Mat. Aplic. Comp., V. 6, n.º 1, pp. 47-56, 1987

A MAXIMIZATION PROBLEM INVOLVING CRITICAL SOBOLEV EXPONENTS

HAÏM BRÉZIS

Département de Mathématiques
Université Paris VI
75252 Paris Cedex 05, France

LUC OSWALD

Department of Mathematics
Purdue University
W. Lafayette, IN 47907, USA

Dedicated to Professor L.A. Medeiros on his 60th birthday

ABSTRACT: *We consider the functional*

$$J(u) = \int_\Omega u^{p+1}dx + \lambda \int_\Omega u^2 dx \qquad (*)$$

where λ is a given positive constant, $p = (N+2)/(N-2)$, Ω is a bounded domain of R^N, $N \geq 3$ and u varies on the set $M = \{u \in H_0^1(\Omega), \int_\Omega \Delta u^2 dx = 1\}$. The problem we discuss in this work answers the question of whether the supremum $\sup\{J(u), u \in M\}$ is achieved or not.

KEY WORDS: Critical Sobolev exponent • maximization problem • best Sobolev constant

RESUMO: *Consideramos o funcional (*), onde λ é uma constante positiva dada, $p = (N+2)/(N-2)$, Ω é um domínio limitado do R^N, $N \geq 3$ e u varia no conjunto $M = \{u \in H_0^1(\Omega), \int_\Omega \Delta u^2 dx = 1\}$. O problema discutido neste trabalho responde à pergunta: é o valor $\sup\{J(u), u \in M\}$ atingido ou não?*

PALAVRAS-CHAVE: Expoente crítico de Sobolev • problema de maximização • melhor constante de Sobolev

Recebido em 9/II/87.

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$. Let us define the set M and the functional $J(u)$ by

$$M = \{u \in H_0^1(\Omega) \; ; \int |\nabla u|^2 = 1\} \tag{1}$$

and

$$J(u) = \int |u|^{p+1} + \lambda \int |u|^2 \tag{2}$$

where λ is a given positive constant and $p = (N+2)/(N-2)$ so that $p+1 = 2N/(N-2)$ is the critical Sobolev exponent.

Let

$$J = \sup_{u \in M} J(u) \; . \tag{3}$$

We would like to know whether the supremum in (3) is achieved or not. Of course the main difficulty comes from the fact that the Sobolev imbedding $H_0^1 \subset L^{p+1}$ is not compact. Note that if $u \geq 0$ is a maximizer in (3), then u satisfies the Euler equation.

$$-m \, \Delta u = u^p + \frac{2\lambda}{p+1} u$$

for some positive constant m (Lagrange multiplier). For this reason our problem is closely related to the one studied by Brézis-Nirenberg [4], which corresponds to the case $m = 1$.

Our main results are the following:

<u>Theorem 1</u>. Assume $N \geq 4$. The supremum in (3) is achieved for every $\lambda > 0$.

<u>Theorem 2</u>. Assume $N = 3$. Then, there exists a constant $\lambda^* > 0$ (depending on Ω) such that the supremum in (3) is achieved for $\lambda > \lambda^*$ and is not achieved for $\lambda < \lambda^*$.

<u>Remark 1</u>. When $N = 3$ we do not know whether the supremum is achieved or not for $\lambda = \lambda^*$. In view of the results of [4] for the ball we suspect that it is not achieved.

As pointed by Th. Aubin [2] the best Sobolev constant plays an important role in such kind of problems.

Let

$$\sum = \sup_{u \in M} \left(\int |u|^{p+1}\right)^{2/(p+1)}$$

be the inverse of the best Sobolev constant S. Note that, by definition of J, we have

$$J \geq \sum^{(p+1)/2}$$

Adapting a technique of [4] we shall base the proofs of Theorems 1 and 2 on the following two lemmas:

Lemma 3. Assume $N \geq 3$ and

$$J > \sum^{(p+1)/2} \quad ; \tag{4}$$

then the supremum in (3) is achieved.

Lemma 4. If $N \geq 4$, then (4) holds for every $\lambda > 0$. If $N = 3$, then (4) holds for $\lambda > \lambda^*$ (large enough).

