Integrability for the Jacobian of Orientation Preserving Mappings

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Assume $F$ is a map from $\Omega$ into $\mathbb{R}^n$ where $\Omega$ is a bounded domain in $\mathbb{R}^n$ such that $|Df| \in L^s(\log L)^{-\varepsilon}$ with $0 \leq s \leq 1$, i.e., $\int_{\Omega} |Df|^s \left( \log(1 + |Df|) \right)^{-\varepsilon} < \infty$, then $J \in L^{(\log L)^{-\varepsilon}}(K)$ for any compact subset $K \subset \Omega$, where $J = \det(Df)$. When $s = 0$ we recover a well-known result of Müller (J. Reine Angew. Math. 412 (1990), 20–34), while the case $s = 1$ was obtained by Iwaniec and Sbordone (Arch. Rational Mech. Anal., to appear). © 1993 Academic Press, Inc.

1. Introduction

Throughout this paper we assume that $\Omega$ is a bounded open set in $\mathbb{R}^n$ and that $f : \Omega \to \mathbb{R}^n$, $n \geq 2$, is a mapping with nonnegative Jacobian $J = J(x, f) = \det Df$.

In [M] S. Müller proved that, if $|Df| \in L^n(\log L)^{-1}(\Omega)$, then $J \in L \log L(K)$ for any compact subset $K \subset \Omega$ (see also [CLMS]).

In [IS] T. Iwaniec and C. Sbordone proved that

(A) if $|Df| \in L^n (\log L)^{-1} (\Omega)$, then $J \in L^1_{\text{loc}}(\Omega)$

(B) if $|Df| \in L^{n,\infty}(\Omega)$, then $J \in L^1_{\text{loc}}(\Omega)$.

In this paper we interpolate between Müller's result and (A); namely, we prove:

THEOREM 1. If $|Df| \in L^n(\log L)^{-s}(\Omega)$, $0 \leq s \leq 1$, then $J \in L(\log L)^{1-s}(K)$ for any $K \subset \Omega$.

This theorem is proved in Section 3.
It would be also natural to interpolate between Müller's result and (B), and to try to prove that if \(|Df|\) belongs to the Lorentz space \(L^{n,q} (\Omega)\), \(n < q < \infty\), then \(J \in L (\log L)^{n/q} (K)\) for any \(K \subset \Omega\). Unfortunately this is not true. However, the maximal function \(MJ\) of \(J\) satisfies \(MJ \in L^{1,q/n}_{\text{loc}}\), which is a weaker statement than saying \(J \in L (\log L)^{n/q} (K)\).

This is discussed in Section 4.

2. SOME PRELIMINARY RESULTS

Denote by \(g^*\) the decreasing rearrangement on the interval \((0, + \infty)\) of a measurable function \(g\) defined on some open set \(\Omega \subset \mathbb{R}^n\).

The Lorentz space \(L^{p,q} (\Omega)\) \((1 < p < \infty, 1 \leq q \leq \infty)\) consists of all measurable functions \(g\) on \(\Omega\) for which

\[
\|g\|_{p,q} = \begin{cases} 
\left\{ \int_0^\infty \left[ t^{1/p} g^*(t) \right]^q \frac{dt}{t} \right\}^{1/q} & q < \infty \\
\sup_{t > 0} t^{1/p} g^*(t) & q = \infty 
\end{cases}
\]

is finite. The spaces \(L^{p,q}\) increase with \(q\); in particular,

\(L^{p,1} \subset L^{p,p} = L^p \subset L^{p,\infty}\)

(see, e.g., [Z]).

The Zygmund space \(L^p (\log L)^z (\Omega)\) \((1 \leq p < \infty, -\infty < z < \infty)\) consists of all measurable functions \(g\) on \(\Omega\) for which

\[
\int_{\Omega} |g|^p \log^z \left( e + \frac{|g|}{|g|_{\Omega}} \right) dx < \infty,
\]

where \(|g|_{\Omega} = (1/|\Omega|) \int_{\Omega} |g| dx\).

It can be shown (see [BS, p. 252]) that \(g \in L^p (\log L)^z (\Omega)\) iff

\[
\int_0^\infty (1 + |\log t|)^z g^*(t)^p dt < \infty.
\]

We denote by \(M_{\Omega}\) the local Hardy–Littlewood maximal function, defined for \(g \in L^1_{\text{loc}} (\Omega)\) by

\[
M_{\Omega} g(x) = \sup_{x \in \Omega \subset \Omega} \int_{\Omega} |g| dy.
\]

The following result is well known (see e.g. [BR]).
LEMMA 1. For \( p > 1, \ 0 \leq s \leq 1, \) \( M_\Omega \) maps boundedly \( L^p(\log L)^{-s} \) into itself, namely

\[
\| M_\Omega \ g \|_{L^p(\log L)^{-s}(\Omega)} \leq c \| g \|_{L^p(\log L)^{-s}(\Omega)}
\]

with \( c = c(n, p, s) \).

