Partial Differential Equations
and the Calculus of Variations, Volume I

Essays in Honor of Ennio De Giorgi

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Edited by
F. Colombini
A. Marino
L. Modica
S. Spagnolo

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ASYMPTOTICS FOR ELLIPTIC EQUATIONS INVOLVING CRITICAL GROWTH

HAÏM BREZIS
LAMBERTUS A. PELETIER

Dedicated to Ennio De Giorgi on his sixtieth birthday

1. Introduction. Consider the problem

\begin{align*}
\begin{cases}
-\Delta u - \lambda u &= 3u^{5-\epsilon} \quad \text{in } \Omega \\
 u &= 0 \quad \text{on } \partial\Omega \\
 u &> 0 \quad \text{in } \Omega 
\end{cases}
(1.1)
\end{align*}

where $\Omega$ is the unit ball in $\mathbb{R}^3$, $\lambda \geq 0$ and $\epsilon \geq 0$. It is well known that if $\epsilon > 0$, Problem (I) has a solution $u_\epsilon$ for any $\lambda < \lambda_1 = \pi^2$. On the other hand, if $\epsilon = 0$ Problem (I) has a solution if and only if $\pi^2/4 < \lambda < \pi^2$ (See [BN]).

In this paper we return to the question, first studied in [AP2], of the asymptotic behaviour of $u_\epsilon$ as $\epsilon \to 0$. There it was shown that if $\lambda = 0$, any solution $u_\epsilon$ of Problem (I) has the following limiting behaviour:

\begin{align*}
\lim_{\epsilon \to 0} \epsilon u_\epsilon^2(0) &= \frac{32}{\pi} 
(1.2)
\end{align*}

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* This work was done whilst the first author held the Kloosterman Chair of the Mathematical Institute of the University of Leiden.
and at any \( x \in \Omega \setminus \{0\} \):

\[
\lim_{\varepsilon \to 0} \varepsilon^{-1/2} u_\varepsilon(x) = \frac{1}{4} \sqrt{\frac{\pi}{2}} \left( \frac{1}{|x|} - 1 \right).
\]

We return to the study of \( u_\varepsilon \) for two reasons. First we shall give a different proof of (1.2) and (1.3), which is mainly based on PDE methods and – we hope – will enable us in due course to handle non-spherical domains. The second reason is that the method of [AP2], which readily extends to the case \( 0 < \lambda < \pi^2/4 \), cannot be applied when \( \lambda = \pi^2/4 \). Here, a new phenomenon occurs, which was first discovered by Budd [Bu]. A formal computation, based on the method of matched asymptotic expansions, suggests that

\[
\lim_{\varepsilon \to 0} \varepsilon^{-1} u_\varepsilon^4(0) = 8\pi^2.
\]

The method presented here does apply to the case \( \lambda = \pi^2/4 \) and provides a rigorous proof of (1.4).

As in [AP2], the method we use to prove (1.2) – (1.4) is based on estimating the different terms in the Pohozaev identity for Problem (1):

\[
\frac{3\varepsilon}{6 - \varepsilon} \int_\Omega u_\varepsilon^{6-\varepsilon} = J(u_\varepsilon),
\]

where

\[
J(w) = \int_{\partial\Omega} (x, n) \left( \frac{\partial w}{\partial n} \right)^2 - 2\lambda \int_\Omega w^2.
\]

Here \( n \) denotes the outward normal on \( \partial\Omega \).

Writing \( \mu = \left( u_\varepsilon(0) \right)^{-2} \), we shall establish that

\[
\mu^{-1/2} u_\varepsilon \to 4\pi G_\lambda \quad \text{as} \quad \varepsilon \to 0,
\]

where \( G_\lambda \) is the Green’s function of \(-\Delta - \lambda \), i.e. \( G_\lambda \) solves

\[
-\Delta G - \lambda G = \delta_0 \quad \text{in} \quad \Omega
\]

\[
G = 0 \quad \text{on} \quad \partial\Omega
\]

in which \( \delta_0 \) is the Dirac mass centered at the origin. By elliptic regularity theory (1.6) implies that

\[
\frac{1}{\mu} J(u_\varepsilon) \to J(4\pi G_\lambda).
\]
On the other hand, we shall establish in Section 4 that

\[ J(G_\lambda) = -g_\lambda(0) \]

so that

\[ J(u_\epsilon) = -16\pi^2 g_\lambda(0)\mu + o(\mu) \quad \text{as} \quad \epsilon \to 0. \]

Here

\[ g_\lambda(x) = G_\lambda(x) - \frac{1}{4\pi|x|}, \]

i.e. \( g_\lambda \) is the regular part of the Green's function.

Finally we check that

\[ \int u_\epsilon^{\delta - \epsilon} \to \frac{\pi^2}{4} \quad \text{as} \quad \epsilon \to 0 \]

and so, putting (1.9) and (1.11) into (1.5) we obtain that

\[ \frac{\epsilon}{\mu} \to -\frac{32}{\pi} \cdot 4\pi g_\lambda(0) \quad \text{as} \quad \epsilon \to 0 \]

or, since

\[ g_\lambda(x) = \frac{1}{4\pi|x|} \left\{ \cos(\sqrt{\lambda}|x|) - \frac{\sin(\sqrt{\lambda}|x|)}{\tan(\sqrt{\lambda})} - 1 \right\}, \]

this yields our first result:

**Theorem 1.** Let \( u_\epsilon \) be a solution of Problem (I).

(a) If \( 0 \leq \lambda \leq \pi^2/4 \), then

\[ \lim_{\epsilon \to 0} \epsilon u_\epsilon^2(0) = \frac{32}{\pi} \frac{\sqrt{\lambda}}{\tan(\sqrt{\lambda})}. \]

(b) If \( 0 \leq \lambda < \pi^2/4 \), then at any \( x \neq 0 \),

\[ \lim_{\epsilon \to 0} \epsilon^{-1/2} u_\epsilon(x) = \left( \frac{\pi^3}{2} \cdot \frac{\tan(\sqrt{\lambda})}{\sqrt{\lambda}} \right)^{1/2} G_\lambda(x), \]

where we define \( \sqrt{\lambda}/\tan(\sqrt{\lambda}) = 1 \) if \( \lambda = 0 \). If \( \lambda = \pi^2/4 \), the right hand side of (1.13) vanishes and all we can conclude is that

\[ \epsilon u_\epsilon^2(0) = o(1) \quad \text{as} \quad \epsilon \to 0. \]
To obtain a precise estimate, we need a better global asymptotic approximation of $u_\varepsilon$ than is given by (1.6). Hereby the following family of functions plays a central role

\begin{equation}
U_\mu(x) = \left( \frac{\mu}{\mu^2 + |x|^2} \right)^{1/2};
\end{equation}

it satisfies the equation

\[-\Delta u = 3u^5 \text{ in } \mathbb{R}^3.\]

The function

\[\phi_\mu = U_\mu + 4\pi \mu^{1/2} g_\lambda\]

turns out to be the required approximation of $u_\varepsilon$. In Section 6 we shall establish that

\begin{equation}
J(u_\varepsilon) = -16\pi^2 \ g_\lambda(0)\mu + 4\pi^2 \mu^2 + O(\varepsilon \mu |\log \mu| + \mu^3 |\log \mu|) \text{ as } \varepsilon \to 0.
\end{equation}

Putting (1.15), together with (1.11), in (1.5) again and using the fact that $g_\lambda(0) = 0$ when $\lambda = \pi^2/4$ we obtain our second result.

**Theorem 2.** Let $u_\varepsilon$ be a solution of Problem (I) in which $\lambda = \pi^2/4$. Then

\[(a) \quad \lim_{\varepsilon \to 0} \varepsilon u_\varepsilon^4(0) = 8\pi^2.\]

\[(b) \quad \lim_{\varepsilon \to 0} \varepsilon^{-1/4} u_\varepsilon(x) = (8\pi^2)^{-1/4} \frac{\cos\left(\frac{\pi}{2} |x|\right)}{|x|}, \ x \neq 0.\]

An important ingredient in the proof of Theorem 2, which is of some interest in its own right, is a Pohozaev-type identity for the Green's function. This identity, which is valid for arbitrary bounded domains $\Omega$ in $\mathbb{R}^3$ is derived in Section 4. Some other integral identities involving the Green's function are also established in this section. The proof of Theorem 2 is subsequently given in Sections 5 and 6.

In Section 7, we consider a different, but related problem:

\[(II) \quad \begin{cases} 
-\Delta u - \left(\frac{\pi^2}{4} + \varepsilon\right) u = 3u^5 & \text{in } \Omega \\
u > 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega,
\end{cases}\]
where \( \Omega \) is again the unit ball in \( \mathbb{R}^3 \). It is known [BN] that if \( 0 < \varepsilon < 3\pi^2/4 \), Problem (II) has a solution \( u_\varepsilon \), and that, as with Problem (I), \( u_\varepsilon(0) \to \infty \) as \( \varepsilon \to 0 \).

With the machinery built for dealing with Problem (I), we can now readily establish the asymptotic properties of \( u_\varepsilon \) both at the origin, and away from the origin, as \( \varepsilon \to 0 \).

**Theorem 3.** Let \( u_\varepsilon \) be a solution of Problem (II). Then

\[
\begin{align*}
(a) \quad \lim_{\varepsilon \to 0} \varepsilon u_\varepsilon^2(0) &= \frac{\pi^3}{2}.
(b) \quad \lim_{\varepsilon \to 0} \varepsilon^{-1/2} u_\varepsilon(x) &= 4\sqrt{2} G_{\pi^2/4}(x), \quad x \neq 0.
\end{align*}
\]

Many of the arguments and results in this paper continue to hold when the domain \( \Omega \) is not a ball. They furnish insight in the behaviour of solutions \( u_\varepsilon \) of Problems (I) and (II) as \( \varepsilon \to 0 \) when \( \Omega \) is a general domain in \( \mathbb{R}^3 \) or \( \mathbb{R}^N (N > 2) \) and enable us to formulate a number of conjectures about the behaviour. This is done in Section 8.

2. Preliminary bounds. Since uniqueness for Problem (I) is not known, we shall always take \( u_\varepsilon \) to be any solution of Problem (I). Because \( u_\varepsilon > 0 \), it is known to be radially symmetric and decreasing [GNN].

As a first observation, note that if \( 0 \leq \lambda \leq \pi^2/4 \), \( u_\varepsilon(0) \to \infty \) as \( \varepsilon \to 0 \). For suppose to the contrary that there exists a sequence \( \{\varepsilon_n\} \), \( \varepsilon_n \to 0 \) as \( n \to \infty \) such that \( u_{\varepsilon_n}(0) \) remains bounded as \( n \to \infty \). Then \( u_{\varepsilon_n} \) remains bounded in \( L^\infty(\Omega) \) and, in view of the elliptic regularity theory applied to (1.1), \( u_{\varepsilon_n} \) remains bounded in \( C^1(\bar{\Omega}) \). So we can extract a subsequence, still denoted by \( \{u_{\varepsilon_n}\} \), which converges uniformly to a limit \( v \), which satisfies

\[
\begin{cases}
-\Delta v - \lambda v = 3v^5, & v \geq 0 \quad \text{in } \Omega \\
v = 0 & \text{on } \partial\Omega.
\end{cases}
\]

This implies that \( v = 0 \) because \( 0 \leq \lambda \leq \pi^2/4 \) [BN].

On the other hand, we assert that

\[(2.1) \quad \| u_\varepsilon \|_{\infty} \geq \kappa > 0 \]

for some constant \( \kappa > 0 \) independent of \( \varepsilon \), which contradicts the conclusion drawn above. To prove (2.1), we multiply (1.1) by the
principal eigenfunction \( \phi_1 \) of \(-\Delta\) (chosen positive) and integrate by parts. This leads to

\[
(\lambda_1 - \lambda) \int_{\Omega} u_\varepsilon \phi_1 = 3 \int_{\Omega} u_\varepsilon^{5-\varepsilon} \phi_1 \\
\leq 3 \| u_\varepsilon \|_{\infty}^{4-\varepsilon} \int_{\Omega} u_\varepsilon \phi_1,
\]

where \( \lambda_1 \) is the principal eigenvalue of \(-\Delta\), and thus

\[
\| u_\varepsilon \|_{\infty}^{4-\varepsilon} \geq \frac{1}{3} (\lambda_1 - \lambda) \geq \frac{\pi^2}{4},
\]

because \( \lambda_1 = \pi^2 \) and \( \lambda \leq \pi^2/4 \).

