A sup + inf Inequality for Some Nonlinear Elliptic Equations Involving Exponential Nonlinearities

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*Communicated by the Editors*

Received September 1, 1992

Let $u$ be a solution of the equation $-\Delta u = V(x) e^u$ in a domain $\Omega \subset \mathbb{R}^2$, where $0 \leq a < V \leq b$ and $V$ is Lipschitz continuous. We prove that $\sup u$ can be controlled in terms of $\inf u$. More precisely, $\sup_K u + \inf_K u \leq C(a, b, K, \Omega, \|V\|_{L^2})$ for any compact subset $K \subset \Omega$. This extends an earlier result of Shafrir who obtained a similar conclusion when $V \equiv 1$. © 1993 Academic Press, Inc.

1. INTRODUCTION

In this paper we are concerned with the equation

$$-\Delta u = V(x) e^u \quad \text{in } \Omega,$$

(1)

where $\Omega$ is a domain in $\mathbb{R}^2$ and $V$ satisfies

$$0 < a \leq V(x) \leq b < \infty$$

(2)

0022-1236/93 $\$5.00

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for some constants $a$ and $b$. This type of nonlinear equation appears in connection with scalar curvature problems and has been extensively studied (see, e.g., [3, 4, 8, 9]). However, the estimate we present here seems to be of a new kind.

The starting point of our analysis is a result from [1] which asserts that for every compact subset $K \subset \Omega$, $\sup_{K} u$ can be controlled in terms of $\inf_{\Omega} u$; that is,

$$\sup_{K} u \leq C(a, b, K, \Omega, \inf_{\Omega} u).$$

However, no explicit dependence of $C$ in terms of $\inf_{\Omega} u$ was established. It was conjectured in [1] that the dependence is linear. In the case in which $V(x) \equiv a$ is a constant, it was proved in [13] that indeed any solution of (1) satisfies

$$\sup_{K} u + \inf_{\Omega} u \leq C(a, K, \Omega).$$

The proof of (3) given in [13] relies heavily on the Liouville representation formula; i.e., if $V$ is constant any solution of (1) may be written (locally) as

$$u = \log \frac{|f'(z)|^2}{(1 + |f(z)|^2)^2} + C$$

for some constant $C$ and some holomorphic function $f$ with $f' \neq 0$. For the general case (2) the linear dependence was also proved in [13]; i.e., there exist two constants $C_1 > 1$ and $C_2$ depending on $a$, $b$, $K$, and $\Omega$ such that

$$\sup_{K} u + C_1 \inf_{\Omega} u \leq C_2.$$  

Moreover, inequality (4) fails if we take $C_1 = 1$ and if we consider the full class of all functions $V$ satisfying (2).

The purpose of this paper is to prove that (3) holds under an additional Lipschitz condition on $V$. More precisely, our main result is the following.

**Theorem 1.** Assume $V$ is a Lipschitz function satisfying (2) and let $K$ be a compact subset of $\Omega$. Then any solution $u$ of (1) satisfies

$$\sup_{K} u + \inf_{\Omega} u \leq C(a, b, \|V\|_{L^\infty}, K, \Omega).$$

One of the ingredients in the proof is the blow-up analysis from [1, Theorem 3]. Another ingredient is the Alexandrov moving plane technique (as developed by Serrin [12] and Gidas, Ni, and Nirenberg [6]).
Let us note that Schoen [11] originally used this method in proving that any positive $C^2$ solution $u$ of
\[ -\Delta u = V(x) u^{(n+2)/(n-2)} \quad \text{in} \quad \Omega \subset \mathbb{R}^n, \text{ with } n = 3, 4, \tag{6} \]

satisfies
\[ \sup_K u \cdot \inf_{\Omega} u \leq C(a, b, \|\nabla V\|_{L^\infty}, K, \Omega) \quad \text{when } n = 3. \]

and
\[ \sup_K u \cdot \inf_{\Omega} u \leq C(a, b, \|\nabla V\|_{L^\infty}, \|\nabla^2 V\|_{L^\infty}, K, \Omega) \quad \text{when } n = 4. \]

However, there is a major difference between problem (1) and problem (6); in both problems we are led, after a blow-up, to classical entire solutions of the equations
\[ -\Delta u = e^u \quad \text{in } \mathbb{R}^2 \tag{7} \]

and
\[ -\Delta u = u^{(n+2)/(n-2)} \quad \text{in } \mathbb{R}^n, \quad u > 0. \tag{8} \]

