NEW DEVELOPMENTS ON THE GINZBURG-LANDAU MODEL

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Dedicated to Jean Leray with admiration

The starting point of the new developments around the Ginzburg-Landau equation is a "frustrating" lemma. Suppose $\Omega \subset \mathbb{R}^2$ is a smooth, bounded domain, simply connected. Fix a smooth boundary condition $g : \partial\Omega \to S^1$ (S^1 = the unit circle in $\mathbb{R}^2 = \mathbb{C}$) and consider the class of functions

$$H^1_g = H^1_g(\Omega, S^1) = \{u : \Omega \to S^1, \nabla u \in L^2 \text{ and } u = g \text{ on } \partial\Omega\}.$$

Lemma 1. *The class $H^1_g \neq \phi$ if and only if*

$$\deg(g, \partial\Omega) = 0.$$

Here deg refers to the usual (Brouwer) degree, also called winding number, of g, considered as a map from $\partial\Omega(\simeq S^1)$ into S^1. It is a pleasure to acknowledge the pioneering role played by J. Leray in the development of degree theory and its use in analysis (see J. Leray and J. Schauder [1]).

The proof of Lemma 1 is not straightforward, especially the implication $\Rightarrow$. One method consists of taking some $u \in H^1_g$ and using it to homotopy g to a constant, for example via its restriction to circles when Ω is a disc. Of course u need not be continuous and thus one cannot use standard degree theory. Instead one relies on the $H^{1/2}$ degree theory—a notion introduced by L. Boutet de Monvel and O. Gabber (see A. Boutet de Monvel-Berthier, V. Georgescu and R. Purice [1] for maps of S^1 into S^1; see also H. Brezis and L. Nirenberg [1] for the higher dimensional case and general manifolds. Another method consists of using the formula

$$\deg(g, \partial\Omega) = \frac{1}{\pi} \int_\Omega \tilde{u}_x \wedge \tilde{u}_y, \tag{1}$$

which holds for any smooth $\tilde{u} : \Omega \to \mathbb{R}^2$ such that $\tilde{u} = g$ on $\partial\Omega$. A density argument shows that (1) still holds for any $\tilde{u} \in H^1_g(\Omega, \mathbb{R}^2)$. In particular, if $H^1_g \neq \phi$, one may use (1) with some $u \in H^1_g$ and since $u_x \wedge u_y = 0$, it follows that $\deg(g, \partial\Omega) = 0$.

We are now led to a dichotomy:

Case 1: $d = \deg(g, \partial\Omega) = 0$,

Case 2: $d = \deg(g, \partial\Omega) \neq 0$.

In Case 1, $H^1_g \neq \phi$ and then one may prove

Lemma 2. *$H^1_g = \{u = e^{i\varphi},\ \varphi \in H^1(\Omega, \mathbb{R})$ and $\varphi = \varphi_0$ on $\partial\Omega\}$ where φ_0 is a smooth lifting of g, i.e., $\varphi_0 : \partial\Omega \to \mathbb{R}$ is a smooth function such that $g = e^{i\varphi_0}$ on $\partial\Omega$.*

Lemma 2 is somewhat subtle. For example, it fails if H^1 is replaced by the Sobolev space $W^{1,p}$, $p < 2$. However Lemma 2 holds for any smooth domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$; see F. Bethuel and X. Zheng [1] and the elegant presentation due to P. Mironescu in H. Brezis [1].

Theorem 1. *If $d = 0$, the minimization problem*

$$\operatorname*{Min}_{u \in H^1_g} \int |\nabla u|^2 \tag{2}$$

has a unique solution $u_\star = e^{i\varphi}$, where φ is the harmonic extension in Ω of φ_0.

In Case 2 the analogue of problem (2) is meaningless since $H^1_g = \phi$. In other words, any extension u of g in Ω with values in S^1 must have infinite energy. Recently, we have discovered with F. Bethuel and F. Hélein an approach showing that problem (2) makes sense even when $d \neq 0$. In a sense we establish that some extensions have "less infinite energy" than others. We summarize here the main ideas developed in F. Bethuel, H. Brezis and F. Hélein [1], as well as other progress.

