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UNIFORM ESTIMATES AND BLOW-UP BEHAVIOR FOR SOLUTIONS OF $-\Delta u = V(x)e^u$ IN TWO DIMENSIONS

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Introduction

In this paper we deal with the equation

$$(*)\qquad \begin{cases} -\Delta u = V(x)e^u & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain (except in Section II.3) and $V(x)$ is a given function in $L^p(\Omega)$ for some $1 < p \le \infty$. We assume that $u \in L^1(\Omega)$ and $e^u \in L^{p'}(\Omega)$ (where p' is the conjugate exponent of p) so that $(*)$ has a meaning in the sense of distributions.

A first question is whether one can conclude that $u \in L^\infty(\Omega)$. As we will see in Section II the answer is positive. Next we turn, in Section III, to a more delicate issue, namely the question of uniform estimates. Suppose we have a sequence (u_n) of solutions of

$$(**)\qquad \begin{cases} -\Delta u_n = V_n(x)e^{u_n} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

with

$$\|V_n\|_{L^p} \le C_1$$

and

$$\|e^{u_n}\|_{L^{p'}} \leq C_2.$$

Can one conclude that

$$\|u_n\|_{L^\infty} \leq C_3$$

where C_3 depends only on C_1, C_2 and Ω? We prove that the answer is positive under a smallness condition, namely $C_1C_2 < 4\pi/p'$ (see Corollary 3). The answer is also positive under a domination condition, namely $|V_n| \leq W$ for a fixed $W \in L^p(\Omega)$, $1 < p < \infty$ (and then C_3 depends also on W, see Corollary 5).

A deeper result (see Corollary 6) asserts that if $V_n \geq 0$ then (u_n) is bounded in $L^\infty_{loc}(\Omega)$, i.e. for every compact subset K of Ω we have

$$\|u_n\|_{L^\infty(K)} \leq C_3$$

where C_3 depends only on C_1, C_2 and K. Surprisingly such an estimate does not hold up to the boundary. Given any $1 < p \leq \infty$ we construct in Example 6 (Section III.3) sequences (u_n) and (V_n) satisfying (**) with $V_n \geq 0$

$$\|V_n\|_{L^p} \leq C_1$$

$$\|e^{u_n}\|_{L^{p'}} \leq C_2$$

and $\|u_n\|_{L^\infty} \to +\infty$.

A corollary of our methods also yields the following. Suppose u_n satisfies

$$-\Delta u_n = V_n e^{u_n} \quad \text{in} \quad \Omega$$

with

$$0 < a \leq V_n \leq b < \infty$$

and

$$\operatorname*{Inf}_{\Omega} u_n \geq -M > -\infty$$

(here no boundary condition is imposed). Then for every compact subset K of Ω, $\operatorname{Sup}_K u_n$ can be estimated just in terms of a,b,M,K and Ω (see Corollary 8).

Finally we turn to the general case where no boundary condition is imposed and (u_n) is not bounded below. More precisely let (u_n) be a sequence of solutions of

$$-\Delta u_n = V_n e^{u_n} \quad \text{in} \quad \Omega$$

with

$$V_n \geq 0 \quad \text{in} \quad \Omega, \ \|V_n\|_{L^p} \leq C_1 \quad \text{and} \quad \|e^{u_n}\|_{L^{p'}} \leq C_2,$$

for some $1 < p \leq \infty$.

Then we have the following alternative (see Theorem 3):

either

(i) (u_n) is bounded in $L^\infty_{loc}(\Omega)$

or

(ii) $u_n \to -\infty$ **uniformly on compact subsets of** Ω

or

(iii) **there is a finite nonempty set** S **such that** $u_n \to -\infty$ uniformly on compact subsets of $\Omega\backslash S$ and $u_n \to +\infty$ on S (in a sense to be precised later). In this case $V_n e^{u_n}$ converges to a finite sum of Dirac masses $\Sigma\alpha_i\delta_{a_i}$ with coefficients $\alpha_i \geq 4\pi/p'$.

Such behavior is well illustrated by the sequence

$$u_n(x) = \log \frac{8n^2}{(1+n^2|x|^2)^2}$$

which satisfies $-\Delta u_n = e^{u_n}$, $\|e^{u_n}\|_{L^1} \leq C$, $u_n(x) \to -\infty$ for all $x \neq 0$ and $u_n(0) \to +\infty$. Here e^{u_n} converges to $8\pi\delta_0$.

We thank Congming Li for raising questions which led us to Theorem 2 and Corollary 3 (Theorem 2 is used in [3]). Some of our results (in particular Corollary 4 and Theorem 4) are connected to earlier works of Nagasaki and Suzuki (see [6] and [7]) who consider mostly the case where the V_n's are constants. A. Chang and P. Yang [2] have also studied blow–up sequences for related equations on S^2 (see e.g. their Concentration Lemma). However their approach involves H^1 norms and is quite different from ours.

In a forthcoming work we shall consider similar issues for the equation $-\Delta u = V(x)u^p$ in $\Omega \subset \mathbb{R}^N$, $N \geq 3$. The plan of the paper is the following:

I. A basic inequality

Assume $\Omega \subset \mathbb{R}^2$ is a bounded domain and let u be a solution of

$$\text{(1)} \qquad \begin{cases} -\Delta u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $f \in L^1(\Omega)$. Set $\|f\|_1 = \int_\Omega |f(x)|\,dx$.

<u>Theorem 1</u>. For every $\delta \in (0,4\pi)$ we have

$$\text{(2)} \qquad \int_\Omega \exp\left[\frac{(4\pi-\delta)|u(x)|}{\|f\|_1}\right] dx \leq \frac{4\pi^2}{\delta} (\operatorname{diam} \Omega)^2.$$

<u>Proof</u>. Let $R = \frac{1}{2} \operatorname{diam} \Omega$ so that $\Omega \subset B_R$ for some ball of radius R. Extend f to be zero outside Ω and set, for $x \in \mathbb{R}^2$,

$$\bar{u}(x) = \frac{1}{2\pi} \int_{B_R} \log\left(\frac{2R}{|x-y|}\right)|f(y)|\,dy$$

so that

$$-\Delta\bar{u} = |f| \quad \text{on} \quad \mathbb{R}^2.$$

Note that $\bar{u}(x) \geq 0$ for $x \in B_R$ since $\frac{2R}{|x-y|} \geq 1$ $\forall x, y \in B_R$. It follows from the maximum principle that $|u| \leq \bar{u}$ on Ω and thus

$$\int_\Omega \exp\left[\frac{(4\pi-\delta)|u(x)|}{\|f\|_1}\right]dx \leq \int_{B_R} \exp\left[\frac{(4\pi-\delta)\bar{u}(x)}{\|f\|_1}\right]dx. \tag{3}$$

We now estimate the right-hand side of (3) using Jensen's inequality

$$F\left(\int w(y)\varphi(y)dy\right) \leq \int w(y)F(\varphi(y))dy$$

with $F(t) = \exp t$, $w(y) = \frac{|f(y)|}{\|f\|_1}$ and $\varphi(y) = \frac{(4\pi-\delta)}{2\pi}\log\left(\frac{2R}{|x-y|}\right)$. We obtain

$$\int_{B_R} \exp\left[\frac{(4\pi-\delta)\bar{u}(x)}{\|f\|_1}\right]dx \leq \int_{B_R} dx \int_{B_R} \left(\frac{2R}{|x-y|}\right)^{2-\frac{\delta}{2\pi}} \frac{|f(y)|}{\|f\|_1}\,dy$$

$$= \int_{B_R} \frac{|f(y)|}{\|f\|_1} \left[\int_{B_R} \left(\frac{2R}{|x-y|}\right)^{2-\frac{\delta}{2\pi}} dx\right] dy.$$

But, for $y \in B_R$, we have

$$\int_{B_R} \left(\frac{2R}{|x-y|}\right)^{2-\frac{\delta}{2\pi}} dx \leq \int_{B_R} \left(\frac{2R}{|x|}\right)^{2-\frac{\delta}{2\pi}} dx = \frac{4\pi^2}{\delta}(\operatorname{diam}\Omega)^2$$

and the estimate (2) follows.