Whereas \( u_\varepsilon(0) \to \infty \) as \( \varepsilon \to 0 \), \( u_\varepsilon(x) \to 0 \) as \( \varepsilon \to 0 \) at any point \( x \neq 0 \). This follows from the upper bound which we shall present next. Define the function

\[
(2.2) \quad W_\mu(x) = \left( \frac{\mu}{\mu^2 + \alpha|x|^2} \right)^{1/2},
\]

where

\[
(2.3) \quad \alpha = \mu^{\varepsilon/2} + \frac{\lambda}{3} \mu^2.
\]

It satisfies the equation

\[
(2.4) \quad -\Delta u = \gamma^{-5} f(\gamma) u^5,
\]

where \( \gamma = \mu^{-1/2} \) and

\[
(2.5) \quad f(s) = \lambda s + 3s^{5-\varepsilon}.
\]

**Lemma 2.1.** Let \( u_\varepsilon \) be any solution of Problem (I). Then

\[
u_\varepsilon(x) \leq W_\mu(x) \quad \text{for} \quad x \in \bar{\Omega},
\]

where \( \mu \) is given by

\[
(2.6) \quad \mu = \{u_\varepsilon(0)\}^{-2}.
\]
Remark. In view of the observations made at the beginning of this section, (2.6) implies that \( \mu \to 0 \) as \( \varepsilon \to 0 \). Thus, we shall always think of \( \mu \) as a small quantity.

Remark. It follows from Lemma 2.1 that if \( x \neq 0 \),
\[ u_\varepsilon(x) = O(\mu^{1/2-\varepsilon/2}) \quad \text{as} \quad \mu \to 0. \]

For the proof of Lemma 2.1 we refer to [AP1, Lemma 1(iii)].

The following elliptic estimates will be needed. Consider the problem
\[
(P) \begin{cases}
-\Delta u - \lambda u = f & \text{in } \Omega \\
u = b & \text{on } \partial \Omega,
\end{cases}
\]
where \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \), with smooth boundary \( \partial \Omega \) and \( \lambda < \lambda_1 \), \( \lambda_1 \) being the principal eigenvalue of \( -\Delta \) in \( \Omega \).

Lemma 2.2. Let \( u \) be a solution of Problem \( (P) \) in which \( \lambda < \lambda_1 \). Then
\[
\| u \|_{W^{1,p}(\Omega)} + \| \nabla u \|_{C^{0,\alpha}(\partial \Omega)} \leq C(\| f \|_{L^1(\Omega)} + \| f \|_{L^\infty(\omega)} + \| b \|_{C^{0,\alpha}(\partial \Omega)}),
\]
for any \( q < 3/2 \), any \( \alpha \in (0,1) \) and any neighbourhood \( \omega \) of the boundary \( \partial \Omega \) in \( \Omega \).

Remark. As a consequence of Lemma 2.2, we have
\[
(i) \quad \| u \|_{L^2(\Omega)} + \| \nabla u \|_{L^2(\partial \Omega)} \leq C(\| f \|_{L^1(\Omega)} + \| f \|_{L^\infty(\omega)} + \| b \|_{C^{0,\alpha}(\partial \Omega)}).
\]
\[
(ii) \quad \text{If } \{f_n\} \text{ is a bounded sequence in } L^1(\Omega) \text{ and in } L^\infty(\omega), \text{ and } \{b_n\} \text{ is a bounded sequence in } C^{2,\alpha}(\partial \Omega), \text{ then the corresponding sequence of solutions } \{u_n\} \text{ has a compact closure in } L^2(\Omega), \text{ whilst the sequence } \{\nabla u_n\}, \text{ restricted to } \partial \Omega, \text{ has a compact closure in } L^2(\partial \Omega).
\]

Proof of Lemma 2.2. First, in view of the classical Schauder estimates, we may always assume that \( b = 0 \). Next, we claim that for all \( q < 3/2 \),
\[
\| u \|_{W^{1,q}(\Omega)} \leq C \| f \|_{L^1(\Omega)}.
\]
This follows easily, by duality, from the fact that if $v$ satisfies

\[
\begin{aligned}
-\Delta v - \lambda v &= f_0 + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} f_i \quad \text{in } \Omega \\
v &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where $f_0, f_1, f_2 \in L^p(\Omega)$, then

\[
\| v \|_{L^\infty(\Omega)} \leq C_p \sum_{i=0}^{3} \| f_i \|_{L^p(\Omega)}
\]

for any $p > 3$ (see e.g. [GT], Theorem 8.15); in other words $(-\Delta - \lambda)^{-1}$ maps $W^{-1,p}(\Omega)$ into $L^\infty(\Omega)$ and, by duality, it also maps $L^1(\Omega)$ into $W_0^{1,p'}(\Omega)$ with $1/p + 1/p' = 1$.

Next, we claim that

\[
\| \nabla u \|_{C^{0,\alpha}(\partial\Omega)} \leq C(\| f \|_{L^1(\Omega)} + \| f \|_{L^\infty(\omega)}).
\]

Indeed let $\chi$ denote the characteristic function of $\omega$ and write $f = f_1 + f_2$ with $f_1 = f \chi$ and $f_2 = f(1 - \chi)$. For $i = 1, 2$, let $u_i$ be the solutions of the problems

\[
\begin{aligned}
-\Delta u_i - \lambda u_i &= f_i \quad \text{in } \Omega \\
u_i &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

so that $u = u_1 + u_2$.

By the $L^p$ regularity theory (see e.g. [GT], Chapter 9) we have

\[
\| u_1 \|_{W^{3,q}(\Omega)} \leq C \| f_1 \|_{L^\infty(\Omega)} = C \| f \|_{L^\infty(\omega)}
\]

for any $q < \infty$, and consequently

\[
\| u_1 \|_{C^{1,\alpha}(\overline{\Omega})} \leq C \| f \|_{L^\infty(\omega)}
\]

for any $\alpha < 1$. On the other hand we have, as above,

\[
\| u_2 \|_{W^{1,q}(\Omega)} \leq C \| f_1 \|_{L^1(\Omega)} \leq C \| f \|_{L^1(\Omega)}
\]

and thus, by the Sobolev imbedding,

\[
\| u_2 \|_{L^2(\Omega)} \leq C \| f \|_{L^1(\Omega)}.
\]
Finally, we note that \( u_2 \) satisfies
\[
\begin{cases}
-\Delta u_2 - \lambda u_2 = 0 & \text{in } \omega \\
u_2 = 0 & \text{on } \partial \Omega.
\end{cases}
\]

It follows from standard elliptic regularity that
\[
\| u_2 \|_{C^{1,\alpha}(\omega')} \leq C \| u_2 \|_{L^2(\omega)}
\]
for any neighbourhood \( \omega' \) of \( \partial \Omega \), strictly smaller than \( \omega \). In particular we have
\[
\| \nabla u_2 \|_{C^{0,\alpha}(\partial \Omega)} \leq C \| f \|_{L^1(\Omega)}.
\]

3. A first estimate. We shall proceed in two steps. First we shall show that when \( 0 \leq \lambda \leq \pi^2/4 \),
\[
(3.1) \quad \epsilon \leq C \mu^{1-\epsilon/2},
\]
where \( C \) is some positive constant, and then we shall establish the desired limits.

To prove (3.1), note that (1.5) implies that
\[
(3.2) \quad \frac{3\epsilon}{6 - \epsilon} \int_\Omega u_e^{5-\epsilon} \leq \int_{\partial \Omega} (x, n) \left( \frac{\partial u_e}{\partial n} \right)^2.
\]
In the next two lemmas, we shall estimate the two integrals in (3.2).

Lemma 3.1. Let \( u_e \) be a solution of Problem (I) in which \( \lambda < \pi^2 \). Then
\[
\int_{\partial \Omega} (x, n) \left( \frac{\partial u_e}{\partial n} \right)^2 \leq C \mu^{1-\epsilon/2},
\]
where \( C \) is some positive constant.

Proof. According to Lemma 2.2 it is sufficient to estimate the right hand side of (1.1) in \( L^1(\Omega) \) and \( L^\infty(\omega) \), where \( \omega \) is some neighbourhood of \( \partial \Omega \) in \( \Omega \).

By Lemma 2.1
\[
\int_\Omega u_e^{5-\epsilon} \leq \int_\Omega W_\mu^{5-\epsilon}
\]
\[
= 4\pi \mu^{(6-\epsilon)/2} \int_0^1 \frac{r^2 dr}{(\mu^2 + \alpha r^2)^{(5-\epsilon)/2}}.
\]
If we now set \( r = \mu \alpha^{-1/2}s \), and observe that \( \alpha \geq \mu^{\epsilon/2} \), we obtain

\[
\int_{\Omega} u_{\epsilon}^{5-\epsilon} \leq 4 \pi \mu^{1/2-\epsilon/4} \int_{0}^{\infty} \frac{s^2 ds}{(1 + s^2)(5-\epsilon)/2} \leq C \mu^{1/2-\epsilon/4}
\]

for some positive constant \( C \), because the integral is convergent.

On the other hand, let \( \omega \) be a neighbourhood of \( \partial \Omega \) in \( \Omega \), which does not contain \( x = 0 \). Then, by Lemma 2.1, \( u_{\epsilon} \leq C \mu^{1/2-\epsilon/4} \) in \( \omega \) and hence

\[
u_{\epsilon}^{5-\epsilon} \leq C \mu^{(5-\epsilon)(1/2-\epsilon/4)} \text{ in } \omega.
\]

Putting (3.3) and (3.4) into (2.7), the desired estimate follows.

**Lemma 3.2.** We have

\[
\liminf_{\epsilon \to 0} \int_{\Omega} u_{\epsilon}^{5-\epsilon} \geq \frac{\pi^2}{4}.
\]

**Proof.** We first prove that

\[
\liminf_{\epsilon \to 0} \int_{\Omega} u_{\epsilon}^{6-\epsilon} > 0.
\]

Suppose, to the contrary, that there exists a subsequence \( \{\epsilon_n\}, \epsilon_n \to 0 \) as \( n \to \infty \), but still denoted by \( \epsilon \), such that

\[
\lim_{\epsilon \to 0} \int_{\Omega} u_{\epsilon}^{6-\epsilon} = 0.
\]

Multiplying equation (1.1) by \( u_{\epsilon} \) we deduce from (3.6), since \( \lambda < \pi^2 \), that

\[
u_{\epsilon} \to 0 \text{ as } \epsilon \to 0 \text{ in } H^1(\Omega).
\]

This implies by the Sobolev injection that

\[
u_{\epsilon} \to 0 \text{ as } \epsilon \to 0 \text{ in } L^6(\Omega).
\]

Next, we multiply (1.1) by \( u_{\epsilon}^5 \) and integrate. This yields

\[
\frac{5}{9} \int_{\Omega} |\nabla u_{\epsilon}^3|^2 = \lambda \int_{\Omega} u_{\epsilon}^6 + 3 \int_{\Omega} u_{\epsilon}^{10-\epsilon},
\]
and hence, using Sobolev's injection again, in view of (3.7),

\[(3.8) \quad \alpha \| u_\varepsilon \|_{18}^6 \leq \int_\Omega u_\varepsilon^{10-\varepsilon} + o(1) \quad \text{as} \quad \varepsilon \to 0,\]

where \(\alpha\) is some positive number. However, using Hölder's inequality, we find that

\[(3.9) \quad \cdot \int_\Omega u_\varepsilon^{10-\varepsilon} \leq \| u_\varepsilon \|_{18}^6 \| u_\varepsilon \|_{6-(3\varepsilon)/2}^{4-\varepsilon} = o(1) \| u_\varepsilon \|_{18}^6.\]

Combining (3.8) and (3.9) we find that

\[(3.10) \quad \| u_\varepsilon \|_{18} \to 0 \quad \text{as} \quad \varepsilon \to 0.\]

Going back to (1.1) and using standard elliptic regularity theory, we see that (3.10) implies that

\[\| u_\varepsilon \|_{H^2(\Omega)} \to 0 \quad \text{as} \quad \varepsilon \to 0\]

and thus, since \(N = 3\),

\[\| u_\varepsilon \|_{\infty} \to 0 \quad \text{as} \quad \varepsilon \to 0.\]

This contradicts the observation made at the beginning of Section 2, that \(u_\varepsilon(0) \to \infty\) as \(\varepsilon \to 0\). Thus (3.5) is established.

