A Minimization Problem with Critical Exponent and Nonzero Data

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Dedicated to Luigi Radicati on his 70th birthday

Let Ω be a smooth bounded domain in $\mathbb{R}^n$ with $n \geq 3$. Given $\varphi \in L^q(\Omega)$, consider the following minimization problem

$$J = \inf_{u \in H_0^1} \int_\Omega |\nabla u|^2 \tag{1}$$

$$\|u + \varphi\|_q = \gamma$$

where $\| \ \|_q$ denotes the norm in $L^q(\Omega)$, $\gamma > 0$ is a constant and $q = 2n/(n-2)$ is the limiting exponent for the Sobolev embedding. It is well-known that the infimum in (1) is not achieved if $\varphi = 0$ (see e.g. [B]). Our main result is:

THEOREM 1. *Assume* $\varphi \not\equiv 0$. *Then the infimum in* (1) *is achieved.*

COROLLARY. *Let* $f \in H^{-1}(\Omega)$, $f \not\equiv 0$ *then*

$$\min_{\substack{v \in H_0^1 \\ \|v\|_q = \gamma}} \int_\Omega \left[\frac{1}{2} |\nabla v|^2 - fv \right] \tag{2}$$

is achieved.

Indeed if φ is the solution in H_0^1 of $-\Delta\varphi = f$, then the problem (2) with $v - \varphi = u$ corresponds to the problem (1).

REMARK 1. Any minimizer for (1) satisfies the equation $-\Delta u = \mu |u + \varphi|^{q-2}(u + \varphi)$ for some Lagrange multiplier μ. In case $\varphi \geq 0$, $\|\varphi\|_q < \gamma$, every minimizer u is ≥ 0 (otherwise replace u by $t|u|$ for a suitable positive $t < 1$). The corresponding μ is positive. If $\varphi \geq 0$ and $\|\varphi\|_q > \gamma$ then every minimizer is ≤ 0 (otherwise replace u by $-tu^-$ for a suitable positive $t < 1$). Moreover

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we can always find a minimizer u for which the corresponding μ is negative; we do not know whether this is true for every minimizer.

REMARK 2. The conclusion of Theorem 1 had been obtained earlier by Caffarelli-Spruck [CS] when $\varphi > 0$, φ is harmonic and in $C^{1+\beta}(\Omega)$ - using delicate estimates developed in [CGS]. Positivity is essential in their approach. Our procedure is quite different from theirs. It is related to the one we used in [BN]; it makes use of the best Sobolev constant:

$$S = \inf_{u \in H_0^1} \int_\Omega |\nabla u|^2$$

$$\|u\|_q = 1$$

and the extremal functions for the Sobolev inequality in $\mathbb{R}^n$:

$$U_\epsilon(x) = \frac{\epsilon^{(n-2)/2}}{(\epsilon^2 + |x|^2)^{(n-2)/2}}, \quad \epsilon > 0.$$

Let (u_j) be a minimizing sequence for (1). We may always assume that $u_j \rightharpoonup u_0$ weakly in H^1 and $u_j \to u_0$ a.e. Set $u_0 + \varphi = w$. Clearly, we have $\|w\|_q \leq \gamma$; to prove Theorem 1 it suffices to show that

$$\|w\|_q = \gamma.$$

This will be proved with the aid of several lemmas.

LEMMA 1. *We have*

$$J - \int |\nabla u_0|^2 \geq S \left[\gamma^q - \int |u_0 + \varphi|^q\right]^{2/q}. \tag{3}$$

PROOF. Set $v_j = u_j - u_0$ so that $v_j \rightharpoonup 0$ weakly in H_0^1 and a.e. As $j \to \infty$ we have

$$J + o(1) = \int |\nabla u_j|^2 = \int |\nabla u_0|^2 + \int |\nabla v_j|^2 + o(1). \tag{4}$$

