
UNIFORM ESTIMATES FOR SOLUTIONS OF \(- \Delta u = V(x)u^p\)

Haïm Brezis

Université Paris VI

and

Rutgers University

Let \( u \) be a weak solution of the nonlinear equation

\[
(1) \quad - \Delta u = V(x)|u|^{p-1}u \quad \text{in} \quad \Omega \subset \mathbb{R}^n, \quad n \geq 3
\]

possibly with a boundary condition

\[
(2) \quad u = 0 \quad \text{on} \quad \partial\Omega.
\]

Assume \( V \in L^\infty(\Omega) \) (\( \Omega \) smooth and bounded) and suppose we start with the information \( u \in L^\alpha(\Omega) \) for some \( \alpha \). We are concerned with the following questions:

**Question 1**: Can one conclude that \( u \in L^\infty(\Omega) \)?

**Question 2**: Can one estimate \( \|u\|_\infty \) in terms of \( \|V\|_\infty \) and \( \|u\|_\alpha \) ? More precisely, suppose we have a sequence \((u_j)\) of solutions of

\[
\begin{cases}
-\Delta u_j = V_j(x)|u_j|^{p-1}u_j \quad \text{in} \quad \Omega \\
u_j = 0 \quad \text{on} \quad \partial\Omega 
\end{cases}
\]

with \( \|V_j\|_\infty \leq C_1 \) and \( \|u_j\|_\alpha \leq C_2 \).

Can one find a constant \( C_3 \) (depending on \( C_1, C_2, \alpha, p \) and \( \Omega \)) such that

\( \|u_j\|_\infty \leq C_3 \) ?
A natural approach is to use **bootstrap.** Since \( V(x)|u|^{p-1}u \in L^{\alpha/p}(\Omega) \) (assuming \( \alpha \geq p \)) we deduce from the elliptic regularity theory (see [1]) that \( u \in W^{2,\alpha/p}(\Omega) \). Using the Sobolev imbedding (see e.g. [4]) we conclude that
\[
u \in L^{\beta}(\Omega) \quad \text{with} \quad \frac{1}{\beta} = \frac{\alpha}{p} - \frac{2}{n}\]
provided \( \frac{\alpha}{p} - \frac{2}{n} > 0 \) (otherwise we see easily that \( u \in L^\alpha(\Omega) \)).

We now distinguish three cases:

**Case 1:** \( \beta > \alpha \), i.e. \( \alpha > \frac{n}{2}(p - 1) \),

**Case 2:** \( \beta < \alpha \), i.e. \( \alpha < \frac{n}{2}(p - 1) \),

**Case 3:** \( \beta = \alpha \), i.e. \( \alpha = \frac{n}{2}(p - 1) \).

In Case 1 bootstrap works well. After a finite number of steps (iterating the above argument) we reach the conclusion that \( u \in L^\alpha(\Omega) \). Moreover the answer to Question 2 is also positive: one can estimate \( \|u\|_\infty \) in terms of \( \|V\|_\infty \) and \( \|u\|_\alpha \).

In Case 2 bootstrap works "backwards" and yields nothing. In general one cannot conclude that \( u \in L^\alpha(\Omega) \). For example, if \( p > n/(n-2) \) the function \( u(x) = |x|^{-2/(p-1)} \) satisfies (1) in the sense of distributions with \( V = \frac{2}{p-1} \left( n-2 - \frac{2}{p-1} \right) > 0 \). Moreover \( u \in L^\alpha(\Omega) \) for any \( \alpha < \frac{n}{2} (p-1) \). However \( u \notin L^\alpha(\Omega) \).

Case 3 is interesting because it is a **border line case.** Bootstrap does not yield any improvement of regularity. However the answer to Question 1 is affirmative and the proof relies on the following result of [2]:

**Theorem 0.** Assume \( u \in H^1_0(\Omega) \) satisfies the linear equation
\[
(3) \quad - \Delta u = a(x) u \quad \text{in} \quad \Omega
\]
with \( a \in L^{n/2}(\Omega) \), \( n \geq 3 \).
Then

\[ u \in L^q(\Omega), \quad \forall q < \infty. \]

Applying Theorem 0 to (1) with \( a(x) = V(x)|u|^{p-1} \) we see that Case 3 corresponds precisely to the assumption that \( a \in L^{n/2}(\Omega) \). Hence we deduce that \( u \in L^q(\Omega) \) \( \forall q < \infty \) and going back to (1) we conclude that \( u \in L^\infty(\Omega) \). Thus the answer to Question 1 is affirmative.