To complete the proof of Lemma 3.2, we multiply (1.1) by \(u_\varepsilon\), use Sobolev's inequality and the observation that

\[\int_\Omega u_\varepsilon^2 \leq \int_\Omega W_\mu^2 \leq 4\pi \mu^{1-\varepsilon/2}.\]

This yields

\[S \| u_\varepsilon \|_{6}^2 \leq 3 \int_\Omega u_\varepsilon^{6-\varepsilon} + o(1) \quad \text{as} \quad \varepsilon \to 0,\]

where \(S\) is the best Sobolev constant. On the other hand, by Hölder's inequality,

\[\| u_\varepsilon \|_{6-\varepsilon} \leq \| u_\varepsilon \|_{6} (1 + o(1)) \quad \text{as} \quad \varepsilon \to 0.\]
So

\[(3.11) \quad \left( S + o(1) \right) \| u_\varepsilon \|_{\delta - \varepsilon}^2 \leq 3 \| u_\varepsilon \|_{\delta - \varepsilon}^{6 - \varepsilon} + o(1) \quad \text{as} \quad \varepsilon \to 0. \]

Set

\[ L = \liminf_{\varepsilon \to 0} \int_{\Omega} u_\varepsilon^{\delta - \varepsilon}. \]

We have shown that \( L > 0 \). If \( L = \infty \), the lemma is proved, so suppose that \( L < \infty \). Then, if we let \( \varepsilon \to 0 \) through a sequence chosen so that \( \| u_\varepsilon \|_{\delta - \varepsilon}^{\delta - \varepsilon} \to L \), we deduce from (3.11) that

\[ SL^{1/3} \leq 3L \]

or

\[ L \geq \left( \frac{1}{3} S \right)^{3/2} = \frac{\pi^2}{4}, \]

(see for instance [T]). This finishes the proof.

If we use the Lemmas 3.1 and 3.2 in (3.2), we obtain the following first estimate.

**Lemma 3.3.** Suppose \( 0 \leq \lambda \leq \pi^2/4 \). Then there exists a constant \( C > 0 \) such that

\[ \varepsilon \leq C \mu^{1-\varepsilon/2} \]

for \( \varepsilon \) sufficiently small.

**Corollary 3.4.** Suppose \( 0 \leq \lambda \leq \pi^2/4 \). Then

\[ |\mu^\varepsilon - 1| = O(\mu |\log \mu|) \quad \text{as} \quad \varepsilon \to 0. \]

We now return to the Pohozaev identity

\[(3.12) \quad \frac{3\varepsilon}{6 - \varepsilon} \int_{\Omega} u_\varepsilon^{\delta - \varepsilon} = \int_{\partial \Omega} (x, n) \left( \frac{\partial u_\varepsilon}{\partial n} \right)^2 - 2\lambda \int_{\Omega} u_\varepsilon^2 \]

for a precise estimate. With Lemma 3.2 it is easy to establish the limit of the first integral.
Lemma 3.5. We have

\[
\lim_{\epsilon \to 0} \int_{\Omega} u_{\epsilon}^{6-\epsilon} = \frac{\pi^2}{4}.
\]

Proof. By Lemma 2.1

\[
\int_{\Omega} u_{\epsilon}^{6-\epsilon} \leq \int_{\Omega} W_{\mu}^{6-\epsilon} \\
\leq 4\pi\mu^{-\epsilon/4} \int_{0}^{\infty} \frac{s^2 \, ds}{(1 + s^2)^{3-(\epsilon/2)}} \\
\to \frac{\pi^2}{4} \quad \text{as} \quad \epsilon \to 0
\]

by Corollary 3.4. Remembering Lemma 3.2, this completes the proof.

To estimate the right hand side of (3.12), we investigate the behaviour of \(u_{\epsilon}\) as \(\epsilon \to 0\).

Lemma 3.6. We have

\[
\mu^{1/2} u_{\epsilon}(\mu x) \to U_1(x) = \frac{1}{(1 + |x|^2)^{1/2}} \quad \text{as} \quad \epsilon \to 0
\]

uniformly on \(\mathbb{R}^3\), where \(u_{\epsilon}\) is extended by 0 outside \(B_{1/\mu}\).

Proof. We use a scaling argument, and define the family of functions

\[
v_{\epsilon}(x) = \mu^{1/2} u_{\epsilon}(\mu x).
\]

They satisfy

\[
-\Delta v_{\epsilon} - \lambda \mu^2 v_{\epsilon} = 3\mu^{\epsilon/2} v_{\epsilon}^{5-\epsilon} \quad \text{on} \quad B_{1/\mu} \\
v_{\epsilon}(0) = 1,
\]

and, in addition, by Lemma 2.1,

\[
v_{\epsilon}(x) \leq \mu^{1/2} W_{\mu}(\mu x) \leq \frac{1}{(1 + \mu^{\epsilon/2}|x|^2)^{1/2}} \quad \text{on} \quad B_{1/\mu}.
\]
Thus, in view of Corollary 3.4, the family \( \{v_\epsilon\} \) is bounded. The elliptic regularity theory implies that \( \{v_\epsilon\} \) is equicontinuous on every compact subset of \( B_{1/\mu} \). Hence, by Arzéla–Ascoli there exists a sequence, also denoted by \( v_\epsilon \), which converges to some function \( V \) uniformly on compact sets, but since \( v_\epsilon(x) \to 0 \) as \( x \to \infty \), uniformly with respect to \( \epsilon \), also on \( \mathbb{R}^3 \). Taking the limit in (3.13), using Corollary 3.4 again, we conclude that \( V \) satisfies

\[
-\Delta V = 3V^5 \quad \text{in} \quad \mathbb{R}^3 \\
V(0) = 1.
\]

In addition, \( V \) depends only on \( |x| \). Thus \( V = U_1 \). Since \( V \) is determined uniquely it follows that the entire family \( v_\epsilon \) converges to \( U_1 \) as \( \epsilon \to 0 \).

We use this limit to estimate the right hand side of (1.1).

**Lemma 3.7.** We have

\[
\lim_{\epsilon \to 0} \mu^{-1/2} \int_\Omega u_\epsilon^{5-\epsilon} = \frac{4\pi}{3}.
\]

**Proof.** Defining \( v_\epsilon \) as in (3.13), we can write the integral as

\[
\mu^{-1/2} \int_{B_1} u_\epsilon^{5-\epsilon}(x) dx = \mu^{-3+\epsilon/2} \int_{B_1} v_\epsilon^{5-\epsilon} \left( \frac{x}{\mu} \right) dx
\]

\[
= \mu^{\epsilon/2} \int_{B_{1/\mu}} v_\epsilon^{5-\epsilon}(y) dy
\]

\[
\to \int_{\mathbb{R}^3} U_1^5(y) dy \quad \text{as} \quad \epsilon \to 0
\]

\[
= \frac{4\pi}{3}.
\]

(The last integral can easily be computed by recalling that \( -\Delta U_1 = 3U_1^5 \) on \( \mathbb{R}^3 \)).

**Remark.** Because at any \( x \neq 0 \),

\[
\mu^{-1/2} u_\epsilon^{5-\epsilon}(x) \leq \frac{C}{|x|^5} \mu^2 \to 0 \quad \text{as} \quad \epsilon \to 0,
\]
Lemma 3.7 implies that

\begin{equation}
\mu^{-1/2} u_\varepsilon^{5-\varepsilon} \to \frac{4\pi}{3} \delta_0 \quad \text{as} \quad \varepsilon \to 0,
\end{equation}

where \( \delta_0 \) is the Dirac mass centered at the origin.

Now, define the function

\[ w_\varepsilon = \mu^{-1/2} u_\varepsilon. \]

It satisfies

\[
\begin{cases}
-\Delta w_\varepsilon - \lambda w_\varepsilon = 3\mu^{-1/2} u_\varepsilon^{5-\varepsilon} & \text{in } \Omega \\
 w_\varepsilon = 0 & \text{on } \partial\Omega.
\end{cases}
\]

By Lemma 2.2 and the subsequent Remark, it follows from (3.15) that

\begin{equation}
(3.16) \quad w_\varepsilon \to 4\pi G_\lambda \quad \text{as} \quad \varepsilon \to 0 \quad \text{in} \quad L^2(\Omega),
\end{equation}

and also uniformly away from the origin. Here \( G_\lambda \) is the Green's function defined by (1.7) and (1.8). In addition, restricted to \( \partial\Omega \) we have

\[ \nabla w_\varepsilon \to 4\pi \nabla G_\lambda \quad \text{as} \quad \varepsilon \to 0 \quad \text{in} \quad L^2(\partial\Omega). \]

Thus, writing the right hand side of (3.12) again as \( J(u_\varepsilon) \), we conclude that

\[ \frac{1}{\mu} J(u_\varepsilon) \to 16\pi^2 J(G_\lambda) \quad \text{as} \quad \varepsilon \to 0. \]

We now use a result, which will be proved in the next section:

**Lemma 3.8.** Let \( G_\lambda \) be the Green's function defined by (1.7) and (1.8). Then for any domain \( \Omega \subset \mathbb{R}^3 \) with smooth boundary \( \partial\Omega \),

\[ \int_{\partial\Omega} (x, n) \left( \frac{\partial G_\lambda}{\partial n} \right)^2 - 2\lambda \int_{\Omega} G_\lambda^2 = -g_\lambda(0), \]

where

\[ g_\lambda(x) = G_\lambda(x) - \frac{1}{4\pi |x|}. \]
Thus, we find that the right hand side of (3.12) behaves asymptotically as

(3.17) \[ J(u_\varepsilon) = 16\pi^2 g_\lambda(0)\mu[1 + o(1)] \quad \text{as} \quad \varepsilon \to 0. \]

Remembering the behaviour of the integral on the left hand side of (3.12), given in Lemma 3.5, we arrive at the final result

(3.18) \[ \lim_{\varepsilon \to 0} \frac{\varepsilon}{\mu} = -\frac{32}{\pi} \cdot 4\pi g_\lambda(0). \]

This completes part (a) of the proof of the first part of Theorem 1. The proof of part (b) follows immediately from (3.16) and (3.18) and the observation that

\[ -4\pi g_\lambda(0) = \frac{\sqrt{\lambda}}{\tan \sqrt{\lambda}} > 0 \quad \text{for} \quad \lambda < \pi^2 / 4. \]

4. A Pohozaev-type identity for the Green's function.
Throughout this section, \( \Omega \) is an arbitrary bounded domain in \( \mathbb{R}^N \) \((N > 2)\) with smooth boundary \( \partial \Omega \), and \( \lambda_1 \) the first eigenvalue of \(-\Delta \) in \( \Omega \) with Dirichlet boundary conditions.

Let \( G_\lambda(x, y) \) be the Green's function of \(-\Delta - \lambda \) in \( \Omega \), i.e. for any \( y \in \Omega \), \( G_\lambda(\cdot, y) \) satisfies

\[
\begin{align*}
-\Delta G_\lambda - \lambda G_\lambda &= \delta_y \quad \text{in} \quad \Omega \\
G_\lambda &= 0 \quad \text{on} \quad \partial \Omega,
\end{align*}
\]

where \( \delta_y \) denotes the Dirac mass concentrated at \( x = y \), and let \( g_\lambda(x, y) \) denote the regular part of \( G_\lambda \),

(4.3) \[ g_\lambda(x, y) = G_\lambda(x, y) - \frac{1}{(N - 2)\sigma_N|x - y|^{N-2}}, \]

where \( \sigma_N \) denotes the area of the unit sphere in \( \mathbb{R}^N \).