A simple consequence of Theorem 1 is

Corollary 1. Let u be a solution of (1) with $f \in L^1(\Omega)$. Then for every constant $k > 0$

$$e^{k|u|} \in L^1(\Omega).$$

Proof. Let $0 < \epsilon < 1/k$. We may split f as $f = f_1 + f_2$ with $\|f_1\|_1 < \epsilon$ and $f_2 \in L^\infty(\Omega)$. Write $u = u_1 + u_2$ where u_i are the solutions of

$$\begin{cases} -\Delta u_i = f_i & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial\Omega . \end{cases}$$

Choosing, for example, $\delta = (4\pi - 1)$ in Theorem 1 we find $\int_\Omega \exp\left[\frac{|u_1(x)|}{\|f_1\|_1}\right] < \infty$ and thus $\int_\Omega \exp[k|u_1|] < \infty$. The conclusion follows since $|u| \le |u_1| + |u_2|$ and $u_2 \in L^\infty(\Omega)$.

Remark 1. The conclusion of Theorem 1 could also be deduced from BMO estimates and the John–Nirenberg inequality [4].

Remark 2. There is a local form of Corollary 1, namely if $u \in L^1_{loc}(\Omega)$ and $\Delta u \in L^1_{loc}(\Omega)$, then for every $k > 0$, $e^{k|u|} \in L^1_{loc}(\Omega)$. [Here we use the well-known fact that $u \in L^1_{loc}(\Omega)$ and $\Delta u \in L^1_{loc}(\Omega)$ imply $\nabla u \in L^1_{loc}(\Omega)$.]

Remark 3. In Corollary 1, $e^{k|u|} \in L^1$ but $\|e^{k|u|}\|_1$ can not be estimated in terms of k and $\|f\|_1$. For example, we may have a sequence (f_n) such that $\|f_n\|_1 \le 1$, $f_n \to \delta_{x_0}$ and then $u_n \to u$ with $u(x) \simeq \frac{1}{2\pi} \log \frac{1}{|x-x_0|}$ as $x \to x_0$ so that $\int e^{k|u|} = \infty$ for $k \ge 4\pi$.

II. L^∞-boundedness for a single solution of $-\Delta u = Ve^u$.

II.1. The case of a bounded domain.

Let u satisfy the nonlinear equation

$$(4) \qquad \begin{cases} -\Delta u = V(x)e^u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega , \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^2$ and $V(x)$ is a given function on Ω.

Corollary 2. Suppose u is a solution of (4) with $V \in L^p(\Omega)$ and $e^u \in L^{p'}(\Omega)$ for some $1 < p \leq \infty$. Then $u \in L^\infty(\Omega)$.

Proof. By Corollary 1 we know that $e^{ku} \in L^1(\Omega)$ $\forall k > 0$, i.e., $e^u \in L^r(\Omega)$ $\forall r < \infty$. It follows that $Ve^u \in L^{p-\delta}$ $\forall \delta > 0$ if $p < \infty$, and $Ve^u \in L^r(\Omega)$ $\forall r < \infty$ if $p = \infty$. Standard elliptic estimates imply that $u \in L^\infty(\Omega)$.

Remark 4. The conclusion of Corollary 2 still holds for a solution u of

$$\begin{cases} -\Delta u = V(x)e^u + f(x) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

with $g \in L^\infty(\partial\Omega)$ and $f \in L^q(\Omega)$ for some $q > 1$. Indeed let w be the solution of

$$\begin{cases} -\Delta w = f & \text{in } \Omega, \\ w = g & \text{on } \partial\Omega, \end{cases}$$

so that $w \in L^\infty(\Omega)$. The function $\tilde{u} = u-w$ satisfies

$$\begin{cases} -\Delta \tilde{u} = (Ve^w)e^{\tilde{u}} & \text{in } \Omega, \\ \tilde{u} = 0 & \text{on } \partial\Omega, \end{cases}$$

and we are reduced to the assumptions of Corollary 2.

Remark 5. There is a local version of Corollary 2, namely if $u \in L^1_{loc}(\Omega)$ satisfies

$$-\Delta u = Ve^u$$

with $V \in L^p_{loc}(\Omega)$ and $e^u \in L^{p'}_{loc}(\Omega)$ for some $1 < p \leq \infty$, then $u \in L^\infty_{loc}(\Omega)$. This follows easily from Remark 2.

II.2. Some variants and counterexamples.

1. The conclusion of Corollary 2 fails when $p = 1$ (we may only say that $u^+ \in L^\infty(\Omega)$). Here is an example:

Example 1. Let $0 < a < 1$. The function

$$u = -a \log(\log \tfrac{e}{r}) \quad \text{with} \quad r = |x| \quad \text{satisfies}$$

$$(5) \qquad \begin{cases} -\Delta u = Ve^u & \text{in } \Omega = B_1 \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $V = -\frac{a}{r^2(\log \frac{e}{r})^{2-a}}$. Note that $V \in L^1(\Omega)$, $e^u \in L^\infty(\Omega)$ and nevertheless $u \notin L^\infty(\Omega)$ since $u(x) \to -\infty$ as $x \to 0$. The same function u with $a < 0$ provides an example where u satisfies (5) with $V \in L^1(\Omega)$, $Ve^u \in L^1(\Omega)$ and nevertheless $u^+ \notin L^\infty(\Omega)$ since $u(x) \to +\infty$ as $x \to 0$.

2. The function e^u is in some sense the "critical nonlinearity" for which a statement such as Corollary 2 holds. Suppose, for example, that u satisfies

$$\begin{cases} -\Delta u = V(x)e^{u^\alpha} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $u \geq 0$, $\alpha > 1$, $V \in L^p(\Omega)$ and $e^{u^\alpha} \in L^{p'}(\Omega)$, $1 < p \leq \infty$. In general, we may not infer that $u \in L^\infty(\Omega)$.

Example 2. Consider first the case $p = \infty$. Fix $1 < \gamma < 2 - (1/\alpha)$. In $\Omega = B_1$ set

$$u(x) = |\log(r^2(\log \tfrac{e}{r})^\gamma)|^{1/\alpha}.$$

For r small we have

$$e^{u^\alpha} = \frac{1}{r^2(\log \frac{e}{r})^\gamma}$$

and therefore $e^{u^\alpha} \in L^1(\Omega)$. On the other hand u satisfies $-\Delta u = Ve^{u^\alpha}$ where V is defined by $V = (-\Delta u)e^{-u^\alpha}$. An easy computation shows that

$$V \sim |\log r|^{\gamma-2+(1/\alpha)} \quad \text{as} \quad r \to 0$$

and hence $V \in L^\infty(\Omega)$. Nevertheless $u \notin L^\infty(\Omega)$.