The first natural approach is to relax the constraint that u takes its values in S^1; instead one considers the class of all testing functions in

$$H^1_g(\Omega, \mathbb{C}).$$

This is *never* empty, even if $d \neq 0$. The S^1 constraint reappears in the energy in the form of a "penalty". Namely, one works with the Ginzburg-Landau functional

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_\Omega (|u|^2 - 1)^2.$$

As $\varepsilon \to 0$ the second term "forces" u to take its values into S^1.

It is easy to see that

$$\operatorname*{Min}_{u \in H^1_g(\Omega, \mathbb{C})} E_\varepsilon(u) \tag{3}$$

is always achieved. Let u_ε be a minimizer. Incidentally, the *uniqueness* of u_ε is a delicate matter; in general uniqueness does not hold (see F. Bethuel, H. Brezis and F. Hélein [1]), however, uniqueness is conjectured in some special situations, for instance where $\Omega = B_1$ and $g(\theta) = e^{i\theta}$.

Clearly, u_ε satisfies an Euler equation, namely the Ginzburg-Landau equation,

$$-\Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2).$$

Using the maximum principle one shows easily that

$$|u_\varepsilon| \leq 1 \quad \text{in } \Omega.$$

This suggests that u_ε converges in some appropriate sense to a limit as $\varepsilon \to 0$.

This is indeed true:

Theorem 2. *Assume $d > 0$, for simplicity. Then, there is a subsequence $\varepsilon_n \to 0$ and d distinct points $a_1, a_2, \dots, a_d$ in Ω such that*

$$u_{\varepsilon_n}(z) \longrightarrow u_\star(z) \quad \text{in } C^k(\Omega \backslash \{a_1, a_2, \dots, a_d\}).$$

Moreover,

$$u_\star(z) = \frac{(z-a_1)}{|z-a_1|} \cdots \frac{(z-a_d)}{|z-a_d|} e^{i\varphi}, \tag{4}$$

where φ is a smooth harmonic functions in Ω and its boundary value is determined via (4) using the fact that $u_\star = g$ on $\partial\Omega$.

Theorem 2 was initially proved by F. Bethuel, H. Brezis and F. Hélein [1], [2] for starshaped domains. This assumption is used to show, via the Pohozaev identity,

$$\frac{1}{2\varepsilon^2} \int_\Omega (|u_\varepsilon|^2 - 1)^2 + \frac{1}{2} \int_{\partial\Omega} (x \cdot \nu) \left| \frac{\partial u_\varepsilon}{\partial \nu} \right|^2 = \frac{1}{2} \int_{\partial\Omega} (x \cdot \nu) \left| \frac{\partial g}{\partial \tau} \right|^2 - \int_{\partial\Omega} (x \cdot \tau) \frac{\partial u_\varepsilon}{\partial \nu} \frac{\partial g}{\partial \tau}$$

(where ν and τ denote the normal and tangential directions on $\partial\Omega$), that

$$\frac{1}{\varepsilon^2} \int_\Omega (|u_\varepsilon|^2 - 1)^2 \le C \tag{5}$$

with C independent of ε. Using a local form of the Pohozaev identity, M. Struwe [1],[2] was later able to remove the starshapedness assumption.

Estimate (5) plays a crucial role in the proof of Theorem 2. It allows to isolate a finite number of "bad" discs in Ω

$$B(x_i, \lambda\varepsilon), \quad i \in J_\varepsilon$$

with the property that

$$\text{card} J_\varepsilon \le C, \quad \text{with } C \text{ and } \lambda \text{ independent of } \varepsilon, \tag{6}$$

$$|x_i - x_j| \ge 8\lambda\varepsilon \quad \forall i, j \in J_\varepsilon \tag{7}$$

$$|u_\varepsilon| \ge 1/2 \quad \text{outside the bad discs.} \tag{8}$$

Another basic ingredient in the proof of Theorem 2 consists of finding a *lower bound for the energy* of a map on a domain Ω with holes, *in terms of the degree of u on the bounadries of the holes.* This study has been initiated in F. Bethuel, H. Brezis and F. Hélein [1] and pursued by H. Brezis, F. Merle and T. Rivière [1]. The best estimate so far is due to Z. C. Han and I. Shafrir [1]. We describe their result, because it is of interest *independently* of the Ginzburg-Landau problem.