2. PROOF OF LEMMA 3

Let (u_n) be a maximizing sequence for (3). By passing to a subsequence we may assume $u_n \rightharpoonup u$ weakly in H_0^1. We write

$$u_n = u + v_n$$

so that $v_n \rightharpoonup 0$ weakly in H_0^1 and

$$1 = \int |\nabla u_n|^2 = \int |\nabla u|^2 + \int |\nabla v_n|^2 + o(1) \quad . \tag{5}$$

On the other hand, using a result of Brézis-Lieb [3] we find that

$$\int |u_n|^{p+1} = \int |u|^{p+1} + \int |v_n|^{p+1} + o(1) \quad . \tag{6}$$

Since (u_n) is a maximizing sequence we have

$$\int |u_n|^{p+1} + \lambda \int |u_n|^2 = J + o(1) \quad . \tag{7}$$

Combining (6) and (7) we obtain

$$\int |u|^{p+1} + \int |v_n|^{p+1} + \lambda \int |u|^2 = J + o(1) \ . \tag{8}$$

Set

$$t = \int |\nabla u|^2$$

so that $0 \le t \le 1$. It suffices to show that $t = 1$ (since this implies that $u_n \to u$ strongly in L^{p+1}).

We claim that

$$\int |u|^{p+1} + \lambda \int |u|^2 \le Jt \ . \tag{9}$$

Indeed $\tilde{u} = \tilde{u}/\sqrt{t}$ satisfies $\int |\nabla \tilde{u}|^2 = 1$ and thus

$$\int |\tilde{u}|^{p+1} + \lambda \int |\tilde{u}|^2 \le J$$

i.e.

$$\frac{1}{t^{(p+1)/2}} \int |u|^{p+1} + \frac{\lambda}{t} \int |u|^2 \le J$$

which implies (9) since $0 \le t \le 1$.

From (8) and (9) we deduce that

$$J(1-t) \le \int |v_n|^{p+1} + o(1) \ . \tag{10}$$

On the other hand, by the definition of $\sum$ and by (5) we have

$$\int |v_n|^{p+1} \le \sum^{(p+1)/2} \left(\int |\nabla v_n|^2\right)^{(p+1)/2} = \sum^{(p+1)/2} (1-t)^{(p+1)/2} + o(1) \ . \tag{11}$$

From (10) and (11) we obtain

$$J(1-t) \le \sum^{(p+1)/2} (1-t)^{(p+1)/2} \tag{12}$$

and since $J \ge \sum^{(p+1)/2}$ we conclude that either $t = 0$ or $t = 1$.
Under the assumption (4) $t = 0$ is excluded by (12).

3. PROOF OF LEMMA 4

Following Th. Aubin [2] and Brézis-Nirenberg [4] we choose as test function

$$u_\varepsilon(x) = \frac{\phi(x)\ \varepsilon^{(N-2)/2}}{(\varepsilon^2 + |x|^2)^{(N-2)/2}} \tag{13}$$

where $\phi(x)$ is a smooth function with compact support in Ω such that $\phi(x) = 1$ near 0 (assuming $0 \in \Omega$). We find that

$$\int |\nabla u_\varepsilon|^2 = K_1 + O(\varepsilon^{N-2})$$

$$\int |u_\varepsilon|^{p+1} = K_2 + O(\varepsilon^N)$$

where

$$K_1 = (N-2)^2 \int_{\mathbb{R}^N} \frac{y^2}{(1+|y|^2)^N}\, dy$$

$$K_2 = \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^N}\, dy$$

and

$$\int u_\varepsilon^2 = \varepsilon^2 K_3 + O(\varepsilon^{N-2}) \quad \text{with}$$

$$K_3 = \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{N-2}}\, dy \quad \text{for } N \geq 5$$

and

$$\int u_\varepsilon^2 = \varepsilon^2 \log\left(\frac{1}{\varepsilon}\right) K_3 + O(\varepsilon^2)$$

where K_3 equals the area of S^3 for $N = 4$. Therefore

$$\begin{cases} J(u_\varepsilon) = K_2 + \lambda\, \varepsilon^2\, K_3 + O(\varepsilon^{N-2}) \\ \int |\nabla u_\varepsilon|^2 = K_1 + O(\varepsilon^{N-2}) \quad \text{for} \quad N \geq 5 \end{cases}$$

and

$$\begin{cases} J(u_\varepsilon) = K_2 + \lambda\, \varepsilon^2 \log\left(\frac{1}{\varepsilon}\right) K_3 + O(\varepsilon^2) \\ \int |\nabla u_\varepsilon|^2 = K_1 + O(\varepsilon^2) \quad \text{for} \quad N = 4\ . \end{cases}$$

Normalizing u_ε we introduce a function v_ε such that $\int |\nabla v_\varepsilon|^2 = 1$ and then we can see that for ε small enough

$$J(v_\varepsilon) > \Sigma^{(p+1)/2} .$$

(Note that $\Sigma = \dfrac{K_2^{\frac{2}{p+1}}}{K_1}$ see [1], [5] or [6]).