When $1 < p < \infty$ we may use the function u above and write

$$-\Delta u = (Ve^{\frac{1}{p}u^\alpha})e^{\frac{1}{p'}u^\alpha}$$

The function $\tilde{u} = (p')^{-1/\alpha}u$ satisfies $-\Delta\tilde{u} = \tilde{V}e^{\tilde{u}^\alpha}$ with $\tilde{V} = (p')^{-1/\alpha}Ve^{\frac{1}{p}u^\alpha}$ so that $\tilde{V} \in L^p(\Omega)$ and $e^{\tilde{u}^\alpha} \in L^{p'}(\Omega)$.

3. There is a version of Corollary 2 for subsolutions. Assume u satisfies

$$\begin{aligned} -\Delta u &\le V(x)e^u \quad \text{in } \Omega ,\\ u &\le 0 \quad \text{on } \partial\Omega , \end{aligned}$$

with $V \in L^p(\Omega)$ and $e^u \in L^{p'}(\Omega)$ for some $1 < p \le \infty$. Then $u^+ \in L^\infty(\Omega)$.

II.3. The case $\Omega = \mathbb{R}^2$.

The main result is the following.

Theorem 2. Suppose $u \in L^1_{loc}(\mathbb{R}^2)$ satisfies

$$-\Delta u = V(x)e^u \quad \text{in } \mathbb{R}^2$$

with $V \in L^p(\mathbb{R}^2)$ and $e^u \in L^{p'}(\mathbb{R}^2)$ for some $1 < p \le \infty$. Then $u \in L^\infty(\mathbb{R}^2)$.

Proof. Fix $0 < \epsilon < 1/p'$ and split Ve^u as $Ve^u = f_1 + f_2$ with $\|f_1\|_{L^1(\mathbb{R}^2)} < \epsilon$ and $f_2 \in L^\infty(\mathbb{R}^2)$. Let B_r be the ball of radius r centered at x_0. We denote by C various constants independent of x_0 (but possibly depending on ϵ). Let u_i be the solution of

$$\begin{cases} -\Delta u_i = f_i & \text{in } B_1 ,\\ u_i = 0 & \text{on } \partial B_1 . \end{cases}$$

By Theorem 1 (applied with $\delta = 4\pi-1$) we have

$$\int_{B_1} \exp\left[\frac{1}{\epsilon}\,|u_1|\right] \le C$$

and in particular $\|u_1\|_{L^1(B_1)} \le C$. We also have $\|u_2\|_{L^\infty(B_1)} \le C$. Let $u_3 = u - u_1 - u_2$ so that $\Delta u_3 = 0$ on B_1. The mean value theorem for harmonic functions implies that

$$\|u_3^+\|_{L^\infty(B_{1/2})} \le C\|u_3^+\|_{L^1(B_1)}. \tag{6}$$

On the other hand we have

$$u_3^+ \le u^+ + |u_1| + |u_2|$$

and since

$$p' \int_{\mathbb{R}^2} u^+ \le \int_{\mathbb{R}^2} e^{p'u} \le C$$

we see that $\|u_3^+\|_{L^1(B_1)} \le C$. Combining this with (6) we find that $\|u_3^+\|_{L^\infty(B_{1/2})} \le C$. Finally we write

$$-\Delta u = Ve^u = (Ve^{u_1})e^{u_2+u_3} = g \tag{7}$$

with $\|g\|_{L^{1+\delta}(B_{1/2})} \le C$ for some $\delta > 0$ (since $e^{u_2+u_3} \in L^\infty(B_{1/2})$, $V \in L^p(B_1)$ and $e^{u_1} \in L^{1/\epsilon}(B_1)$ with $1/\epsilon > p'$). Using once more the mean value theorem and standard elliptic estimates we deduce from (7) that

$$\|u^+\|_{L^\infty(B_{1/4})} \le C\|u^+\|_{L^1(B_{1/2})} + C\|g\|_{L^{1+\delta}(B_{1/2})} \le C.$$

Since C is independent of x_0 we conclude that $u^+ \in L^\infty(\mathbb{R}^2)$.

III. Uniform L^∞ bounds and blow-up behavior for solutions of $-\Delta u = V(x)e^u$.

In this section we consider a sequence (u_n) of solutions of

$$-\Delta u_n = V_n(x)e^{u_n} \quad \text{in } \Omega \tag{8}$$

where Ω is a bounded domain in $\mathbb{R}^2$. We seek a uniform bound for $\|u_n\|_{L^\infty}$

(resp. $\|u_n\|_{L^\infty_{loc}}$) under various assumptions. We start with:

III.1. Some easy cases

There are two different kinds of assumptions which lead easily to uniform bounds:

a) **Smallness assumption.**

b) **Uniform domination.**

a) Smallness assumption

Corollary 3. Assume (u_n) is a sequence of solutions of (8) with $u_n = 0$ on $\partial\Omega$, such that

$$\|V_n\|_{L^p} \le C \quad \text{for some} \quad 1 < p \le \infty \tag{9}$$

and

$$\int_\Omega |V_n| e^{u_n} \le \epsilon_0 < 4\pi/p' \qquad \forall n. \tag{10}$$

Then $\|u_n\|_{L^\infty} \le C$.

Proof. Fix $\delta > 0$ such that $4\pi - \delta > \epsilon_0(p'+\delta)$. By Theorem 1 we have

$$\int_\Omega e^{(p'+\delta)|u_n|} \le C.$$

Therefore e^{u_n} is bounded in $L^{p'+\delta}(\Omega)$ and so $V_n e^{u_n}$ is bounded in $L^q(\Omega)$ for some $q > 1$. Hence u_n is bounded in $L^\infty(\Omega)$.

Remark 6. The smallness condition (10) is sharp. Given any $1 < p \le \infty$ one can construct a sequence (u_n) of solutions of (8) satisfying (9) and

$$\int |V_n| e^{u_n} = 4\pi/p' \tag{11}$$

such that $\|u_n\|_{L^\infty} \to \infty$:

Example 3. Set

$$f_n(x) = \begin{cases} \frac{4}{p'}\, n^2 & \text{if } |x| < 1/n, \\ 0 & \text{otherwise} \end{cases}.$$

Let u_n be the solution of

$$\begin{cases} -\Delta u_n = f_n & \text{in } B_1, \\ u_n = 0 & \text{on } \partial B_1 . \end{cases}$$

Note that u_n satisfies (8) with V_n being defined by $V_n = f_n e^{-u_n}$. An easy computation shows that (9) and (11) hold. Moreover $\|u_n\|_{L^\infty} = u_n(0) = \frac{1}{p'}(2 \log n + 1)$.

Here is a variant of Corollary 3 where no boundary condition is imposed.

Corollary 4. Assume (u_n) is a sequence of solutions of (8) such that, for some $1 < p \leq \infty$,

$$\|V_n\|_{L^p} \leq C_1, \tag{12}$$

$$\|u_n^+\|_{L^1} \leq C_2 \tag{13}$$

and

$$\int_\Omega |V_n| e^{u_n} \leq \epsilon_0 < 4\pi/p'. \tag{14}$$

Then (u_n^+) is bounded in $L^\infty_{loc}(\Omega)$.