Let $\Omega \subset \mathbb{R}^2$ be any domain and set

$$G = \Omega \backslash \bigcup_{j=1}^{n} B(x_i, \rho)$$

where $x_1, x_2, \dots, x_n$ are n distinct points in G and $\rho > 0$. Assume further that, for some $\mu > 0$,

$$\operatorname{dist}(x_j, \partial\Omega) \geq \max\{\rho, 2\mu\} \quad \forall j \tag{9}$$

and

$$|x_i - x_j| \geq 4\rho \quad \forall i \neq j. \tag{10}$$

Consider the class $\mathcal{C}$ of maps u satisfying

$$u \in C^0(\overline{G}, \mathbb{R}^2) \cap C^1(G, \mathbb{R}^2), \tag{11}$$

$$0 < a \leq |u| \leq 1 \quad \text{in } G \tag{12}$$

$$\frac{1}{\rho^2} \int_G (|u|^2 - 1)^2 \leq K \tag{13}$$

and

$$\deg(u, \partial B(x_i, \rho)) = d_i \ , \quad \forall i \tag{14}$$

where a, K are constants and $d_i \in \mathbb{Z}$.

Theorem 3. *Assume (9)–(14). Then*

$$\int_G |\nabla u|^2 \geq 2\pi F(d_1, d_2, \dots, d_n) \log\left(\frac{\mu}{\rho}\right) - C, \tag{15}$$

where

$$F(d_1, d_2, \dots, d_n) = \min\left\{ \sum_{m=1}^{N} |\sum_{j \in J_m} d_j|^2 \right\},$$

this minimum being taken over all possible partitions of the set $\{1, 2, \dots, n\}$ into disjoint subset $J_1, J_2, \dots, J_N$. The constant C in (15) depends only on $a, K, d_1, \dots d_n$. ∎

Theorem 3 is used in the proof of Theorem 2 with $\rho = \lambda\varepsilon$, the constant K being related to C in (5).

The location of the singular points $a_1, \dots, a_d$ in Theorem 2 is determined via a rather simple and explicit procedure, involving a "renormalized energy" discovered by F. Bethuel, H. Brezis and F. Hélein [1],[2]. We describe it using a simplified presentation communicated to us by C. G. Ragazzo.

Given a point $y \in \Omega$ consider the Green's function $G(x,y)$ relative to a Neumann boundary condition:

$$\Delta G = 2\pi\delta_y \quad \text{in } \Omega,$$
$$\frac{\partial G}{\partial \nu} = \frac{1}{d}(g \wedge g_\tau) \quad \text{on } \partial\Omega$$
$$\int_{\partial\Omega} G(\cdot, y)(g \wedge g_\tau) = 0.$$

Note that G is uniquely defined because of the third (normalization) condition. It is not difficult to check that $G(x,y) = G(y,x)$, $\forall x, y$.

Consider the regular part of G,

$$R(x,y) = G(x,y) - \log(x-y),$$

so that $R(x,x)$ makes sense (and is smooth).

Given a configuration $b = (b_1, b_2, \dots, b_d)$ of d distinct points in Ω, set

$$W(b) = -\pi \sum_{i \neq j} \log|b_i - b_j| - \pi \sum_{i,j} R(b_i, b_j). \tag{16}$$

Theorem 4. *The configuration $a = (a_1, a_2, \dots, a_d)$ given by Theorem 2 satisfies*

$$W(a) \leq W(b) \quad \forall b. \tag{17}$$

The proof is somewhat technical (see F. Bethuel, H. Brezis and F. Hélein [1], Chapter VIII). But here is one illuminating observation. Let us return to the "meaningless" problem

$$\operatorname*{Min}_{H^1_g} \int |\nabla u|^2,$$

and let us recall that it made no sense because of the *topological obstruction* stated in Lemma 1. The first approach consisted of removing the topological obstruction by changing the *target space*: instead of S^1 we used $\mathbb{C}(= \mathbb{R}^2)$. A *totally different approach* consists of breaking the topological obstruction in the *domain space*, i.e., Ω, by making holes. Of course, for topological purposes it suffices to make just one hole. It is however convenient, to minimize energy, to make several holes. Set