All solutions of (8) are classified; i.e.,
\[ u = C \left( \frac{\varepsilon}{\varepsilon^2 + |x - x_0|^2} \right)^{(n-2)/2} \]

(see Obata [10], Gidas et al. [6, 7], Caffarelli et al. [2], and Chen and Li [5]). By contrast, the solutions of (7) have a more complicated structure; it is only the solutions of (7) such that $\int_{\mathbb{R}^n} e^u < \infty$ which have a simple form; i.e.,
\[ u(x) = \log \frac{8\varepsilon^2}{(\varepsilon^2 + |x - x_0|^2)^2} \]

(see Chen and Li [5]).

2. PROOF OF THEOREM 1

Set $A = \|\nabla V\|_{L^\infty}$. The proof is divided into 5 steps.

**Step 1.** Reduction to $\Omega = B_2$ (the disc of radius 2 centered at 0) and \[ u(0) + \inf_{B_2} u \leq C(a, b, A). \tag{9} \]

The general case follows from (9). Indeed, suppose (9) holds and let $v$ be a solution of
\[ -\Delta v = V e^v \quad \text{on } B_R. \]
Then
\[ u(x) = v\left(\frac{R}{2} x\right) + 2 \log(R/2) \]
satisfies
\[ -\Delta u = V\left(\frac{R}{2} x\right) e^u \quad \text{on } B_2 \]
and therefore, by (9),
\[ v(0) + \inf_{B_R} v \leq C(a, b, RA/2) - 4 \log(R/2) \]
\[ = C(a, b, A, R). \tag{10} \]

Theorem 1 is clearly implied by (10).

In what follows we argue by contradiction and we assume that (9) fails. More precisely we assume that there is a sequence \((u_n)\) of solutions of
\[ -\Delta u_n = V_n e^{u_n} \quad \text{on } B_2 \tag{11} \]
with
\[ a \leq V_n \leq b, \quad \|\nabla V_n\|_{L^\infty} \leq A \tag{12} \]
such that
\[ u_n(0) + \inf_{B_2} u_n \rightarrow +\infty. \tag{13} \]

By passing to a subsequence we may assume that \(V_n \rightarrow V\) uniformly on \(\bar{B}_2\) with \(V(0) = K \geq a > 0\). Set
\[ \delta_n = e^{-u_n(0)/2}. \tag{14} \]

**Step 2.** We have
\[ \delta_n \rightarrow 0 \tag{15} \]
and
\[ \lim_{\delta_n} \int_{\cdot B_{R\delta_n}} V_n e^{u_n} \leq 8\pi \quad \text{for all } R > 0. \tag{16} \]

**Proof.** We have
\[ u_n(0) + \inf_{B_2} u_n \leq 2u_n(0) \].
and thus, by (13), \( u_n(0) \to +\infty \). As in [13] we introduce the function

\[
G(r) = u_n(0) + \frac{1}{2\pi r} \int_{\partial B_r} u_n \, ds + 4 \log r, \quad 0 < r \leq 2.
\]

Since

\[
G'(r) = \frac{1}{2\pi r} \int_{\partial B_r} \frac{\partial u_n}{\partial r} \, ds + \frac{4}{r}
\]

and

\[
\int_{\partial B_r} \frac{\partial u_n}{\partial r} \, ds = \int_{B_r} \Delta u_n \, dx = -\int_{B_r} V_n e^{u_n},
\]

we conclude that

\[
G'(r) \geq 0 \iff \int_{B_r} V_n e^{u_n} \leq 8\pi \tag{17}
\]

and

\[
G'(r) = 0 \iff \int_{B_r} V_n e^{u_n} = 8\pi. \tag{18}
\]

The function \( G(r) \) achieves its maximum on \([0, 2]\) at some point \( 0 < \mu_n \leq 2 \). If \( \mu_n < 2 \) we have

\[
\int_{B_{\mu_n}} V_n e^{u_n} = 8\pi.
\]

Otherwise, \( \mu_n = 2 \) and we have

\[
\int_{B_{\mu_n}} V_n e^{u_n} \leq 8\pi.
\]

Thus, in all cases,

\[
\int_{B_{\mu_n}} V_n e^{u_n} \leq 8\pi.
\]

