Remarks on Finding Critical Points

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Dedicated with affection to Natascha Brunswick.

In the course of writing a chapter of a book we observed some simple facts dealing with the Palais Smale property and critical points of functions. Some of these facts turned out to be known, though not well-known, and we think it worthwhile to make them more available. In addition, we present some other recent results which we believe will prove to be useful—in particular, a result of Ghoussoub and Preiss; see [9], [8]. There are two useful techniques used in obtaining critical points. One is Ekeland's Principle (see below), the other is based on deformation arguments. We will use versions of both of them. In particular we present a rather general deformation result.

Throughout this paper we consider real C^1 functions F defined on a Banach space X. When looking for critical points of F it has become standard to assume the following "compactness condition":

$$(\mathrm{PS})_a \qquad \begin{cases} \text{any sequence } (u_n) \text{ in } X \text{ such that } F(u_n) \to a \\ \text{and } \|F'(u_n)\| \to 0 \text{ has a convergent subsequence .} \end{cases}$$

If this holds for every $a \in \mathbb{R}$ one says that F satisfies (PS)—a condition originally introduced by Palais and Smale; see [13].

1. Some Applications of Ekeland's Principle

We start with an elementary statement in which (PS) is not assumed.

PROPOSITION 1. *If*

$$\alpha = \liminf_{\|u\| \to \infty} F(u) \quad \textit{is finite} \tag{1}$$

then there exists a sequence (u_n) *in* X *such that* $\|u_n\| \to \infty$, $F(u_n) \to \alpha$, *and* $\|F'(u_n)\| \to 0$.

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The proof relies on Ekeland's Principle (see [7] and [3] Chapter 5.3) which we shall use in the following form:

EKELAND'S PRINCIPLE. Let M be a complete metric space with metric $d(x, y)$. Let $\psi : M \to (-\infty, +\infty]$, $\psi \not\equiv +\infty$, be a lower semi-continuous function bounded from below. Then, given $\varepsilon > 0$ and $z_0 \in M$ there exists a point $z \in M$ such that

$$\psi(x) - \psi(z) + \varepsilon d(x, z) \geqq 0, \qquad \forall x \in M. \tag{2}$$

$$\psi(z) \leqq \psi(z_0) - \varepsilon d(z_0, z). \tag{3}$$

Proof of Proposition 1: Set, for $r \geqq 0$,

$$m(r) := \inf_{\|u\| \geqq r} F(u).$$

Clearly m is a nondecreasing function and $\lim_{r \to \infty} m(r) = \alpha$. Then for any positive $\varepsilon < \frac{1}{2}$ we have

$$\alpha - \varepsilon^2 \leqq m(r) \quad \text{for} \quad r \geqq \bar{r}.$$

We may take $\bar{r} \geqq \varepsilon^{-1}$. Choose z_0 with $\|z_0\| \geqq 2\bar{r}$ such that

$$F(z_0) < m(2\bar{r}) + \varepsilon^2 \leqq \alpha + \varepsilon^2.$$

Applying Ekeland's Principle in the region $\{\|x\| \geqq \bar{r}\}$ we find some z, $\|z\| \geqq \bar{r}$ satisfying

$$F(x) - F(z) + \varepsilon\|x - z\| \geqq 0 \quad \text{provided} \quad \|x\| \geqq \bar{r},$$

$$\alpha - \varepsilon^2 \leqq m(\bar{r}) \leqq F(z) \leqq F(z_0) - \varepsilon\|z - z_0\|.$$

It follows that $\|z - z_0\| \leqq 2\varepsilon$. Hence $\|z\| > \bar{r}$ and we may conclude that $\|F'(z)\| \leqq \varepsilon$.

COROLLARY 1. *If F is bounded below and satisfies* (PS) *then $F(u) \to \infty$ as $\|u\| \to \infty$.*

This result was proved by S. J. Li in [10] using a gradient flow and by Costa and de Silva in [6] using Ekeland's Principle in a similar way.

Remark 1. The conclusion of Proposition 1 can be strengthened in the finite dimensional case. There exists a sequence (u_n), as in the proposition, satisfying in addition: $F'(u_n)$ is a multiple of u_n. This is done by moving a suitable radially symmetric function until it touches the graph of F.

PROPOSITION 2. *Assume F is bounded below and satisfies* (PS). *Then every minimizing sequence has a convergent subsequence.*

Proof: Let (x_n) be a minimizing sequence. For a subsequence, still denoted (x_n), we may assume that $F(x_n) \leqq \inf F + 1/n^2$. By Ekeland's Principle there exists y_n in X such that

$$F(y) - F(y_n) + (1/n)\|y - y_n\| \geqq 0 \qquad \forall y \in X,$$

$$F(y_n) \leqq F(x_n) - (1/n)\|x_n - y_n\| .$$

Thus $\|F'(y_n)\| \leqq 1/n$, $F(y_n) \leqq \inf F + (1/n^2)$ and $\|x_n - y_n\| \leqq 1/n$. By (PS) the sequence (y_n) has a convergent subsequence (y_{n_k}), and (x_{n_k}) also converges.

Here are some other results proved using the same kind of argument:

PROPOSITION 3. *Assume F is bounded below and satisfies* (PS). *Suppose that all the critical points of F lie in $\{\|u\| \leqq R\}$. Set*

$$M(r) := \inf_{\|u\| = r} F(u) .$$

Then, for $r > R$, $M(r)$ is strictly increasing and continuous from the right.

There is a localized version of this for functions $F \in C^1(\{\|u\| \leqq R\})$ assumed to satisfy (PS) in the following sense: any sequence (u_n) such that $\|u_n\| \leqq R' < R$, for all n, with $F(u_n)$ bounded and $\|F'(u_n)\| \to 0$, has a convergent subsequence.

PROPOSITION 4. *Let F be a C^1 function on $\|u\| \leqq R$ satisfying* (PS) *as above. Assume $F(0) = 0$, $F(u) > 0$ for $0 < \|u\| < R$ and F has no critical point in $\|u\| < R$ except $u = 0$. Then there exists $0 < r_0 \leqq R$ such that M is strictly increasing in $[0, r_0)$ and strictly decreasing on $[r_0, R)$.*

Proof of Proposition 4: Fix $R' < R$. It is easy to see that M is upper semicontinuous on $[0, R']$ and so achieves its maximum at some point $r_0 \in [0, R']$. The conclusion of the proposition follows easily with the aid of the following lemma—on letting $R' \to R$.

LEMMA 1. *Let F be a non-negative C^1 function in the ring*

$$R = \{0 \leqq a \leqq \|u\| \leqq b\} .$$

Assume (PS) *in the following sense: any sequence (u_n) such that*

$$a + \delta \leqq \|u_n\| \leqq b - \delta$$

for some $\delta > 0$ with $F(u_n)$ bounded and $\|F'(u_n)\| \to 0$ has a compact subsequence. Assume F has no critical points in $a < \|u\| < b$. Then the function

$$M(r) := \inf_{\|x\| = r} F(x)$$

satisfies: *if* $a < r_1 < r < r_2 < b$

$$M(r) > \min\{M(r_1),\ \ M(r_2)\}\ . \tag{4}$$

Proof of Lemma 1: Suppose for some $r_1 < r < r_2$, (4) does not hold. There is a sequence (x_n), satisfying

$$\|x_n\| = r\,, \qquad F(x_n) < M(r) + \frac{1}{n^2}\,.$$

Applying Ekeland in $\tilde{R} = \{r_1 \leqq \|x\| \leqq r_2\}$, $\exists y_n$ in that region such that

$$F(z) - F(y_n) + \frac{1}{n}\|z - y_n\| \geqq 0 \qquad \text{for } z \in \tilde{R} \tag{5}$$

$$F(y_n) \leqq F(x_n) - \frac{1}{n}\|x_n - y_n\|\ . \tag{6}$$

Thus y_n is not on $\partial\tilde{R}$, for if it were, say $\|y_n\| = r_1$, we would have

$$\begin{aligned} M(r_1) \leqq F(x_n) - \frac{1}{n}\|x_n - y_n\| &\leqq M(r) + \frac{1}{n^2} - \frac{1}{n}(r - r_1) \\ &\leqq M(r_1) + \frac{1}{n^2} - \frac{1}{n}(r - r_1)\,. \end{aligned}$$

This is impossible for n large. Therefore $\|F'(y_n)\| \to 0$. By (PS), (y_n) has a subsequence which converges to a critical point of F in $\tilde{R}$. Impossible.