$$\Omega_\rho = \Omega \setminus \bigcup_{i=1}^{k} B(b_i, \rho),$$

where $b = (b_1, b_2, \dots, b_k)$ is a given configuration of k distinct points in Ω and ρ is a small parameter. Consider the class $\mathcal{E}_\rho$ of maps $u \in H^1(\Omega_\rho, S^1)$ such that $u = g$ on $\partial\Omega$ and $\deg(u, \partial B(b_i, \rho)) = d_i \quad \forall i = 1, \dots, k$ where $d_1, \dots, d_k$ are given in $\mathbb{Z}$. The analogue of Lemma 1 here says that $\mathcal{E}_\rho$ is not empty iff

$$\sum_{i=1}^{k} d_i = d. \tag{18}$$

Assuming (18) one then considers the problem

$$\operatorname*{Min}_{u\in\mathcal{E}_\rho}\int_{\Omega_\rho}|\nabla u|^2. \tag{19}$$

It has a unique solution u_ρ which can be expressed explicitly in terms of harmonic functions as in Theorem 1. It is not difficult to see that, as $\rho\to 0$,

$$\frac{1}{2}\int_{\Omega_\rho}|\nabla u_\rho|^2=\pi\left(\sum_{i=1}^k d_i^2\right)\log\frac{1}{\rho}+0(1). \tag{20}$$

Of course, the right hand side in (20) tends to infinity as $\rho\to 0$. However, in order to make it a "small infinity" it pays to minimize $\sum\limits_{i=1}^k d_i^2$, subject to the constraint (18). Recall that the number of holes, k, is also at our disposal. Since

$$\sum_{i=1}^k d_i^2\geq\sum_{i=1}^k|d_i|\geq|d|=d,$$

it follows that the best one can do is to take $k=d$ and choose $d_i=+1\quad\forall i$. In what follows we make this choice.

Returning to (20) we then have

$$\frac{1}{2}\int_{\Omega_\rho}|\nabla u_\rho|^2=\pi\,d\,\log\frac{1}{\rho}+0(1). \tag{21}$$

Pushing further the expansion one finds, as $\rho\to 0$,

$$\frac{1}{2}\int_{\Omega_\rho}|\nabla u_\rho|^2=\pi\,d\,\log\frac{1}{\rho}+W(b)+0(\rho). \tag{22}$$

In order to minimize the "infinite energy" (as $\rho\to 0$) it is then natural to make the holes centered at a configuration $a=(a_1,a_2,\dots,a_d)$ which minimizes the renormalized energy W. Then, one proves that $u_\rho\to u_\star$ given by (4).

We find it very surprising that these two approaches (via Ginzburg-Landau or via holes), which are quite different in nature, turn out to be consistent. This means that, in some sense, $u_\star$ given by (4), together with (17), provides an *intrinsic solution* to the original meaningless problem.

This assertion is reinforced by the following recent result of R. Hardt and F. H. Lin [1], showing that a third natural approximation method yields the same $u_\star$.

Given any $p<2$ it is easy to see that

$$W_g^{1,p}=W_g^{1,p}(\Omega,S^1)\neq\phi,$$

even when $d\neq 0$; for example, if $\Omega=B_1$, then $g\left(\dfrac{x}{|x|}\right)$ belongs to $W^{1,p}$ for every $p<2$. For any $p<2$, let u_p be a minimizer for

$$\operatorname*{Min}_{u\in W_g^{1,p}}\int|\nabla u|^p.$$

Theorem 5. *A subsequence (u_{p_n}) converges, as $p_n \to 2$, to $u_\star$ given by Theorem 2 with $a = (a_1, a_2, \dots, a_d)$ satisfying (17).*

At this stage the reader may think that any "reasonable" approximation procedure will lead to the same answer. In fact, the situation is more complicated. Roughly speaking, the above $u_\star$ corresponds to a "homogeneous" material. Suppose we introduce a weight function w in the Ginzburg-Landau functional and set

$$\widetilde{E}_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_\Omega (|u|^2 - 1)^2 w,$$

where $w \in C^1(\widetilde{\Omega})$ and $w \geq \alpha > 0$ on Ω.