Proof. Without loss of generality we may assume that $\Omega = B_R$. Split u_n as $u_n = u_{1n} + u_{2n}$ where u_{1n} is the solution of

$$\begin{cases} -\Delta u_{1n} = V_n e^{u_n} & \text{in } \Omega, \\ u_{1n} = 0 & \text{on } \partial\Omega ; \end{cases} \tag{15}$$

so that $\Delta u_{2n} = 0$ in Ω. By the mean value theorem for harmonic functions we have

$$\|u_{2n}^+\|_{L^\infty(B_{R/2})} \le C\|u_{2n}^+\|_{L^1(B_R)} \le C\left[\|u_n^+\|_{L^1(B_R)} + \|u_{1n}\|_{L^1(B_R)}\right]$$
$$\le C.$$

Using (15), the smallness condition (14) and Theorem 1 we see that $(e^{u_{1n}})$ is bounded in $L^{p'+\delta}(B_R)$ for some $\delta > 0$. Therefore $(V_n e^{u_n})$ is bounded in $L^q(B_{R/2})$ for some $q > 1$. Using (15) once more we see that (u_{1n}) is bounded in $L^\infty(B_{R/4})$. Therefore (u_n) is bounded in $L^\infty(B_{R/4})$.

b) Uniform domination

Corollary 5. Assume (u_n) is a sequence of solutions of (8) with $u_n = 0$ on $\partial\Omega$, satisfying, for some $1 < p < \infty$,

(16) $$\|e^{u_n}\|_{L^{p'}} \le C$$

and one of the following conditions:

either

(17) $$|V_n(x)| \le W(x) \quad \forall n, \text{ with } W \in L^p(\Omega)$$

or

(18) $$V_n \to V \quad \text{in } L^p(\Omega).$$

Then $\|u_n\|_{L^\infty} \le C$.

Proof. Assume first that (17) holds. For every $\epsilon > 0$ we have

$$|V_n| e^{u_n} \le W e^{u_n} \le \epsilon e^{p' u_n} + \frac{1}{\epsilon^{1/(p-1)}} W^p.$$

By (16) we may fix $\epsilon > 0$ small enough so that

(19) $$\epsilon \int_\Omega e^{p' u_n} \le \alpha < 4\pi/p' \quad \forall n.$$

We have $|u_n| \le u_{1n} + u_2$ where u_{1n} is the solution of

$$\begin{cases} -\Delta u_{1n} = \epsilon e^{p' u_n} & \text{in } \Omega, \\ u_{1n} = 0 & \text{on } \partial\Omega \end{cases}$$

and u_2 is the solution of

$$\begin{cases} -\Delta u_2 = \frac{1}{\epsilon^{1/(p-1)}} W^p & \text{in } \Omega, \\ u_2 = 0 & \text{on } \partial\Omega . \end{cases}$$

By Theorem 1 and (19) we see that $e^{u_{1n}}$ is bounded in $L^{p'+\delta}(\Omega)$ for some $\delta > 0$ and, by Corollary 1, $e^{u_2} \in L^k(\Omega)$ for every $k \geq 1$. Thus $|V_n| e^{u_n} \leq e^{u_{1n}} (e^{u_{2n}} W)$ remains bounded in some L^q, $q > 1$, and the conclusion follows.

Assume now that (18) holds. Suppose, by contradiction, that $\|u_n\|_{L^\infty}$ is not bounded. We may then extract a subsequence such that $\|u_{n_k}\|_{L^\infty} \to \infty$. By passing to a further subsequence (still denoted n_k) we may assume that $|V_{n_k}| \leq W$ for some $W \in L^p$ (see e.g. [1]), Théorème IV.9). We are therefore reduced to the previous case.

III.2. The main results

We now turn to the study of a sequence (u_n) of solutions of (8) under the assumptions

(20) $$V_n \geq 0 \quad \text{in} \quad \Omega, \ \|V_n\|_{L^p} \leq C_1 \quad \text{and} \quad \|e^{u_n}\|_{L^{p'}} \leq C_2$$

for some $1 < p \leq \infty$. A typical example is the sequence

$$u_n(x) = \log \frac{8n^2}{(1+n^2|x|^2)^2}$$

which satisfies $-\Delta u_n = e^{u_n}$ in $\mathbb{R}^2$ and $\|e^{u_n}\|_{L^1(\mathbb{R}^2)} = 8\pi$. Note that $u_n(x) \to -\infty$ for all $x \neq 0$ and $u_n(0) \to +\infty$. This example provides a very good description of the blow-up mechanism in the general case under the assumption (20). In fact, if a sequence (u_n) becomes unbounded then there is a finite set S (possibly empty) where u_n tends to $+\infty$ and elsewhere u_n tends to $-\infty$.

More precisely, define the "blow-up" set as follows:

$$S = \left\{ x \in \Omega;\ \begin{array}{l} \text{there exists a sequence } x_n \text{ in } \Omega \text{ such that } x_n \to x \\ \text{and } u_n(x_n) \to +\infty \end{array} \right\}.$$

Then we have

<u>Theorem 3</u>. Assume (u_n) is a sequence of solutions of (8) satisfying, for some $1 < p \leq \infty$,

(21) $$V_n \geq 0 \quad \text{in} \quad \Omega,$$

(22) $$\|V_n\|_{L^p} \leq C_1$$

and

(23) $$\|e^{u_n}\|_{L^{p'}} \leq C_2 .$$

Then, there exists a subsequence (u_{n_k}) satisfying the following alternative:

either

(i) (u_{n_k}) is bounded in $L^\infty_{loc}(\Omega)$

or

(ii) $u_{n_k}(x) \to -\infty$ uniformly on compact subsets of Ω

or

(iii) the blow-up set S (relative to (u_{n_k})) is finite, nonempty and $u_{n_k}(x) \to -\infty$ uniformly on compact subsets of $\Omega \backslash S$. In addition $V_{n_k} e^{u_{n_k}}$ converges in the sense of measures on Ω to $\sum_i \alpha_i \delta_{a_i}$ with $\alpha_i \geq 4\pi/p'$ $\forall i$ and $S = \bigcup_i \{a_i\}$.

Before giving the proof of Theorem 3 we mention some Corollaries.

<u>Corollary 6</u>. Assume (u_n) is a sequence of solutions of (8) with $u_n = 0$ on $\partial\Omega$,

satisfying (21), (22), and (23).

Then (u_n) is bounded in $L^\infty_{loc}(\Omega)$.

Proof. By the maximum principle $u_n \geq 0$ on Ω and therefore cases (ii) and (iii) in Theorem 3 are excluded for all subsequences. Therefore the (full) sequence (u_n) is bounded in $L^\infty_{loc}(\Omega)$.

Remark 7. One may wonder whether the conclusion of Corollary 6 holds uniformly up to the boundary (since we impose here the boundary condition $u_n = 0$ on $\partial\Omega$). This is not true as is shown in Section III.3 (Example 6). However it is plausible that a stronger assumption about the V_n's yields an estimate up to the boundary. For example, here is an

Open problem 1: Suppose (u_n) is a sequence of solutions of (8) with $u_n = 0$ on $\partial\Omega$ satisfying (21),

$$V_n \to V \quad \text{in} \quad C^\circ(\overline{\Omega}) \tag{24}$$

and

$$\|e^{u_n}\|_{L^1} \leq C. \tag{25}$$

Can one conclude that $\|u_n\|_{L^\infty} \leq C$?

Remark 8. The conclusion of Corollary 6 also fails if we remove assumption (21) (i.e. $V_n \geq 0$ on Ω); see [8].

Another obvious consequence of Theorem 3 is:

Corollary 7. Assume (u_n) is a sequence of solutions of (8) satisfying (21), (22) and (23). Assume in addition

$$u_n \geq -M \quad \text{in} \quad \Omega, \forall n \tag{26}$$

for some positive constant M, or more generally

$$\|u_n^-\|_{L^1} \leq M \quad \forall n. \tag{27}$$

Then (u_n) is bounded in $L^\infty_{loc}(\Omega)$.