Proof of Proposition 3: That M is strictly increasing follows easily from Lemma 1 and the fact that $M(r) \to +\infty$ as $r \to \infty$ (by Corollary 1). Since M is also upper semicontinuous it must be continuous from the right.

Another immediate consequence of Lemma 1 is the following

COROLLARY 2. *Let F be as in Lemma* 1, *with* $a = 0$, *and assume that the origin is a critical point of F which is not a local minimum. Then $M(r)$ is strictly decreasing on* $[0, b)$.

Remark 2. In Proposition 4 the number r_0 might be R but in general it is less than R. Here is an example in $\mathbb{R}^2$:

$$F(x, y) = x^2 - (x - 1)^3 y^2\,.$$

Choose R in such a way that

$$F(x, y) > 0 \quad \text{for} \quad 0 < x^2 + y^2 < R^2$$

and

$$F(x_0, y_0) = 0 \quad \text{for some} \quad x_0, y_0 \quad \text{with} \quad x_0^2 + y_0^2 = R^2 .$$

This function in $\mathbb{R}^2$ serves also as an example for which the Mountain Pass Lemma (MPL) (see [2]) fails because (PS) is not satisfied. For the convenience of the reader we recall the setting of MPL. Let $F : X \to \mathbb{R}$ be a C^1 function satisfying the condition:

$$\text{(7)} \qquad \begin{aligned} &\text{there is an open neighbourhood } U \text{ of } 0 \text{ and some point } u_0 \notin \bar{U} \\ &\text{such that} \quad F(0), F(u_0) < c_0 \leqq F(u) \quad \forall u \in \partial U . \end{aligned}$$

Consider the family $\mathcal{A}$ of all continuous paths A joining 0 to u_0 and set

$$\text{(8)} \qquad c := \inf_{A \in \mathcal{A}} \max_{u \in A} F(u) .$$

Clearly $c \geqq c_0$ and we would expect that c is a critical value. This seems intuitively obvious, but it is not true in general. The function $F(x, y)$ above satisfies (7) with U = small disc about the origin, $c_0 > 0$ and u_0 = any point outside the disc with $F(u_0) \leqq 0$. It is easy to check that $(0, 0)$ is the only critical point of F and that c of (8), which is positive, cannot be a critical value. The correct statement is the following (see [2]):

STANDARD MPL. Under condition (7) and c given by (8) there exists a sequence (u_n) in X such that

$$F(u_n) \to c \quad \text{and} \quad \|F'(u_n)\| \to 0 .$$

If in addition we assume $(\mathrm{PS})_c$ with c given by (8), then c is a critical value.

In connection with this well-known MPL we would like to call attention to the following two forms: As before F is a real C^1 function on a Banach space X. Let K be a compact metric space and let K^* be a nonempty closed subset $\neq K$. Let

$$\mathcal{A} = \{p \in C(K; X) ; \qquad p = p^* \text{ on } K^*\}$$

where p^* is a fixed continuous map on K. Define

$$\text{(9)} \qquad c = \inf_{p \in \mathcal{A}} \max_{\xi \in K} F(p(\xi)) ,$$

so that

$$c \geqq \max_{\xi \in K^*} F(p^*(\xi)) .$$

THEOREM 1. *Assume that for every $p \in \mathcal{A}$, $\max_{\xi \in K} F(p(\xi))$ is attained at some point in $K \backslash K^*$.*[1] *Then there exists a sequence (u_n) in X, such that*

$$F(u_n) \to c \quad \text{and} \quad \|F'(u_n)\| \to 0 .$$

If in addition F satisfies $(\mathrm{PS})_c$, then c is a critical value. Moreover, if (p_n) is any sequence in $\mathcal{A}$ such that

$$\max_{\xi \in K} F(p_n(\xi)) \to c ,$$

then there exists a sequence (ξ_n) in K such that $F(p_n(\xi_n)) \to c$ and $\|F'(p_n(\xi_n))\| \to 0$.

The standard MPL is clearly a special case of this with $K = [0, 1]$, $K^* = \{0, 1\}$ and with $p^*(t) = tu_0$. We shall present two proofs of the theorem. The first is based on Ekeland's Principle; the second, in the Appendix, uses the Deformation Theorem of Section 2.

Proof of Theorem 1: For $\xi \in K$, set

$$d(\xi) = \min\{\operatorname{dist}(\xi, K^*), 1\} ,$$

and consider for any fixed $\varepsilon > 0$, and $p \in \mathcal{A}$,

$$G(p, \xi) = F(p(\xi)) + \varepsilon d(\xi) .$$

(The idea of perturbing the function is taken from N. Ghoussoub and D. Preiss (see [9]); they perturb F; our perturbation is different.) Set

$$\begin{aligned} \psi_\varepsilon(p) &= \max_{\xi \in K} G(p(\xi), \xi) , \\ c_\varepsilon &= \inf_{p \in \mathcal{A}} \psi_\varepsilon(p) . \end{aligned} \tag{10}$$

Clearly $c \leqq c_\varepsilon \leqq c + \varepsilon$.

For $M = \mathcal{A}$ (equipped with the usual metric) we see easily that $\psi_\varepsilon(p)$ is continuous on M. By Ekeland's Principle, $\exists p \in \mathcal{A}$ such that

$$\begin{aligned} \psi_\varepsilon(q) - \psi_\varepsilon(p) + \varepsilon d(p, q) &\geqq 0 \qquad \forall q \in \mathcal{A} , \\ c \leqq c_\varepsilon \leqq \psi_\varepsilon(p) &\leqq c_\varepsilon + \varepsilon \leqq c + 2\varepsilon . \end{aligned} \tag{11}$$

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[1] It is equivalent to assume that for every $p \in \mathcal{A}$ there is some point $\xi \in K \backslash K^*$ such that $F(p(\xi)) \geqq c$.

By our main hypothesis,

$$\psi_\varepsilon(p) > \max_{\xi \in K^*} F(p(\xi)) . \tag{12}$$

Set

$$B_\varepsilon(p) = \{\xi \in K ; G(p(\xi), \xi) = \psi_\varepsilon(p)\} .$$

We shall prove that there is some $\xi_0 \in B_\varepsilon(p)$ such that

$$\| F'(p(\xi_0)) \| \leqq 2\varepsilon . \tag{13}$$

The conclusion of the first part of the theorem then follows by choosing $\varepsilon = 1/n$ and $u_n = p(\xi_0)$.

We shall use the following result which is proved with the aid of a partition of unity as in the construction of a pseudo-gradient (see, e.g., Theorem A.2 in [15]).