It is instructive to sketch the proof of Theorem 0. Multiplying (3) by \( |u|^{k-1}u \) (\( k \) to be determined) we obtain

\[ k \int |u|^{k-1} |\nabla u|^2 = \int a|u|^{k+1} \]

and thus, using the Sobolev inequality, we find

\[ \frac{4kS}{(k+1)^2} \|a^{(k+1)/2}\|_{2n/(n-2)}^2 \leq \int |a| |u|^{k+1}. \]

Note that the RHS in (4) can be estimated by \( \|a\|_{n/2} \|u^{k+1}\|_{n/(n-2)} \). However, this is useless since the same expression \( \|u^{k+1}\|_{n/(n-2)} \) appears on both sides.

The main idea is to split the integral on the RHS of (4) as

\[ \int |a| |u|^{k+1} + \int |a| |u|^{k+1} \]

where \( A \) is a constant to be determined. The first integral in (5) is bounded by

\[ \left( \int |a|^{2n/2} \right)^{2/n} \|u\|^{k+1} \]

while the second integral is bounded by

\[ A \int |u|^{k+1}. \]

Choosing \( A \) large enough so that
\[
\left( \int_{|a| > A} |a|^{n/2} \right)^{2/n} \leq \frac{2kS}{(k+1)^2}
\]

(this is possible since \(a \in L^{n/2} \)) we are led to

\[
(6) \quad \frac{2kS}{(k+1)^2} \|u\|^{k+1} = \frac{(k+1)n/(n-2)}{k+1} \leq A \|u\|^{k+1}.
\]

We start with some initial value of \(k\) (for example \(k+1 = 2n/(n-2)\)) and then follow the sequence \((k+1)^{\frac{n}{n-2}}, (k+1)^{\frac{n}{n-2}}^2, (k+1)^{\frac{n}{n-2}}^3, \ldots\), until we pass any given \(q\).

Remark 1. At each stage (even at the first step) of the iteration the estimate we obtain for \(u\) depends not only on \(\|a\|_{n/2}\) but also on the distribution function of \(a\). Hence Theorem 0 does not come with an estimate of the form

\[
(7) \quad \|u\|_q \leq F(\|a\|_{n/2}, \|u\|_{2n/(n-2)})
\]

as soon as \(q > 2n/(n-2)\). This means that if we apply Theorem 0 to (1) in the Case 3 we cannot estimate \(\|u\|_{\infty}\) in terms of \(\|u\|_{\alpha}\). This is not just a deficiency of the proof. A typical example of lack of \(L^{\infty}\) estimates is given by the sequence

\[
u_\epsilon(x) = \left(\frac{\epsilon}{\epsilon^2 + |x|^2}\right)^{n-2}/2 \text{ in } \mathbb{R}^n, \ n \geq 3.
\]

It satisfies

\[- \Delta u_\epsilon = B u_\epsilon^{(n+2)/(n-2)}\]

for some positive constant \(B\) (depending only on \(n\)). Moreover it is easy to see that

\[
\|u_\epsilon\|_{2n/(n-2)} \leq C \text{ as } \epsilon \to 0.
\]

Thus we are in Case 3 with \(p = (n+2)/(n-2)\) and \(\alpha = 2n/(n-2)\). However
\[ \| u_\varepsilon \|_\infty = u_\varepsilon(0) = \varepsilon^{-(n-2)/2} \rightarrow \infty \text{ as } \varepsilon \rightarrow 0 \]

so that here the answer to Question 2 is negative.

