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Convergence in $\mathcal{D}'$ and in L^1 under strict convexity

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Dedicated to Enrico Magenes with esteem and affection

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let (u_n) be a sequence in $L^1(\Omega;\mathbb{R}^M)$ which converges "*weakly*" to some limit $u \in L^1(\Omega;\mathbb{R}^M)$. Let $j : \mathbb{R}^M \to \mathbb{R}$ be a convex function such that

$$\limsup_{n\to\infty} \int_\Omega j(u_n) \leq \int_\Omega j(u). \tag{1}$$

Many authors have studied the question whether (u_n) converges *strongly* in L^1 if, in addition, j is assumed to be strictly convex (see [8], [2], [3], [4], [6] and the references therein). In these works it is often assumed that (u_n) converges to u *weakly* in L^1, that is, for the weak $\sigma(L^1, L^\infty)$ topology. Unfortunately, the convergence in $\sigma(L^1, L^\infty)$ is a *very restrictive assumption* and it is desirable to replace it by the *much weaker* and *more natural assumption* that (u_n) converges to u in the sense of distributions

$$u_n \to u \quad \text{in } \mathcal{D}'(\Omega;\mathbb{R}^M). \tag{2}$$

Throughout this paper we shall assume, for convenience, that $j : \mathbb{R}^M \to \mathbb{R}$ is a continuous convex function such that

$$|j(t)| \leq C(|t| + 1) \qquad \forall t \in \mathbb{R}^M \tag{3}$$

for some constant C.

Our main result is the following

THEOREM 1. *Let (u_n) be a sequence in $L^1(\Omega;\mathbb{R}^M)$ and let $u \in L^1(\Omega;\mathbb{R}^M)$ be such that (1) and (2) hold. Assume that*

(4) $$j \text{ is strictly convex.}$$

(i) Then

(5) $$u_n \to u \quad \text{strongly in } L^1_{loc}(\Omega;\mathbb{R}^M).$$

(ii) If, in addition, we suppose that

(6) $$\lim_{|t|\to\infty} j(t) = +\infty$$

then

(7) $$u_n \to u \quad \text{strongly in } L^1(\Omega;\mathbb{R}^M).$$

Remark 1. If we assume that $u_n \rightharpoonup u$ weakly in $\sigma(L^1, L^\infty)$, then (1) and (4) imply (7) *without* having to assume (6) (see [8], Theorem 2). However, if we assume only (1), (2) and (4) *without* (6) then conclusion (7) may fail as the following example shows :

Example 1. Let j be any (smooth) strictly convex function on $\mathbb{R}$ satisfying

(8) $$j(t) > 0 \qquad \forall t \in \mathbb{R}$$

and

(9) $$\lim_{t\to+\infty} j(t) = 0.$$

Let $\Omega = (0,1)$ and let

$$u_n(x) = \begin{cases} 0 & \text{if } \ 0 < x < 1 - \dfrac{1}{n} \\ n^2 & \text{if } \ 1 - \dfrac{1}{n} < x < 1 \end{cases}$$

so that

$$u_n \to 0 \qquad \text{in } \mathcal{D}'(\Omega).$$

We have

$$\int_\Omega j(u_n) = (1 - \frac{1}{n}) j(0) + \frac{1}{n} j(n^2)$$

and thus (1) holds. But (7) fails and we even have $\|u_n\|_{L^1} \to \infty$.

An easy consequence of Theorem 1 is the following :

COROLLARY 1. *Let (u_n) be a sequence in $W^{1,1}(\Omega;\mathbb{R})$ and let $u \in W^{1,1}(\Omega;\mathbb{R})$ be such that*

$$u_n \to u \quad \text{in } L^1_{loc}(\Omega). \tag{10}$$

Assume that $j : \mathbb{R}^N \to \mathbb{R}$ satisfies (3) and (4) and that

$$\limsup_{n\to\infty} \int_\Omega j(\nabla u_n) \le \int_\Omega j(\nabla u). \tag{11}$$

(i) Then we have

$$\nabla u_n \to \nabla u \quad \text{strongly in } L^1_{loc}(\Omega;\mathbb{R}^N). \tag{12}$$

(ii) If, in addition, (6) holds then we have

$$\nabla u_n \to \nabla u \quad \text{strongly in } L^1(\Omega;\mathbb{R}^N). \tag{13}$$

Assertion (ii) in Corollary (1) corresponds essentially to the conclusion of Theorem 8.6 in [1].

The proof of Theorem 1 is divided into 6 steps.