Corollary 8. Assume (u_n) is a sequence of solutions of (8) satisfying (26) and

(28) $$0 < a \le V_n \le b < \infty \quad \text{in } \Omega$$

for some constants a, b.

Then (u_n) is bounded in $L^\infty_{loc}(\Omega)$.

Proof. In view of Corollary 7 we have only to show that (e^{u_n}) is bounded in $L^1_{loc}(\Omega)$. We may always assume that $M = 0$, i.e. $u_n \ge 0$ (this amounts to replace u_n by $u_n + M$). Let φ_1 be the first eigenfunction of $-\Delta$ on Ω with zero Dirichlet conditions and let λ_1 be the corresponding eigenvalue. Multiplying (8) by φ_1 and integrating we obtain

$$\int_\Omega V_n e^{u_n} \varphi_1 = \int_{\partial\Omega} u_n \frac{\partial\varphi_1}{\partial\nu} + \lambda_1 \int_\Omega u_n \varphi_1$$

where ν is the outward normal. Using (28), $u_n \ge 0$ and $\frac{\partial\varphi_1}{\partial\nu} \le 0$ we obtain

$$a \int_\Omega e^{u_n} \varphi_1 \le \lambda_1 \int_\Omega u_n \varphi_1$$

This provides an upper bound for $\int_\Omega e^{u_n} \varphi_1$. Therefore (e^{u_n}) is bounded in $L^1_{loc}(\Omega)$ and the conclusion follows.

Remark 9. There are two natural questions suggested by Corollary 8:

Open problem 2: Suppose (u_n) is a sequence of solutions of (8) with $u_n = 0$ on $\partial\Omega$ satisfying (28). Can one conclude that (u_n) is bounded in $L^\infty(\Omega)$? Is this true if we assume in addition that $\|e^{u_n}\|_{L^1} \le C$?

Open problem 3: Assume (u_n) is a sequence of solutions of (8) satisfying (26) and (28). Let K be a compact subset of Ω. What is the optimal bound for $\operatorname{Sup}_K u_n$ as a function of M? Does one have

(29) $$\operatorname*{Sup}_K u_n \leq C_1 M + C_2$$

for some positive constants C_1, C_2 depending only on a, b, K and Ω? Can one take $C_1 = 1$ if $V_n(x) \equiv 1$?

[Note that (29) holds with $C_1 = 1$ for the special sequence $u_n(x) = \log \frac{8n^2}{(1+n^2|x|^2)^2}$].

Proof of Theorem 3. Since $(V_n e^{u_n})$ is bounded in $L^1(\Omega)$ we may extract a subsequence (still denoted $V_n e^{u_n}$) such that $V_n e^{u_n}$ converges in the sense of measures on Ω to some nonnegative bounded measure μ, i.e.

(30) $$\int V_n e^{u_n} \psi \rightarrow \int \psi d\mu$$

for every $\psi \in C_c(\Omega)$.

Definition: We say that a point $x_0 \in \Omega$ is a regular point if there is a function $\psi \in C_c(\Omega)$, $0 \leq \psi \leq 1$, with $\psi = 1$ in some neighborhood of x_0, such that

(31) $$\int \psi d\mu < 4\pi/p'.$$

It follows from Corollary 4 (applied in a small ball around x_0) that if x_0 is a regular point then there is some $R_0 > 0$ such that

(32) $$(u_n^+) \text{ is bounded in } L^\infty(B_{R_0}(x_0)).$$

[Note that (13) holds since (e^{u_n}) is bounded in $L^{p'}(\Omega)$].

We denote by Σ the set of nonregular points in Ω. Clearly $x_0 \in \Sigma$ iff $\mu(\{x_0\}) \geq 4\pi/p'$. Since μ is a bounded measure (with $\int d\mu \leq C_1 C_2$) it follows that Σ is finite and

$$\text{card}\,(\Sigma) \leq C_1 C_2 p'/4\pi.$$

We now split the proof of Theorem 3 into 3 steps.

Step 1: $S = \Sigma$.

Clearly $S \subset \Sigma$ by (32). Conversely, suppose $x_0 \in \Sigma$. Then we have

(33) $$\forall R > 0, \lim \|u_n^+\|_{L^\infty(B_R(x_0))} = +\infty.$$

Otherwise there would be some $R_0 > 0$ and a subsequence such that $\|u_{n_k}^+\|_{L^\infty(B_{R_0}(x_0))} \le C$. In particular $\|e^{u_{n_k}}\|_{L^\infty(B_{R_0}(x_0))} \le C$ and therefore

$$\int_{B_R(x_0)} V_{n_k} e^{u_{n_k}} \le CC_1 R^{2/p'} \quad \text{for all } R < R_0.$$

This implies (31) for some suitable ψ. Therefore x_0 is regular – a contradiction. Hence we have established (33). Choose $R > 0$ small enough so that $\bar{B}_R(x_0)$ does not contain any other point of Σ. Let $x_n \in B_R(x_0)$ be such that

$$u_n^+(x_n) = \max_{\bar{B}_R(x_0)} u_n^+ \to +\infty.$$

We claim that $x_n \to x_0$. Otherwise there would be a subsequence $x_{n_k} \to \bar{x} \ne x_0$ and $\bar{x} \notin \Sigma$, i.e. $\bar{x}$ is a regular point. This is impossible in view of (32). Hence we have established that $x_0 \in S$. This completes the proof of Step 1.

<u>Step 2</u>: $S = \phi$ implies (i) or (ii) holds.

By (32) (u_n^+) is bounded in $L^\infty_{loc}(\Omega)$ and therefore $f_n = V_n e^{u_n}$ is bounded in $L^p_{loc}(\Omega)$. This implies that $\mu \in L^1(\Omega) \cap L^p_{loc}(\Omega)$. Let v_n be the solution of

$$\begin{cases} -\Delta v_n = f_n & \text{in } \Omega, \\ v_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Clearly, $v_n \to v$ uniformly on every compact subset of Ω, where v is the solution of

$$\begin{cases} -\Delta v = \mu & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $w_n = u_n - v_n$ so that $\Delta w_n = 0$ on Ω and w_n^+ is bounded in $L^\infty_{loc}(\Omega)$. By Harnack's principle we find that:

either

(34) a subsequence (w_{n_k}) is bounded in $L^\infty_{loc}(\Omega)$,

or

(35) (w_n) converges uniformly to $-\infty$ on compact subsets of Ω.

Case (i) corresponds to (34) and case (ii) to (35).

Step 3. $S \neq \phi$ implies (iii) holds.

By (32) (u_n^+) is bounded in $L^\infty_{loc}(\Omega\backslash S)$ and therefore $f_n = V_n e^{u_n}$ is bounded in $L^\infty_{loc}(\Omega\backslash S)$. This implies that μ is a bounded measure on Ω with $\mu \in L^p_{loc}(\Omega\backslash S)$. (A basic difference with Step 2 is that here μ is a measure, not an L^1 function, and, as will be shown later, μ is a sum of Dirac masses). Let v_n, v and w_n be defined as in Step 2. Then $v_n \to v$ uniformly on compact subsets of $\Omega\backslash S$. As above, by Harnack's principle, we find that

either

(36) a subsequence (w_{n_k}) is bounded in $L^\infty_{loc}(\Omega\backslash S)$

or

(37) (w_n) converges to $-\infty$ uniformly on compact subsets of $\Omega\backslash S$.