LEMMA 2. *Let N be a metric space and let $f : N \to X^*$ be a continuous map. Then, given $\varepsilon > 0$, there exists a locally Lipschitz map $v : N \to X$ such that for all $\xi \in N$,*

$$\|v(\xi)\| \leqq 1$$

$$\langle f(\xi), v(\xi)\rangle \geqq \|f(\xi)\| - \varepsilon .$$

Applying Lemma 2 with $N = K$ and $f(\xi) = F'(p(\xi))$ we obtain a continuous map $v : K \to X$ such that for all $\xi \in K$,

$$\|v(\xi)\| \leqq 1 , \qquad \langle F'(p(\xi)) , v(\xi)\rangle \geqq \|F'(p(\xi))\| - \varepsilon . \tag{14}$$

By (12), $B_\varepsilon(p) \subset K \backslash K^*$. Thus there is a continuous non-negative function $\alpha(\xi) \leqq 1$ on K which equals 1 on $B_\varepsilon(p)$, and vanishes on K^*. We shall take for q, in (11), small variations of the path p:

$$q_h(\xi) = p(\xi) - hw(\xi)$$

for $0 < h$ small, and $w(\xi) = \alpha(\xi)v(\xi)$.

In what follows, $\varepsilon > 0$ is fixed while we let $h \to 0$. Observe that

$$\psi_\varepsilon(q_h) = \max_{\xi \in K} G(q_h(\xi), \xi)$$

is attained at some point $\xi_h \in K$. For a suitable sequence $h_n \to 0$, ξ_{h_n} converges to some ξ_0 which belongs to $B_\varepsilon(p)$. By (11), with $q = q_h$, and by (14), we obtain

$$F(p(\xi_h) - hw(\xi_h)) + \varepsilon d(\xi_h) - \psi_\varepsilon(p) + \varepsilon h \geqq 0 . \tag{15}$$

On the other hand, it is easy to check that as $h \to 0$,

$$(16)\qquad F(p(\xi_h) - hw(\xi_h)) = F(p(\xi_h)) - \langle F'(p(\xi_h)), hw(\xi_h)\rangle + o(h)\,.$$

Combining (15) and (16) we see that

$$-h\langle F'(p(\xi_h))\,,\,w(\xi_h)\rangle + \varepsilon h + o(h) \geqq 0$$

(note that $F(p(\xi_h)) + \varepsilon d(\xi_h) \leqq \psi_\varepsilon(p)$). Hence

$$\langle F'(p(\xi_h))\,,\,w(\xi_h)\rangle \leqq \varepsilon + o(1)\,.$$

As $h \to 0$ we find

$$(17)\qquad \langle F'(p(\xi_0))\,,\,v(\xi_0)\rangle \leqq \varepsilon$$

which, by (14), yields (13).

The last assertion of Theorem 1 is established by constructing first, via Ekeland's Principle, a sequence (q_n) in $\mathcal{A}$ such that

$$\psi_{\varepsilon_n^2}(q) - \psi_{\varepsilon_n^2}(q_n) + \varepsilon_n\, d(q, q_n) \geqq 0 \qquad \forall q \in \mathcal{A}$$

$$\psi_{\varepsilon_n^2}(q_n) \leqq \psi_{\varepsilon_n^2}(p_n) - \varepsilon_n\, d(p_n, q_n)\,.$$

Here (ε_n) is a sequence of positive numbers, $\varepsilon_n \to 0$, such that $\max_{\xi \in K} F(p_n(\xi)) \leqq c + \varepsilon_n^2$. It follows that $d(p_n, q_n) \leqq 2\varepsilon_n$. The preceding argument (applied with q_n in place of p) leads to the existence of some $\xi_n \in K$ such that

$$F(q_n(\xi_n)) = c + O(\varepsilon_n^2)\,, \qquad \|F'(q_n(\xi_n))\| \leqq 2\varepsilon_n\,.$$

This is the desired sequence (ξ_n). Indeed, by $(\mathrm{PS})_c$, a subsequence of $q_n(\xi_n)$ converges to a critical point and the corresponding subsequence of $p_n(\xi_n)$ converges to the same limit. A standard argument shows that for the full sequence, $F(p_n(\xi_n)) \to c$ and $\|F'(p_n(\xi_n))\| \to 0$. Theorem 1 is proved.

Next we present a theorem of Ghoussoub (see [8]) which contains earlier results of Pucci and Serrin (see [14]) as special cases. We believe this theorem will prove very useful; in particular we use it in the proof of Theorems 4 and 5 below.

THEOREM 2. *Assume the conditions of Theorem* 1 *and that there is a closed set* Σ *in* X, *disjoint from* $p^*(K^*)$, *on which*

$$(18)\qquad F \geqq c \ (\textit{defined in } (8))$$

and such that

$$\forall p \in \mathcal{A}, \qquad p(K) \textit{ intersects } \Sigma .$$

Then there is a sequence (u_n) *in* X *satisfying*

$$F(u_n) \to c, \quad \|F'(u_n)\| \to 0 \quad \textit{and} \quad \operatorname{dist}(u_n, \Sigma) \to 0 .$$

In general, c is unknown, so the condition (18) may be difficult to verify. This theorem will be used when $c = \max_{K^*} F(p^*(\xi))$.

COROLLARY 3. *Under the condition of Theorem 2, if* F *satisfies in addition* $(\mathrm{PS})_c$, *then there is a critical point* u *in* Σ, *with* $F(u) = c$.

Unlike the previous proof, and the proof of Ghoussoub, our proof of Theorem 2—in Section 2—makes use of a general deformation theorem. Paul Rabinowitz has still another proof of Theorem 2 using the dual max-min principles of [2] (see also Theorem 3.2 in [15]).

2. A General Deformation Theorem

We consider a function $F \in C^1$ in X and set

$$F_a = \{u \in X \,;\, F(u) \leqq a\},$$

$$K_a = \text{set of critical points of } F \text{ where } F = a .$$

THEOREM 3 (Deformation Theorem). *Let* $c \in \mathbb{R}$. *For any given* $\delta < \frac{1}{8}$ *there exists a continuous deformation* $\eta : [0, 1] \times X \to X$ *such that*

$$(19) \qquad \eta(0, u) = u \qquad \forall u \in X$$

$$(20) \qquad \eta(t, \cdot) \textit{ is a homeomorphism of } X \textit{ onto } X, \qquad \forall t \in [0, 1]$$

$$(21) \quad \eta(t, u) = u \quad \forall t \in [0, 1] \quad \textit{if } |F(u) - c| \geqq 2\delta \quad \textit{or if } \|F'(u)\| \leqq \sqrt{\delta}$$

$$(22) \qquad 0 \leqq F(u) - F(\eta(t, u)) \leqq 4\delta \qquad \forall u \in X, \forall t \in [0, 1]$$

$$(23) \qquad \|\eta(t, u) - u\| \leqq 16\sqrt{\delta} \qquad \forall u \in X, \forall t \in [0, 1]$$

(24) *If* $u \in F_{c+\delta}$ *then either*

(i) $\eta(1, u) \in F_{c-\delta}$ *or*
(ii) *for some* $t_1 \in [0, 1]$, *we have*

$$\|F'(\eta(t_1, u))\| < 2\sqrt{\delta} .$$

(25) *More generally, let* $\tau \in [0, 1]$ *and assume that*

$$\text{for all } t \in [0, \tau],\ \eta(t, u) \text{ belongs to the set}$$

$$\tilde{N} = \{v \in X\,;\ |F(v) - c| \leqq \delta \text{ and } \|F'(v)\| \geqq 2\sqrt{\delta}\}\,,$$

then $F(\eta(\tau, u)) \leqq F(u) - \tau/4$.

Before the proof, some corollaries.