Step 1. Assume j is a convex function satisfying (3) and that (2) holds, then

$$\liminf_{n\to\infty} \int_\Omega j(u_n)\zeta \ge \int_\Omega j(u)\zeta \qquad \forall \zeta \in C^\infty(\Omega) \text{ with } 0 \le \zeta \le 1. \tag{14}$$

Proof. Let j^* be the conjugate convex function of j. Then

$$\int_\Omega j(u_n)\zeta \ge \int_\Omega u_n \varphi \zeta - \int_\Omega j^*(\varphi)\zeta \tag{15}$$

for every $\varphi \in \mathcal{D}(\Omega;\mathbb{R}^M)$. Passing to the limit in (15) we see that

$$\liminf_{n\to\infty} \int_\Omega j(u_n\zeta \ge \int_\Omega u\varphi\zeta - \int_\Omega j^*(\varphi)\zeta. \tag{16}$$

Next we observe (as in [5], Proposition 1) that

$$\sup_{\varphi\in\mathcal{D}(\Omega;\mathbb{R}^M)} \{\int_\Omega u\varphi\zeta - \int_\Omega j^*(\varphi)\zeta\} = \int_\Omega j(u)\zeta.$$

Remark 2. The spirit of Step 1 has been essentially known for a long time (see e.g. [7]).

Step 2. Assume (1), (2) and (4). Then, there is a subsequence (u_{n_k}) such that

$$u_{n_k} \to u \quad \text{a.e.} \tag{17}$$

Proof. Set

$$f_n = \frac{1}{2}j(u) + \frac{1}{2}j(u_n) - j\Big(\frac{u+u_n}{2}\Big) \geq 0. \tag{18}$$

By (1) and Step 1 (applied to $\frac{u+u_n}{2}$ and $\zeta \equiv 1$) we have

$$\limsup_{n\to\infty} \int_\Omega f_n \leq \int_\Omega j(u) - \liminf_{n\to\infty} \int_\Omega j\Big(\frac{u+u_n}{2}\Big) \leq 0. \tag{19}$$

Hence $f_n \to 0$ in $L^1(\Omega)$ and thus there is a subsequence n_k such that

$$f_{n_k} \to 0 \quad \text{a.e.} \tag{20}$$

We conclude easily with the help of the following standard

Lemma 1. *Assume j is strictly convex on $\mathbb{R}^M$. Let $a \in \mathbb{R}^M$ and let (b_n) be a sequence in $\mathbb{R}^M$ such that*

$$\frac{1}{2}j(a) + \frac{1}{2}j(b_n) - j\Big(\frac{a+b_n}{2}\Big) \to 0.$$

Then $b_n \to a$.

Step 3. Assume (1), (2) and (4). Then

$$j(u_{n_k}) \to j(u) \quad \text{in } \mathcal{D}'(\Omega). \tag{21}$$

Proof. Since j is convex there exist some $\xi \in \mathbb{R}^M$ and a constant C such that

$$j(t) \geq \xi t - C \qquad \forall t \in \mathbb{R}^M. \tag{22}$$

Set

$$\tilde{j}(t) = j(t) - \xi t + C$$

so that $\tilde{j}$ is convex and $\tilde{j}(t) \geq 0\ \forall t$. Set

$$g_k(x) = \tilde{j}(u_{n_k}(x)) - \left|\tilde{j}(u_{n_k}(x)) - \tilde{j}(u(x))\right|,$$

so that

$$|g_k(x)| \leq \tilde{j}(u(x)) \tag{23}$$

(since $\tilde{j} \geq 0$).

On the other hand, by Step 2, we know that

$$g_k(x) \to \tilde{j}(u(x)) \quad \text{a.e.} \tag{24}$$

We deduce from (23) and (24), by dominated convergence, that

$$g_k \to \tilde{j}(u) \quad \text{in } L^1(\Omega).$$

But

$$g_k - \tilde{j}(u) = -2(\tilde{j}(u_{n_k}) - \tilde{j}(u))^-$$

and thus we conclude that

$$\int_\Omega (\tilde{j}(u_{n_k}) - \tilde{j}(u))^- \to 0. \tag{25}$$

Finally, we observe that

$$|\tilde{j}(u_{n_k}) - \tilde{j}(u)| = \tilde{j}(u_{n_k}) - \tilde{j}(u) + 2(\tilde{j}(u_{n_k}) - \tilde{j}(u))^-. \tag{26}$$