When the exponent in the Zygmund space is \( p = 1 \) the behaviour of the maximal function is different; namely the following result holds (see [S] for \( x = 1 \)).

LEMMA 2. For \( 0 < \alpha \leq 1 \), the following statements are equivalent,

(a) \( g \in L(\log L)^\alpha \)

(b) \( M_\Omega \ g \in L(\log L)^{\alpha - 1} \).

In particular, there exists \( c = c(n, \alpha) \) such that

\[
\int_\Omega |g| \log^2 \left( e + \frac{|g|}{|g|_\Omega} \right) dx \leq c \int_\Omega M_\Omega \ g \log^{\alpha - 1} \left( e + \frac{M_\Omega \ g}{|g|_\Omega} \right) dx. \tag{1}
\]

The equivalence between (a) and (b) is essentially due to Bennett [B, Theorem 4.1]. Here we prove inequality (1).

**Proof.** Suppose \( |g|_\Omega = 1 \). Then it is well known that \( \forall t > 1 \),

\[
\frac{1}{t} \int_{\{|g| > t\}} |g| \leq 2^n \{ |\{M_\Omega \ g > t\}| \}
\]

We have by integration by parts

\[
\int_{\{|g| > 1\}} |g| \log^2(e + |g|) \]

\[
= - \int_1^\infty \log^2(e + t) \left[ \int_{\{|g| > t\}} |g| \ dx \right] dt
\]

\[
= - \left[ \log^2(e + t) \int_{\{|g| > t\}} |g| \ dx \right]_{t = 1}^{t = \infty} \]

\[
+ \alpha \int_1^\infty \frac{\log^{\alpha - 1}(e + t)}{e + t} \int_{\{|g| > t\}} |g| \ dx \ dt
\]

\[
\leq \log^2(1 + e) \int_{\{|g| > 1\}} |g| \ dx + 2^n \alpha \int_1^\infty \log^{\alpha - 1}(e + t) \{ |\{M_\Omega \ g > t\}| \} \ dt.
\]
If we set $\phi(t) = t \log^{a-1}(e + t)$ we have $\phi'(t) \geq \alpha \log^{a-1}(e + t) \forall t > 0$. And so

$$\int_{|g| > 1} |g| \log^a(e + |g|)$$

$$\leq \log^a(1 + e) \int_{|g| > 1} |g| dx + 2^n \int_1^\infty \phi'(t) |\{M_{g \mathcal{R}} g > t\}| dt$$

$$= \log^a(1 + e) \int_{|g| > 1} |g| dx + 2^n \int_{M_{g \mathcal{R}} > 1} \phi(M_{g \mathcal{R}} g).$$

In fact

$$\int_{|h| > 1} \phi(|h|) dx = \int_1^\infty \phi'(t) |\{|h| > t\}| dt.$$

So we have

$$\int_\Omega |g| \log^a(e + |g|) \leq \log^a(1 + e) \int_\Omega |g| dx + 2^n \int_\Omega M_{g \mathcal{R}} g \log^{a-1}(e + M_{g \mathcal{R}} g).$$

From this inequality one gets the result when $|g|_{L^1} = 1$ since, $\forall \epsilon > 0$ and $\forall t > 0,$

$$t \leq \epsilon t \log^a(e + t) + C(\epsilon, a) t \log^{a-1}(e + t).$$

The general case then follows by homogeneity.

**Lemma 3.** If $g \in L^n(\log L)^{-s}(\Omega), 0 < s \leq 1, g \geq 0,$ then

$$\lim_{\epsilon \to 0} \epsilon^s \int_\Omega g^{n-\epsilon}(x) dx = 0.$$

**Proof.** We have

$$\epsilon^s \int_\Omega g^{n-\epsilon}(x) dx \leq \epsilon^s + \frac{1}{|\Omega|} \int_{g \geq 1} \frac{\epsilon^s g^n}{e^{\epsilon \log g}} dx.$$

The second integral converges to zero by dominated convergence. Clearly the integrand converges to zero pointwise and in addition

$$\frac{\epsilon^s g^n}{e^{\epsilon \log g}} \leq \frac{\epsilon^s g^n}{1 + \epsilon \log g} \leq \frac{g^n}{(\log g)^{a'}}.$$
3. Proof of Theorem 1

Let us recall the following inequality which holds for \( f \in W^{1,n-\varepsilon}(\Omega; \mathbb{R}^n), -\infty < \varepsilon < 1 \), and \( Q \subset Q_0/2 \), \( Q_0 \) a cube contained in \( \Omega \).

\[
\frac{1}{|Q|} \int_Q |Df|^{-\varepsilon} J(x,f) \leq c(n) |\varepsilon| \frac{1}{|2Q|} \int_{2Q} |Df|^{n-\varepsilon} + c(n) \left( \frac{1}{|2Q|} \int_{2Q} |Df|^{(n+2)/(n+1)} \right)^{n+1/n}.
\]