On the other hand, we recall (see [BL]) that

$$\gamma^q = \int |w + v_j|^q = \int |w|^q + \int |v_j|^q + o(1). \tag{5}$$

Applying Sobolev's inequality to v_j we have

$$\int |\nabla v_j|^2 \geq S\|v_j\|_q^2 \tag{6}$$

and, therefore, using (4), (5) and (6) we obtain

$$J - \int |\nabla u_0|^2 \geq S\left[\gamma^q - \int |w|^q\right]^{2/q}.$$

LEMMA 2. *Let $u \in H_0^1$ be any function such that $\|u+\varphi\|_q < \gamma$. Then*

$$J - \int |\nabla u|^2 \leq S\left[\gamma^q - \int |u+\varphi|^q\right]^{2/q}.$$

PROOF. We may always assume that $0 \in \Omega$ and let $\zeta \in C_0^\infty(\Omega)$ be any function such that $\zeta \equiv 1$ near $x = 0$. Since $\|u+\varphi\|_q < \gamma$ there is a (unique) constant $c_\epsilon > 0$ such that

$$\|u + \varphi + c_\epsilon \zeta U_\epsilon\|_q = \gamma. \tag{7}$$

Using [BL] once more, we have, as $\epsilon \to 0$,

$$\gamma^q = \int |u+\varphi|^q + c_\epsilon^q \int |\zeta U_\epsilon|^q + o(1).$$

On the other hand recall (see [BN]) that

$$\int |\zeta U_\epsilon|^q = A + O(\epsilon^n) \tag{8}$$

where $A = \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^n} = \int_{\mathbb{R}^n} U_1^q \, dx$ and therefore

$$c_\epsilon^q = \frac{1}{A}\left[\gamma^q - \int |u+\varphi|^q\right] + o(1). \tag{9}$$

Finally we have, since $\zeta U_\epsilon \rightharpoonup 0$ weakly in H^1,

$$J \leq \int |\nabla(u + c_\epsilon \zeta U_\epsilon)|^2 = \int |\nabla u|^2 + c_\epsilon^2 \int |\nabla(\zeta U_\epsilon)|^2 + o(1) \tag{10}$$

and we also recall (see [BN]) that

$$\int |\nabla(\zeta U_\epsilon)|^2 = B + O(\epsilon^{n-2}) \tag{11}$$

where

$$B = \int_{\mathbb{R}^n} |\nabla U_1|^2;$$

so

$$S = BA^{-2/q}.$$

Putting together (9), (10) and (11) we obtain

$$J - \int |\nabla u|^2 \leq \frac{B}{A^{2/q}} \left[\gamma^q - \int |u+\varphi|^q\right]^{2/q} = S\left[\gamma^q - \int |u+\varphi|^q\right]^{2/q}.$$

Throughout the rest of the paper we will argue by contradiction assuming that

$$\|w\|_q = \|u_0 + \varphi\|_q < \gamma. \tag{12}$$

From Lemmas 1 and 2 we deduce that

$$J - \int |\nabla u_0|^2 = S\left[\gamma^q - \int |w|^q\right]^{2/q}. \tag{13}$$

LEMMA 3. *The function u_0 satisfies (in the weak sense)*

$$-\Delta u_0 = \lambda |u_0 + \varphi|^{q-2}(u_0 + \varphi) \tag{14}$$

with

$$\lambda = S\left[\gamma^q - \int |u_0+\varphi|^q\right]^{(2-q)/q}. \tag{15}$$

PROOF. Let $\zeta \in C_0^\infty(\Omega)$. Applying Lemma 2 to $u = (u_0 + t\zeta)$ with $|t|$ sufficiently small we have

$$\begin{aligned} J - \int |\nabla u_0|^2 - 2t \int \nabla u_0 \nabla \zeta + o(t) \\ \leq S\left[\gamma^q - \int |w|^q - qt \int |w|^{q-2} w\zeta + o(t)\right]^{2/q} \\ = S\left[\gamma^q - \int |w|^q\right]^{2/q} \left[1 - \frac{2t}{(\gamma^q - \int |w|^q)} \int |w|^{q-2} w\zeta + o(t)\right] \end{aligned}$$

and therefore, by (13),

$$\int \nabla u_0 \nabla \zeta = S\left[\gamma^q - \int |w|^q\right]^{(2-q)/q} \int |w|^{q-2} w\zeta.$$