**Remark.** When \( \lambda \neq 0 \), then if \( N = 3 \), \( g_\lambda(\cdot, y) \in C(\Omega) \), but if \( N \geq 4 \), \( g_\lambda(x, y) \) has no finite limit as \( x \to y \). Of course, if \( \lambda = 0 \), \( g_0(\cdot, y) \) is a smooth function for any \( N > 2 \).
Conforming with the notation of the previous sections, we shall write \( G_\lambda(x) = G_\lambda(x,0) \) and \( g_\lambda(x) = g_\lambda(x,0) \) if \( 0 \in \Omega \).

**Theorem 4.1.** Suppose \( N = 3 \), and let \( G_\lambda(\cdot,y) \) be defined by (4.1), (4.2) and \( g_\lambda(\cdot,y) \) by (4.3). Then

\[
\int_{\partial\Omega} (x - y, n) \left( \frac{\partial G_\lambda}{\partial n}(x,y) \right)^2 - 2\lambda \int_\Omega G_\lambda^2(x,y) = -g_\lambda(y,y).
\]

In the proof of Theorem 4.1 we shall use the following Lemma about the linear problem

(4.4) \( -\Delta u - \lambda u = f(x) \) in \( \Omega \)
(4.5) \( u = 0 \) on \( \partial \Omega \).

**Lemma 4.2.** Let \( u \) be a solution of (4.4), (4.5) in which \( \lambda < \lambda_1 \) and \( f \in L^\infty(\Omega) \). Then

(4.6) \( \int_{\partial\Omega} (x, n) \left( \frac{\partial u}{\partial n} \right)^2 - 2\lambda \int_\Omega u^2 = -\int_\Omega f((N-2)u + 2(x, \nabla u)). \)

**Proof.** Observe that since \( f \in L^\infty \), \( u \in W^{2,p} \) for any \( p > 1 \), whence the integrals in (4.6) are all well defined.
We first multiply (4.4) by \( u \) and integrate over \( \Omega \). This yields, in view of the boundary condition,

(4.7) \( \int_\Omega |\nabla u|^2 - \lambda \int_\Omega u^2 = \int_\Omega fu. \)

Next, we multiply (4.4) by \( (x, \nabla u) \) and integrate over \( \Omega \) to obtain

\[
-\int_{\partial\Omega} (x, n) \left( \frac{\partial u}{\partial n} \right)^2 + \int_\Omega (\nabla u, \nabla (x, \nabla u)) - \lambda \int_\Omega u(x, \nabla u)
= \int_\Omega f(x, \nabla u),
\]

which we can write as

\[
-\int_{\partial\Omega} (x, n) \left( \frac{\partial u}{\partial n} \right)^2 + \int_\Omega |\nabla u|^2 + \frac{1}{2} \int_\Omega (x, \nabla(|\nabla u|^2)) + \frac{1}{2} \lambda \int_\Omega (x, \nabla u^2) = \int_\Omega f(x, \nabla u)
\]
or
\begin{equation}
-\frac{1}{2} \int_{\partial \Omega} (x, n) \left( \frac{\partial u}{\partial n} \right)^2 + \left(1 - \frac{N}{2}\right) \int_{\Omega} \vert \nabla u \vert^2 + \lambda \frac{N}{2} \int_{\Omega} u^2 = \int_{\Omega} f(x, \nabla u).
\end{equation}

If we now use (4.7) to eliminate the second integral on the left hand side in (4.8) we end up with the required identity.

Proof of Theorem 4.1. Without loss of generality we set \( y = 0 \). Define the family of functions
\[
\delta_\rho = \frac{1}{\vert B_\rho \vert} \chi_{B_\rho}, \quad \rho > 0
\]
with \( \vert B_\rho \vert = \frac{4}{3} \pi \rho^3 \). Plainly the functions \( \delta_\rho \) converge weakly to \( \delta_0 \) as \( \rho \to 0 \).

Let \( v_\rho \) be the solution of the equation
\[
-\Delta v = \delta_\rho \quad \text{in } \mathbb{R}^3
\]
such that \( v_\rho(x) \to 0 \) as \( \vert x \vert \to \infty \). It is readily verified that
\[
v_\rho(x) = \begin{cases} 
\frac{r^2}{8\pi \rho^3} + \frac{3}{8\pi \rho} & \text{if } 0 < r < \rho \\
1 & \text{if } \rho \leq r < \infty 
\end{cases}
\]

Let \( u_\rho \) be the solution of the problem
\[
\begin{cases}
-\Delta u - \lambda u = \delta_\rho & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]
so that by Lemma 4.2,
\begin{equation}
\int_{\partial \Omega} (x, n) \left( \frac{\partial u_\rho}{\partial n} \right)^2 - 2\lambda \int_{\Omega} u_\rho^2 = - \int_{\Omega} \delta_\rho \{ u_\rho + 2(x, \nabla u_\rho) \}.
\end{equation}

Now let \( \rho \to 0 \). Then by Lemma 2.2 and the subsequent Remark, \( u_\rho \to G_\lambda \) and the left hand side of (4.9) converges to
\begin{equation}
\int_{\partial \Omega} (x, n) \left( \frac{\partial G_\lambda}{\partial n} \right)^2 - 2\lambda \int_{\Omega} G_\lambda^2.
\end{equation}
To evaluate the limit of the right hand side we write it as

\[(4.11) \quad -\int_{\Omega} \delta_{\rho}\{(u_{\rho} - v_{\rho}) + 2(x, \nabla(u_{\rho} - v_{\rho}))\} - \int_{\Omega} \delta_{\rho}\{v_{\rho} + 2(x, \nabla v_{\rho})\}.\]

It follows by direct computation that

\[(4.12) \quad \int_{\Omega} \delta_{\rho}\{v_{\rho} + 2(x, \nabla v_{\rho})\} = 0.\]

On the other hand,

\[\begin{cases}
-\Delta (u_{\rho} - v_{\rho}) - \lambda (u_{\rho} - v_{\rho}) = \lambda v_{\rho} \quad \text{in } \Omega \\
u_{\rho} - v_{\rho} = -v_{\rho} \quad \text{on } \partial \Omega
\end{cases}\]

and hence, since \(v_{\rho} \to 1/4\pi r\) as \(\rho \to 0\) in \(L^{2}(\Omega)\), \(u_{\rho} - v_{\rho} \to g_{\lambda}\) as \(\rho \to 0\) in \(H^{2}(\Omega)\), and hence uniformly in \(\Omega\), where \(g_{\lambda}\) is the solution of the problem

\[-\Delta g - \lambda g = \frac{\lambda}{4\pi r} \quad \text{in } \Omega \\
g = -\frac{1}{4\pi r} \quad \text{on } \partial \Omega,
\]

i.e. \(g_{\lambda}\) is the regular part of the Green's function. Therefore

\[(4.13) \quad \int_{\Omega} \delta_{\rho}(u_{\rho} - v_{\rho}) \to g_{\lambda}(0) \quad \text{as } \rho \to 0.\]

Finally,

\[(4.14) \quad \left|\int_{\Omega} \delta_{\rho}(x, \nabla(u_{\rho} - v_{\rho}))\right| \leq \frac{3}{4\pi \rho^{2}} \int_{B_{\rho}} |\nabla(u_{\rho} - v_{\rho})|.\]

But \(\nabla(u_{\rho} - v_{\rho})\) is bounded in \(H^{1}(\Omega)\), and hence in \(L^{6}(\Omega)\). So, by Hölder's inequality,

\[\int_{B_{\rho}} |\nabla(u_{\rho} - v_{\rho})| \leq C\|\nabla(u_{\rho} - v_{\rho})\|_{6}^{5/2}\]

for some positive constant \(C\), which implies by (4.14) that

\[(4.15) \quad \left|\int_{\Omega} \delta_{\rho}(x, \nabla(u_{\rho} - v_{\rho}))\right| = O(\rho^{1/2}) \quad \text{as } \rho \to 0.\]
So, (4.12), (4.13) and (4.15) together imply that the right hand side of (4.9), given by (4.11), tends to

\[(4.16) \quad -g_\lambda(0)\]

as \(\rho \to 0\). Thus we conclude that letting \(\rho \to 0\) in (4.9) yields the desired identity.

In the next two theorems we establish Pohozaev–type identities for the Green's function \(G_\lambda\), when \(\lambda = 0\), in arbitrary domains in \(\mathbb{R}^N(N > 2)\). The second of these will be used in Section 8. For convenience we write \(G(x, y) = G_0(x, y)\) and \(g(x, y) = g_0(x, y)\).

**Theorem 4.3.** We have, for every \(y \in \Omega\),

\[\int_{\partial\Omega} (x - y, n) \left( \frac{\partial G}{\partial n}(x, y) \right)^2 dx = -(N - 2)g(y, y),\]

where \(n = n(x)\) denotes the outward normal to \(\partial\Omega\) at \(x\).

**Proof.** As in the proof of Theorem 4.1 we may assume that \(y = 0\) and consider the family of functions

\[\delta_\rho = \frac{1}{|B_\rho|} \chi_{B_\rho}\]

with \(|B_\rho| = \sigma_N \rho^N / N\). Let \(v_\rho\) be the solution of the equation

\[-\Delta v = \delta_\rho \quad \text{in} \quad \mathbb{R}^N\]

such that \(v_\rho(x) \to 0\) as \(|x| \to \infty\). It is readily verified that

\[v_\rho(x) = \begin{cases} \frac{r^2}{2\sigma_N \rho^N} + \frac{N}{2\sigma_N (N - 2) \rho^{N-2}} & \text{if } 0 < r < \rho, \\ \frac{1}{(N - 2)\sigma_N r^{N-2}} & \text{if } \rho \leq r < \infty. \end{cases}\]

Let \(u_\rho\) be the solution of the problem

\[\begin{cases} -\Delta u = \delta_\rho & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}\]
so that, by Lemma 4.2

\[ \int_{\partial \Omega} (x, n) \left( \frac{\partial u_\rho}{\partial n} \right)^2 = -\int_\Omega \delta_\rho \{(N - 2)u_\rho + 2(x, \nabla u_\rho)\}. \]

Now let \( \rho \to 0 \). Clearly \( u_\rho \to G \) in \( C^k(\omega) \) for any integer \( k \) and any neighbourhood \( \omega \) of \( \partial \Omega \) which does not contain 0, and thus the left hand side converges to

\[ \int_{\partial \Omega} (x, n) \left( \frac{\partial G}{\partial n} \right)^2. \]

To evaluate the right hand side we write it as

\[ -\int_\Omega \delta_\rho \{(N - 2)(u_\rho - v_\rho) + 2(x, \nabla (u_\rho - v_\rho))\} \]

\[ -\int_\Omega \delta_\rho \{(N - 2)v_\rho + 2(x, \nabla v_\rho)\}. \]

It follows by direct computation that

\[ \int_\Omega \delta_\rho \{(N - 2)v_\rho + 2(x, \nabla v_\rho)\} = 0. \]

On the other hand,

\[-\Delta (u_\rho - v_\rho) = 0 \quad \text{in } \Omega \]
\[ u_\rho - v_\rho = -v_\rho \quad \text{on } \partial \Omega \]

and hence, since \( v_\rho(x) \to 1/\{(N - 2)\sigma_N r^{N-2}\} \) as \( \rho \to 0 \), \( u_\rho - v_\rho \to g \) as \( \rho \to 0 \) in \( C^k(\overline{\Omega}) \) for any integer \( k \). Therefore

\[ \int_\Omega \delta_\rho (u_\rho - v_\rho) \to g(0, 0) \]

and

\[ \int_\Omega \delta_\rho (x, \nabla (u_\rho - v_\rho)) \to 0 \]

as \( \rho \to 0 \).
Theorem 4.4. We have for every \( y \in \Omega \),
\[
\int_{\partial \Omega} \left( \frac{\partial G}{\partial n}(x, y) \right)^2 n(x) dx = -\nabla \phi(y),
\]
where \( \phi(y) = g(y, y) \) and \( n(x) \) denotes the outward normal to \( \partial \Omega \) at \( x \).