Since \( u_n \) is superharmonic we have

\[
2(u_n(0) + 2 \log \mu_n) \geq u_n(0) + \frac{1}{2\pi \mu_n} \int_{\partial B_{\mu_n}} u_n \, ds + 4 \log \mu_n
\]

\[
= G(\mu_n) \geq G(2) = u_n(0) + \frac{1}{4\pi} \int_{\partial B_2} u_n \, ds + 4 \log 2
\]

\[
\geq u_n(0) + \inf_{\partial B_2} u_n + 4 \log 2 \geq u_n(0) + \inf_{B_2} u_n + 4 \log 2.
\]
Using (13) we conclude that
\[ u_n(0) + 2 \log \mu_n \to +\infty, \]
i.e.,
\[ \log(\mu_n/\delta_n) \to +\infty \]
so that \( \mu_n/\delta_n \to +\infty \). Hence for any given \( R \) and for \( n \) sufficiently large, \( R\delta_n \leq \mu_n \) and thus
\[ \int_{B_{R\delta_n}} V_n e^{u_n} \leq \int_{B_{\mu_n}} V_n e^{u_n} \leq 8\pi. \]

**STEP 3.** There exist a sequence \( x_n \to 0 \) and a sequence \( R_n > 0 \) such that (for a subsequence)
\[ |x_n| < R_n \leq 1, \tag{20} \]

\( x_n \) is a maximum point of \( u_n \) on \( B_{R_n}(x_n) \),
\[ R_n e^{u_n(x_n)/2} \to \infty \tag{22} \]

and
\[ \lim_{B_{R_n}(x_n)} \int V_n e^{u_n} \leq 8\pi. \tag{23} \]

**Proof.** Set
\[ v_n(x) = u_n(\delta_n x) + 2 \log \delta_n \quad \text{for} \quad |x| \leq 1/\delta_n. \]

Consider \( (v_n) \) restricted to \( B_1 \); it satisfies
\[ -\Delta v_n = V_n(\delta_n x) e^{v_n} \quad \text{on} \quad B_1. \tag{24} \]

From (16) (applied with \( R = 1 \)) and (12) we deduce that
\[ \lim_{B_1} e^{v_n} \leq \frac{8\pi}{a}. \tag{25} \]

We are now in a situation where we may apply the blow-up analysis of [1, Theorem 3], i.e., there are only three possibilities (after choosing a subsequence):

**Case 1.** \( (v_n) \) is bounded in \( L^\infty_{\text{loc}}(B_1) \).

**Case 2.** \( v_n \to -\infty \) uniformly on compact subsets of \( B_1 \).
Case 3. There is a non-empty, finite, blow-up set $S$ in $B_1$ such that $v_n \to -\infty$ uniformly on compact subsets of $B_1 \setminus S$ and for each point $a \in S$ there is a sequence $(a_n)$ such that $a_n \to a$ and $v_n(a_n) \to +\infty$.

Since $v_n(0) = 0$, Case 2 is excluded. We examine Cases 1 and 3 separately.

Case 1. Consider $(v_n)$ restricted to $B_R$ for some fixed $R > 1$. For $n$ sufficiently large, $(v_n)$ satisfies (24) and (25) (with $B_1$ replaced by $B_R$). Applying [1] in $B_R$ we see that $(v_n)$ is bounded in $L^\infty_{\text{loc}}(B_R)$. By elliptic regularity theory we deduce that $(v_n)$ is bounded in $W^{2,p}_{\text{loc}}(B_R)$ for every $p < \infty$. Hence, by passing to a subsequence, we may assume that $(v_n)$ converges in $C^1_{\text{loc}}(\mathbb{R}^2)$ to some $v$ satisfying

\[ v \in L^\infty_{\text{loc}}(\mathbb{R}^2), \]

\[ -\Delta v = Ke^v \text{ on } \mathbb{R}^2 \quad (K = \lim_{n \to \infty} V_n(0)), \]

\[ \int_{\mathbb{R}^2} e^v \leq \frac{8\pi}{a}, \]

and

\[ v(0) = 0. \]