COROLLARY 4. *In Theorem* 3, *if* F *also satisfies* $(\mathrm{PS})_c$, *then given* $\varepsilon > 0$, $\exists\, \delta < \varepsilon$ *and deformation* η *as above, so that in addition*:

$$\text{If} \quad u \in F_{c+\delta} \quad \text{and} \quad F(\eta(1, u)) > c - \delta\,, \qquad \text{then } \|F'(\eta(t, u))\| < \varepsilon \quad \forall t \in [0, 1]\,. \tag{26}$$

Proof of Corollary 4: Observe first that for all $\varepsilon > 0$, $\exists \delta > 0$ such that

$$|F(x) - c| \leqq \delta\,,\ \|F'(x)\| \leqq 2\sqrt{\delta} \quad \text{and} \quad \|x - y\| \leqq 32\sqrt{\delta}\,, \quad \Longrightarrow \|F'(y)\| \leqq \varepsilon\,. \tag{27}$$

Otherwise there would exist $\varepsilon_0 > 0$, and sequences (x_n), (y_n), with $F(x_n) \to c$, $\|F'(x_n)\| \to 0$, $\|x_n - y_n\| \to 0$ and $\|F'(y_n)\| \geqq \varepsilon_0$. Impossible. Choose such $\delta < \frac{1}{8}$ and $\delta < \varepsilon$ for the given ε. Then alternative (24)(ii) must hold, and (26) follows from (23) and (27).

COROLLARY 5. *Assume* F *satisfies* $(\mathrm{PS})_c$. *Given* $\varepsilon > 0$ *and a neighbourhood* $\mathcal{O}$ *of* K_c, *there exist* $\delta < \varepsilon$ *and deformation* η *as in Theorem* 3 *satisfying, in addition*:

$$\text{If } u \in F_{c+\delta} \backslash \mathcal{O}\,, \text{ then alternative (24)(i) holds}\,. \tag{28}$$

Proof: By $(\mathrm{PS})_c$, there exists $\alpha > 0$ such that

$$U = \{u \in X\,;\ |F(u) - c| < \alpha \quad \text{and} \quad \|F'(u)\| < \alpha\} \subset \mathcal{O}\,.$$

We may suppose $\varepsilon < \alpha$ and apply Corollary 4. With δ, η as in that corollary we see that if $u \in F_{c+\delta} \backslash \mathcal{O}$ and (24)(i) does not hold, then $c - \delta < F(\eta(1, u)) \leqq F(u) \leqq c + \delta$, and $\|F'(u)\| \leqq \varepsilon$. This means that $u \in U \subset \mathcal{O}$—contradiction.

COROLLARY 6. *Assume* F *satisfies the conditions of Theorem* 2 *and* $(\mathrm{PS})_c$ *and suppose*

$$c > \max_{\xi \in K^*} F(p^*(\xi))\,.$$

Then $\forall \varepsilon > 0$, $\exists \delta > 0$ *and* $\exists p \in \mathcal{A}$ *such that*

$$\max_{\xi \in K} F(p(\xi)) < c + \delta$$

and

$$|F(p(\xi)) - c| < \delta \Rightarrow \|F'(p(\xi))\| < \varepsilon .$$

Proof: Given $\varepsilon > 0$, choose $\delta > 0$ as in Corollary 4 with

$$2\delta < c - \max_{\xi \in K^*} F(p^*(\xi)).$$

Let $p_0 \in \mathcal{A}$ such that $\max_{\xi \in K} F(p_0(\xi)) < c + \delta$. Then the path $p(\xi) = \eta(1, p_0(\xi))$ has the desired properties.

Corollary 5 is an extension of a well-known deformation theorem; see Theorem A.4 in [15]. Paul Rabinowitz pointed out to us that Corollary 6 also follows easily from a variant of Theorem A.4 in [15]. The form of Theorem 3 presented here was suggested by an unpublished result due to H. Berestycki and C. Taubes. Indeed, the existence of a special path, as in Corollary 6, was proved by Taubes in Lemma 5.2 of [17] for the Yang-Mills functional.

Like all deformation theorems, to obtain the deformation the idea is to follow negative gradient flow. But since F is only in C^1, one replaces F' by a pseudo-gradient on the set $\{F'(u) \neq 0\}$. This is a locally Lipschitz vector field $v(u)$ satisfying

$$\|v(u)\| \leqq 2\|F'(u)\| \quad \text{and} \quad \langle F'(u), v(u)\rangle \geqq \|F'(u)\|^2 .$$

See Lemma A.2 in [15].

Proof of Theorem 3: In addition to the set $\tilde{N}$ in (25) we shall make use of the set

$$N = \{u \in X ; |F(u) - c| < 2\delta \quad \text{and} \quad \|F'(u)\| > \sqrt{\delta}\} .$$

Since $\tilde{N}$ and N^c are disjoint closed sets there is a locally Lipschitz non-negative function $g \leqq 1$ satisfying

$$g = \begin{cases} 1 & \text{on} \quad \tilde{N} \\ 0 & \text{outside} \quad N . \end{cases}$$

For example,

$$g(u) = \frac{\operatorname{dist}(u, N^c)}{\operatorname{dist}(u, N^c) + \operatorname{dist}(u, \tilde{N})} .$$

Consider the vector field

$$V(u) = \begin{cases} -g(u)\dfrac{v(u)}{\|v(u)\|^2} & \text{on } N \\ 0 & \text{outside } N \end{cases}$$

where v is a pseudogradient defined on $\{F'(u) \neq 0\}$. Clearly V is locally Lipschitz on X and $\|V(u)\| \leqq 1/\sqrt{\delta}$, for all $u \in X$. Consider the flow $\eta(t) = \eta(t, u)$ defined by

$$\frac{d\eta}{dt} = V(\eta), \qquad \eta|_{t=0} = u .$$

Clearly, η is defined for $t \in [0, 1]$ and satisfies (19)-(21) and

$$\frac{d}{dt} F(\eta(t)) \leqq -\frac{1}{4} g(\eta(t)) \qquad \forall u \in X, \forall t \in [0, 1] .$$

In particular, we have

$$\int_0^t g(\eta(s))\, ds \leqq 4(F(u) - F(\eta(t))) . \tag{29}$$

Proof of (22): If $|F(u) - c| \geqq 2\delta$, then $\eta(t, u) = u$, for all $t \in [0, 1]$ and the conclusion is obvious. Hence we may assume that $|F(u) - c| < 2\delta$. If $F(\eta(1)) \geqq c - 2\delta$, the proof is finished. Suppose that $F(\eta(1)) < c - 2\delta$. Then $\exists\, t_1 \in [0, 1]$ such that $F(\eta(t_1)) = c - 2\delta$ and since $\eta(t) = \eta(t_1)$ for $t \geqq t_1$ it follows that $F(u) - F(\eta(1)) = F(u) - F(\eta(t_1)) \leqq c + 2\delta - (c - 2\delta) = 4\delta$.

Proof of (23):

$$\begin{aligned} \|\eta(t) - u\| &\leqq \int_0^t \left\| \frac{d\eta}{dt}(s) \right\| ds = \int_0^t \|V(\eta(s))\|\, ds \\ &\leqq \int_{[0,t]\cap\{s:\eta(s)\in N\}} \frac{g(\eta(s))\, ds}{\|v(\eta(s))\|} \leqq \int_{[0,t]\cap\{s:\eta(s)\in N\}} \frac{g(\eta(s))}{\|F'(\eta(s))\|} \leqq \frac{1}{\sqrt{\delta}} \int_0^t g(\eta(s))\, ds \\ &\leqq \frac{4}{\sqrt{\delta}}(F(u) - F(\eta(t))) \leqq 16\sqrt{\delta} \qquad \text{by (29) and (22)} . \end{aligned}$$

Proof of (25): If $\eta(t) \in \tilde{N}$ for $0 \leqq t \leqq \tau$, then $g(\eta(t)) = 1$ and the assertion of (25) follows from (29).

Theorem 2 follows easily from Theorem 3 by suitable deformation of "paths" $p \in \mathcal{A}$ on which max F is close to c. There is a new ingredient however: the time t for which we consider the deformed path $\eta(t, p(\xi))$ will vary with ξ.

