_1. In addition, Lemma 1 below asserts that $(\tilde{3})$ always holds if Y_1 is closed and has finite codimension. We pose the following

Question. Let Y be a Banach space with a direct sum decomposition $Y = Y_1 \oplus Y_2, Y_1, Y_2$ closed subspaces. For any given $\varepsilon > 0$, is there a Lipschitz map Q of Y_2 into Y_1 satisfying

$$||y_2 - Q(y_2)|| \leq (1+\varepsilon) \text{ dist } (y_2, Y_1) \quad \forall y_2 \in Y_2 \ ? \tag{8}$$

LEMMA 1. *The answer to the question is yes in case dim $Y_2 = k < \infty$.*

PROOF. It suffices to construct Q on the unit sphere S in Y_2 with the property (8). We may then extend Q to all of Y_2 as homogeneous of degree one. For any $y_2 \in S$, there is a vector $\tilde{y}_1(y_2) \in Y_1$ such that

$$||y_2 - \tilde{y}_1(y_2)|| < (1+\varepsilon) \text{ dist } (y_2, Y_1).$$

By continuity there is a neighbourhood V of y_2 on S such that

$$||y - \tilde{y}_1(y_2)|| < (1+\varepsilon) \text{ dist } (y, Y_1) \quad \forall y \in V.$$

Cover S by a finite number of such neighbourhoods V_j of points $y_2^j, j = 1, \dots, N$, such that for every j, there exists $\tilde{y}_j \in Y_1$ with the property that

$$||y - \tilde{y}_j|| < (1 + \varepsilon) \text{ dist } (y, Y_1) \quad \forall y \in V_j.$$

For $j = 1, \dots, N$, let φ_j be a Lipschitz continuous partition of unity on S which is subordinate to the V_j. Then

$$Q(y) = \Sigma \varphi_j(y) \tilde{y}_j$$

has the desired property.

□

Section 2

To prove theorem 2 we use a convenient form of the Implicit Function theorem, (IFT), in which X, Λ, Z are Banach spaces. Consider a map G of $V = \overline{B}_a \times \overline{B}_b$ in $X \times \Lambda$ into Z, where

$$B_a = \{u \in X; ||x|| < a\}, \quad B_b = \{\lambda \in \Lambda; ||y|| < b\}.$$

For every λ in $\overline{B}_b$ we wish to solve

$$G(u, \lambda) = 0, \tag{9}$$

for u in $\overline{B}_a$.

LEMMA 2. (IFT). *Assume that there is a bounded linear injective map A of Z into X, and a positive constant $\gamma < 1$ such that*

$$\begin{aligned} ||A(G(u, \lambda) - G(v, \lambda)) - (u - v)|| \leq \gamma ||u - v|| \\ \textit{for } (u, \lambda), (v, \lambda) \in V. \end{aligned} \tag{10}$$

Assume also that

$$||AG(0, \lambda)|| \leq (1 - \gamma) a \quad \forall \lambda \in \overline{B}_b. \tag{11}$$

Then, for every λ in $\overline{B}_b$ there is a unique solution $u = u(\lambda)$ of (9) in $\overline{B}_a$. If G is continuous in V then $u(\lambda)$ is continuous. If, furthermore, G is uniformly Hölder, or Lipschitz continuous in λ, i.e., for some $k, \alpha > 0, \alpha \leq 1$,

$$||G(v, \lambda) - G(v, \mu)|| \leq k ||\lambda - \mu||^\alpha \textit{ for } (v, \lambda), (v, \mu) \in V,$$

then $u(\lambda)$ is Hölder continuous in λ, with α as Hölder exponent.

Lemma 2, which is taken from [B-N], is easily proved if one writes (9) in the form

$$u = u - AG(u, \lambda) =: T_\lambda(u).$$

One checks that for λ in $\bar{B}_b$, T_λ is a contracting map of $\overline{B}_a$ into itself, so that it has a unique fixed point. The remaining part of the conclusion is also readily verified.