In addition we make use of:

LEMMA 4. *Assume $1 \leq q \leq 3$. Then there is a constant C (depending on q) such that, for all $\alpha, \beta \in \mathbb{R}$,*

$$\left||\alpha+\beta|^q - |\alpha|^q - |\beta|^q - q\alpha\beta(|\alpha|^{q-2} + |\beta|^{q-2})\right| \leq \begin{cases} C|\alpha|\,|\beta|^{q-1} & \text{if } |\alpha| \geq |\beta| \\ C|\alpha|^{q-1}|\beta| & \text{if } |\alpha| \leq |\beta|. \end{cases}$$

When $q \geq 3$ there is a constant C (dependending on q) such that for all $\alpha, \beta \in \mathbb{R}$,

$$\left| |\alpha+\beta|^q - |\alpha|^q - |\beta|^q - q\alpha\beta(|\alpha|^{q-2} + |\beta|^{q-2}) \right| \leq C(|\alpha|^{q-2}\beta^2 + \alpha^2|\beta|^{q-2}).$$

PROOF. If $1 \leq q \leq 3$ it suffices to check the inequality

$$\left| |1+t|^q - 1 - |t|^q - qt(|t|^{q-2} + 1) \right| \leq C|t|^{q-1}, \quad \forall t \in [-1, +1],$$

which is easily established since $|t|^2 \leq |t|^{q-1}$. When $q \geq 3$ the inequality reduces to

$$\left| |1+t|^q - 1 - |t|^q - qt(|t|^{q-2} + 1) \right| \leq C(|t|^{q-2} + |t|^2), \ \forall t \in [-1, +1]$$

which clearly holds.

PROOF OF THEOREM 1. We cannot have $u_0 + \varphi \equiv 0$, for then it would follow from (14) that $u_0 \equiv 0$ and so $\varphi \equiv 0$; but $\varphi \not\equiv 0$. Since we may replace φ by $-\varphi$, and hence u_0 by $-u_0$, we may assume, without loss of generality, that $w > 0$ in a set Σ of positive measure in a ball $B(0, \rho/2) \subset B(0, \rho) \subset \Omega$. Choose $\zeta \in C_0^\infty(\Omega)$ such that $\zeta \equiv 1$ on $B(0, \rho/2)$ and $\zeta \equiv 0$ outside $B(0, \rho)$. Later we will fix a point a in Σ; for any such a, denote $U_\epsilon(x-a)$ by $U_{\epsilon,a}(x)$ and $\zeta(x-a)$ by ζ_a.

Fix $c_\epsilon > 0$ such that $\|w + c_\epsilon \zeta_a U_{\epsilon,a}\|_q = \gamma$. Clearly c_ϵ also depends on a but we will not indicate that explicitly; $c'' \geq c_\epsilon \geq c' > 0$ independent of ϵ and a. Our goal is to show that for suitable $a \in \Sigma$, as $\epsilon \to 0$,

$$\begin{aligned} J &\leq \int |\nabla(u_0 + c_\epsilon \zeta_a U_{\epsilon,a})|^2 \\ &= \int |\nabla u_0|^2 + S\left[\gamma^q - \int |w|^q\right]^{2/q} - \delta\epsilon^{(n-2)/2} + o(\epsilon^{(n-2)/2}) \end{aligned} \tag{16}$$

for some positive constant δ. This will contradict (3) in Lemma 1 and complete the proof of Theorem 1.