We claim that (36) does not happen. Fix some point $x_0 \in S$ and $R > 0$ small so that x_0 is the only point of S in $\bar{B}_R(x_0)$. Assume (36) holds, so that (w_{n_k}) is bounded in $L^\infty(\partial B_R(x_0))$ and similarly for (v_n). Therefore (u_{n_k}) is bounded in $L^\infty(\partial B_R(x_0))$, say by C. Let z_{n_k} be the solution of

$$\begin{cases} -\Delta z_{n_k} = f_{n_k} & \text{in } B_R(x_0), \\ z_{n_k} = -C & \text{on } \partial B_R(x_0). \end{cases}$$

By the maximum principle $u_{n_k} \geq z_{n_k}$ in $B_R(x_0)$.

In particular

$$\int e^{p' z_{n_k}} \leq \int e^{p' u_{n_k}} \leq C_2^{p'}. \tag{38}$$

On the other hand $z_{n_k} \to z$ a.e. (even uniformly on compact subsets of $B_R(x_0)\backslash\{x_0\}$) where z is the solution of

$$\begin{cases} -\Delta z = \mu & \text{in } B_R(x_0), \\ z = -C & \text{on } \partial B_R(x_0). \end{cases}$$

Finally note that since $x_0 \in S$ is not a regular point we have $\mu(\{x_0\}) \geq 4\pi/p'$. This implies that $\mu \geq \frac{4\pi}{p'} \delta_{x_0}$ and therefore

$$z(x) \geq \frac{2}{p'} \log \frac{1}{|x-x_0|} + 0(1) \text{ as } x \to x_0.$$

Thus $e^{p'z} \geq C/|x-x_0|^2$ with $C > 0$. Hence $\int_{B_R(x_0)} e^{p'z} = \infty$. On the other hand, by (38) and Fatou's Lemma we find that

$$\int e^{p'z} \leq C_2^{p'}.$$

A contradiction. Hence we have shown that (37) holds. Consequently (u_n) converges to $-\infty$ uniformly on compact subsets of $\Omega\backslash S$. Therefore $V_n e^{u_n} \to 0$ in $L^p_{loc}(\Omega\backslash S)$ and hence μ is supported on S. This means that $\mu = \sum_i \alpha_i \delta_{a_i}$ with $S = \bigcup_i \{a_i\}$. The argument above gives that $\alpha_i \geq 4\pi/p'$ for each i.

Remark 10. The conclusion (iii) in Theorem 3 involves a finite sum of Dirac masses $\sum_i \alpha_i \delta_{a_i}$ with coefficients α_i: $\geq 4\pi/p'$. The α_i's as well as the a_i's can be chosen arbitrarily. More precisely given any finite set $S = \bigcup_{i=1}^k \{a_i\}$ and any $\alpha_i > 4\pi/p'$ there exist sequences (u_n) and (V_n) as in Theorem 3 such that $V_n e^{u_n}$ converges to $\sum_{i=1}^k \alpha_i \delta_{a_i}$.

To construct such sequences we proceed as follows. Set, for $1 \leq i \leq k$,

$$v_{i,n} = \begin{cases} -\frac{A_i}{4} n^{2\beta_i} |x-a_i|^2 + \frac{A_i}{4} & \text{if } |x-a_i| < 1/n^{\beta_i}, \\ \frac{A_i}{2} \log(\frac{1}{n^{\beta_i}|x-a_i|}) & \text{if } |x-a_i| \geq 1/n^{\beta_i} \end{cases}$$

where $A_i = \alpha_i/\pi > 4/p'$ and β_i is defined by the relation $\beta_i(\frac{A_i}{2} - \frac{2}{p'}) = 1$. Let $u_n = \sum_{i=1}^k v_{i,n} + \sigma_n$ where $\sigma_n = ((k-1) + \frac{2}{p'} \sum_{i=1}^k \beta_i)\log n$. A direct computation shows that $V_n = (-\Delta u_n)e^{-u_n}$ satisfies (21) and (22); moreover (e^{u_n}) is bounded in $L^{p'}$ and $V_n e^{u_n}$ converges to $\sum_{i=1}^{k} \alpha_i \delta_{a_i}$.

We believe that under additional conditions on the V_n's the α_i's in Theorem 3 cannot take arbitrary values ($> 4\pi/p'$):

<u>Open problem 4</u>: Assume (u_n) is a sequence of solutions of (8) satisfying $V_n \geq 0$ on Ω, $V_n \to V$ uniformly in $\overline{\Omega}$ with $V_n, V \in C^0(\overline{\Omega})$ and $\|e^{u_n}\|_{L^1} \leq C$. Assume $S \neq \phi$ so that case (iii) holds. Can one conclude that $V_{n_k} e^{u_{n_k}}$ converges to $8\pi \Sigma m_i \delta_{a_i}$ with $m_i \in \mathbb{N}$?

Evidence in favor of a positive answer comes from the fact that after a blow-up near a_i we are led to a solution of $-\Delta v = ce^v$ on $\mathbb{R}^2$ with $c = V(a_i)$ and $\int_{\mathbb{R}^2} e^v < \infty$. It follows from the result of [3] that $\int_{\mathbb{R}^2} ce^v = 8\pi$. On the other hand, the blow-up analysis gives (formally) $\alpha_i = \int_{\mathbb{R}^2} ce^v$.

In Theorem 3 the assumption $\|e^{u_n}\|_{L^{p'}} \leq C$ provides some kind of <u>bound from above</u> for (u_n) and plays an important role in proving that the blow-up set S is finite. If we drop that assumption little can be said in the general case. For instance, we may have a sequence (u_n) of solutions of

$$-\Delta u_n = e^{u_n} \quad \text{on } \Omega$$

(with $\|e^{u_n}\|_{L^1} \to \infty$) such that

$$\begin{cases} u_n \to +\infty & \text{on a line } S, \\ u_n \to -\infty & \text{in } \Omega \backslash S. \end{cases}$$

<u>Example 4</u>. The sequence

$$u_n(x,y) = 2nx - 2 \log (1+e^{2nx}) + \log 8n^2$$

satisfies $-\Delta u_n = e^{u_n}$, $u_n(0,y) \to +\infty$ and $u_n(x,y) \to -\infty$ for $x \neq 0$.

However, if we assume some <u>bound from below</u> for the u_n's then there are only two possibilities: either $S = \Omega$ (total blow-up) or S is (locally) finite.

Theorem 4. Assume (u_n) is a sequence of solutions of (8) satisfying, for some $1 < p \leq \infty$, (21), (22) and

(39) $$\|u_n^-\|_{L^1} \leq C$$

Then, there exists a subsequence (u_{n_k}) satisfying the following alternative:

either

(i) $u_{n_k} \longrightarrow +\infty$ uniformly on compact subsets of Ω

or

(ii) the blow-up set S (relative to (u_{n_k})) is locally finite (i.e. for each $x \in \Omega$ there is some neighborhood $N(x)$ of x such that $N(x) \cap S$ is finite). Moreover (u_{n_n}) is bounded in $L^\infty_{loc}(\Omega \backslash S)$.

Remark 11. **Both cases in the alternative may occur:**

Example of (i). Let v be any solution of $-\Delta v = e^v$ in $\mathbb{R}^2$. Then $u_n = v+n$ satisfies $-\Delta u_n = V_n e^{u_n}$ with $V_n = e^{-n}$ and $u_n \longrightarrow +\infty$ everywhere.