Proof of Theorem 2: For any given $\delta > 0$ we shall show that there is a point $\hat{u}$ such that $c \leqq F(\hat{u}) < c + \delta$, $\|F'(\hat{u})\| < 2\sqrt{\delta}$ and $\operatorname{dist}(\hat{u}, \Sigma) \leqq 32\sqrt{\delta}$. Letting $\delta \to 0$ through a sequence δ_n, the corresponding $\hat{u}_n$ have the desired properties.

We take $\delta < \frac{1}{8}$ and so that $32\sqrt{\delta} < \operatorname{dist}(\Sigma, p^*(K^*))$. Let η be a deformation in Theorem 3. Let $p \in \mathcal{A}$ be such that $\max_{\xi \in K} F(p(\xi)) < c + \delta$. Let $0 \leqq \zeta(v) \leqq 1$ be a continuous function on X which equals 1 if $\operatorname{dist}(v, \Sigma) \leqq 16\sqrt{\delta}$, and vanishes if $\operatorname{dist}(v, \Sigma) \geqq 32\sqrt{\delta}$.

Consider the "path"

$$q(\xi) = \eta(\zeta(p(\xi)), p(\xi)) .$$

Clearly $q \in \mathcal{A}$. Let $\tilde{u} \in q(K) \cap \Sigma$. So $\tilde{u} = \eta(\zeta(p(\tilde{\xi})), p(\tilde{\xi}))$ for some $\tilde{\xi} \in K$. Set $u = p(\tilde{\xi})$. By property (23)

$$\|\eta(t, p(\tilde{\xi})) - p(\tilde{\xi})\| \leqq 16\sqrt{\delta} \qquad \forall t \in [0, 1] ,$$

and so $\zeta(p(\tilde{\xi})) = 1$. Hence $\tilde{u} = \eta(1, u)$, and $c \leqq F(\eta(t, p(\tilde{\xi}))) < c + \delta$, for all $t \in [0, 1]$. So alternative (24)(ii) in Theorem 3 must hold, and hence, for some $t_1 \in [0, 1]$, $\hat{u} = \eta(t_1, p(\tilde{\xi}))$ satisfies

$$\|F'(\hat{u})\| < 2\sqrt{\delta} .$$

Furthermore $\|\hat{u} - \tilde{u}\| \leqq 32\sqrt{\delta}$, by (23).

3. Critical Points in the Presence of Splitting

We are going to apply Theorem 2 to functions which are bounded below and satisfy (PS). Let X be a Banach space with a direct sum decomposition

$$X = X_1 \oplus X_2$$

with $k = \dim X_2 < \infty$. We write any $u \in X$ as $u = u_1 + u_2 = (I - P)u + Pu$ where P is the projection onto X_2 along X_1.

THEOREM 4. *Let F be a C^1 function on X with $F(0) = 0$, satisfying* (PS) *and assume that, for some $R > 0$*

$$\begin{cases} F(u) \geqq 0 , & \text{for} \quad u \in X_1 , \quad \|u\| \leqq R , \\ F(u) \leqq 0 , & \text{for} \quad u \in X_2 , \quad \|u\| \leqq R . \end{cases} \tag{30}$$

Assume also that F is bounded below and $\inf_X F < 0$. *Then F has at least two nonzero critical points.*

This theorem is related to results of K. C. Chang in [5] and later results of J. Q. Liu and S. J. Li in [12] and J. Q. Liu in [11] (which also contains other references). Their arguments rely on Morse theory while ours uses Theorem 2 together with an idea of K. C. Chang (personal communication) involving a negative gradient flow, and linking. In [16] E. Silva has extended results of [12]. He assumes (30) and replaces the assumption that F is bounded from below by conditions like those of [12]. One of the ingredients of our proof is the following extension of a result of Rabinowitz (see, e.g., [15]).

LEMMA 3. *Assume that in the decomposition,* $0 < \dim X_1 \leqq \infty$, *and let v be a fixed unit vector in X_1. Set*

$$K = \{ u = sv + u_2 \, ; \, u_2 \in X_2 \, , \, \|u\| \leqq 1 \quad \text{and} \quad s \geqq 0 \} \, .$$

Consider any continuous map $p : K \to X$ satisfying

$$\begin{aligned} & p(u_2) = u_2 && \textit{if} \quad u_2 \in X_2 \quad \textit{and} \quad \|u_2\| \leqq 1 \\ & \|p(u)\| \geqq r > 0 && \textit{if} \quad u \in K \quad \textit{and} \quad \|u\| = 1 \, . \end{aligned} \tag{31}$$

Then, for any $r > 0$, the image $p(\partial K)$ "links" the set of points in X_1 with norm $\rho < r$. That is, for any $0 < \rho < r$, there exists $\bar{u} \in K$ such that

$$Pp(\bar{u}) = 0$$

$$\|p(\bar{u})\| = \rho \, .$$

Proof of Lemma 3: In $X_3 = X_2 \oplus \operatorname{span}\{v\}$ consider the map $T : K \to X_3$,

$$T(u) = Pp(u) + \|(I - P)p(u)\| v \, .$$

To prove the lemma it suffices to show that for some point $\bar{u} \in K$, $T(\bar{u}) = \rho v$. We use finite dimensional degree to do this. Since $\rho < r$ it follows from conditions (31) that for all $u \in \partial K$, $T(u) \neq \rho v$. Consequently $\deg(T, K, \rho v)$ is defined. We shall prove that it equals 1, and this yields the desired result. As we know, the degree depends only on the boundary values of T. So we may consider only $T|_{\partial K}$. Set

$$A = \{ (u_2, 0) \, ; \, u_2 \in X_2 \quad \text{and} \quad \|u_2\| \leqq 1 \}$$

and

$$B = \{ u \in K \, ; \, \|u\| = 1 \} \, .$$

We have $\partial K = A \cup B$. Clearly $Tu = u$ for $u \in A$ and $\|Tu\| \geqq r' > 0$ for $u \in B$. On ∂K define

$$\tilde{T}u = \begin{cases} u & \text{if} \quad u \in A \\ Tu/\|Tu\| & \text{if} \quad u \in B\,. \end{cases}$$

Using (31) we see that T and $\tilde{T}$ are homotopic in $X_3 \backslash \rho v$ through

$$T_t u = tTu + (1-t)\tilde{T}u,\, t \in [0,\,1]\,.$$

Note that $\tilde{T}(B) \subset B$ and that $\tilde{T} = id$ on ∂B. Since B is homeomorphic to a ball there is a continuous deformation $\tilde{T}_t$ connecting $\tilde{T}$ to the identity in B with $\tilde{T}_t = id$ on ∂B for all $t \in [0, 1]$. It follows that $T|_{\partial K}$ is homotopic to the identity in $X_3 \backslash \rho v$ and thus $\deg(T, K, \rho v) = \deg(id, K, \rho v) = 1$.

Our proof of Theorem 4 also makes use of the following

LEMMA 4. *Let F be a C^1 function defined on a Banach space X satisfying* (PS) *and: for some $u_0 \in X$*

$$F(u) > F(u_0) \qquad \forall u \neq u_0\,.$$

Let v be a pseudo-gradient for F on the set $\{u \in X;\ F'(u) \neq 0\}$ (see, e.g., Lemma A.2 [15]). *Let $y \neq u_0$ be such that $F'(y) \neq 0$, and F has no critical value in $(F(u_0), F(y))$. Then the "negative gradient flow" starting at y, defined by*

$$\frac{dx}{dt} = -\frac{v(x)}{\|v(x)\|^2}, \qquad x(0) = y\,, \tag{32}$$

exists for a maximal finite time $0 \leqq t \leqq T(y)$ and $x(T(y)) = u_0$.