Let $\zeta \in \mathcal{D}(\Omega)$ with $0 \leq \zeta \leq 1$ and write

$$\begin{aligned} \int_\Omega \tilde{j}(u_{n_k})\zeta &= \int_\Omega [j(u_{n_k})\zeta - \xi u_{n_k}\zeta + C\zeta] \\ &= \int_\Omega j(u_{n_k}) - \int_\Omega j(u_{n_k})(1-\zeta) - \int_\Omega \xi u_{n_k}\zeta + C\int_\Omega \zeta. \end{aligned} \tag{27}$$

Passing to the limit in (27) with the help of (1) and Step 1 (applied with $1-\zeta$ in place of ζ) we are led to

$$\limsup_{k\to\infty} \int_\Omega \tilde{j}(u_{n_k})\zeta \leq \int_\Omega \tilde{j}(u)\zeta. \tag{28}$$

Combining (25), (26) and (28) we obtain

$$\limsup_{k\to\infty} \int_\Omega |\tilde{j}(u_{n_k}) - \tilde{j}(u)|\zeta \leq 0. \tag{29}$$

In particular, $\tilde{j}(u_{n_k}) \to \tilde{j}(u)$ in $\mathcal{D}'(\Omega)$ and consequently $j(u_{n_k}) = \tilde{j}(u_{n_k}) + \xi u_{n_k} - C$ converges in $\mathcal{D}'(\Omega)$ to $j(u)$.

Step 4. We shall need the following :

LEMMA 2. *Let (ψ_n) be a sequence in $L^1(\Omega;\mathbb{R})$ and let $\psi \in L^1(\Omega;\mathbb{R})$ such that*

$$\psi_n \geq 0 \quad \text{a.e.}, \forall n, \tag{30}$$

$$\psi_n \to \psi \quad \text{a.e.} \tag{31}$$

and

$$\psi_n \to \psi \quad \text{in } \mathcal{D}'(\Omega). \tag{32}$$

Then

$$\psi_n \to \psi \quad \text{in } L^1_{loc}(\Omega). \tag{33}$$

Proof. Note that, by (30),

$$\psi - \psi_n \leq \psi$$

and thus

$$(\psi - \psi_n)^+ \leq \psi.$$

By dominated convergence we deduce that

$$(\psi - \psi_n)^+ \to 0 \quad \text{in } L^1(\Omega). \tag{34}$$

But

$$(\psi - \psi_n)^- = (\psi - \psi_n)^+ - (\psi - \psi_n)$$

and thus

$$\int_\Omega (\psi - \psi_n)^- \zeta = \int_\Omega (\psi - \psi_n)^+ \zeta - \int_\Omega (\psi - \psi_n)\zeta \to 0 \tag{35}$$

for every $\zeta \in \mathcal{D}(\Omega)$, by (32) and (34). The conclusion (33) follows from (34) and (35).

Step 5. We shall need the following :

LEMMA 3. *Let K be a closed convex set in $\mathbb{R}^M$, $K \neq \mathbb{R}^M$, and K strictly convex. Let (v_n) be a sequence in $L^1(\Omega; \mathbb{R}^M)$ and let $v \in L^1(\Omega; \mathbb{R}^M)$. Assume*

$$v_n(x) \in K \quad \text{a.e.}, \forall n,$$

$$v_n \to v \quad \text{a.e.}$$

and

$$v_n \rightharpoonup v \quad \text{in } \mathcal{D}'(\Omega; \mathbb{R}^M). \tag{38}$$

Then

$$v_n \hookrightarrow v \quad \text{in } L^1_{loc}(\Omega; \mathbb{R}^M). \tag{39}$$

Proof. Let I_K^* denote the conjugate function of the indicator function I_K of K, i.e.

$$I_K^*(y) = \sup_{x \in K} yx, \text{ for } y \in \mathbb{R}^M.$$

Note that $I_K^*(0) = 0$, $I_K^*(y) \in [0, \infty]$ $\forall y$ and $I_K^*(\lambda y) = \lambda I_K^*(y)$ $\forall \lambda > 0, \forall y$. Hence

$$D(I_K^*) = \{y \in \mathbb{R}^M; I_K^*(y) < \infty\}$$

is a convex cone with vertex at 0. We claim that

$$D(I_K^*) \quad \text{has non empty interior.} \tag{40}$$

For otherwise $D(I_K^*)$ would be contained in some hyperplane, say $y_M = 0$. Then

$$I_K(x) = \sup_{y \in \mathbb{R}^M} \{xy - I_K^*(y)\} = \sup_{[y_M = 0]} \{xy - I_K^*(y)\}$$

and consequently

$$I_K(x + te_M) = I_K(x) \qquad \forall t \in \mathbb{R}, \forall x$$

where e_M denote the unit vector normal to the hyperplane $y_M = 0$. This means that K is a cylinder of the form

$$K = Q \times \mathbb{R}$$

where $Q = K \cap [y_M = 0]$. This is impossible since K is assumed to be strictly convex and $K \neq \mathbb{R}^M$. Hence we have proved the claim (40).