It follows that (see [5]) $v$ is of the form

\[ v(x) = \log \left\{ \frac{8\lambda^2/K}{\left(1 + \lambda^2|x - y_0|^2\right)^2} \right\} \]

for some point $y_0 \in \mathbb{R}^2$ and some constant $\lambda > 0$. For any fixed $\rho > |y_0|$ the maximum of $v_n$ on $\overline{B}_\rho$ is achieved at some $y_n$. Clearly $y_n \to y_0$ since $v_n \to v$ uniformly on $B_\rho$. In particular, for each integer $k$ (large) we have some $(v_{n_k})$ and $(y_{n_k})$ such that

\[ \text{Max } v_{n_k} \text{ is achieved at } y_{n_k}, \]

and

\[ y_{n_k} \to y_0. \]

Since $(\mu_n/\delta_n) \to +\infty$ we may in addition require that

\[ k\delta_{n_k} \leq \frac{1}{2} \mu_{n_k}. \]
Set
\[ x_{n_k} = \delta_{n_k} y_{n_k} \quad \text{and} \quad R_{n_k} = (k - |y_{n_k}|) \delta_{n_k}. \]

It is easily seen that the corresponding subsequence satisfies (20)--(23).

\textbf{Case 3.} Clearly \( 0 \in S \) (for otherwise we would have \( v_n(0) \to -\infty \), but \( v_n(0) = 0 \)). We may choose some \( r_0 \in (0, 1) \) such that \( (v_n) \) has no other blow-up point in \( B_{r_0} \) except the origin. For each \( n \) let \( y_n \) be a maximum point of \( v_n \) on \( B_{r_0} \). Then, by the blow-up assumption, \( v_n(y_n) \to +\infty \) and \( y_n \to 0 \). Set \( x_n = \delta_n y_n \) and \( R_n = \frac{1}{2} r_0 \delta_n \). It is easily seen that properties (20)--(23) are satisfied.

To summarize, if we set, on \( B_1 \),
\[ \bar{u}_n(x) = u_n(x + x_n) \quad \text{and} \quad \bar{v}_n(x) = V_n(x + x_n), \]
then we have
\[ -\Delta \bar{u}_n = \bar{v}_n e^{\delta_n} \quad \text{on} \ B_1, \quad (26) \]

\( 0 \) is a maximum point of \( \bar{u}_n \) on \( B_{R_n} \), with \( 0 < R_n \leq 1 \),
\[ R_n e^{\delta_n(0)/2} \to +\infty, \quad (27) \]

\[ \lim_{B_{R_n}} \int \bar{v}_n e^{\delta_n} \leq 8\pi, \quad (29) \]

and
\[ \bar{u}_n(0) + \inf_{B_1} \bar{u}_n \to +\infty. \quad (30) \]

\textbf{Step 4.} Set
\[ \eta_n = e^{-\delta_n(0)/2}, \quad \text{so that} \quad \eta_n \to 0, \]
\[ \bar{v}_n(x) = \bar{u}_n(\eta_n x) + 2 \log \eta_n, \quad \text{for} \quad |x| < 1/\eta_n, \]

and
\[ \bar{w}_n(x) = \bar{v}_n(x) + 2 \log |x|, \quad \text{for} \quad |x| < 1/\eta_n. \]

Clearly, \( \bar{w}_n \) satisfies
\[ -\Delta \bar{w}_n = V_n(\eta_n x) e^{\delta_n} \quad \text{for} \quad |x| < 1/\eta_n, \]
\[ \bar{w}_n(0) = 0, \]
and for each $R$, 

$$\max_{B_R} \bar{\varphi}_n \text{ is achieved at 0, for } n \text{ sufficiently large},$$

$$\overline{\lim}_{B_R} \int e^{\bar{\varphi}_n} \leq 8\pi/a \quad \text{(by (28) and (29)).}$$

We may use once more Theorem 3 of [1] to conclude that $\bar{\varphi}_n$ is bounded in $L^{\infty}_{\text{loc}}(\mathbb{R}^2)$. By standard elliptic estimates, we find that $\bar{\varphi}_n$ is also bounded in $C^{1,2}_{\text{loc}}(\mathbb{R}^2)$. Therefore, for a subsequence, $\bar{\varphi}_n$ converges in $L^{\infty}_{\text{loc}}(\mathbb{R}^2)$ to a function $\bar{\varphi}$ satisfying