PROOF OF THEOREM 2. In view of Lemma 1 it suffices to prove the theorem using the assumption $(\tilde{3})$ in place of $(3)''$. Fix $\varepsilon > 0$ so that $(1+\varepsilon)\theta < 1$. Then, choose Y_2 and Q as in $(\tilde{3})$. Let P be the projection of Y onto Y_1 along Y_2. For x_2 in $X_2, \|x_2\|$ small, we shall solve the following equation for $x_1 = u(x_2)$:

$$G(x) = G(x_1 + x_2) := PF(x) + Q[(I-P)F(x)] = 0. \tag{12}$$

G maps into Y_1. As a map from X_1 into Y_1 the operator $F'(0)$ has a bounded inverse $A = F'(0)^{-1}$.

To solve (12) we use Lemma 2. Note that near the origin, G is Lipschitz continuous. Now

$$F(x) = F'(0)x + 0(\|x\|^2) \tag{13}$$

and, for x, x' near the origin,

$$\|F(x) - F(x') - F'(0)(x - x')\| = 0(\|x\| + \|x'\|) \cdot \|x - x'\|, \tag{14}$$

so that

$$\|(I-P)(F(x) - F(x'))\| \le C\|x - x'\|(\|x\| + \|x'\|). \tag{15}$$

It follows from (14) and (15) and the Lipschitz continuity of Q_2, that for x_1, x_1' near the origin in X_1, x_2 near the origin in X_2,

$$\|A(G(x_1+x_2) - G(x_1'+x_2)) - (x_1 - x_1')\| \le C\|x_1 - x_1'\|(\|x_1\| + \|x_1'\| + \|x_2\|). \tag{16}$$

We now apply Lemma 2 with X_1 as X, X_2 as Λ and Y_1 as Z. From (13) and (16) we see that for $0 < a, b$ small, the conditions of the lemma are satisfied. We conclude that there is a unique Lipschitz continuous solution $x_1 = u(x_2)$ with $u(0) = 0$, of $G(x_1 + x_2) = 0$—for every x_2 near the origin in X_2.

Claim: $x = x_2 + u(x_2)$ satisfies $F(x) = 0$.

PROOF. By (12),

$$\begin{aligned} F(x) &= F(x) - G(x) \\ &= (I-P)F(x) - Q[(I-P)F(x)]. \end{aligned}$$

Since $(I-P)F(x)$ is in Y_2 it follows from (8), taking $y_2 = (I-P)F(x)$, that

$$\begin{aligned} \|f(x)\| &= \|(I-P)F(x) - Q[(I-P)F(x)]\| \\ &\le (1+\varepsilon) \text{ dist } ((I-P)F(x), Y_1) \\ &= (1+\varepsilon) \text{ dist } (F(x), Y_1) \\ &\le (1+\varepsilon)\theta\|F(x)\| \qquad \text{by (6)}. \end{aligned}$$

Since $(1+\varepsilon)\theta < 1$, the claim follows.

Using a standard argument we now complete the proof of Theorem 2 by showing that $u(x_2)$ is smooth. We know that u is Lipschitz continuous. If we perturb x_2 by $\xi \in X_2$, we have

$$F(x_2 + \xi + u(x_2 + \xi)) - F(x_2 + u(x_2)) = 0.$$

Thus, for $x = x_2 + u(x_2)$, if $\|\xi\|$ is small,

$$F'(x)(\xi + u(x_2 + \xi) - u(x_2)) = o(\|\xi\|).$$

Apply P, the projection operator of Y onto Y_1 along Y_2. Then, with

$$M(x) = PF'(x)_{|X_1}$$

we obtain

$$M(x)(u(x_2 + \xi) - u(x_2)) = -PF'(x)\xi + o(\|\xi\|).$$

For x near the origin, $M(x)$ is close to $F'(0)_{|X_1}$, and so is invertible as a map from X_1 to Y_1, and $M(x)^{-1}$ is smooth in x. Thus

$$u(x_2 + \xi) - u(x_2) = -M(x)^{-1}PF'(x)\xi + o(\|\xi\|).$$

It follows that $u'(x_2)$ exists and is given by

$$\begin{aligned} u'(x_2) &= -M(x)^{-1}PF'(x) \\ &= -M(x_2 + u(x_2))^{-1}PF'(x_2 + u(x_2)). \end{aligned} \tag{17}$$

Thus $u'(x_2)$ is continous, in fact Lipschitz continuous, in x_2. Consequently, the right hand side of (17) is continuously differentiable in x, which means that u is in C^2—and so on.