**Proof.** Without loss of generality we may assume that \( y = 0 \). We use the same notation as in the proof of Theorem 4.1. Multiplying the equation
\[
-\Delta u_\rho = \delta_\rho \quad \text{in} \quad \Omega
\]
through by \( \partial u_\rho / \partial x_i \) and integrating over \( \Omega \) we obtain
\[
(4.17) \quad -\frac{1}{2} \int_{\partial \Omega} \left( \frac{\partial u_\rho}{\partial n} \right)^2 (n, e_i) = \int_{\Omega} \delta_\rho \frac{\partial u_\rho}{\partial x_i} dx,
\]
where \( e_i \) denotes the unit vector along the \( x_i \)-axis. But
\[
\int_{\Omega} \delta_\rho \frac{\partial u_\rho}{\partial x_i} dx = \int_{\Omega} \delta_\rho \frac{\partial}{\partial x_i} (u_\rho - v_\rho) dx + \int_{\Omega} \delta_\rho \frac{\partial v_\rho}{\partial x_i} dx
\]
and, since the last integral is zero by symmetry, we have
\[
(4.18) \quad \int_{\Omega} \delta_\rho \frac{\partial u_\rho}{\partial x_i} dx = \int_{\Omega} \delta_\rho \frac{\partial}{\partial x_i} (u_\rho - v_\rho) dx.
\]
On the other hand \( (u_\rho - v_\rho)(x) \to g(x, 0) \) in \( C^k(\Omega) \) for any integer \( k \), as \( \rho \to 0 \) and thus from (4.18) we deduce that
\[
\lim_{\rho \to 0} \left. \int_{\Omega} \frac{\partial u_\rho}{\partial x_i} \right| \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_i} g(x, 0)|_{x=0}.
\]
Passing to the limit in (4.17) we are led to
\[
-\frac{1}{2} \int_{\Omega} \left( \frac{\partial G}{\partial n}(x, 0) \right)^2 (n, e_i) = \frac{\partial}{\partial x_i} g(x, 0).
\]
Finally, we differentiate the relation \( \phi(x) = g(x, x) \) and we find
\[
\frac{\partial \phi}{\partial x_i}(0) = \frac{\partial g}{\partial x_i}(0, 0) + \frac{\partial g}{\partial y_i}(0, 0) = 2 \frac{\partial g}{\partial x_i}(0, 0).
\]
by the symmetry of the function $g(x, y)$.

5. A good approximation for $u_\varepsilon$. To study the properties of $u_\varepsilon$ as $\varepsilon \to 0$ when $\lambda = \pi^2/4$, we need a better estimate for $J(u_\varepsilon)$ than is given in (3.17). Thus it is necessary to establish with greater precision the behaviour of $u_\varepsilon(x)$ as $\varepsilon \to 0$. Writing as before

$$
(5.1) \quad \mu = (u_\varepsilon(0))^{-2},
$$

We shall prove in this section that in all of $\Omega, u_\varepsilon$ can be very well approximated – in some adequate norms – by the function

$$
(5.2) \quad \phi_\mu = U_\mu + 4\pi \sqrt{\mu} g_\lambda,
$$

where $g_\lambda$ is the regular part of the Green’s function defined in (1.10) and $0 \leq \lambda \leq \pi^2/4$. Consider the “error term” $\eta$ defined by

$$
(5.3) \quad \eta = u_\varepsilon - \phi_\mu.
$$

Then an easy calculation shows that

$$
(5.4) \quad -\Delta \eta - \lambda \eta = f \quad \text{in} \quad \Omega
$$

$$
(5.5) \quad \eta = b \quad \text{on} \quad \partial \Omega,
$$

where

$$
(5.6) \quad f = \lambda \left( U_\mu - \frac{\sqrt{\mu}}{|x|} \right) + 3 \left( u_\varepsilon^{5-\varepsilon} - U_\mu^{5} \right),
$$

$$
(5.7) \quad b = -(U_\mu + 4\pi \sqrt{\mu} g_\lambda).
$$

Recall, moreover, from Section 3 that

$$
(5.8) \quad \varepsilon = O(\mu) \quad \text{as} \quad \mu \to 0.
$$

Our main estimate for $\eta$ will be the following.

**Proposition 5.1.** Let $\eta$ be defined by (5.3). Then

$$
\|\eta\|_{L^2(\Omega)} + \|\nabla \eta\|_{L^2(\partial \Omega)} \leq C(\mu^{5/2} |\log \mu| + \varepsilon \mu^{1/2} |\log \mu|),
$$

where $\nabla \eta$ denotes the full gradient of $\eta$, and not just the tangential gradient.
The proof of Proposition 5.1 is based on Lemma 2.2 which requires estimates for $f$ in $L^1(\Omega)$ and $L^\infty(\omega)$, where $\omega$ is a suitable neighbourhood of $\partial \Omega$ and for $b$ in $C^{2,\alpha}(\partial \Omega)$. We shall now derive these bounds in succession.

**Lemma 5.2.** Let $\alpha \in (0,1)$. Then

$$\|b\|_{C^{2,\alpha}(\partial \Omega)} = O(\mu^{5/2}) \text{ as } \mu \to 0.$$ 

**Proof.** On $\partial \Omega$, $G_\lambda = 0$, and so

$$b(x) = -\sqrt{\mu} \left( \frac{1}{\sqrt{\mu^2 + |x|^2}} - \frac{1}{|x|} \right).$$

Since $|x|$ is bounded away from zero on $\partial \Omega$, it readily follows that

$$\|b\|_{C^k(\partial \Omega)} = O(\mu^{5/2}) \text{ as } \mu \to 0$$

for any $k \geq 0$.

**Lemma 5.3.** Let $\omega$ be a neighbourhood of $\partial \Omega$ in $\Omega$, which does not contain the origin. Then

$$\|f\|_{L^\infty(\omega)} = O(\mu^{5/2}) \text{ as } \mu \to 0.$$ 

**Proof.** As in the proof of Lemma 5.2, one proves that the first term in (5.6) is $O(\mu^{5/2})$ as $\mu \to 0$. As to the second term, plainly

$$\|U_\mu\|_{L^\infty(\omega)} \leq C\sqrt{\mu},$$

where $C$ is some generic constant, and, since by Lemma 2.1 $u_\epsilon \leq W_\mu$, we have

$$\|u_\epsilon\|_{L^\infty(\omega)} \leq |W_\mu|_{L^\infty(\omega)} \leq C\sqrt{\mu}$$

Thus

$$\|U_\mu^5\|_{L^\infty(\omega)} \leq C\mu^{5/2}$$

and

$$\|u_\epsilon^5\|_{L^\infty(\omega)} \leq C\mu^{5/2} \mu^{-\epsilon/2} \leq C\mu^{5/2}.$$
in view of (5.8).

To estimate \( f \) in \( L^1(\Omega) \), we split it into three pieces:

\[
f = f_1 + f_2 + f_3,
\]

where

\[
\begin{align*}
(5.9) & \quad f_1 = \lambda \left( U_\mu - \frac{\sqrt{\mu}}{|x|} \right), \\
(5.10) & \quad f_2 = 3(u_\varepsilon^{5-\varepsilon} - W_\mu^{5-\varepsilon}), \\
(5.11) & \quad f_3 = 3(W_\mu^{5-\varepsilon} - U_\mu^{5}).
\end{align*}
\]

**Lemma 5.4.** Let \( f_1 \) be given by (5.9). Then

\[
||f_1||_{L^1(\Omega)} = O(\mu^{5/2}|\log \mu|) \quad as \quad \mu \to 0.
\]

**Proof.** By the expression for \( U_\mu \), we have

\[
\int_\Omega |f_1| = 4\pi \lambda \sqrt{\mu} \int_0^1 \frac{1}{r} - \frac{1}{\sqrt{\mu^2 + r^2}} \mid r^2 dr,
\]

or, if we set \( r = \mu s \),

\[
\int_\Omega |f_1| = 4\pi \lambda \mu^{5/2} \int_0^{1/\mu} \frac{1}{s} - \frac{1}{\sqrt{1 + s^2}} \mid s^2 ds \\
\leq C\mu^{5/2} \int_0^{1/\mu} \frac{sds}{1 + s^2} \\
< C\mu^{5/2}|\log \mu| \quad as \quad \mu \to 0.
\]

The splitting of the second term in the function \( f \) into \( f_2 \) and \( f_3 \) was inspired by the fact that \( u_\varepsilon \leq W_\mu \) in \( \Omega \). Thus \( f_2 \leq 0 \) and to estimate it we merely need a lower bound for \( u_\varepsilon \). This bound is given in the following Lemma. It is very similar to a bound given in [AP3], but for completeness, we give the proof in the Appendix.
Lemma 5.5. Let \( u_\epsilon \) be any solution of (1.1) in which \( 0 \leq \lambda < \pi^2 \). Then, for \( x \in \bar{\Omega} \),

\[
u_\epsilon(x) \geq W_\mu(x) \left( 1 - \frac{\lambda}{2} |x|^2 \right) - \frac{1}{\sqrt{\mu}} \left\{ (\sqrt{\mu} W_\mu(x))^{-\epsilon} - 1 \right\}.
\]

Lemma 5.6. Let \( f_2 \) be given by (5.10). Then for \( \mu \) small

\[
\| f_2 \|_{L^1(\Omega)} \leq C(\epsilon \mu^{1/2} |\log \mu| + \mu^{5/2} |\log \mu|),
\]

where \( C \) is some positive constant.

PROOF. By Lemmas 2.1 and 5.5

\[
|f_2| \leq CW_\mu^{4-\epsilon}(W_\mu - u_\epsilon)
\]

\[
\leq C \mu^{-1/2} W_\mu^{4-\epsilon} \left\{ (\sqrt{\mu} W_\mu)^{-\epsilon} - 1 \right\} + C|x|^2 W_\mu^{5-\epsilon},
\]

where, like throughout this proof, \( C \) is some positive generic constant.

Observe that

\[
\left\{ (\sqrt{\mu} W_\mu(x))^{-\epsilon} = \left( \frac{\mu^2 + \alpha |x|^2}{\mu^2} \right)^{\epsilon/2},
\]

where \( \alpha = \mu^{\epsilon/2} + \frac{1}{3} \lambda \mu^2 \). Hence, because \( |x| \leq 1 \),

\[
\left\{ (\sqrt{\mu} W_\mu(x))^{-\epsilon} - 1 \leq (2\mu^{-2})^{\epsilon/2} - 1
\]

\[
= C\epsilon |\log \mu|
\]

in view of (5.8). Also, note that since \( \alpha \geq \mu^{\epsilon/2} \) and \( \mu^{\epsilon/2} < 1 \),

\[
W_\mu(x) \leq \left( \frac{\mu}{\mu^2 + \mu^{\epsilon/2} |x|^2} \right)^{1/2} < \mu^{-\epsilon/4} U_\mu(x).
\]

Thus, (5.13) and (5.14), as well as (5.8) enable us to simplify (5.12) to the following bound:

\[
|f_2| \leq C\epsilon \mu^{-1/2} |\log \mu| U_\mu^{4-\epsilon} + C|x|^2 U_\mu^{5-\epsilon}
\]

\[
\leq C\epsilon \mu^{-1/2} |\log \mu| U_\mu^{4} + C|x|^2 U_\mu^{5}.
\]
since \( U_{\mu}^{-\epsilon} \leq C \). However
\[
\int_{\Omega} U_{\mu}^4 = 4\pi\mu^2 \int_0^{1/\mu} \frac{r^2 dr}{(\mu^2 + r^2)^2} = 4\pi\mu \int_0^{1/\mu} \frac{s^2 ds}{(1 + s^2)^2},
\]
where we have set \( r = \mu s \). Because the last integral is convergent if we let \( \mu \) tend to zero, we may conclude that
\[
\int_{\Omega} U_{\mu}^4 \leq C\mu.
\]
(5.16)

Similarly, we obtain
\[
\int_{\Omega} U_{\mu}^5 |x|^2 = 4\pi\mu^{5/2} \int_0^{1/\mu} \frac{s^4 ds}{(1 + s^2)^{5/2}} \leq C\mu^{5/2}|\log \mu|.
\]
(5.17)

Thus, integrating (5.15) over \( \Omega \), and using (5.16) and (5.17), we end up with the desired bound
\[
\int_{\Omega} |f_2| \leq C(\epsilon\mu^{1/2} |\log \mu| + \mu^{5/2} |\log \mu|).
\]

Lemma 5.7. Let \( f_3 \) be given by (5.11). Then for \( \mu \) small
\[
\|f_3\|_{L^1(\Omega)} \leq C(\epsilon\mu^{1/2} |\log \mu| + \mu^{5/2}),
\]
where \( C \) is some positive constant.