To establish (16) we need an expansion for c_ϵ as $\epsilon \to 0$. Applying Lemma 4 with $\alpha = w$ and $\beta = c_\epsilon \zeta_a U_{\epsilon,a}$ we obtain

$$\begin{aligned} \gamma^q &= \int |w + c_\epsilon \zeta_a U_{\epsilon,a}|^q \\ &= \int |w|^q + c_\epsilon^q \int |\zeta_a U_{\epsilon,a}|^q + qc_\epsilon \int |w|^{q-2} w \zeta_a U_{\epsilon,a} + \\ &\quad + qc_\epsilon^{q-1} \int w|\zeta_a U_{\epsilon,a}|^{q-1} + (R_\epsilon + S_\epsilon), \end{aligned} \tag{17}$$

where R_ϵ and S_ϵ (which also depend on a) are estimated as follows:

CASE 1. $1 < q \leq 3$ (i.e. $n \geq 6$)

$$|R_\epsilon| \leq C \int_{\{|w| \geq c_\epsilon \zeta_a U_{\epsilon,a}\}} |w| c_\epsilon^{q-1} |\zeta_a U_{\epsilon,a}|^{q-1}$$

and

$$|S_\epsilon| \leq C \int_{\{|w| < c_\epsilon \zeta_a U_{\epsilon,a}\}} |w|^{q-1} c_\epsilon \zeta_a U_{\epsilon,a}$$

In this case we claim that for almost all a in Σ,

$$R_\epsilon = o(\epsilon^{(n-2)/2}) \tag{18}$$

and

$$S_\epsilon = o(\epsilon^{(n-2)/2}). \tag{19}$$

CASE 2. $q > 3$ (i.e. $n < 6$)

$$|R_\epsilon| \leq C \int |w|^{q-2} c_\epsilon^2 \zeta_a^2 U_{\epsilon,a}^2$$

and

$$|S_\epsilon| \leq C \int |w|^2 c_\epsilon^{q-2} \zeta_a^{q-2} U_{\epsilon,a}^{q-2}.$$

In this case we claim that for almost every a in Σ,

$$R_\epsilon = \begin{cases} O(\epsilon) & \text{if } n = 3 \\ O(\epsilon^{2\theta}) \quad \forall \theta < 1 & \text{if } n = 4 \\ O(\epsilon^2) & \text{if } n = 5 \end{cases} \tag{20}$$

and

$$S_\epsilon = \begin{cases} O(\epsilon) & \text{if } n = 3 \\ O(\epsilon^{2\theta}) \quad \forall \theta < 1 & \text{if } n = 4 \\ O(\epsilon^2) & \text{if } n = 5. \end{cases} \tag{21}$$

It follows that, in *all cases*, for almost all $a \in \Sigma$,

$$R_\epsilon, S_\epsilon = o(\epsilon^{(n-2)/2}). \tag{22}$$

PROOF OF (18).

$$|R_\epsilon| \leq C \int_{\{|w| \geq c_\epsilon \zeta_a U_{\epsilon,a}\}} |w| \ |U_{\epsilon,a}|^{q-1}.$$

Dividing the region of integration into $|x-a| < \rho/2$ and $\rho/2 \le |x-a|$ we see that

$$|R_\epsilon| \le C \int_{\{|w| \ge c' U_{\epsilon,a}\}} |w|\ |U_{\epsilon,a}|^{q-1} + C \int_\Omega |w| \epsilon^{(n+2)/2}.$$

Write

$$|U_{\epsilon,a}|^{q-1} = U_{\epsilon,a}^\alpha U_{\epsilon,a}^\beta \text{ with } \alpha+\beta = q-1,\ \alpha,\beta > 0.$$

Then we have

$$|R_\epsilon| \le C \int_\Omega |w|^{1+\alpha} U_{\epsilon,a}^\beta + C\ \epsilon^{(n+2)/2}$$

$$\le C \int_\Omega |w|^{1+\alpha}\ \frac{\epsilon^{\beta(n-2)/2}}{|x-a|^{(n-2)\beta}} + C\ \epsilon^{(n+2)/2}.$$

Choose any $0 < \beta < n/(n-2)$. It follows that $1+\alpha = q-\beta < q$ and thus $|w|^{1+\alpha}$ and $|x|^{(2-n)\beta}$ are in L^1. Since the convolution of two L^1 functions is finite a.e. we have

$$\int |w|^{1+\alpha}\ \frac{1}{|x-a|^{(n-2)\beta}} < \infty \text{ for almost every } a$$

and hence

$$R_\epsilon = O(\epsilon^{(n/2)\theta})\ \forall \theta < 1, \tag{23}$$

proving (18).