□

Section 3

With the aid of Theorem 2 we now derive a result which is closer to Theorem 1 but which replaces the cosymmetry condition (4) by an inequality.

THEOREM 3. *Let X, Y, F be as above and assume that (2) and* $(3)''$ *hold. Assume that there exists a k-dimensional subspace L of $Y^\star$ such that for some constant $\sigma < 1/3$,*

$$|\langle \eta, F(x)\rangle| \le \sigma \|F(x)\|\,\|\eta\| \quad \forall x \in U, \quad \forall \eta \in L. \tag{18}$$

Then there is a smooth map u of a ball $B_r(0) \subset X_2$ into X_1, with $u(0) = 0$, such that

$$F(u(x_2) + x_2) = 0 \quad \forall x_2 \in B_r(0).$$

The proof makes use of the following

LEMMA 3. *Let Y be a Banach space and Y_1 a closed subspace of codimension $k < \infty$. Assume that for some positive number $\theta < 1$,*

$$|\langle \eta, y_1\rangle| \le \frac{\theta}{2+\theta}\|\eta\|\,\|y_1\| \quad \forall \eta \in L, \quad \forall y_1 \in Y_1. \tag{19}$$

Suppose also that for some vector $y \in Y$,

$$|\langle \eta, y\rangle| \le \frac{\theta}{2+\theta}\|\eta\|\,\|y\| \quad \forall \eta \in L. \tag{20}$$

Then

$$d := \ dist\ (y, Y_1) \le \theta\|y\|.$$

Using Lemma 3 we first give the

PROOF OF THEOREM 3. We will prove that for $\theta = \frac{2\sigma}{1-\sigma} < 1$,

$$\text{dist}\ (F(x), Y_1) \leq \theta \|F(x)\| \quad \forall x \in U. \tag{21}$$

The desired conclusion then follows from Theorem 2.

For $\|x\|$ small, we have, by (18): for every unit vector $\eta \in L$,

$$|\langle \eta, F'(0)x \rangle| \leq \sigma \|F'(0)x\| + 0(\|x\|^2).$$

Consequently

$$|\langle \eta, y_1 \rangle| \leq \sigma \|y_1\| \quad \forall y_1 \in Y_1,$$

i.e., (19) holds, since $\sigma = \theta/(2+\theta)$. By (18), (20) holds with $y = F(x)$. Lemma 3 then yields (21).

□

To prove Lemma 3 we rely on the following standard lemma, whose proof uses Borsuk's theorem (see e.g. Lemma 2.3 in [K]).

LEMMA 4. *Let Y be a Banach space, and $\overline{Y}$ a closed linear subspace of finite codimension $k-1$. Let L be a linear subspace of $Y^\star$ of dimension k. Then there is a unit vector $y^\star \in L$ such that*

$$\sup_{\substack{y \in \overline{Y} \\ \|y\|=1}} \langle y^\star, y \rangle = 1. \tag{22}$$

PROOF OF LEMMA 3. We may suppose that $\|y\| = 1$ and that $y \notin Y_1$— otherwise there is nothing to prove. Let $\overline{Y}$ be the space spanned by Y_1 and y; it has codimension $(k-1)$. According to Lemma 4, there is a unit vector $\eta \in L$ such that

$$\sup_{\substack{z \in \overline{Y} \\ \|z\|=1}} \langle \eta, z \rangle = 1 \tag{23}$$

Any unit vector z in $\overline{Y}$ has the form

$$z = ay + y_1 \quad \text{with } y_1 \in Y_1.$$

Clearly, for $d = \text{ dist } (y, Y_1)$,

$$|a|d = \text{ dist } (z, Y_1) \leq 1.$$

Furthermore, $\|y_1\| \leq 1 + |a|$. Then, by (19) and (20),

$$\begin{aligned} \langle \eta, z \rangle &= \langle \eta, ay + y_1 \rangle \\ &\leq \frac{\theta}{2+\theta}(1 + 2|a|). \end{aligned}$$

Since this holds for any z in $\overline{Y}$, it follows from (24) that

$$1 \leq \frac{\theta}{2+\theta}(1 + 2|a|) \leq \frac{\theta}{2+\theta}(1 + \frac{2}{d});$$

this implies that $d \leq \theta$.

□

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