It is a surprising fact that the answer to Question 2 in Case 3 is positive, at least locally, when \( p \leq n/(n-2) \). The main result is the following:

**Theorem 1.** Assume \( 1 + \frac{2}{n} < p \leq \frac{n}{n-2} \).

Let \( (u_j) \) be a sequence of (smooth) solutions of

\[ -\Delta u_j = V_j u_j^p \text{ in } \Omega \]

with \( u_j \geq 0, V_j \geq 0 \text{ in } \Omega \),

(8) \[ \| V_j \|_\infty \leq C_1 \]

and

(9) \[ \| u_j \|_\alpha \leq C_2 \text{ where } \alpha = \frac{n}{2} \cdot (p-1) \]

Then \( (u_j) \) is bounded in \( L^\infty_{\text{loc}}(\Omega) \).

**Proof.** It is inspired by some of the techniques introduced in [3]. Let \( K \) be a compact subset of \( \Omega \) and consider the "concentration function":

\[ \varphi_j(t) = \max_{a \in K} \int_{B_t(a)} V_j^{\alpha/p} u_j^\alpha \]

defined for \( t < \text{dist}(K, \partial \Omega) = d \), where \( B_t(a) = \{ x \in \mathbb{R}^n; |x-a| < t \} \).

**Lemma 1.** With the same assumptions as in Theorem 1, given any \( \varepsilon > 0 \) there is some \( t \in (0, d) \) such that

(10) \[ \limsup_{j \rightarrow \infty} \varphi_j(t) < \varepsilon \].
Proof of Lemma 1.

We argue by contradiction. If (10) does not hold, there is some \( \varepsilon_0 > 0 \) and sequences \( R_j \to 0 \), \( (a_j) \) in \( K \) such that

\[
\int_{B_{R_j}(a_j)} v_j^{\alpha/p} u_j^\alpha \geq \varepsilon_0
\]

(along a subsequence still denoted \( u_j \)).

Set \( g_j = v_j u_j^p \chi_{B_{R_j}(a_j)} \), where \( \chi \) denotes the characteristic function. Let \( v_j \) be the solution of

\[
\begin{cases}
- \Delta v_j = g_j & \text{in } \Omega, \\
v_j = 0 & \text{on } \partial \Omega.
\end{cases}
\]

By the maximum principle \( u_j \geq v_j \).

On the other hand

\[
v_j \geq \frac{C}{|x|^{n-2}} \ast g_j + O(1)
\]

so that

\[
v_j(x) \geq C \int_{B_{R_j}(a_j)} \frac{1}{|x-y|^{n-2}} g_j(y) dy + O(1)
\]

and in particular

\[
v_j(x) \geq \frac{C}{(|x-a_j|+R_j)^{n-2}} \int_{B_{R_j}(a_j)} g_j(y) + O(1).
\]

Since \( p \leq n/(n-2) \) we have \( \alpha \leq p \) and by Hölder's inequality

\[
\int_{B_{R_j}(a_j)} v_j^{\alpha/p} u_j^{\alpha} \leq \left( \int_{B_{R_j}(a_j)} v_j^{\alpha/p} \right)^{\alpha/p} \left( \int_{B_{R_j}(a_j)} u_j^p \right)^{1-\alpha/p} C R_j^{n(1-\alpha/p)}.
\]

It follows that
\[ \int_{B_{R_j}(a_j)} V_j u_j^p \geq C \varepsilon_0^{p/\alpha} R_j^{n(1-p/\alpha)} . \]

We conclude that
\[ u_j(x) \geq \frac{C R_j^{n(1-p/\alpha)}}{(|x-a_j|+R_j)^{n-2}} + O(1) \]
for some positive constant \( C \) (depending on \( \varepsilon_0 \)). It follows that
\[ \int_{B_d(a_j)} u_j^\alpha \geq C \int_0^d R_j^{n(\alpha-p)} \frac{r^{n-1}}{(r+R_j)^{\alpha(n-2)}} dr + O(1) . \]

But
\[ \int_0^d R_j^{n(\alpha-p)} \frac{r^{n-1}}{(r+R_j)^{\alpha(n-2)}} dr = \int_0^{d/R_j} R_j^{np+\alpha-2} \frac{s^{n-1}}{(s+1)^{\alpha(n-2)}} ds \]
\[ = \begin{cases} \log \frac{1}{R_j} + O(1) & \text{if } \alpha = n/(n-2) \\ CR_j^{np+n\alpha} + O(1) & \text{if } \alpha < n/(n-2) \end{cases} \]

In both cases the integral tends to \( +\infty \) (note that \( \alpha \leq n/(n-2) \) since \( p \leq n/(n-2) \) and either \( p = \alpha = n/(n-2) \) or \( \alpha < p < n/(n-2) \)). This is impossible since \( \| u_j \|_\alpha \leq C_2 \) by assumption (9). Hence we have proved Lemma 1.