Proof. We split \( f_3 \) as follows:
\[
f_3 = 3(W_{\mu}^{5-\epsilon} - U_{\mu}^{5-\epsilon}) + 3(U_{\mu}^{5-\epsilon} - U_{\mu}^5)
\]
\[
= f_{31} + f_{32}
\]
in an obvious notation. By the Mean Value Theorem
\[
|f_{31}| \leq C\mu^{(5-\epsilon)/2} \frac{\mu^{\epsilon/2} - 1 + \lambda\mu^2}{(\mu^2 + \mu^{5/2}|x|^2)(7-\epsilon)/2} \cdot |x|^2,
\]
and hence, because $\epsilon = O(\mu)$ by (5.8),

$$|f_{31}| \leq C \mu^{5/2}(\epsilon |\log \mu| + \mu^2) \frac{|x|^2}{(\mu^2 + |x|^2)^{7/2}}.$$ 

This yields upon integration over $\Omega$,

$$\int_{\Omega} |f_{31}| \leq C(\epsilon \mu^{1/2}|\log \mu| + \mu^{5/2}). \tag{5.18}$$

As to the second term,

$$|f_{32}| = 3U_{\mu}^{5}|U_{\mu}^{-\epsilon} - 1|$$

$$= 3U_{\mu}^{5}\left|\left(\frac{\mu^2 + |x|^2}{\mu}\right)^{\epsilon/2} - 1\right|$$

$$\leq C\epsilon |\log \mu| U_{\mu}^5,$$

and so

$$\int_{\Omega} |f_{32}| \leq C\epsilon \mu^{1/2}|\log \mu|. \tag{5.19}$$

Together (5.18) and (5.19) yield the required estimate.

From Lemmas 5.4, 5.6 and 5.7 we finally conclude that for $\mu$ small

$$\int_{\Omega} |f| \leq C(\epsilon \mu^{1/2}|\log \mu| + \mu^{5/2}|\log \mu|),$$

$C$ being some positive constant. This estimate, together with earlier estimates for $f$ near $\partial \Omega$ and $b$ on $\partial \Omega$ given in Lemmas 5.3 and 5.2 suffice to prove Proposition 5.1 with the help of Lemma 2.2.

The following two bounds for $\phi_{\mu}$ will prove useful later.

**Lemma 5.8.** Let $\phi_{\mu}$ be given by (5.2). Then

(a) \[ \int_{\Omega} \phi_{\mu}^2 \leq C\mu, \]

(b) \[ \int_{\partial \Omega} |\nabla \phi_{\mu}|^2 \leq C\mu, \]
where \( C \) is some positive constant.

**Proof.** Both estimates follow immediately from the definition of \( U_\mu \) and the fact that \( \lambda \) does not depend on \( \mu \).

6. The main results. We now return again to Pohozaev’s identity for Problem (I):

\[
\frac{3\varepsilon}{6-\varepsilon} \int_\Omega u^{-\varepsilon}_{\varepsilon} = \int_{\partial\Omega} (x, n) \left( \frac{\partial u_{\varepsilon}}{\partial n} \right)^2 - 2\lambda \int_\Omega u_{\varepsilon}^2,
\]

and use the approximation

\[
u_{\varepsilon} = \phi_{\mu} + \eta,
\]

where

\[
\phi_{\mu} = U_\mu + 4\pi \sqrt{\mu} g_\lambda,
\]

which was discussed in the previous section to obtain an expansion for the right hand side \( J(u_{\varepsilon}) \) of (6.1).

**Theorem 6.1.** We have the expansion, as \( \varepsilon \to 0 \)

\[
J(u_{\varepsilon}) = -16\phi^2 g_\lambda(0) \mu + 4\pi^2 \lambda \mu^2 + O(\varepsilon \mu \log \mu + \mu^3 |\log \mu|).
\]

The proof of Theorem 6.1 is given in two lemmas.

**Lemma 6.2.** Let \( \phi_{\mu} \) be defined by (6.3). Then

\[
J(u_{\varepsilon}) = J(\phi_{\mu}) + R_{\varepsilon},
\]

where

\[
|R_{\varepsilon}| \leq C(\varepsilon \mu |\log \mu| + \mu^3 |\log \mu|),
\]

in which \( C \) is some positive constant.

**Proof.** By (6.2),

\[
R_{\varepsilon} = J(\phi_{\mu} + \eta) - J(\phi_{\mu})
\]

\[
= \int_{\partial\Omega} (x, n) \left\{ 2 \frac{\partial \phi_{\mu}}{\partial n} \cdot \frac{\partial \eta}{\partial \eta} + \left( \frac{\partial \eta}{\partial \eta} \right)^2 \right\} - 2\lambda \int_\Omega (2\phi_{\mu} \eta + \eta^2)
\]
and hence, by Cauchy–Schwarz's inequality and Lemma 5.8
\[ |R_e| \leq C \{ \mu^{1/2} (\| \nabla \eta \|_{L^2(\Omega)} + \| \eta \|_{L^2(\Omega)}) + \| \nabla \eta \|^2_{L^2(\Omega)} + \| \eta \|^2_{L^2(\Omega)} \}. \]

If we now use Proposition 5.1 to estimate \( \eta \), we arrive at the required bound.

The function \( \phi_\mu \) is explicitly given in terms of the parameter \( \mu \), and closely related to the Green's function \( G_\lambda \). Indeed
\[
\phi_\mu (x) = 4\pi \sqrt{\mu} \left( \frac{1}{4\pi \sqrt{\mu^2 + |x|^2}} + g_\lambda (x) \right) \\
\approx 4\pi \sqrt{\mu} \left( \frac{1}{4\pi |x|} + g_\lambda (x) \right) \\
= 4\pi \sqrt{\mu} G_\lambda (x).
\]

In the next Lemma we shall exploit this relationship, and our knowledge of \( J(G_\lambda) \) from Theorem 4.1, to determine the first two terms of the asymptotic expansion of \( J(\phi_\mu) \) as \( \mu \to 0 \).

**Lemma 6.3.** Let \( \phi_\mu \) be given by (6.3). Then, as \( \mu \to 0 \),
\[
J(\phi_\mu) = -16\pi^2 g_\lambda (0) \mu + 4\lambda \pi^2 \mu^2 + O(\mu^3 \log \mu).
\]

**Proof.** Recall that
\[
(6.4) \quad \phi_\mu = 4\pi \sqrt{\mu} G_\lambda + \left( U_\mu - \frac{\sqrt{\mu}}{|x|} \right)
\]

Hence,
\[
\frac{\partial \phi_\mu}{\partial n} = 4\pi \sqrt{\mu} \frac{\partial G_\lambda}{\partial n} + \sqrt{\mu} \frac{\partial}{\partial n} \left( \frac{1}{\sqrt{\mu^2 + |x|^2}} - \frac{1}{|x|} \right),
\]
and therefore, since \( \Omega = B_1 \),
\[
\left. \frac{\partial \phi_\mu}{\partial n} \right|_{\partial \Omega} = 4\pi \sqrt{\mu} \left. \frac{\partial G_\lambda}{\partial n} \right|_{\partial \Omega} + O(\mu^{5/2}) \text{ as } \mu \to 0.
\]
Thus
\[(6.5)\]
\[\int_{\partial \Omega} (x, n) \left( \frac{\partial \phi_\mu}{\partial n} \right)^2 = 16\pi^2 \mu \int_{\Omega} (x, n) \left( \frac{\partial G_\lambda}{\partial n} \right)^2 + O(\mu^3) \quad \text{as} \quad \mu \to 0.\]

Next, by (6.3) we can write the second integral in \(J(\phi_\mu)\) as
\[\int_{\Omega} \phi_\mu^2 = \int_{\Omega} \frac{\mu}{\mu^2 + |x|^2} + 8\pi \mu \int_{\Omega} \frac{g_\lambda}{\sqrt{\mu^2 + |x|^2}} + 16\pi^2 \mu \int_{\Omega} g_\lambda^2\]
\[= 16\pi^2 \mu \int_{\Omega} G_\lambda^2 + X_1 + X_2,\]
where
\[X_1 = 8\pi \mu \int_{\Omega} \left( \frac{1}{\sqrt{\mu^2 + |x|^2}} - \frac{1}{|x|} \right) g_\lambda\]
and
\[X_2 = \mu \int_{\Omega} \left( \frac{1}{\mu^2 + |x|^2} - \frac{1}{|x|^2} \right).\]

Plainly, because \(g_\lambda \in L^\infty(\Omega),\)
\[|X_1| \leq C \mu \int_{0}^{1} \left| \frac{1}{\sqrt{\mu^2 + r^2}} - \frac{1}{r} \right| r^2 dr\]
\[\leq C \mu^3 |\log \mu|.
\]

On the other hand
\[X_2 = -4\pi \mu^3 \int_{0}^{1} \frac{dr}{\mu^2 + r^2} = -2\pi^2 \mu^2 + O(\mu^3).\]

Thus
\[(6.6)\]
\[\int_{\Omega} \phi_\mu^2 = 16\pi^2 \mu \int_{\Omega} G_\lambda^2 - 2\pi^2 \mu^2 + O(\mu^3 |\log \mu|) \quad \text{as} \quad \mu \to 0.\]

Putting (6.5) and (6.6) together, we obtain
\[J(\phi_\mu) = 16\pi^2 \mu J(G_\lambda) + 4\pi^2 \mu \lambda \mu^2 + O(\mu^3 |\log \mu|) \quad \text{as} \quad \mu \to 0.\]

Since \(J(G_\lambda) = -g_\lambda(0)\) according to Theorem 4.1, the proof is complete.
Plainly, Lemma's 6.2 and 6.3 yield Theorem 6.1.

We are now ready to prove Theorem 2. We have \( \lambda = \pi^2/4 \), and so \( g_\lambda(0) = 0 \). This implies, according to Theorem 6.1, that \( J(u_\epsilon) = O(\mu^2) \) as \( \epsilon \to 0 \). Therefore, we divide (6.1) by \( \mu^2 \) and let \( \epsilon \) tend to zero. Remembering the limit of the integral on the left hand side of (6.1), given in Lemma 3.5, we obtain

\[
(6.7) \quad \frac{\pi^2}{8} \lim_{\epsilon \to 0} \frac{\epsilon}{\mu^2} = \pi^4,
\]

or, since \( \mu = (u_\epsilon(0))^{-2} \),

\[
\lim_{\epsilon \to 0} \epsilon u_\epsilon^4(0) = 8\pi^2
\]

which is the content of Part (a).

As to Part (b), we recall from (3.16) that

\[
\mu^{-1/2} u_\epsilon(x) \to 4\pi G_{\pi^2/4}(x) \quad \text{as} \quad \epsilon \to 0
\]

in \( L^2(\Omega) \), and pointwise for \( x \neq 0 \). If we now use (6.7) to eliminate \( \mu \), we find the limit

\[
\epsilon^{-1/4} u_\epsilon(x) \to 2^{5/4} \sqrt{\pi} G_{\pi^2/4}(x) \quad \text{as} \quad \epsilon \to 0
\]

in \( L^2(\Omega) \), which is the content of Part (b).