$$-\Delta \bar{\varphi} = K e^{\bar{\varphi}} \quad \text{on } \mathbb{R}^2,$$

$$\bar{\varphi}(0) = 0,$$

and

$$\int_{\mathbb{R}^n} e^{\bar{\varphi}} < \infty.$$

Thus, by [5], $\bar{\varphi}$ is given by

$$\bar{\varphi}(x) = \log \left\{ \frac{1}{(1 + \gamma^2 |x|^2)^2} \right\}$$

with $\gamma = (K/8)^{1/2}$. It follows that

$$\bar{u}_n - \bar{u} \to 0 \quad \text{in } C^{2}_{\text{loc}}(\mathbb{R}^2), \quad (31)$$

where

$$\bar{u}(x) = \log \left\{ \frac{|x|^2}{(1 + \gamma^2 |x|^2)^2} \right\}.$$

Next, it is convenient to work in polar coordinates $(r, \theta)$ and to set $t = \log r$. Set, for $t > 0$ and $\theta \in [0, 2\pi]$, 

$$\bar{w}_n(t, \theta) = \bar{u}_n(e^t \cos \theta, e^t \sin \theta) + 2t. \quad (32)$$

Clearly $\bar{w}_n$ satisfies

$$-\Delta \bar{w}_n = \bar{v}_n(t, \theta) e^{\bar{\varphi}_n} \quad \text{in } Q,$$

where

$$Q = \{(t, \theta); \ t \leq 0 \text{ and } 0 \leq \theta \leq 2\pi\},$$

$$\Delta = \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial \theta^2}.$$
and
\[ \tilde{V}_n(t, \theta) = \tilde{V}_n(e^\epsilon \cos \theta, e^\epsilon \sin \theta). \]

We also introduce, for \( s \in \mathbb{R} \),
\[ \tilde{w}(s) = \log \left\{ \frac{e^{2s}}{(1 + \gamma^2 e^{2s})^2} \right\} = 2s - \log(1 + \gamma^2 e^{2s}). \]

Note that \( \tilde{w} \) achieves its maximum at \( s = -\log \gamma \), \( \tilde{w}'(s) > 0 \) for \( s < -\log \gamma \), and \( \tilde{w}(s) \) is symmetric about \( s = -\log \gamma \). We shall also use the fact that
\[ \tilde{w}(s) \leq 2s \quad \forall s \in \mathbb{R}. \] (33)

Clearly we have
\[ \tilde{w}_n(s + \log \eta_n, \theta) - \tilde{w}(s) = \tilde{w}_n(e^\epsilon \cos \theta, e^\epsilon \sin \theta) - \tilde{w}(e^\epsilon \cos \theta, e^\epsilon \sin \theta). \]

In the new variables property (31) implies that, for every fixed \( x \in \mathbb{R} \), as \( n \to \infty \),
\[ \| \tilde{w}_n(s + \log \eta_n, \theta) - \tilde{w}(s) \|_{L^\infty([x, x + 2\pi])} \to 0. \]

In particular we may choose \( n_0 \) sufficiently large so that, for \( n \geq n_0 \), we have
\[ |\tilde{w}_n(t, \theta) - \tilde{w}(t - \log \eta_n)| \leq 1 \quad \text{if} \quad t \leq 4 - \log \gamma + \log \eta_n, 0 \leq \theta \leq 2\pi, \] (34)

and
\[ \tilde{w}_n(-\log \gamma + \log \eta_n, \theta) > \tilde{w}_n(-\log \gamma + \log \eta_n + 4, \theta) \quad \text{if} \quad 0 \leq \theta \leq 2\pi. \] (35)

Finally, we introduce
\[ \tilde{w}_n(t, \theta) = \tilde{w}_n(t, \theta) - \frac{A}{a} e^\epsilon \quad \text{in} \ Q. \] (36)

We claim that
\[ \frac{\partial}{\partial t} \left\{ \tilde{V}_n(t, \theta) e^{\epsilon e^t} + \frac{A}{a} e^t \right\} \geq 0 \quad \forall (t, \theta) \in Q, \forall \xi \in \mathbb{R}. \] (37)

This follows easily from the fact that \( \tilde{V}_n \geq a \) and the estimate
\[ \| (\partial/\partial t) \tilde{V}_n(t, \theta) \| \leq Ae^t. \]

**Step 5.** (*Conclusion via the reflection method*). We now follow the standard reflection method (see [12, 6]). For \( \lambda < 0 \) and \( \lambda \leq t \leq 0 \) we set
\[ t^\prime = 2\lambda - t \]
and
\[ \hat{w}_n(t, \theta) = \hat{w}(r^t, \theta). \]

We have
\[ -A(\hat{w}_n - \hat{w}_n) = \hat{V}_n(t, \theta) e^{\hat{w}_n} - \hat{V}_n(t, \theta) e^w + \frac{A}{a} (e^{\nu} - e^t), \tag{38} \]
where \( \hat{V}_n(t, \theta) = \hat{V}_n(t, \theta) e^{\lambda_\alpha/\alpha} \) and \( \hat{V}_n(t, \theta) = \hat{V}_n(t^\gamma, \theta) \).