**Lemma 2.** Let \( (u_j) \) be a sequence of (smooth) solutions of
\[ \tag{11} -\Delta u_j = V_j u_j^p \text{ in } B_R \]
such that
\[ u_j \geq 0 \text{ in } B_R , \]
\[ \| V_j \|_{L^\infty(B_R)} \leq C_1 , \]

and

$$\|u_j\|_{L^\alpha(B_R)} \leq C_2 \quad \text{with} \quad \alpha = \frac{n}{2}(p-1).$$

Then there is some \( \varepsilon_0 > 0 \) (depending only on \( n, C_1 \) and \( p \)) such that

$$\int V_j^{\alpha/p} u_j^\alpha \leq \varepsilon_0$$

implies

$$\|u_j\|_{L^{\alpha}(B_{R/2})} \leq C_3$$

where \( C_3 \) depends on \( C_1, C_2, R, n, p \) and \( \varepsilon_0 \).

**Proof.** This is a localized version of the proof of Theorem 0. Fix a function \( \zeta \in C_0^\infty(B_R) \) with \( \zeta = 1 \) on \( B_{R/2} \). Multiplying (11) by \( \zeta^2 u_j^k \) (\( k \) to be determined later) we obtain

$$I = \int \nabla u_j \nabla (\zeta^2 u_j^k) = \int \nabla_j u_j^{p-1} \zeta^2 u_j^{k+1} = II.$$

We have

$$I = k \int |\nabla u_j|^2 u_j^{k-1} \zeta^2 + \frac{1}{k+1} \int \nabla(u_j^{k+1}) \nabla \zeta^2$$

$$= \frac{4k}{(k+1)^2} \int |\nabla u_j^{(k+1)/2}|^2 \zeta^2 - \frac{1}{k+1} \int u_j^{k+1} \Delta \zeta^2$$

$$= \frac{4k}{(k+1)^2} \int |\nabla (\zeta u_j^{(k+1)/2})|^2 + \frac{1}{(k+1)^2} \int ((k-1)\Delta \zeta^2 - 4k|\nabla \zeta|^2)u_j^{k+1}.$$

Therefore, using Sobolev's inequality, we obtain

$$I \geq \frac{4k}{(k+1)^2} S \| \zeta u_j^{(k+1)/2} \|_{2n/(n-2)} - C \int u_j^{k+1}$$

with a constant \( C \) depending only on \( R \). We estimate \( II \) by Hölder's inequality:

$$II \leq \|V_j u_j^{p-1}\|_{n/2} \|\zeta^2 u_j^{k+1}\|_{n/(n-2)}.$$

Note that
\[ \|V_j u_j^{n-1}\|_{n/2}^{n/2} = \left\| V_j^{n/2} u_j^{n/2} \right\| \leq C_1^{(n/2)-\alpha/p} \int V_j^{\alpha/p} u_j^\alpha \leq \epsilon_0 \| C_1^{(n/2)-\alpha/p} . \]

Hence we find
\[ \frac{4k}{(k+1)^2} \| \zeta^2 u_j^{k+1} \|_{n/(n-2)} \leq C_1^1/p \epsilon_0^{2/n} \| \zeta^2 u_j^{k+1} \|_{n/(n-2)} + C \int u_j^{k+1} . \]

We start with \( k = \alpha-1 \); using the smallness of \( \epsilon_0 \) we obtain an estimate for \( \| \zeta^2 u_j^\alpha \|_{n/(n-2)} \). Iterating we find local estimates for \( u_j \) in \( L^q \), \( q = \alpha(n/n-2) \), \( \alpha(n/n-2)^2 \), etc... After a finite number of steps we reach some \( q > \frac{n}{n-2} \). Therefore we have an estimate for \( u_j^p \) in \( L^q \) and thus (by (11) we have an estimate for \( u_j \) in \( L^q \).