PROOF OF (19).

$$|S_\epsilon| \le \int_{\{|w| \le c'' U_{\epsilon,a}\}} |w|^{q-1} U_{\epsilon,a}.$$

Write

$$|w|^{q-1} = |w|^\alpha\ |w|^\beta \text{ with } \alpha+\beta = q-1,\ \alpha,\beta > 0.$$

Then

$$|S_\epsilon| \le C \int |w|^\alpha U_{\epsilon,a}^{1+\beta}$$

$$\le \int |w|^\alpha\ \frac{\epsilon^{((n-2)/2)(1+\beta)}}{|x-a|^{(n-2)(1+\beta)}}.$$

If we choose $\beta > 0$ with $1+\beta < n/(n-2)$, then $\alpha = q-1-\beta < q$, and so w^α and $|x|^{(2-n)(1+\beta)}$ are in L^1. Thus for almost every a,

$$\int |w|^\alpha\ \frac{dx}{|x-a|^{(n-2)(1+\beta)}} < \infty,$$

and

$$S_\epsilon = O(\epsilon^{(n/2)\theta}) \quad \forall \theta < 1, \tag{24}$$

proving (19).

PROOF OF (20) AND (21).

(a) $n = 3$, so $q = 6$.

$$|R_\epsilon| \le C\epsilon \int |w|^4 \frac{\zeta^2(x-a)}{|x-a|^2}\, dx.$$

Since the integral is the convolution of two L^1 functions it follows that

$$R_\epsilon = O(\epsilon)$$

for almost every a in $B(0, \rho/2)$.

Next

$$|S_\epsilon| \le C\epsilon \int_{R^n} w^2 \frac{\epsilon}{(\epsilon^2 + |x-a|^2)^2}$$

(it is understood that $w = 0$ outside Ω). For a suitable constant $c > 0$ the kernel

$$\frac{c\epsilon}{(\epsilon^2 + |x-a|^2)^2}$$

corresponds to that of a mollifer (its integral = 1). Hence for almost all a (see e.g. [F], p. 235)

$$\int_{R^n} w^2 \frac{c\epsilon}{(\epsilon^2 + |x-a|^2)^2} \to w^2(a),$$

and

$$S_\epsilon = O(\epsilon).$$

(b) $n = 4$, so $q = 4$.

$$|R_\epsilon|,\ |S_\epsilon| \le C \int w^2 \frac{\epsilon^2}{(\epsilon^2 + |x-a|^2)^2}.$$

Write

$$\frac{1}{(\epsilon^2 + |x-a|^2)^2} = \frac{1}{(\epsilon^2 + |x-a|^2)^\alpha} \frac{1}{(\epsilon^2 + |x-a|^2)^\beta}$$

with $\alpha + \beta = 2$, $\alpha, \beta > 0$; then

$$\frac{1}{(\epsilon^2 + |x-a|^2)^2} \le \frac{1}{\epsilon^{2\alpha}} \frac{1}{|x-a|^{2\beta}}.$$

Consequently

$$|R_\epsilon|,\ |S_\epsilon| \le \epsilon^{2-2\alpha} \int \frac{w^2}{|x-a|^{2\beta}}.$$

As before, for almost every a the integral is finite, and so for almost every a,

$$|R_\epsilon|,\ |S_\epsilon| = O(\epsilon^{2\theta}) \quad \forall \theta < 1.$$

(c) $n = 5,\ q = 10/3.$

$$|R_\epsilon| \le C\epsilon^2 \int |w|^{q-2} \frac{\epsilon}{(\epsilon^2 + |x-a|^2)^3}\, dx.$$