For \( \lambda \) very negative (depending on \( n \)), we have
\[ \hat{w}_n(t, \theta) - \hat{w}_n(t, \theta) < 0 \quad \text{for} \quad \lambda < t \leq 0, 0 \leq \theta \leq 2\pi. \tag{39} \]

To prove (39) we just use the fact that, for fixed \( n \), we have by (32) and (36),
\[ \hat{w}_n(t, \theta) = 2t + a_n + O_n(e^t) \quad \text{as} \quad t \to -\infty \]
and
\[ \frac{\partial \hat{w}_n}{\partial t}(t, \theta) = 2 + O_n(e^t) \quad \text{as} \quad t \to -\infty. \]

Define
\[ \lambda_n = \sup \{ \lambda < 0; \hat{w}_n(t, \theta) - \hat{w}_n(t, \theta) < 0 \text{ for } \lambda < t \leq 0, 0 \leq \theta \leq 2\pi \}. \]

We claim that
\[ \lambda_n \leq -\log \gamma + \log \eta_n + 2. \tag{40} \]

Indeed, if we choose \( \lambda = -\log \gamma + \log \eta_n + 2 \) and \( t = -\log \gamma + \log \eta_n + 4 \)
then \( t^\gamma = -\log \gamma + \log \eta_n \) and, by (35), \( \hat{w}_n(t, \theta) > \hat{w}_n(t, \theta) \), \( \forall \theta \in [0, 2\pi] \).

On the other hand, we have, by (38), (37), and the definition of \( \lambda_n \),
\[ -A(\hat{w}_n(t, \theta) - \hat{w}_n(t, \theta)) \leq 0 \quad \text{for} \quad \lambda \leq t \leq 0, \lambda \leq \lambda_n, \text{ and } 0 \leq \theta \leq 2\pi. \]

Now, we claim that,
\[ \min_{0 \leq \theta \leq 2\pi} \hat{w}_n(0, \theta) \leq \max_{0 \leq \theta \leq 2\pi} \hat{w}_n(2\lambda_n, \theta). \tag{41} \]

Suppose not, that
\[ \max_{0 \leq \theta \leq 2\pi} \hat{w}_n(2\lambda_n, \theta) < \min_{0 \leq \theta \leq 2\pi} \hat{w}_n(0, \theta); \]
then, by the maximum principle,
\[ \hat{w}_n(t, \theta) - \hat{w}_n(t, \theta) < 0 \quad \text{for} \quad \lambda_n < t \leq 0, 0 \leq \theta \leq 2\pi. \]
and by Hopf's Lemma

\[ \frac{\partial}{\partial t} \{ \hat{w}^n_{\lambda_n}(t, \theta) - \tilde{w}_n(t, \theta) \} \big|_{t = \lambda_n} < 0, \quad 0 \leq \theta \leq 2\pi. \]

This contradicts the definition of \( \lambda_n \).

Using (34) we have

\[ \max_{0 \leq \theta \leq 2\pi} \hat{w}_n(2\lambda_n, \theta) \leq \hat{w}(2\lambda_n - \log \eta_n) + 1 \leq 4\lambda_n - 2 \log \eta_n + 1 \quad \text{by (33)}. \]

Hence, by (40), we obtain

\[ \max_{0 \leq \theta \leq 2\pi} \hat{w}_n(2\lambda_n, \theta) \leq 2 \log \eta_n + C(\gamma). \quad (42) \]

Putting together (41) and (42) we see that

\[ \min_{0 \leq \theta \leq 2\pi} \hat{w}_n(0, \theta) \leq 2 \log \eta_n + C(\gamma). \]

Going back to the definition of \( \hat{w}_n \) (see (36) and (32)), we have

\[ \min_{\partial B_1} \bar{u}_n \leq 2 \log \eta_n + C(a, A, \gamma). \quad (43) \]

Recall that \( \eta_n = e^{-\omega(0)^2} \), and thus, by (43),

\[ \bar{u}_n(0) + \min_{\partial B_1} \bar{u}_n \leq C(a, A, \gamma), \]

which contradicts (30).