Example of (ii). Recall that $v_n(x) = \log \dfrac{8n^2}{(1+n^2|x|^2)^2}$ satisfies $-\Delta v_n = e^{v_n}$. Thus $u_n = v_n + \log n^2$ satisfies $-\Delta u_n = V_n e^{u_n}$ with $V_n = 1/n^2$. Note that $u_n(0) \longrightarrow +\infty$ while $u_n(x)$ remains bounded for $x \neq 0$.

Proof of Theorem 4. Without loss of generality we may assume that

(40) $$u_n \geq 0 \quad \text{in } \Omega.$$

Indeed, by Kato's inequality [5] we have

(41) $$\Delta u_n^- \geq -(\Delta u_n)\chi([u_n \leq 0]) = V_n e^{u_n}\chi([u_n \leq 0]) \geq -|V_n|.$$

It follows from (39), (41) and standard elliptic estimates that (u_n^-) is bounded in

$L^\infty_{loc}(\Omega)$. Passing to a smaller domain and adding a constant to (u_n) we may always assume that (40) holds.

We now split the proof into 3 cases.

<u>Case 1</u>: There exists a compact subset $K \subset \Omega$ and a subsequence (u_{n_k}) such that

$$\int_K V_{n_k} e^{u_{n_k}} \longrightarrow +\infty . \tag{42}$$

Then (i) holds.

Indeed, let K' be any compact subset of Ω. Using (40) we obtain

$$u_{n_k}(x) \geq \int_\Omega G(x,y) V_{n_k}(y) e^{u_{n_k}(y)} du,$$

where G is the Green's function of $-\Delta$ with Dirichlet condition on $\partial\Omega$. Since $G(x,y) \geq \alpha > 0$ $\forall x \in K'$, $\forall y \in K$ we see that, for $x \in K'$,

$$u_{n_k}(x) \geq \alpha \int_K V_{n_k} e^{u_{n_k}} \longrightarrow +\infty .$$

<u>Case 2</u>. $(V_n e^{u_n})$ is bounded in $L^1_{loc}(\Omega)$ and there exists a compact subset $K \subset \Omega$ such that, for a subsequence,

$$\int_K u_{n_k} \longrightarrow +\infty .$$

Then (*i*) holds.

Indeed, let K' be any compact subset of Ω. Let ω be an open set such that $K \cup K' \subset \omega \subset\subset \Omega$. In ω, split u_n as $u_n = u_{1n} + u_{2n}$ where u_{1n} is the solution of

$$\begin{cases} -\Delta u_{1n} = V_n e^{u_n} & \text{in } \omega, \\ u_{1n} = 0 & \text{on } \partial\omega . \end{cases}$$

Note that (u_{1n}) is bounded in $L^1(\omega)$ and u_{2n} satisfies

$$\begin{cases} -\Delta u_{2n} = 0 & \text{in } \omega, \\ u_{2n} \geq 0 & \text{on } \partial\omega \end{cases}$$

Thus $u_{2n} \geq 0$ in ω and by Harnack's principle

(43) $$\operatorname*{Sup}_{K \cup K'} u_{2n} \leq C \operatorname*{Inf}_{K \cup K'} u_{2n} \leq C \operatorname*{Inf}_{K'} u_n.$$

On the other hand

$$\int_K u_{2n} \leq C \operatorname*{Sup}_{K} u_{2n} \leq C \operatorname*{Sup}_{K \cup K'} u_{2n}$$

and

$$\int_K u_{2n} = \int_K u_n - \int_K u_{1n} \geq \int_K u_n - C.$$

It follows that $\operatorname*{Inf}_{K'} u_{n_k} \to +\infty$ and thus (i) holds.

We are left with:

Case 3: $(V_n e^{u_n})$ and (u_n) are bounded in $L^1_{loc}(\Omega)$. Then (ii) holds.

We proceed here as in the proof of Theorem 3. We extract a subsequence (still denoted $V_n e^{u_n}$) such that $V_n e^{u_n}$ converges in the sense of measures to some nonnegative (possibly unbounded) measure μ, i.e.

$$\int V_n e^{u_n} \psi \to \int \psi d\mu$$

for every $\psi \in C_c(\Omega)$. We say that a point $x_0 \in \Omega$ is a regular point if there is a function $\psi \in C_c(\Omega)$, $0 \leq \psi \leq 1$, with $\psi = 1$ in some neighborhood of x_0, such that

$$\int \psi d\mu < 4\pi/p'.$$

It follows from Corollary 4 (applied in a small ball around x_0) that if x_0 is a regular point then there is some $R_0 > 0$ such that

(44) $$(u_n) \text{ is bounded in } L^\infty(B_{R_0}(x_0)).$$

We denote by Σ the set of nonregular points in Ω. Clearly $x_0 \in \Sigma$ if $\mu(\{x_0\}) \geq 4\pi/p'$. It follows that Σ is locally finite and for every compact subset K of Ω

$$\operatorname{card}(\Sigma \cap K) \leq (p'/4\pi) \int_K d\mu.$$

We have $S = \Sigma$ as in the proof of Theorem 3 (Step 1). Thus S is locally finite and by (44) (u_n) is bounded in $L^\infty_{loc}(\Omega\backslash S)$, i.e. (ii) holds.

III.3. Variants and counterexamples

1. Suppose that instead of a sequence of solutions of (8) we have a sequence of subsolutions, i.e.

$$-\Delta u_n \leq V_n(x)e^{u_n} \quad \text{in} \quad \Omega.$$

It is easy to adapt the arguments of Section III.1 to obtain estimates for $\|u_n^+\|_{L^\infty}$ under smallness or uniform domination assumption. However the analogue of Corollary 6 for subsolutions does not hold as may be seen from the following:

Example 5. There is a sequence (u_n) satisfying

$$\begin{cases} -\Delta u_n \leq e^{u_n} & \text{in} \quad \Omega = B_1, \\ u_n = 0 & \text{on} \quad \partial\Omega \end{cases}$$

with

$$\int_\Omega e^{u_n} \leq C$$

and such that $u_n(0) \longrightarrow +\infty$. First, note that the function

$$\varphi_\epsilon(x) = \log \frac{8\epsilon^2}{(\epsilon^2+|x|^2)^2}$$

satisfies

$$-\Delta\varphi_\epsilon = e^{\varphi_\epsilon} \quad \forall\epsilon > 0$$

and

$$\int_{\mathbb{R}^2} e^{\varphi_\epsilon} = 8\pi \quad \forall\epsilon > 0.$$

Hence the function $u_n = \varphi_{1/n}^+$ has all the required properties. The same example can be used to produce sequences (v_n) and (V_n) such that

$$\begin{cases} -\Delta v_n \leq V_n e^{v_n} & \text{in} \quad \Omega = B_1 \\ v_n = 0 & \text{on} \quad \partial\Omega \end{cases}$$

such that $V_n \geq 0$, $\|V_n\|_{L^p} \leq C$, $\|e^{v_n}\|_{L^{p'}} \leq C$, $1 < p < \infty$, and $v_n(0) \to +\infty$. It suffices to take $v_n = \frac{1}{p'} u_n$ and $V_n = \frac{1}{p'} e^{\frac{1}{p} u_n}$.

2. The same kind of example shows that the conclusion of Theorem 2 does not hold uniformly. More precisely there are sequences (u_n) and (V_n) such that

$$-\Delta u_n = V_n e^{u_n} \quad \text{on} \quad \mathbb{R}^2$$

with $\|V_n\|_{L^p(\mathbb{R}^2)} \leq C$, $\|e^{v_n}\|_{L^{p'}(\mathbb{R}^2)} \leq C$, $1 < p \leq \infty$, such that $u_n(0) \to +\infty$. One may take for instance $u_n = \frac{1}{p'} \varphi_{1/n}$ and $V_n = \frac{1}{p'} \exp(\frac{1}{p} \varphi_{1/n})$.