For a suitable constant c the kernel

$$\frac{c\epsilon}{(\epsilon^2 + |x-a|^2)^3}$$

corresponds to a mollifier. Hence for almost all a the last integral converges to $c^{-1}|w(a)|^{q-2}$, and so for almost all a,

$$R_\epsilon = O(\epsilon^2).$$

Next

$$|S_\epsilon| \le C\epsilon^2 \int |w|^2 \frac{\varsigma^{4/3}(x-a)}{|x-a|^4}\, dx.$$

Thus as in case (a) we see that

$$S_\epsilon = O(\epsilon^2)$$

for almost every a. (20) and (21) are proved.

Next we deal with the terms $\int |w|^{q-2} w \varsigma_a U_{\epsilon,a}$ and $\int w|\varsigma_a U_{\epsilon,a}|^{q-1}$ which appear in (17). For almost every a

$$\int |w|^{q-1} \frac{\varsigma(x-a)}{|x-a|^{n-2}}\, dx < \infty$$

and thus, for almost every a, we have (by dominated convergence)

$$\int |w|^{q-2} w \frac{\varsigma(x-a)}{(\epsilon^2 + |x-a|^2)^{(n-2)/2}} \longrightarrow \int |w|^{q-2} w \frac{\varsigma(x-a)}{|x-a|^{n-2}}\, dx$$

as $\epsilon \to 0$. Hence for every such a in Σ,

$$\int |w|^{q-2} w \varsigma_a U_{\epsilon,a} = \epsilon^{(n-2)/2} \int |w|^{q-2} w \frac{\varsigma_a}{|x-a|^{n-2}} + o(\epsilon^{(n-2)/2}). \tag{25}$$

Finally

$$\int w|\varsigma_a U_{\epsilon,a}|^{q-1} = \epsilon^{(n-2)/2} \int w \varsigma_a^{q-1} \frac{\epsilon^2\, dx}{(\epsilon^2 + |x-a|^2)^{(n+2)/2}}.$$

For $D = \int_{R^n} \frac{dx}{(1+|x|^2)^{(n+2)/2}}$ we claim that for almost every $a \in \Sigma$,

$$\int w|\zeta_a U_{\epsilon,a}|^{q-1} = \epsilon^{(n-2)/2} w(a) D + o(\epsilon^{(n-2)/2}). \tag{26}$$

Indeed the integral equals

$$\int w(\zeta^{q-1}(x-a) - 1)\ U_\epsilon^{q-1} + \epsilon^{(n-2)/2} \int w\ \frac{\epsilon^2}{(\epsilon^2 + |x-a|^2)^{(n+2)/2}}.$$

The first of these integrals is $O(\epsilon^{(n+2)/2})$, while the second integral, as before, tends to $w(a)D$ for almost every a. So (26) is proved.

Combining (17), (8), (22), (25) and (26) we obtain for almost all a in Σ,

$$\gamma^q = \int |w|^q + c_\epsilon^q A + q\epsilon^{(n-2)/2} \cdot \left[c_\epsilon \int |w|^{q-2} w\ \frac{\zeta_a}{|x-a|^{n-2}} + c_\epsilon^{q-1} w(a) D \right] + o(\epsilon^{(n-2)/2}). \tag{27}$$

Writing

$$c_\epsilon = c_0(1-\delta) \text{ with } c_0^q = \frac{1}{A}\left[\gamma^q - \int |w|^q\right] \tag{28}$$

we deduce from (27) that

$$\delta A c_0^q = \epsilon^{(n-2)/2} \left[c_0 \int |w|^{q-2} w\ \frac{\zeta_a}{|x-a|^{n-2}} + c_0^{q-1} w(a) D \right] + o(\epsilon^{(n-2)/2}). \tag{29}$$