3. SOME OPEN PROBLEMS

In connection with Theorem 1 there are several natural questions:

**Question 1.** Can one replace in Theorem 1 the Lipschitz assumption on \( V \) by a uniform Hölder condition \( |V(x) - V(y)| \leq A |x - y|^\alpha, \quad 0 < \alpha < 1? \) We do not know the answer.

**Question 2.** Can one replace in Theorem 1 the Lipschitz assumption on \( V \) by a uniform modulus of continuity? The answer is negative as may be seen by the following example.

**Example.** There is a sequence \( (\bar{u}_n) \) of solutions of

\[ -\Delta \bar{u}_n = \tilde{V}_n e^{\omega_n} \quad \text{on } B_1, \quad (44) \]
such that
\[ V_n \to V \equiv 2 \quad \text{in } C(\bar{B}_1) \]
but
\[ \tilde{u}_n(0) + \inf_{B_1} \tilde{u}_n \to +\infty. \]

Choose a sequence $\beta_n > 1$ and define the sequence $(u_n)$ by
\[
u_n(r) = \begin{cases} 
2 \log \left( \frac{2\beta_n(nr)^{\beta_n-1}}{1+(nr)^{2\beta_n}} \right) + 2 \log n & \text{if } 1/n < r \leq 1 \\
2 \log \beta_n + 2 \log \left( \frac{2}{1+(nr)^2} \right) + 2 \log n & \text{if } 0 \leq r \leq 1/n.
\end{cases}
\]

It is easy to see that $u_n$ satisfies
\[-\Delta u_n = V_n e^{u_n} \quad \text{on } B_1\]
with
\[ V_n = \begin{cases} 
2 & \text{if } 1/n < r \leq 1, \\
\frac{2}{(\beta_n)^2} & \text{if } 0 \leq r \leq 1/n.
\end{cases}
\]
Clearly,
\[ u_n(0) + u_n(1) \geq 4 \log n + 2 \log \left( \frac{n^{\beta_n-1}}{1+n^{2\beta_n}} \right) - C \]
\[ = 2 \log \left( \frac{1}{n^{-\beta_n-1} + n^{\beta_n-1}} \right) - C. \]
Thus $u_n(0) + u_n(1) \to +\infty$ if we choose $(\beta_n)$ such that $(1-\beta_n) \log n \to +\infty$.

We may now smooth the sequences $(u_n)$ and $(V_n)$. To do so we choose, for each fixed $n$, a sequence $(V'_n)$ of smooth functions such that
\[ V'_n \to V_n \quad \text{a.e. on } B_1 \]
and
\[ 2 \leq V'_n \leq \frac{2}{\beta_n^2} \quad \text{on } B_1. \]
Let \( u_n^j \) be the solution of the linear equation

\[
-\Delta u_n^j = V_n^j e^{u_n^j} \quad \text{on} \quad B_1,
\]
\[
u_n^j = u_n \quad \text{on} \quad \partial B_1.
\]

Clearly, \( u_n^j \to u_n \), in \( C(\overline{B}_1) \), as \( j \to \infty \). Therefore, we may choose \( j = j(n) \) such that

\[
||u_n^j - u_n||_{L^\infty} \leq 1/n.
\]

Since

\[
-\Delta u_n^j = (V_n^j e^{u_n^j} - u_n^j) e^{u_n^j}
\]

we see that \( \tilde{u}_n = u_n^{j(n)} \) and \( \tilde{V}_n = V_n^{j(n)} e^{u_n^{j(n)}} - u_n^{j(n)} \) have all the required properties.

**Question 3.** Assume \( V \) is a fixed positive continuous (or even Hölder continuous) function on \( \Omega \). Let \( K \) be a compact subset of \( \Omega \) and let \( u \) be a solution of (1). Does one have

\[
sup_K u + \inf_{\Omega} u \leq C(V, K, \Omega)?
\]

We do not know the answer.

**ACKNOWLEDGMENTS**

The research of the second author (YYL) was partially supported by NSF Grant DMS-9104293. Part of this work was done while the second author (YYL) and third author (IS) were visiting the ENS Cachan and the third author (IS) was visiting Rutgers University. They thank the hosting institutions for their support and hospitality.

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