Next, let $\xi_1, \xi_2, ...\xi_M$ be a collection of unit vectors in $D(I_K^*)$ which are linearly independent (such a collection exists by (40)). Set

$$C_i = I_K^*(\xi_i) < \infty.$$

For each fixed $i = 1, 2, ...M$, consider the function

$$\psi_n^i(x) = C_i - v_n(x)\xi_i.$$

It is easy to see, using (36)-(38), that ψ_n^i satisfies (30)-(32) and therefore, by Lemma 2,

$$\psi_n^i \underset{n-\infty}{\longrightarrow} \psi_i \quad \text{in } L^1_{\text{loc}}, \ \forall i.$$

Since the directions ξ_i are linearly independent we conclude that

$$v_n \to v \quad \text{in } L^1_{\text{loc}}(\Omega; \mathbb{R}^M).$$

Step 6. Proof of Theorem 1.

Part (i). Let $K = epi\ j = \{[t, \lambda] \in \mathbb{R}^M \times \mathbb{R}; \lambda \geq j(t)\}$, so that K is a closed convex set in $\mathbb{R}^{M+1}$, $K \neq \mathbb{R}^{M+1}$, and K is strictly convex (because j is strictly convex). Set

$$v_n(x) = [u_n(x), j(u_n(x))].$$

Clearly, $v_n(x) \in K$ a.e., $\forall n$. By Step 2 we know that

$$v_{n_k} \to v = [u, j(u)] \quad \text{a.e.}$$

By assumption (2) and by Step 3 we know that

$$v_{n_k} \to v \quad \text{in } \mathcal{D}'(\Omega; \mathbb{R}^{M+1}).$$

Applying Lemma 3 (with $(M+1)$ instead of M) we conclude that

$$v_{n_k} \to v \quad \text{in } L^1_{\text{loc}}(\Omega; \mathbb{R}^{M+1})$$

and in particular

$$u_{n_k} \to u \quad \text{in } L^1_{\text{loc}}(\Omega; \mathbb{R}^M).$$

The uniqueness of the limit implies, as usual, that

$$u_n \to u \quad \text{in } L^1_{\text{loc}}(\Omega; \mathbb{R}^M).$$

Part (ii). The additional assumption (6) implies that

$$j(t) \geq \alpha|t| - C \qquad \forall t$$

for some constants $\alpha > 0$ and C. Adding a constant to j we may always assume that

(41) $$j(t) \geq \alpha|t| \geq 0 \qquad \forall t.$$

Applying Step 1 with $\zeta \equiv 1$ and combining this with assumption (1) we see that

(42) $$\int_\Omega j(u_n) \to \int_\Omega j(u).$$

We write once more (as in the proof of Lemma 2)

$$(j(u) - j(u_n))^+ \leq j(u),$$

so that, by Step 2 and dominated convergence

(43) $$\int_\Omega (j(u) - j(u_{n_k}))^+ \to 0.$$

Finally we recall that

$$(j(u) - j(u_n))^- = (j(u) - j(u_n))^+ - (j(u) - j(u_n))$$

and consequently (using (42) and (43)) we conclude that

(44) $$\int_\Omega (j(u) - j(u_{n_k}))^- \to 0.$$

From (43) and (44) we deduce that

$$j(u_{n_k}) \to j(u) \quad \text{in } L^1$$

Passing to a further subsequence we may always assume that

(45) $$|j(u_{n_k})| \leq f \qquad \forall k. \text{ a.e.}$$

for some fixed function $f \in L^1(\Omega)$. Combining (41) and (45) we conclude that

$$|u_{n_k}| \leq \frac{1}{\alpha} f \qquad \forall k. \text{ a.e.}$$

From Step 2 and dominated convergence we infer that

$$u_{n_k} \to u \quad \text{in } L^1(\Omega;\mathbb{R}^M).$$

Again, the uniqueness of the limit implies the convergence of the full sequence.

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