Proof: We may suppose $u_0 = 0$, $F(u_0) = 0$. On the integral curve of (32),

$$\frac{dF}{dt} \leqq -\frac{1}{4}$$

by the standard properties of the pseudo-gradient. Thus the solution $x(t)$ of (32) exists on a maximal open interval $(0, T)$ with $T \leqq 4F(y)$, and $0 < F(x(t)) < F(y)$ on $(0, T)$. We will show that $x(t) \to 0$ as $t \to T$.

Case 1. There exists $\delta > 0$ such that $\|F'(x(t))\| \geqq \delta$ on $(0, T)$. Then $\|v(x(t))\| \geqq \|F'(x(t))\| \geqq \delta$, and

$$\int_0^T \frac{dx}{dt}\, dt \quad \text{exists}\,.$$

Consequently $\lim_{t \to T} x(t) = x(T)$ exists. Moreover $x(T)$ is necessarily 0 for otherwise the solution $x(t)$ could be continued beyond $t = T$.

Case 2. If we are not in Case 1, there exists a sequence $t_j \to T$ such that $\|F'(x(t_j))\| \to 0$. By (PS), a subsequence, $x(t_j)$ converges to a critical point of F, which can only be 0. Therefore $\lim_{t\to T} F(x(t)) = 0$ and by Proposition 2, $x(t) \to 0$ as $t \to T$.

Proof of Theorem 4: We know that F achieves its minimum at some point u_0. Supposing 0 and u_0 to be the only critical points we will be led to a contradiction. We consider first the case that k and $\dim X_1$ are positive. We may suppose $R = 1 < \|u_0\|$.

Since at every point $y \in X_2$ with $\|y\| = 1$, $F'(y) \neq 0$, we may apply the preceding Lemma 4 to conclude that the flow starting at y, described by (32), exists on a maximal open interval $0 < t < T(y) < -4F(u_0)$, and $x(t) \to u_0$ as $t \to T(y)$.

Applying Proposition 2, we see that $\exists \delta > 0$ such that the set $\{F(u) < F(u_0) + \delta\}$ lies in $\|u - u_0\| < \|u_0\|/2$. By choosing δ sufficiently small there is a unique value $t = t(y) < T(y)$ such that $F(x(t(y))) = F(u_0) + \delta$. It is a simple exercise to verify that $t(y)$ is continuous in y.

For v a unit vector in X_1, let K denote the set

$$K = \{u = sv + u_2\,;\, u_2 \in X_2\,,\, s \geqq 0 \quad \text{and} \quad \|u\| \leqq 1\}\,. \tag{33}$$

We now define a continuous map p^* of $\partial K = K^*$ into X. Any $u \neq v$ on ∂K, with $\|u\| = 1$, has the unique representation

$$u = sv + \sigma y \tag{34}$$

with $0 \leqq s \leqq 1$, $y \in X_2$, $\|y\| = 1$, $0 < \sigma \leqq 1$, i.e., s, σ, y are unique. Define

$$p^*(u) = u \quad \text{for } u \in X_2\,,\ \|u\| \leqq 1\,, \quad \text{and} \quad p^*(v) = u_0\,. \tag{35}$$

For u given by (34), define

$$p^*(sv + \sigma y) = x(2st(y)) \quad \text{for } 0 \leqq s \leqq \tfrac{1}{2} \tag{36}$$

where $x(t)$ is our solution of (32). So $p^*(\frac{1}{2}v + \sigma y) = x(t(y))$ and it lies in $\|x - u_0\| < \frac{1}{2}\|u_0\|$. Finally define

$$p^*(sv + \sigma y) = (2s - 1)u_0 + (2 - 2s)x(t(y)) \quad \text{for } \tfrac{1}{2} \leqq s < 1\,. \tag{37}$$

As s goes from $\frac{1}{2}$ to 1, the right-hand side traverses the straight segment from $x(t(y))$ to u_0, and so for $s \geqq \frac{1}{2}$,

$$\|p^*(sv + \sigma y) - u_0\| \leqq \|u_0\|/2\,.$$

The mapping p^* is clearly continuous and on its image we have $F \leqq 0$. In addition we see that

$$\|p^*(u)\| \geqq r > 0 \quad \text{for } \|u\| = 1 .$$

We have $r \leqq 1$. Fix $0 < \rho < r$. We are now going to use Lemma 3, according to which the image of p^* links the set $\Sigma = \{u \in X_1 ; \|u\| = \rho\}$, i.e., for any continuous extension p of p^* to all of K, the image of p intersects Σ.

Denoting by $\mathcal{A}$ the set of all such maps p, we are now in a position to apply a min max argument, namely Theorem 2. According to that theorem the non-negative number

$$c = \inf_{p \in \mathcal{A}} \max_{u \in K} F(p(u))$$

is a critical value of F. If $c > 0$ we have obtained a second nonzero critical point. If $c = 0$, according to Theorem 2, there is a critical point on Σ, and so different from the origin, where $F = 0$.

Before considering the other cases in the theorem, a remark:

Remark 3. The last part of the argument proves the following:

Let F be a C^1 function on X satisfying (PS). Assume there is a continuous map p^* of the boundary of the half ball:

$$K = \{u = sv + u_2 ; u_2 \in X_2 , s \geqq 0 , \|u\| \leqq R\}$$

where v is a fixed unit vector in X_1, into X, with the following properties: $p^*(u) = u$ for $u \in X_2$, $\|u\| \leqq R$, $\|p^*(u)\| \geqq r_0 > 0$ for $\|u\| = R$, and $F(p^*(u)) \leqq 0$ $\forall u \in \partial K$. Assume furthermore that for some positive $\rho < r_0$,

$$F(u) \geqq 0 \quad \text{for } u \in X_1 , \|u\| = \rho .$$

Then F has a nonzero critical point where $F \geqq 0$.

Returning to the theorem, suppose $k = 0$. In that case, assuming u_0, the minimum point of F, is the only nonzero stationary point, we see that in a neighborhood of the origin, $F(u) > 0$ for $u \neq 0$. By Proposition 4 we have

$$F(u) \geqq c_0 > 0 \quad \text{on } \|u\| = r \text{ small} .$$

Applying MPL we find F has a critical value $\geqq c_0$.

The last case to consider is when $\dim X_1 = 0$. In this case we may even permit $k = \infty$. Applying Proposition 4 again (to $-F$) we see that $F(u) \leqq -c_0 < 0$ for $\|u\| = r$ small, and we recall that $F(u) \to \infty$ as $\|u\| \to \infty$, and so we may again apply MPL to $-F$, considering paths joining 0 to a point w where $F(w) > 0$ and such that u_0 is not on the segment joining 0 to w.

Remark 4. In Theorem 4, in case F is a C^2 function in a finite dimensional space and 0 is a *nondegenerate* critical point, i.e., the Hessian $F''(0)$ is nonsingular, then conditions (30) automatically hold. If 0 is a degenerate critical point the conclusion of Theorem 4 need not hold. Here is an example in the plane. Consider the function F defined in polar coordinates r, θ

$$F(r, \theta) = (1 + \cos\theta)f(r) + (1 - \cos\theta)r^2$$

where $f(r)$ is a smooth strictly increasing function on $[0, \infty)$ satisfying (i) $f(r) = r^2$ for $r < 1$, (ii) $f(r) > r^2$ for $r > 1$, (iii) f has only one positive critical point, say $r = 2$, and $f''(2) = 0$. The function F tends to $+\infty$ at infinity, achieves its minimum at $(0, 0)$ and has only one other critical point namely $(2, 0)$. At that point one eigenvalue of $F''(0)$ is negative and the other is zero. If (ii) is replaced by

$$\text{(ii)}' \qquad f(r) < r^2 \quad \text{for } r > 1 \quad \text{and} \quad f(r) \to \infty \quad \text{as } r \to \infty\,,$$

the function F has again critical points only at $(0, 0)$ and $(2, 0)$. At the latter, $F''(0)$ has one positive eigenvalue and the other is zero. The index of F' at $(2, 0)$ is zero.