**Proof of Theorem 1 completed.** Let \( \epsilon_0 > 0 \) be as in Lemma 2. By Lemma 1 there is some \( t > 0 \) such that
\[ \lim_{j \to \infty} \sup \varphi_j(t) < \epsilon_0 . \]

Hence for \( j \geq j_0 \) we have
\[ \int_{B_t(a)} V_j^\alpha/p u_j^\alpha \leq \epsilon_0 \quad \forall a \in K . \]

By Lemma 2 we have an estimate for \( u_j \) in \( L^q(B_{t/2}(a)) \). Therefore we have a bound for \( (u_j) \) in \( L^q(K) \).

**Some counterexamples.**

1. **No estimate up to the boundary.**

The conclusion of Theorem 1 is sharp. In general \( (u_j) \) cannot be estimated in \( L^\infty \) up to the boundary, even if \( u_j = 0 \) on \( \partial \Omega \).

Using an idea of [3] we shall construct sequences \( (u_j) \) and \( (V_j) \) such that
\[- \Delta u_j = V_j u_j^p \quad \text{in} \quad \Omega \]
\[u_j = 0 \quad \text{on} \quad \partial \Omega \]

with \( u_j \geq 0, \quad V_j \geq 0 \) in \( \Omega \) such that
\[\|V_j\|_{L^\infty(\Omega)} \leq C_1, \quad \|u_j\|_{L^\alpha(\Omega)} \leq C_2, \quad \alpha = \frac{n}{2}(p-1)\]

and
\[\|u_j\|_{L^\infty(\Omega)} \to \infty.\]

Let \( \Omega \) be the unit ball centered at \((1, 0, 0, \cdots)\).

Let \( a_\varepsilon = (d_\varepsilon, 0, 0, \cdots) \) with \( d_\varepsilon = 2\varepsilon < 1 \) and let
\[f_\varepsilon = \begin{cases} \frac{2N}{\varepsilon^\theta} & \text{in } B_\varepsilon(a_\varepsilon) \\ 0 & \text{otherwise} \end{cases},\]

with \( \theta = 2p/(p-1) \).

Let \( u_\varepsilon \) be the solution of
\[- \Delta u_\varepsilon = f_\varepsilon \quad \text{in} \quad \Omega ,
\[u_\varepsilon = 0 \quad \text{on} \quad \partial \Omega.\]

Set
\[V_\varepsilon = f_\varepsilon / u_\varepsilon^p,\]

so that \( u_\varepsilon \) satisfies
\[- \Delta u_\varepsilon = V_\varepsilon u_\varepsilon^p \quad \text{in} \quad \Omega ,
\[u_\varepsilon = 0 \quad \text{on} \quad \partial \Omega.\]

We claim that

(12) \[\|V_\varepsilon\|_{\infty} \leq C_1\]

(13) \[\|u_\varepsilon\|_{\alpha} \leq C_2\]
and

\[(14) \quad \|u_\epsilon\|_\infty \to \infty \text{ as } \epsilon \to 0.\]

**Verification of (12).** Let \(v_\epsilon\) be the solution of

\[
\begin{cases}
- \Delta v_\epsilon = f_\epsilon & \text{in } B_{d_\epsilon}(a_\epsilon) \\
v_\epsilon = 0 & \text{on } \partial B_{d_\epsilon}.
\end{cases}
\]

By the maximum principle \(v_\epsilon \leq u_\epsilon\) in \(B_{d_\epsilon}(a_\epsilon)\) so that \((1/u_\epsilon^p) \leq (1/v_\epsilon^p)\). But \(v_\epsilon\) is given explicitly by

\[
v_\epsilon(x) = \begin{cases}
\frac{1}{\theta} r^2 + \frac{n}{(n-2)\epsilon^{\theta-2}} - \frac{2}{(n-2)d_\epsilon^{n-2}} & \text{if } r < \epsilon \\
\frac{2}{n-2} \epsilon^{n-\theta} \left(\frac{1}{\epsilon^{n-2}} - \frac{1}{d_\epsilon^{n-2}}\right) & \text{if } \epsilon < r < d_\epsilon
\end{cases}
\]

with \(r = |x-a_\epsilon|\).