7. A related problem. Consider the problem

\[
(II) \quad \begin{cases}
-\Delta u - \left( \frac{\pi^2}{4} + \epsilon \right) u = 3u^5 & \text{in } \Omega \\
 u > 0 & \text{in } \Omega \\
 u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( \Omega \) is the unit ball in \( \mathbb{R}^3 \) and \( \epsilon \) a small positive number. As in Problem (I), Problem (II) has a solution \( u_\epsilon \) if \( \epsilon > 0 \) (and small enough), but it has no solution if \( \epsilon = 0 \) [BN]. It is again our objective to study the behaviour of \( u_\epsilon \) as \( \epsilon \to 0 \), and specifically, to prove Theorem 3.
It is readily shown, as in Section 2, that \( u_\varepsilon(0) \to \infty \) as \( \varepsilon \to 0 \), and from [AP1] one has the upper bound

\[
(7.1) \quad u_\varepsilon(x) \leq W_\mu(x) = \left( \frac{\mu}{\mu^2 + \alpha|x|^2} \right)^{1/2} \quad \text{in} \quad \Omega,
\]

where

\[
(7.2) \quad \mu = (u_\varepsilon(0))^{-2}
\]

and

\[
(7.3) \quad \alpha = 1 + \frac{1}{3} \left( \frac{\pi^2}{4} + \varepsilon \right) \mu^2.
\]

One also has from [AP3] the lower bound

\[
(7.4) \quad u_\varepsilon(x) \geq W_\mu(x) \left\{ 1 - \frac{1}{2} \left( \frac{\pi^2}{4} + \varepsilon \right) |x|^2 \right\}.
\]

The two bounds (7.1) and (7.4) relate \( u_\varepsilon(x) \) via the parameter \( \mu \) to its central value \( u_\varepsilon(0) \). To relate \( \mu \) to \( \varepsilon \) we use Pohozaev's identity again. For Problem (II) it becomes

\[
(7.5) \quad J(u_\varepsilon) = \int_{\partial\Omega} (x,n) \left( \frac{\partial u_\varepsilon}{\partial n} \right)^2 - 2 \left( \frac{\pi^2}{4} + \varepsilon \right) \int_\Omega u_\varepsilon^2 = 0.
\]

We now estimate the two terms in \( J(u_\varepsilon) \) by finding a good approximation for \( u_\varepsilon \). As before we use for this purpose the function

\[
\phi = U_\mu + 4\pi \sqrt{\mu} g_{(\pi^2/4)+\varepsilon},
\]

where \( U_\mu \) has been defined by (1.14) and \( g_\lambda \) in (1.10). To see that \( \phi \) is indeed a good approximation, we note that the remainder term

\[
\eta = u_\varepsilon - \phi,
\]

is a solution of the problem

\[
(7.6) \quad -\Delta \eta - \left( \frac{\pi^2}{4} + \varepsilon \right) \eta = f
\]

(7.7) \quad \eta = b,
where
\[
f = \left( \frac{\pi^2}{4} + \epsilon \right) \left( U_\mu - \frac{\sqrt{\mu}}{|x|} \right) + 3 \left( u_\varepsilon^5 - U_\mu^5 \right)
\]
and hence that it can be estimated by
\[
(7.8) \quad \| \eta \|_{L^2(\Omega)} + \| \nabla \eta \|_{L^2(\partial \Omega)} \leq C \mu^{5/2} |\log \mu|.
\]

The proof of (7.8) is very similar to that of Proposition 5.1 (actually it is a little simpler) and we therefore omit it.

It follows from (7.8) that \( J(\varepsilon) \) can be expressed as
\[
(7.9) \quad J(\varepsilon) = J(\phi) + R_\varepsilon,
\]
where
\[
(7.10) \quad R_\varepsilon = O(\mu^3 |\log \mu|) \quad \text{as} \quad \varepsilon \to 0.
\]

For the proof of (7.9) and (7.10) we refer to Lemma 6.2.

If we now use Lemma 6.3 to evaluate \( J(\phi) \), setting \( \lambda = (\pi^2/4) + \epsilon \), we conclude from (7.9) that
\[
-16\pi^2 g(\pi^2/4) + \epsilon(0) \mu + 4\pi^2 \left( \frac{\pi^2}{4} + \epsilon \right) \mu^2 = O(\mu^3 |\log \mu|) \quad \text{as} \quad \varepsilon \to 0
\]
or, if we divide by \(-16\pi^2\mu\),
\[
(7.11) \quad g(\pi^2/4) + \epsilon(0) = \frac{1}{4} \left( \frac{\pi^2}{4} + \epsilon \right) \mu + O(\mu^2 |\log \mu|) \quad \text{as} \quad \varepsilon \to 0.
\]

However, \( g(\lambda) \) is given by
\[
g(\lambda) = -\frac{1}{4\pi} \sqrt{\lambda} \cot \sqrt{\lambda},
\]
and so, if \( \lambda = (\pi^2/4) + \epsilon \),
\[
(7.12) \quad g(\pi^2/4) + \epsilon(0) = \frac{\epsilon}{8\pi} + O(\epsilon^2) \quad \text{as} \quad \epsilon \to 0.
\]
Inserting (7.12) into (7.11) we conclude first that \( \mu = O(\varepsilon) \) as \( \varepsilon \to 0 \) and subsequently that

\[
\frac{\varepsilon}{\mu} = \frac{\pi^3}{2} + O(\varepsilon|\log \varepsilon|) \quad \text{as} \quad \varepsilon \to 0.
\]

As to the limiting behaviour of \( u_\varepsilon \) as \( \varepsilon \to 0 \), we recall that by (7.8),

\[
\mu^{-1/2} u_\varepsilon \to 4\pi G_{x^2/4} \quad \text{as} \quad \varepsilon \to 0
\]

in \( L^2(\Omega) \), and pointwise away from the origin. Hence, by (7.13)

\[
\varepsilon^{-1/2} u_\varepsilon(x) \to 4\sqrt{2/\pi} G_{x^2/4} \quad \text{as} \quad \varepsilon \to 0.
\]

This completes the proof of Theorem 3.

8. Conjectures for general domains. We shall now formulate various conjectures for general domains in \( \mathbb{R}^N \). They are motivated partly by the results in the previous sections and partly by some recent results of [R1,2]. We shall also present evidence in support of these conjectures.

Conjecture 1. Let \( \Omega \subset \mathbb{R}^N, N \geq 3 \), be a bounded domain with smooth boundary. Let \( u_\varepsilon \) be a solution of

\[
(8.1) \quad -\Delta u_\varepsilon = N(N-2)u_\varepsilon^{p-\varepsilon} \quad \text{in} \quad \Omega
\]

\[
(8.2) \quad u_\varepsilon > 0 \quad \text{in} \quad \Omega
\]

\[
(8.3) \quad u_\varepsilon = 0 \quad \text{on} \quad \partial \Omega,
\]

where \( p = (N+2)/(N-2) \).

We denote again by \( G(x,y) = G_0(x,y) \) the Green's function of \(-\Delta\) and by \( g(x,y) = g_0(x,y) \) its regular part i.e.

\[
g(x,y) = G(x,y) - \frac{1}{(N-2)\sigma_N|x-y|^{N-2}},
\]

where \( \sigma_N \) is the area of the unit sphere in \( \mathbb{R}^N \):

\[
\sigma_N = \frac{2\pi^{N/2}}{\Gamma(N/2)}.
\]
Recall that \(g(x, y)\) is smooth on \(\Omega \times \Omega\).

**Conjecture 1.** Assume \(u_\varepsilon\) is a minimizing sequence for the Sobolev inequality, i.e.

\[
(8.4) \quad \frac{\int_\Omega |\nabla u_\varepsilon|^2}{\|u_\varepsilon\|^2_{p+1}} = S_N + o(1),
\]

where \(S_N\) is the best Sobolev constant in \(\mathbb{R}^N\):

\[
S_N = \pi N(N - 2) \left\{ \frac{\Gamma(N/2)}{\Gamma(N)} \right\}^{2/N}.
\]

Then

\[
\lim_{\varepsilon \to 0} \varepsilon \|u_\varepsilon\|^2_{L^\infty} = 2\sigma_N^2 \left[ \frac{N(N - 2)}{S_N} \right]^{N/2} |g|,
\]

where \(g\) is a critical value of the function \(\phi(x) = g(x, x)\) i.e. \(g = \phi(x_0)\) for some point \(x_0 \in \Omega\) such that \(\nabla \phi(x_0) = 0\).

**Evidence.** Recall that Pohozaev's identity says that if \(u\) is a solution of the problem

\[
\begin{cases}
-\Delta u = f(u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

then,

\[
(8.5) \quad \left(1 - \frac{N}{2}\right) \int_\Omega uf(u) + N \int \Omega F(u) = \frac{1}{2} \int_{\partial \Omega} (x - y, n) \left( \frac{\partial u}{\partial n} \right)^2,
\]

where \(F(u) = \int_0^u f(t) dt\) and \(y\) is any point in \(\mathbb{R}^N\). Applying this identity to (8.1) - (8.3) we obtain

\[
\frac{N(N - 2)^3}{2N - \varepsilon(N - 2)} \varepsilon \int_\Omega u_\varepsilon^{p+1-\varepsilon} = \int_{\partial \Omega} (x - y, n) \left( \frac{\partial u_\varepsilon}{\partial n} \right)^2,
\]

or

\[
(8.6) \quad \frac{1}{2} (N - 2)^3 \varepsilon \|u_\varepsilon\|^2_{p+1} \simeq \int_{\partial \Omega} (x - y, n) \left( \frac{\partial u_\varepsilon}{\partial n} \right)^2.
\]
If we multiply (8.1) by \( u_\epsilon \), integrate over \( \Omega \) and use (8.4) we obtain
\[
\|u_\epsilon\|_{p+1}^{p-1} \sim \frac{S_N}{N(N-2)},
\]
and therefore
\[
\|u_\epsilon\|_{p+1}^{p+1} \sim \left[ \frac{S_N}{N(N-2)} \right]^{N/2}.
\]
In view of (8.6) we thus find that
\[
(8.7) \quad \int_\Omega (x - y, n) \left( \frac{\partial u_\epsilon}{\partial n} \right)^2 \simeq \frac{1}{2} (N-2)^3 \left[ \frac{S_N}{N(N-2)} \right]^{N/2} \epsilon.
\]
On the other hand we know that \( u_\epsilon \) "concentrates" around some point \( x_0 \) (see [S], [L]) and near \( x_0 \) we have
\[
(8.8) \quad u_\epsilon(x) \simeq \frac{\mu^{(N-2)/2}}{(\mu^2 + |x - x_0|^2)^{(N-2)/2}}
\]
for some appropriate \( \mu = \mu_\epsilon \) which tends to 0 as \( \epsilon \to 0 \). Then, we have
\[
(8.9) \quad \|u_\epsilon\|_{L^\infty(\Omega)} \simeq \mu^{-(N-2)/2}.
\]
From (8.8) we deduce that, near \( x = x_0 \),
\[
N(N-2)u_\epsilon^{p-\epsilon} \simeq N(N-2) \frac{\mu^{(N+2)/2}}{(\mu^2 + |x - x_0|^2)^{(N+2)/2}}
\]
\[
= \mu^{(N-2)/2} \frac{N(N-2)\mu^2}{(\mu^2 + |x - x_0|^2)^{(N+2)/2}}
\]
\[
\simeq \mu^{(N-2)/2} K_N \delta_{x_0},
\]
where
\[
K_N = N(N-2) \int_{\mathbb{R}^N} \frac{dx}{(1 + |x|^2)^{(N+2)/2}} = N(N-2) \int_{\mathbb{R}^N} U^p(x)dx
\]
and
\[
U(x) = \frac{1}{(1 + |x|^2)^{(N-2)/2}}.
\]
Since $U$ satisfies the equation $-\Delta U = N(N-2)U^p$ we see that
\[
N(N-2) \int_{\mathbb{R}^N} U^p \, dx = \int_{\mathbb{R}^N} (-\Delta U) = -\lim_{r \to \infty} r^{N-1} \sigma_N U'(r)
\]
\[
= (N-2)\sigma_N,
\]
and therefore
\[
(8.11) \quad K_N = (N-2)\sigma_N.
\]

Going back to (8.1) – (8.3) we conclude that, globally on $\Omega$,
\[
(8.12) \quad u_\varepsilon(x) \simeq \mu^{(N-2)/2} K_N G(x, x_0)
\]
and therefore
\[
(8.13) \quad \int_{\partial \Omega} (x - y, n) \left( \frac{\partial u_\varepsilon}{\partial n} \right)^2 \, dx \simeq \int_{\partial \Omega} (x - y_0, n) \mu^{N-2} K_N^2 \left( \frac{\partial G}{\partial n}(x, x_0) \right)^2 \, dx.
\]