The proof of Theorem 4 yields in fact a somewhat stronger result:

THEOREM 5. *Let F be as in Theorem 4—with $k > 1$. Assume that F has only a finite number of critical points where $F < 0$ and that each one is a local minimum point of F. Then there is another critical point $\neq 0$.*

In particular, if F is an even function satisfying all the conditions of Theorem 4, and if $k > 1$, then F has at least two pairs of nonzero critical points.

Proof: Supposing the conclusion of the theorem is false we will obtain a contradiction. Let $u_0, u_1, \cdots, u_m$ be the critical points of F where $F < 0$, and fix $r > 0$ so that the balls $B_j = \{\|u - u_j\| \leqq r\}, j = 0, \cdots, m$, are disjoint and do not contain the origin. We may suppose R, in (30), so small that $B_R = \{\|u\| \leqq R\}$ does not touch any B_j. Using Proposition 4 we see that there is a number $\delta > 0$ such that the component $\mathcal{C}_j$, containing u_j, of the set $\{F(u) < F(u_j) + \delta\}$ lies in B_j.

Let y be a point in X_2 with $\|y\| = R$. Then $F'(y) \neq 0$ and, as in the proof of Lemma 4, we find that the "negative gradient flow" starting at y, defined by (32) exists for a finite time $T(y) \leqq -4 \min F$, and that there is a unique value $t = t(y)$ such that $x(t(y))$ first encounters $\bigcup_j (\partial\mathcal{C}_j)$. Suppose $x(t(y)) \in \partial\mathcal{C}_0$. Then for $t(y) < t < T(y)$, $x(t)$ lies in $\mathcal{C}_0$. Thus the flow curve starting at y ends up in $\mathcal{C}_0$. It follows that for $z \in X_2$, lying close to y on ∂B_R, the "negative gradient flow" curve from z also ends up in $\mathcal{C}_0$. Since $k > 1$, the set $X_2 \cap \partial B_R$ is connected and it follows that the flow curve starting at every point in that set ends up in $\mathcal{C}_0$.

If $\dim X_1 > 0$ we may follow the remainder of the proof of Theorem 4 and obtain a nonzero critical point of F where, of course, $F \geqq 0$.

Suppose $\dim X_1 = 0$. Here we may assume $1 < k \leqq \infty$. We apply the standard MPL to $-F$, to obtain a critical value $c > 0$ of $-F$ and a sequence of curves (p_n) such that $\max_{p_n}(-F) \to c$. By our assumption, $-F$ equals c at some of the (u_j). Consider such u_j and a ball $B(u_j)$ of small radius such that on ∂B_j, $-F \leqq c - \varepsilon$, with $\varepsilon > 0$. We construct a new sequence of paths (q_n) as follows. Replace each arc of (p_n) lying in B_j by one on ∂B_j with the same end points. Clearly $\max_{q_n}(-F) \to c$, and by Theorem 1 there exists a sequence of points x_n on q_n such that $-F(x_n) \to c$ and $\|F'(x_n)\| \to 0$. By (PS) a subsequence of x_n converges to one of our u_j—impossible.

Theorem 5 is a generalization of Lemma 2.2 of Ambrosetti and Lupo (see [1]) which is proved using the Morse inequalities.

Here is a simple application of Theorem 4. Consider the problem

$$\begin{cases} -\Delta u + a(x)u = \lambda g(u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{38}$$

We assume that $a \in L^\infty$, g is smooth with

$$g(0) = g'(0) = 0\,, \tag{39}$$

and

$$\overline{\lim_{|u|\to\infty}} \frac{g(u)}{u} < 0\,. \tag{40}$$

In addition we assume that

$$G(u) = \int_0^u g(t)\,dt > 0 \quad \text{for some } u\,.$$

In case 0 is an eigenvalue of $-\Delta + a$ we also assume that

$$G(u) \geqq 0 \quad \text{for } |u| \leqq \delta \text{ small}\,. \tag{41}$$

THEOREM 6. *For every λ sufficiently large there are at least two nontrivial solutions of* (38).

Example. $g(u) = u^3 + u^4 - u^5$.

Proof: Since, for $\lambda >$ some $\bar{\lambda}$, there is an a priori estimate for the solution (by the maximum principle), we may also assume that $g(u) = bu + c$ with $b < 0$, for

$|u|$ large. Solutions of the modified problem will still be solutions of the original problem. The functional

$$F(u) = \frac{1}{2}\int |\nabla u|^2 + au^2 - \lambda \int G(u)$$

is well defined on $X = H_0^1$ and we claim that it satisfies the conditions of Theorem 4.

First, it is clear that $\inf F < 0$ for λ large enough. Indeed, since $G(s) > 0$ for some s there exists a function $u_0 \in H_0^1$ such that $\int G(u_0) > 0$. Using (40) we readily see that $\inf F > -\infty$ for λ large.

Next we turn to the splitting $X = X_1 \oplus X_2$. For X_2 we choose the (finite dimensional) space spanned by the eigenfunctions corresponding to nonpositive eigenvalues of $-\Delta + a$, and X_1 is its orthogonal complement in X. Using (39) and the fact that $G(t) < 0$ for $|t|$ large we have, for any $\varepsilon > 0$,

$$G(t) \leqq \varepsilon t^2 + C_\varepsilon |t|^{2n/(n-2)} \quad \forall t . \tag{42}$$

We deduce that on X_1, for some $\alpha > 0$,

$$F(u) \geqq \alpha \|u\|^2 - \frac{\alpha}{2}\int u^2 - C\|u\|^{2n/(n-2)} \geqq 0$$

for $\|u\|_{H^1} = \|u\| \leqq R$ small. On X_2, which is finite dimensional, we have

$$F(u) \leqq 0 \quad \text{for } \|u\| \text{ small}$$

by (42)—and (41) in case 0 is an eigenvalue of $-\Delta + a$. The (PS) condition is easily verified for λ large. Theorem 4 yields the conclusion.

As another application of Theorem 4 we prove the existence of nontrivial time-periodic solutions of a system of ordinary differential equations for a vector-valued function of time $x(t)$, taking its values in $\mathbb{R}^N$:

$$\ddot{x} = \nabla_x V(t, x) . \tag{43}$$

Here V is a smooth function defined on $\mathbb{R}^1 \times \mathbb{R}^N$ which is periodic in t of period, say, 2π. Assume

(i) $V(t, 0) \equiv 0$, $\nabla_x V(t, 0) \equiv 0$,
(ii) $V(t, x) \to +\infty$ as $|x| \to \infty$ uniformly in t.
(iii) For some constant vector x_0,

$$\int_0^{2\pi} V(t, x_0)\, dt < 0 .$$

(iv) For $|x| \leqq r$ small, and some integer $k \geqq 0$,

$$-\tfrac{1}{2}(k+1)^2|x|^2 \leqq V(t,x) \leqq -\tfrac{1}{2}k^2|x|^2 .$$

THEOREM 7. *Under the conditions above,* (43) *has at least two nonzero solutions of period* 2π.