It follows that

\[\|V_\epsilon\|_\infty \leq \frac{C}{\epsilon^{\theta-2}} = C.\]

**Verification of (13).** Let \(G\) be the half-space

\[G = \{(x_1, x'); \ x_1 > 0\}\]

and let \(w_\epsilon\) be the solution of

\[
\begin{cases}
- \Delta w_\epsilon = f_\epsilon & \text{in } G \\
w_\epsilon = 0 & \text{on } \partial G.
\end{cases}
\]

By the maximum principle \(u_\epsilon \leq w_\epsilon\) and thus

\[\|u_\epsilon\|_{L^\alpha(\Omega)} \leq \|w_\epsilon\|_{L^\alpha(\Omega)}.\]

But \(w_\epsilon\) is given explicitly by
\[ w_\varepsilon(x) = \begin{cases} 
-\frac{1}{\varepsilon^\theta} r^2 + \frac{n}{(n-2)\varepsilon^{\theta-2}} - \frac{2 \varepsilon^{n-\theta}}{(n-2)|x-a_\varepsilon'|^{n-2}} & \text{if } |x-a_\varepsilon| < \varepsilon \\
2 \varepsilon^{n-\theta} \left( \frac{1}{|x-a_\varepsilon|^n} - \frac{1}{|x-a_\varepsilon'|^n} \right) & \text{otherwise}
\end{cases} \]

where \( r = |x-a_\varepsilon| \) and \( a_\varepsilon' = -a_\varepsilon \). Note that

\[ w_\varepsilon(x) \leq \frac{C}{\varepsilon^{\theta-2}} \quad \text{if } |x-a_\varepsilon| < \varepsilon \]

so that

\[ \int_{|x-a_\varepsilon| < \varepsilon} w_\varepsilon^\alpha \leq \frac{C}{\varepsilon^{(\theta-2)\alpha}} \varepsilon^n = C \]

since

\[ (\theta-2)\alpha = \frac{2}{p-1} \alpha = n. \]

On the other hand we claim that

(15) \[ w_\varepsilon(x) \leq \frac{C \varepsilon^{n-\theta+1}}{|x-a_\varepsilon|^{n-1}} \quad \text{if } x \in G \text{ and } |x-a_\varepsilon| > \varepsilon. \]

Indeed we have

\[ \frac{1}{|x-a_\varepsilon|^n} - \frac{1}{|x-a_\varepsilon'|^n} \leq \frac{C}{|x-a_\varepsilon|^{n-1}} (|x-a_\varepsilon'| - |x-a_\varepsilon|) \leq \frac{C \delta_\varepsilon}{|x-a_\varepsilon|^{n-1}} \]

which implies (15). Therefore

\[ \|w_\varepsilon\|_\alpha^\alpha \leq C \varepsilon^{(n-\theta+1)\alpha} \int_{\varepsilon}^{2} \frac{r^{n-1}dr}{r^{(n-1)\alpha}} \leq C \]

since \( (n-1)\alpha > n \) (this follows from the assumption \( p > (n+1)/(n-1) \)) and \( \alpha(\theta-2) = 1 \).

Verification of (14). We have

\[ u_\varepsilon(a_\varepsilon) \geq v_\varepsilon(a_\varepsilon) = \frac{C}{\varepsilon^{\theta-2}} \rightarrow \infty \quad \text{as } \varepsilon \to 0. \]
2. No $L^\infty$ local estimate when $p > n/(n-2)$.

As we have already pointed out, there are no $L^\infty$ local estimates when $p = (n+2)/(n-2)$ and $\alpha = 2n/(n-2)$. In fact, the same holds for any $p > n/(n-2)$ and $\alpha = \frac{n}{2}(p-1)$.