Recall that (see Theorem 4.3)
\[
\int_{\partial \Omega} (x - x_0, n) \left( \frac{\partial G}{\partial n}(x, x_0) \right)^2 \, dx = -(N-2)g(x_0, x_0).
\]

Hence, if we put $y = x_0$ in (8.13) we obtain using (8.11)
\[
(8.14) \quad \int_{\partial \Omega} (x - x_0, n) \left( \frac{\partial u_\varepsilon}{\partial n} \right)^2 \, dx = -\mu^{N-2} \sigma_N^2 (N-2)^3 g(x_0, x_0).
\]

Putting (8.7) and (8.14) together we are led to
\[
\frac{1}{2} (N-2)^3 \left[ \frac{S_N}{N(N-2)} \right]^{N/2} \varepsilon \simeq -\mu^{(N-2)} \sigma_N^2 (N-2)^3 g(x_0, x_0).
\]
Consequently, using (8.9), we conjecture that
\[
\|u_\varepsilon\|_{L^\infty(\Omega)} \simeq \mu^{-(N-2)} \simeq \frac{2}{\varepsilon} \sigma_N^2 \left[ \frac{N(N-2)}{S_N} \right] |g(x_0, x_0)|
\]
since $g(x, y) < 0$ (by the maximum principle).
Finally, we claim that the point of concentration $x_0$ is a critical point of the function $\phi(x) = g(x, x)$. First, note that in Pohozaev's identity (8.5) the point $y$ is arbitrary and thus we have

$$\int_{\partial \Omega} \left( \frac{\partial u_\varepsilon}{\partial n} \right)^2 n dx = 0. \tag{8.15}$$

From (8.12) we deduce that

$$\int_{\partial \Omega} \left( \frac{\partial G}{\partial n} (x, x_0) \right)^2 n(x) dx = 0. \tag{8.16}$$

To complete the argument it now suffices to apply Theorem 4.4.

**Remark.** Conjecture 1 is consistent with the results obtained in [AP2] when $\Omega$ is a ball.

**Conjecture 2.** Let $\Omega \subset \mathbb{R}^N, N \geq 4$, be a bounded domain with smooth boundary. Let $u_\varepsilon$ be a solution of

$$\left\{ \begin{array}{ll} -\Delta u_\varepsilon = N(N-2)u_\varepsilon^p + \varepsilon u_\varepsilon & \text{in } \Omega, \\ u_\varepsilon > 0 & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial \Omega, \end{array} \right.$$  

where $p = (N + 2)/(N - 2)$.

**Conjecture 2.** Assume $\{u_\varepsilon\}$ is a minimizing sequence for the Sobolev inequality. Then

$$\lim_{\varepsilon \to 0} \varepsilon \|u_\varepsilon\|_{L^2(\Omega)}^{2(N-4)/N-2} = \frac{(N-2)^3 \sigma_N}{2a_N} |g| \quad \text{if } N > 4$$

$$\lim_{\varepsilon \to 0} \varepsilon \log \|u_\varepsilon\|_{L^\infty(\Omega)} = 4\sigma_4 |g| \quad \text{if } N = 4,$$

where $g$ is a critical value of the function $\phi(x) = g(x, x)$ and

$$a_N = \int_0^\infty \frac{r^{N-1} dr}{(1 + r^2)^{N-2}} = 2 \frac{N - 1}{N - 4} \frac{\{\Gamma(N/2)\}^2}{\Gamma(N)}.$$

**Evidence.** From Pohozaev's identity we obtain

$$\varepsilon \int_{\Omega} u_\varepsilon^2 = \frac{1}{2} \int_{\partial \Omega} (x - y, n) \left( \frac{\partial u_\varepsilon}{\partial n} \right)^2 \tag{8.17}$$
for any point \( y \in \mathbb{R}^N \). As above, we have (8.8) at the point of concentration \( x_0 \) and thus

\[
(8.18) \quad \int_{\Omega} u_{\varepsilon}^2 \simeq \mu^2 \int_{\mathbb{R}^N} \frac{dx}{(1 + |x|^2)^{N-2}} = \mu^2 \sigma_N a_N \quad \text{if} \quad N > 4
\]

and

\[
(8.19) \quad \int_{\Omega} u_{\varepsilon}^2 \simeq \mu^2 |\log \mu| \sigma_4 \quad \text{if} \quad N = 4.
\]

On the other hand, we have as in Conjecture 1,

\[
\int_{\Omega} (x - x_0, n) \left( \frac{\partial u_{\varepsilon}}{\partial n} \right)^2 \simeq \int_{\Omega} (x - x_0, n) \mu^{N-2} K_N^2 \left( \frac{\partial G}{\partial n}(x, x_0) \right)^2 dx \]

\[
= -\mu^{N-2} (N-2)^3 \sigma_N^2 g(x_0, x_0).
\]

Putting together (8.17) with \( y = x_0 \), (8.18), (8.19) and (8.20) we are led to Conjecture 2. The argument for showing that \( x_0 \) as a critical point of \( \phi \) is the same as in Conjecture 1. Part of this programme has been made rigorous by O. Rey [R2].

**Remark.** Conjecture 2 is consistent with the results for the ball in [AP3].

**Conjecture 3.** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with smooth boundary. Let \( u_{\varepsilon} \) be the solution of

\[
\begin{cases}
-\Delta u_{\varepsilon} = 3u_{\varepsilon}^{5-\varepsilon} + \lambda u_{\varepsilon} & \text{in } \Omega, \\
u_{\varepsilon} > 0 & \text{in } \Omega, \\
u_{\varepsilon} = 0 & \text{on } \partial \Omega.
\end{cases}
\]

As before, we denote by \( G_\lambda(x, y) \) the Green's function of \(-\Delta - \lambda \) on \( \Omega \) and by \( g_\lambda(x, y) \) its regular part. It is not difficult to see that \( \phi_\lambda(x) = g_\lambda(x, x) \) is smooth on \( \Omega \).

**Conjecture 3.** (i) Assume \( \phi_\lambda(x) \leq 0 \) on \( \Omega \) and \( \{u_{\varepsilon}\} \) is a minimizing sequence for the Sobolev inequality. Then

\[
\lim_{\varepsilon \to 0} \varepsilon \|u_{\varepsilon}\|_{L^\infty(\Omega)}^2 = 128|g_\lambda|,
\]

where \( g_\lambda \) is a critical value of the function \( \phi_\lambda \), i.e \( g_\lambda = \phi_\lambda(x_0) \) for some point \( x_0 \in \Omega \) such that \( \nabla \phi_\lambda(x_0) = 0 \).

(ii) If \( \phi_\lambda(x_0) = 0 \), then

\[
\lim_{\varepsilon \to 0} \varepsilon \|u_{\varepsilon}\|_{L^\infty(\Omega)}^4 = 32\lambda.
\]

**Remark.** This would be consistent with Theorems 1 and 2.
Appendix.

Let $u_\varepsilon$ be a positive radial solution of

\begin{align}
-\Delta u - \lambda u &= 3u^{5-\varepsilon} \quad \text{in} \quad \Omega \\
 u(0) &= \mu^{-1/2}.
\end{align}

Then, according to [AP1], we have the upper bound

\begin{equation}
 u_\varepsilon(x) \leq W_\mu(x), \tag{A.3}
\end{equation}

where

\begin{equation}
 W_\mu(x) = \left( \frac{\mu}{\mu^2 + \alpha |x|^2} \right)^{1/2} \tag{A.4}
\end{equation}

and

\begin{equation}
 \alpha = \mu^{\varepsilon/2} + \frac{1}{3} \lambda \mu^2. \tag{A.5}
\end{equation}

In addition we have the lower bound:

Lemma 5.5. We have

\begin{equation}
 u_\varepsilon(x) \geq W_\mu(x) \left( 1 - \frac{\lambda}{2} |x|^2 \right) - \mu^{-1/2} \left[ \mu^{1/2} W_\mu(x) \right]^{-\varepsilon} - 1. \tag{A.6}
\end{equation}

Proof. We follow [AP3], exploiting the radial symmetry of $u_\varepsilon$. Setting

\begin{equation}
 t = \frac{1}{|x|} \quad \text{and} \quad y(t) = u_\varepsilon(x),
\end{equation}

we transform (A.1), (A.2) to

\begin{equation}
 y'' + t^{-4} f(y) = 0 \tag{A.7}
\end{equation}

\begin{equation}
 y(t) \to \gamma \quad \text{as} \quad t \to \infty, \tag{A.8}
\end{equation}

where $f$ is given by

\begin{equation}
 f(s) = \lambda s + 3s^{5-\varepsilon}.
\end{equation}
and $\gamma = \mu^{-1/2}$. Rephrasing (A.3) we have

(A.9) \quad y(t) \leq z(t) \quad \text{for} \quad 1 \leq t < \infty,

where $z(t) = W_\mu(x)$, and is given explicitly by

$$z(t) = \gamma t \left( t^2 + \frac{1}{3} \frac{f(\gamma)}{\gamma} \right)^{-1/2}.$$

Note that $z$ is a solution of the problem

(A.10) \quad z'' + t^{-4}\gamma^{-5} f(\gamma) z^5 = 0

(A.11) \quad z(t) \rightarrow \gamma \quad \text{as} \quad t \rightarrow \infty.

We now integrate the differential equation (A.7) for $y$ twice. That yields, if we use (A.8),

$$y(t) = \gamma - \int_t^\infty (s - t) s^{-4} f(y(s)) ds \geq \gamma - \int_t^\infty (s - t) s^{-4} f(z(s)) ds = \gamma - \lambda I_1 - 3 I_2,$$

(A.12)

where

$$I_1 = \int_t^\infty (s - t) s^{-4} z(s) ds$$

$$I_2 = \int_t^\infty (s - t) s^{-4} z^{5-\epsilon}(s) ds.$$

By the concavity and positivity of $z$, we have

$$\frac{z(s)}{s} < \frac{z(t)}{t} \quad \text{if} \quad s > t,$$

and hence we can estimate $I_1$ by

(A.13) \quad I_1 < \frac{z(t)}{t} \int_t^\infty (s - t) s^{-3} ds = \frac{z(t)}{2t^2}. \quad \text{(A.13)}
To estimate $I_2$, we note that since $z$ is increasing,

$$z^{-\epsilon}(s) < z^{-\epsilon}(t) \quad \text{if} \quad s > t$$

and so

$$I_2 < z^{-\epsilon}(t) \int_t^\infty (s-t)s^{-4}z^5(s)ds.$$  \hspace{1cm} (A.14)

If we integrate the differential equation (A.10) for $z$ twice and use (A.11), we obtain

$$z(t) = \gamma - \gamma^{-5} f(\gamma) \int_t^\infty (s-t)s^{-4}z^5(s)ds.$$  

Using this relation to eliminate the integral from (A.14) we arrive at the estimate

$$I_2 < z^{-\epsilon}(t) \frac{\gamma - z(t)}{\gamma^{-5} f(\gamma)}.$$  

This yields, remembering the definition of $f$,

$$I_2 < \frac{1}{3} \{\gamma - z(t)\} \left(\frac{\gamma}{z(t)}\right)^\epsilon.$$  \hspace{1cm} (A.15)

Putting the estimates (A.13) and (A.15) for $I_1$ and $I_2$ into (A.12) finally yields the lower bound

$$y(t) > z(t)\left(1 - \frac{\lambda}{2t^2}\right) - \{\gamma - z(t)\} \left[\left(\frac{\gamma}{z(t)}\right)^\epsilon - 1\right]$$

$$< z(t)\left(1 - \frac{\lambda}{2t^2}\right) - \gamma\left[\left(\frac{\gamma}{2(t)}\right)^\epsilon - 1\right]$$

because $\gamma > z(t)$. This reads, in terms of the original variables,

$$u_\epsilon(x) \geq W_\mu(x)\left(1 - \frac{\lambda}{2}|x|^2\right) - \mu^{-1/2}[[\mu^{1/2}W_\mu(x)]^{-\epsilon} - 1],$$

i.e. the lower bound we set out to prove.
References


Département de Mathématiques
Université Paris VI
4 Place Jussieu
F-75230 PARIS, cedex 05

Mathematical Institute
University of Leiden
The Netherlands