Proof: $x(t) \equiv 0$ is a solution, and we seek two others as stationary points of the functional

$$F(x) = \int_0^{2\pi} \left[\frac{1}{2}|\dot{x}|^2 + V(t, x(t))\right] dt .$$

We work in the Hilbert space X of vector functions $x(t)$ having period 2π and belonging to H^1 on $[0, 2\pi]$, with the standard norm

$$\|x\| = \left[\int_0^{2\pi} |\dot{x}|^2 + |x|^2\right]^{1/2} .$$

It is easy to verify that F satisfies (PS) and is bounded below; by (iii), $\inf F < 0$. Writing any $x \in X$ in Fourier series

$$x(t) = \sum_{-\infty}^{\infty} a_j e^{ijt}, \qquad a_{-j} = \bar{a}_j ,$$

we set

$$X_2 = \left\{x = \sum_{-k}^{k} a_j e^{ijt}\right\}$$

$$X_1 = X_2^{\perp} = \left\{x \in X;\, x = \sum_{|j|>k} a_j e^{ijt}\right\}.$$

Theorem 4 will give the desired result once condition (30) is verified. Choose $R > 0$ so small that

$$\|x\| \leqq R \Rightarrow |x|_{L\infty} \leqq r .$$

Then for $x \in X_2$, $\|x\| \leqq R$,

$$F(x) = \int \frac{1}{2} |\dot{x}|^2 + V(t, x)$$

$$\leqq \pi \sum_{|j| \leqq k} j^2 |a_j|^2 - \tfrac{1}{2} k^2 \int |x|^2$$

$$= \pi \sum_{|j| \leqq k} |a_j|^2 (|j|^2 - k^2) \leqq 0 \, .$$

Similarly one verifies the first inequality of (30), and the conclusion follows from Theorem 4.

The same proof yields a slightly more general result:

THEOREM 7′. *Theorem 7 holds if condition* (iv) *is replaced by the condition*:

(iv)′ *For some* $r > 0$, *some integer* l, $0 \leqq l \leqq N$, *and for some non-negative integers* $k_1, \cdots, k_l$,

$$-\frac{1}{2} \sum_{n=1}^{l} (k_m + 1)^2 x_m^2 \leqq V(t, x_1, \cdots, x_l, 0, \cdots, 0) \leqq -\frac{1}{2} \sum_{m=1}^{l} k_m^2 x_m^2$$

$$\textit{if} \quad \sum_{1}^{l} x_m^2 \leqq r^2 \, , \quad \textit{and also}$$

$$0 \leqq V(t, 0, \cdots, 0, x_{l+1}, \cdots, x_N) \quad \textit{if} \quad \sum_{m > l} x_m^2 \leqq r^2 \, .$$

Proof: We write any Fourier coefficient a_j of x as

$$a_j = \sum_{m=1}^{N} a_j^m e_m$$

where e_m is the unit vector pointing in the positive x_m direction. Then take

$$X_2 = \left\{ x = \sum_{m=1}^{l} e_m \sum_{|j| \leqq k_m} a_j^m e^{ijt} \right\}$$

and $X_1 = X_2^{\perp}$; the proof proceeds as before.

Appendix

A Proof of Theorem 1 Based on (Deformation) Theorem 3

(i) To prove the first statement we wish to show that

$$\forall \delta < \tfrac{1}{8},\ \exists u \quad \text{such that} \quad |F(u) - c| > \delta \ \text{and}\ \|F'(u)\| < 2\sqrt{\delta}\,. \tag{A.1}$$

Taking a sequence of such $\delta_n \to 0$, the corresponding u_n have the desired properties.

Set $d(u) = \operatorname{dist}(u, p^*(K^*))$. There exist $a, C > 0$, such that

$$F(u) \leqq c + Cd(u) \quad \text{if} \quad d(u) \leqq a\,. \tag{A.2}$$

Suppose (A.1) is false. Then for some $\delta < \frac{1}{8}$, $|F(u) - c| < \delta \Rightarrow \|F'(u)\| \geqq 2\sqrt{\delta}$. Let η be the deformation in Theorem 3 corresponding to this δ. Let $p \in \mathcal{A}$ be such that $\max_{\xi \in K} F(p(\xi)) < c + \delta$. By (24) we see that

$$\eta(1, p(\xi)) \in F_{c-\delta} \qquad \forall \xi \in K\,. \tag{A.3}$$

Let $\zeta(\xi) = \min\{\max(\alpha d(p(\xi)), \operatorname{dist}(\xi, K^*)), 1\}$ where α is any constant $> \max(4C, a^{-1})$.

Consider the "path" $q \in \mathcal{A}$:

$$q(\xi) = \eta(\zeta(\xi), p(\xi)) \qquad \forall \xi \in K\,.$$

By the main condition in Theorem 1, there exists $\tilde{\xi} \in K \backslash K^*$—hence $\zeta(\tilde{\xi}) > 0$—such that

$$c \leqq F(q(\tilde{\xi})) \leqq F(p(\tilde{\xi})) < c + \delta\,.$$

By (A.3) we see that $\zeta(\tilde{\xi}) < 1$, so that $d(p(\tilde{\xi})) < \alpha^{-1} < a$. If we apply (25) of Theorem 3, with $\tau = \zeta(\tilde{\xi})$ we find

$$c \leqq F(q(\tilde{\xi})) \leqq F(p(\tilde{\xi})) - \tfrac{1}{4}\zeta(\tilde{\xi}) \leqq c + Cd(p(\tilde{\xi})) - \tfrac{1}{4}\zeta(\tilde{\xi}) \quad \text{by} \quad (\text{A.2})\,.$$

But this implies $\zeta(\tilde{\xi}) \leqq 4Cd(p(\tilde{\xi}))$ and thus $d(p(\tilde{\xi})) = 0$; so $\zeta(\tilde{\xi}) = 0$. Contradiction.

(ii) We turn to the last statement in Theorem 1. Assuming $(\text{PS})_c$, clearly c is a critical value. Consider a sequence $p_n \in \mathcal{A}$ such that $\max_{\xi \in K} F(p_n(\xi)) \to c$.

We have to show that there is a sequence $(\xi_n) \in K$ such that $F(p_n(\xi_n)) \to c$ and $\|F'(p_n(\xi_n))\| \to 0$. The desired conclusion follows easily from the following

Claim. Given any open neighbourhood $\mathcal{O}$ of K_c, there exists δ such that if $p \in \mathcal{A}$ with $\max_{\xi \in K} F(p(\xi)) < c + \delta$, then $p(K) \cap \mathcal{O} \neq \emptyset$.

The proof makes use of Corollary 5.

Proof of Claim: We will make use again of (A.2). Choose $\delta > 0$ and a deformation η to satisfy the conditions of Corollary 5, (with $\varepsilon = \frac{1}{8}$), so that (28) holds. Suppose there is a "path" $p \in \mathcal{A}$ with $\max_{\xi \in K} F(p(\xi)) < c + \delta$ and $p(K) \cap \mathcal{O} = \emptyset$. Let $0 \leqq \zeta(\xi) \leqq 1$ be the same function on K as above. By (28),

$$\eta(1, p(\xi)) \in F_{c-\delta} \qquad \forall \xi \in K. \tag{A.4}$$

Consider once more the "path" in $\mathcal{A}$:

$$q(\xi) = \eta(\zeta(\xi), p(\xi)) \qquad \forall \xi \in K.$$

As before, there exists $\tilde{\xi} \in K \backslash K^*$ (so $\zeta(\tilde{\xi}) > 0$) such that

$$c \leqq F(q(\tilde{\xi})) \leqq F(p(\tilde{\xi})).$$

By (A.4), we have $\zeta(\tilde{\xi}) < 1$, and so $d(p(\tilde{\xi})) < a$. Applying once more (25) with $\tau = \zeta(\tilde{\xi})$ we find

$$c \leqq F(q(\tilde{\xi})) \leqq F(p(\tilde{\xi})) - \tfrac{1}{4}\zeta(\tilde{\xi}) \leqq c + Cd(p(\tilde{\xi})) - \tfrac{1}{4}\zeta(\tilde{\xi}) \quad \text{by} \quad (A.2).$$

This leads to a contradiction as above.

Some time after completion of this paper we learned that Corollary 1 of Proposition 1, and a more general form, were proved using Ekeland's principle by L. Caklovic, S. J. Li, and M. Willem; see [18]. In addition we learned that in [19], M. Willem had presented a variant of our Theorem 3.

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