Using an idea of [3] we shall construct in $\Omega = \{x \in \mathbb{R}^n; |x| < 1\}$ sequences $(u_j)$ and $(V_j)$ satisfying

$$
\begin{cases}
-\Delta u_j = V_j u_j^p \quad \text{in } \Omega, \quad \text{with } p > n/(n-2) \\
u_j = 0 \quad \text{on } \partial\Omega
\end{cases}
$$

$$
u_j \geq 0, V_j \geq 0 \quad \text{in } \Omega$$

$$
\|V_j\|_{L^\infty(\Omega)} \leq C_1, \quad \|u_j\|_{L^\alpha(\Omega)} \leq C_2, \quad \alpha = \frac{n}{2}(p-1)
$$

and

$$u_j(0) \to \infty.$$

Indeed, set, as above, $\theta = 2p/(p-1)$ and let

$$f_\varepsilon(x) = \begin{cases}
\frac{2n}{\varepsilon^\theta} & \text{if } |x| < \varepsilon, \\
0 & \text{otherwise}
\end{cases}.$$

Let $u_\varepsilon$ be the solution of

$$
\begin{cases}
-\Delta u_\varepsilon = f_\varepsilon \quad \text{in } \Omega = \{x \in \mathbb{R}^n; |x| < 1\}, \\
u_\varepsilon = 0 \quad \text{on } \partial\Omega
\end{cases}
$$

Set

$$V_\varepsilon = f_\varepsilon / u_\varepsilon^p$$

so that $u_\varepsilon$ satisfies
\[
\begin{cases}
- \Delta u_\varepsilon = V_\varepsilon u_\varepsilon^p & \text{in } \Omega, \\
u_\varepsilon = 0 & \text{on } \partial \Omega.
\end{cases}
\]

We claim that

\begin{align*}
(16) & \quad \|V_\varepsilon\|_\infty \leq C_1 \\
(17) & \quad \|u_\varepsilon\|_\alpha \leq C_2
\end{align*}

and

\begin{equation}
(18) \quad u_\varepsilon(0) \to \infty \text{ as } \varepsilon \to 0.
\end{equation}

**Verification of (16).** As above we have an explicit formula for \( u_\varepsilon \):

\[
u_\varepsilon(x) = \begin{cases}
- \frac{1}{\varepsilon^\theta} \frac{2}{r^2} + \frac{n}{(n-2}\varepsilon^{\theta-2} - \frac{2}{n-2} \varepsilon^{n-\theta} & \text{if } r < \varepsilon \\
\frac{2}{n-2} \varepsilon^{n-\theta} \left( \frac{1}{r^{n-2}} - 1 \right) & \text{if } \varepsilon < r < 1
\end{cases}
\]

with \( r = |x| \). It follows that

\[
\|V_\varepsilon\|_\infty \leq \frac{C}{\varepsilon^\theta \varepsilon^{(\theta-2)p}} = C.
\]

**Verification of (17).** We have

\[
\|u_\varepsilon\|_\alpha \leq \frac{C}{\varepsilon^{(\theta-2)\alpha}} \varepsilon^n + C \varepsilon^{(n-\theta)\alpha} \int_\varepsilon^1 \frac{r^{n-1}}{r^{(n-2)\alpha}} dr.
\]

But \( \alpha > n/(n-2) \) (since \( p > n/(n-2) \)) and thus

\[
\|u_\varepsilon\|_\alpha \leq \frac{C \varepsilon^n}{\varepsilon^{(\theta-2)\alpha}} = C.
\]

**Verification of (18).** We have

\[
u_\varepsilon(0) = \frac{n}{(n-2)\varepsilon^{\theta-2}} (1 + o(1))
\]

and thus \( u_\varepsilon(0) \to \infty \) since \( \theta > 2 \).
3. No $L^\infty$ local estimate if $V$ changes sign.

S. Wang [5] has given an example showing that when $p = n/(n-2)$, the assumption $V_j \geq 0$ in Theorem 1 is essential. He constructs sequences $(u_j)$ and $(V_j)$ in $\Omega$, the unit ball of $\mathbb{R}^n$, such that

\[
\begin{cases}
- \Delta u_j = V_j u_j^p & \text{in } \Omega, \\
u_j = 0 & \text{on } \partial \Omega,
\end{cases}
\]

$u_j \geq 0$ in $\Omega$, $\|V_j\|_\alpha \leq C_1$, $\|u_j\|_\alpha \leq C_2$

and

$u_j(0) \to \infty$

with

$p = \alpha = \frac{n}{n-2}$.

On the other hand, S. Wang [5] also proves that, when $1 + \frac{2}{n} < p < \frac{n}{n-2}$, the conclusion of Theorem 1 still holds without the assumption $V_j \geq 0$. 
References