Theorem 6. *Let $\tilde{u}_\varepsilon$ be a minimizer of $\widetilde{E}_\varepsilon$. Then, there is a subsequence $\varepsilon_n \to 0$ such that $u_{\varepsilon_n} \to u_\star$ given by (4). In addition, the configuration $a = (a_1, a_2, \dots, a_d)$ minimizes another renormalized energy; more precisely*

$$\widetilde{W}(a) \leq \widetilde{W}(b) \quad \forall b$$

where $\widetilde{W}(b) = W(b) + \frac{\pi}{2}\sum_{j=1}^{d} \log w(b_j)$, and W is given by (16).

For the proof of Theorem 6, see C. Lefter and V. Radulescu [1] and also M. C. Hong [1].∎ As a consequence one sees that the configuration of singularities $a = (a_1, \dots, a_d)$ may be driven to any desired location by choosing a weight w which is almost zero near certain points (this may be related to the "pinning of vortices" observed in the physics of super-conductors). It would be interesting to determine whether other natural approximation techniques could yield some limits which are not of the form (4).

An interesting line of current research consists of describing the behavior of u_ε *near* its singularities. For example one knows that $\int_\Omega |\nabla u_\varepsilon|^2 \to \infty$ as $\varepsilon \to 0$. However certain related quantities remain bounded, for example

$$\int_\Omega \Big|\nabla |u_\varepsilon|\Big|^2 \leq C;$$

this was proved in F. Bethuel, H. Brezis and F. Hélein [1]. We had made some conjectures which have been solved by P. Mironescu and M. Comte:

Theorem 7. *Let u_ε be a minimizer of E_ε. Then*

$$\int_\Omega (1 - |u_\varepsilon|)^\alpha |\nabla u_\varepsilon|^2 \leq C_\alpha \quad \text{for any } \alpha > 0, \tag{23}$$

$$\int_\Omega (1 - |u_\varepsilon|)^\alpha |u_\varepsilon|^\alpha \left|\nabla\left(\frac{u_\varepsilon}{|u_\varepsilon|}\right)\right|^2 \leq C_\alpha \quad \text{for any } \alpha > 0 \tag{24}$$

and

$$\int_\Omega |\det \nabla u_\varepsilon| \leq C. \tag{25}$$

For the proofs we refer to P. Mironescu [2], M. Comte and P. Mironescu [1] [2]. On a related matter, one knows that for ε sufficiently small, u_ε has precisely d zeroes. *Far away from these zeroes* formula (4) provides a good approximation for u_ε, but how about *near* these zeroes?

Here is an interesting result of I. Shafrir [2]. For simplicity, we state it when $d = 1$.

Theorem 8. *Let φ be as in Theorem 2. There is a smooth complex-valued function F defined on $\mathbb{R}^2$ such that,*

$$\lim_{\varepsilon_n \to 0} \left\| u_{\varepsilon_n}(z) - F\left(\frac{z - a_{\varepsilon_n}}{\varepsilon_n}\right) e^{i\varphi(z)} \right\|_{L^\infty(\Omega)} = 0. \tag{26}$$

where a_ε is the zero of u_ε.

The function F satisfies $F(0) = 0$,

$$-\Delta F = F(1 - |F|^2) \quad \text{on } \mathbb{R}^2 \tag{27}$$

and F is energy minimizing (for the natural energy associated to (27)).

Further properties of F have been obtained by I. Shafrir [1]; for example, 0 is the only point where F vanishes and it is a zero of index $+1$. In addition

$$\lim_{|z| \to \infty} \left| F(z) - \frac{z}{|z|} \right| = 0.$$

A very interesting *open problem* is to determine whether F has the precise form

$$F(z) = f(|z|)\frac{z}{|z|} \tag{28}$$

where f is a real valued function satisfying

$$-f'' - \frac{1}{r}f' + \frac{1}{r^2}f = f(1 - f^2) \quad \text{on } (0, \infty)$$

with

$$f(0) = 0 \quad \text{and } f(\infty) = 1.$$

We call attention to a nice result of P. Mironescu [1] on a related subject.

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