3. The conclusion of Corollary 6 cannot be strengthened to $\|u_n\|_{L^\infty} \leq C$. There are sequences (u_n) and (V_n) satisfying

$$-\Delta u_n = V_n e^{u_n} \quad \text{in} \quad \Omega = B_1$$

$$u_n = 0 \quad \text{on} \quad \partial\Omega$$

$$V_n \geq 0 \quad \text{in} \quad \Omega$$

$$\|V_n\|_{L^p} \leq C,$$

$$\|e^{u_n}\|_{L^{p'}} \leq C,$$

with $1 < p \leq \infty$ and such that $\|u_n\|_{L^\infty} \to \infty$. It suffices to construct such an example when $p = \infty$. For a general $1 < p < \infty$ we may use the $p = \infty$ example and note that $\tilde{u}_n = \frac{1}{p'} u_n$ satisfies $-\Delta \tilde{u}_n = \tilde{V}_n e^{\tilde{u}_n}$ with $\tilde{V}_n = \frac{1}{p'} V_n \exp(\frac{1}{p'} u_n)$ so that $\|\tilde{V}_n\|_{L^p} \leq C$ and $\|e^{\tilde{u}_n}\|_{L^{p'}} \leq C$.

Example 6. Let Ω be the unit disc centered at $(1,0)$. Set $a_\epsilon = (d_\epsilon, 0)$ with

$\epsilon < d_\epsilon < 1$. Let $A > 1$ be a constant and let

$$f_\epsilon = \begin{cases} \frac{4A}{\epsilon^2} & \text{in } B_\epsilon(a_\epsilon) \\ 0 & \text{otherwise}. \end{cases}$$

Let u_ϵ be the solution of

$$\begin{cases} -\Delta u_\epsilon = f_\epsilon & \text{in } \Omega, \\ u_\epsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Let V_ϵ be defined by

$$V_\epsilon = f_\epsilon e^{-u_\epsilon}$$

so that $-\Delta u_\epsilon = V_\epsilon e^{u_\epsilon}$. We claim that, for an appropriate choice of d_ϵ, we have

$$\|V_\epsilon\|_{L^\infty} \le C \tag{45}$$

$$\int_\Omega e^{u_\epsilon} \le C. \tag{46}$$

while $u_\epsilon(a_\epsilon) \to +\infty$.

<u>Verification of (45)</u>. Let v_ϵ be the solution of

$$\begin{cases} -\Delta v_\epsilon = f_\epsilon & \text{in } B_{d_\epsilon}(a_\epsilon), \\ v_\epsilon = 0 & \text{on } \partial B_{d_\epsilon}(a_\epsilon). \end{cases}$$

By the maximum principle we have $v_\epsilon \le u_\epsilon$ in $B_{d_\epsilon}(a_\epsilon)$ so that

$$\|V_\epsilon\|_{L^\infty} = \|f_\epsilon e^{-u_\epsilon}\|_{L^\infty} \le \frac{4A}{\epsilon^2} \|e^{-v_\epsilon}\|_{L^\infty(B_\epsilon(a_\epsilon))}.$$

But v_ϵ is given explicitly by

$$v_\epsilon = \begin{cases} -\frac{4A}{\epsilon^2} r^2 + \alpha_\epsilon & 0 \le r < \epsilon \\ 2A \log(\frac{d_\epsilon}{r}) & \epsilon < r < d_\epsilon \end{cases}$$

where $r = |x-a_\epsilon|$ and $\alpha_\epsilon = A + 2A\log(\frac{d_\epsilon}{\epsilon})$. Thus $\|e^{-v_\epsilon}\|_{L^\infty(B_\epsilon(a_\epsilon))} = e^{A-\alpha_\epsilon} = (\frac{\epsilon}{d_\epsilon})^{2A}$. Hence (45) holds provided

$$\frac{1}{\epsilon^2}(\frac{\epsilon}{d_\epsilon})^{2A} \le C. \tag{47}$$

<u>Verification of (46)</u>. Let G be the half-plane

$$G = \{(x_1,x_2) \in \mathbb{R}^2;\ \ x_1 > 0\}.$$

Let w_ϵ be the solution of

$$\begin{cases} -\Delta w_\epsilon = f_\epsilon & \text{in } G, \\ w_\epsilon = 0 & \text{on } \partial G. \end{cases}$$

By the maximum principle we have $u_\epsilon \le w_\epsilon$ in Ω and thus

$$\int_\Omega e^{u_\epsilon} \le \int_\Omega e^{w_\epsilon}.$$

But w_ϵ is given explicitly by

$$w_\epsilon = \begin{cases} -\frac{4A}{\epsilon^2}|x-a_\epsilon|^2 + \beta_\epsilon + 2A\log|x-a'_\epsilon| & \text{if } |x-a_\epsilon| < \epsilon \\ 2A\log\left[\frac{|x-a'_\epsilon|}{|x-a_\epsilon|}\right] & \text{otherwise} \end{cases}$$

where $a'_\epsilon = -a_\epsilon$ and $\beta_\epsilon = A - 2A\log\epsilon$. We have

$$w_\epsilon(x) \le C + 2A\log(\frac{d_\epsilon}{\epsilon}) \quad \text{if } |x-a_\epsilon| < \epsilon$$

(since $|x-a'_\epsilon| < |x-a_\epsilon| + 2d_\epsilon \le \epsilon + 2d_\epsilon \le 3d_\epsilon$),

$$w_\epsilon(x) \le C + 2A\log(\frac{d_\epsilon}{|x-a_\epsilon|}) \quad \text{if } \epsilon \le |x-a_\epsilon| < d_\epsilon$$

(since $|x-a'_\epsilon| < 3d_\epsilon$) and

$$w_\epsilon \le C \quad \text{if } |x-a_\epsilon| \ge d_\epsilon$$

(since $|x-a'_\epsilon| < |x-a_\epsilon| + 2d_\epsilon \le 3|x-a_\epsilon|$).

It follows that

$$\int_\Omega e^{w_\epsilon} \le C\, \epsilon^2 \left(\frac{d_\epsilon}{\epsilon}\right)^{2A} + C \int_\epsilon^{d_\epsilon} \left(\frac{d_\epsilon}{r}\right)^{2A} r\, dr + C$$

$$\le C\, \epsilon^2 \left(\frac{d_\epsilon}{\epsilon}\right)^{2A} + C.$$

Hence (47) and (46) can be achieved by choosing $d_\epsilon = \epsilon^{1-(1/A)}$. Finally we have $u_\epsilon(a_\epsilon) \ge v_\epsilon(a_\epsilon) = \alpha_\epsilon \ge 2A \log\left(\frac{d_\epsilon}{\epsilon}\right)^{2A} \to +\infty$ as $\epsilon \to 0$. Note that in this Example $\int_\Omega V_\epsilon e^{u_\epsilon} = 4A\pi$ can be made arbitrarily close to 4π, showing once more that assumption (10) in Corollary 3 is sharp.

4. One may combine the techniques of Sections III.1 and Section III.2. Assume for example that all the assumptions of Corollary 6 hold with $1 < p < \infty$ and in addition

$$|V_n(x)| \le W(x) \quad \text{in some fixed neighbourhood of } \partial\Omega$$

with $W \in L^p$. Then $\|u_n\|_{L^\infty} \le C$.

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