(see [IS]). Since \( |Df| \in L^n(\log L)^{-s} \), if \( 0 < s \leq 1 \) we can pass to the limit as \( \varepsilon \to 0 \) and use Lemma 3 to obtain, for any cube \( Q \subset Q_0/2 \),

\[
\frac{1}{|Q|} \int_Q J \leq c(n) \left( \frac{1}{|2Q|} \int_{2Q} |Df|^{n/(n+1)} \right)^{n+1/n}.
\]

If we denote by \( M \) the local maximal function associated to the cube \( Q_0/2 \), i.e., \( M = M_{Q_0/2} \), and by \( \mathcal{M} \) the local maximal function associated to the cube \( Q_0 \), i.e., \( \mathcal{M} = M_{Q_0} \), by (1) we deduce for \( x \in Q_0/2 \),

\[
MJ(x) \leq c(n) \mathcal{M}(|Df|^{n/(n+1)}(x))^{n+1/n}.
\]

Since \( |Df| \in L^n(\log L)^{-s} \), it is easy to check that \( |Df|^{n/(n+1)} \) belongs to \( L^{(n+1)/n}(\log L)^{-s} \) and then, by Lemma 1, \( \mathcal{M}(|Df|^{n/(n+1)}) \) belongs also to the same space. From this it follows that \( \mathcal{M}(|Df|^{n/(n+1)})^{n+1/n} \) and also \( MJ \), by (3), belongs to \( L(\log L)^{-s} (Q_0/2) \). Finally, from Lemma 2 we deduce that \( J \in L(\log L)^{1-s}(K) \forall K \subset \Omega \).

Remark. Using inequalities (1) and (3) and the fact that

\[
\frac{|\mathcal{M}(|h|)|^{(n+1)/n}}{Q_{0/2}} \log^s(e + \mathcal{M}(|h|)) \leq c \frac{|h|^{(n+1)/n}}{Q_{0/2}} \log^s(e + |h|)
\]

for any \( h \in L^{(n+1)/n}(\log L)^{-s}(Q_0/2) \) (see [FS], Prop. 1.2), one can easily prove the inequality

\[
\int_{Q_{0/2}} J \log^{1-s}(e + \frac{J}{J_{Q_{0/2}}}) \leq c \int_{Q_0} |Df|^{n} \log^{-s} \left( e + \frac{|Df|}{|Df|_{Q_0}} \right)
\]

\( \forall Q_0 \subset \Omega \).

4. Inequalities in Lorentz Spaces

Here is one positive result.

Theorem 2. If \( n < q < \infty \), \( |Df| \in L^{n,q}(\Omega) \), and \( M \) denotes the local maximal function associated to a cube \( Q \subset \Omega \), then we have \( MJ \in L^{1,q/n}(Q) \).
Proof. First we recall (see, e.g., [BR]) that the maximal function maps $L^{p,r}$ into itself and the following inequality holds:

$$\|Mh\|_{L^{p,r}} \leq c \|h\|_{L^{p,r}}.$$  

(4)

We will also use the relation

$$\|h^p\|_{L^{p,r}} = \|h\|^p_{L^{p^*,r^*}}.$$  

(5)

As in the proof of Theorem 1 we have (2), since $|Df|$ belongs to $L^{n,q}$ and $L^{n,q} \subset L^n / \log L$ (see [BR, Theorem 9.3]). Hence we have also (3). Next we apply (4) to $h = |Df| n^{n/(n+1)}$, with $p = (n+1)/n, \quad r = (n+1)q/n^2$, and rewrite the right-hand side, using (5) with $x = n^2/(n+1)$. So this proves that

$$MJ \in L^{1,q/n}(Q).$$  

(6)

In view of inequality (6) it would be natural to expect that $J \in L(\log L)^{n/q}(K)$ for $K \subset \Omega$. However this is not true as it can be seen by the following.

Example. Consider the function

$$f(x) = \frac{x}{|x|} |\log |x||^{-1/q} (\log |\log |x||)^{-1/n},$$

where $|x| < a < 1, q > n$. We claim that

$$|Df| \in L^{n,q}$$  

(7)

$$J \notin L(\log L)^{n/q}.$$  

(8)

Verification of (7). It is easy to check that $|Df|$ is equivalent to $(1/|x|) |\log |x||^{-1/q} (\log |\log |x||)^{1/n}$. Then the claim follows since

$$\int_0^a \left[ \frac{1}{\log r^{-1/q} (\log |\log r|)^{1/n}} \right]^q \frac{dr}{r} < \infty.$$  

Verification of (8). $J$ is now equivalent to $(1/|x|^n) |\log |x||^{-1-n/q} (\log |\log |x||)^{-1}$ and this function does not belong to $L(\log L)^{n/q}$.

References