Finally we estimate

$$T_\epsilon = \int |\nabla(u_0 + c_\epsilon \zeta_a U_\epsilon)|^2 = \int |\nabla u_0|^2 - 2\int (\Delta u_0) c_\epsilon \zeta_a U_\epsilon + c_\epsilon^2 \int |\nabla(\zeta_a U_\epsilon)|^2.$$

By (14), $\Delta u_0 \in L^1$. Applying (11), (28) and (29) we find, as in the proof of (25), for some a in Σ, which we may now fix,

$$T_\epsilon = \int |\nabla u_0|^2 - 2c_0\epsilon^{(n-2)/2} \int (\Delta u_0)\ \frac{\zeta_a}{|x-a|^{n-2}} + c_0^2(1-2\delta)B + o(\epsilon^{(n-2)/2}).$$

Thus, by (29), we have

$$T_\epsilon = \int |\nabla u_0|^2 + S\left[\gamma^q - \int |w|^q\right]^{2/q} - \delta\epsilon^{(n-2)/2} + o(\epsilon^{(n-2)/2})$$

with

$$\delta = 2c_0 \int (\Delta u_0) \frac{\zeta_a}{|x-a|^{n-2}} + \\ + 2c_0^{2-q}\frac{B}{A}\left[c_0 \int |w|^{q-2}w \frac{\zeta_a}{|x-a|^{n-2}} + c_0^{q-1}w(a)D\right] \\ = 2c_0 \frac{BD}{A}w(a) > 0$$

since $-\Delta u_0 = \lambda |w|^{q-2}w$ and $\lambda = c_0^{2-q}\frac{B}{A}$ by Lemma 3. This proves (16) and hence Theorem 1.

REMARK 3. Problem (1) is clearly related to the boundary value problem

$$(30) \qquad \begin{aligned} -\Delta u &= \mu|u+\varphi|^{q-2}(u+\varphi) && \text{in } \Omega \\ u &= 0 && \text{in } \partial\Omega \end{aligned}$$

where μ is fixed (and $\|u+\varphi\|_q$ is not prescribed). For this problem we have the following results in case $\varphi \geq 0$, $\varphi \not\equiv 0$ (and, for convenience, φ sufficiently smooth):

(a) For every $\mu \leq 0$, there is a unique solution $u = u(\mu)$ of (30). It is obtained as the unique minimizer of an associated convex Lagrangian. One easily sees that $u \leq 0$. Moreover, the function $f(\mu) = \|u(\mu)+\varphi\|_q$ is continuous, strictly increasing on $(-\infty, 0]$, and $\lim_{\mu \to -\infty} f(\mu) = 0$, $f(0) = \|\varphi\|_q$. This implies the last assertion of Remark 1.

(b) For $\mu > 0$ we look only for positive solutions - though we believe others exist. We have the following:

There is a positive $\mu^* < \infty$ such that for every $\mu \in [0, \mu^*]$ there is a least positive (smooth) solution $\underline{u}(\mu)$. There is no positive solution of (30) for $\mu > \mu^*$. This curve of solutions is obtained by applying the Implicit Function Theorem starting at $\underline{u}(0) = 0$. Moreover for $\mu \in [0, \mu^*)$, the first eigenvalue of the linearized problem at $\underline{u}(\mu)$ is positive, and $\underline{u}(\mu)$ is increasing in μ. Therefore, for $\gamma > \|\underline{u}(\mu^*)+\varphi\|_q$ a solution of (1) is not among the $\underline{u}(\mu)$. Hence, for some values of $\mu \in (0, \mu^*]$ there is more than one positive solution of (30). In fact, this is true for every $\mu \in (0, \mu^*)$:

(c) For every $\mu \in (0, \mu^*)$ there exists a smooth solution u of (30) with $u > \underline{u}(\mu)$ in Ω. This follows from the fact that the first eigenvalue of the linearized problem at $\underline{u}(\mu)$ is positive, using the Mountain Pass Lemma as in [BN], Section 2. The proof of smoothness, also of $\underline{u}(\mu^*)$, is nontrivial.

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