Case of Dimension 3. Taking u_ε as in (13) we find:

$$\int |\nabla u_\varepsilon|^2 = K_1 + \varepsilon \int \frac{|\nabla\phi|^2}{|x|^2}\, dx + O(\varepsilon^2)$$

$$\int |u_\varepsilon|^6 = K_2 + O(\varepsilon^2)$$

$$\int u_\varepsilon^2 = \varepsilon \int \frac{\phi^2}{|x|^2}\, dx + O(\varepsilon^2)$$

where $K_1 = 3 \int_{\mathbf{R}^3} \dfrac{dy}{(1+|y|^2)^3}$ and $K_2 = \int_{\mathbf{R}^3} \dfrac{dy}{(1+|y|^2)^3}$.

Hence

$$J(u_\varepsilon) = K_2 + \lambda\, \varepsilon \int \frac{\phi^2}{|x|^2}\, dx + O(\varepsilon^2) .$$

Stretching u_ε into v_ε so that $\int |\nabla v_\varepsilon|^2 = 1$, we find that

$$J(v_\varepsilon) = \frac{K_2}{K_1^3} + \varepsilon \left(\frac{\lambda}{K_1} \int \frac{\phi^2}{|x|^2} - \frac{3K_2}{K_1^4} \int \frac{|\nabla\phi|^2}{|x|^2} \right) + O(\varepsilon^2) .$$

Since $\Sigma^3 = K_2/K_1^3$ (cf. [1], [5] or [6]), we see that for λ large enough, we can find ε small enough so that $J > \Sigma^{(p+1)/2}$.

4. PROOF OF THEOREM 2

Step 1. For λ small enough the supremum is not achieved.

Let us suppose by contradiction that the supremum in (3) is achieved for arbitrarily small values of λ. Then we would have a positive solution to the

equation

$$\begin{cases} -\Delta u = u^p + \frac{2\lambda}{p+1} u & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega . \end{cases}$$

Multiplying by u and integrating over Ω we obtain

$$m = m \int |\nabla u|^2 = \int u^{p+1} + \frac{2\lambda}{p+1} \int u^2 \geq \frac{2}{p+1} J \geq \frac{2}{p+1} \sum^{(p+1)/2} .$$

Hence m is bounded from below. Letting $u = \alpha v$ with $\alpha^{p-1} = m$ we obtain a positive solution of

$$\begin{cases} -\Delta u = v^p + \tilde{\lambda} v & \text{on } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

in dimension 3 and with $\tilde{\lambda} = \frac{2\lambda}{(p+1)m}$ arbitrarily small. This is a contradiction with a result of Brézis-Nirenberg [4] (Theorem 1.2 and 1.2").

Step 2. The set of all λ for which the supremum in (3) is achieved is an interval.

We shall show that if the supremum is achieved for some λ_o, then it is achieved for every λ greater than λ_o. By assumption, there exists some $u_o \in M$ such that

$$J_{\lambda_o} = \sup J_{\lambda_o}(u) = J_{\lambda_o}(u_o)$$

where

$$J_\lambda(u) = \int u^{p+1} + \lambda \int u^2 .$$

On the other hand, we always have

$$J_\lambda \geq \sum^{(p+1)/2}$$

so that

$$J_{\lambda_o}(u_o) \geq \sum^{(p+1)/2} .$$

Therefore

$$J_\lambda(u_o) = J_{\lambda_o}(u_o) + (\lambda - \lambda_o) \int u_o^2 > \Sigma^{(p+1)/2}$$

By Lemma 3 we deduce that the supremum in (3) is achieved.

Remark 2. Suppose Ω is the unit ball in $\mathbb{R}^3$. The expansion given in Section 3 for the case $N = 3$ shows that if

$$\lambda > 3\Sigma^3 \inf_{\substack{\phi'(0)=0 \\ \phi(1)=0}} \left(\int_0^1 \phi'^2(r)dr \Big/ \int_0^1 \phi^2(r)dr \right) = 3\Sigma^3 \frac{\pi^2}{4}$$

then the supremum in (3) is achieved. This says that $\lambda^* \leq 3\Sigma^3 \frac{\pi^2}{4}$. We conjecture that $\lambda^* = 3\Sigma^3 \frac{\pi^2}{4}$.

More generally, suppose Ω is any bounded domain in $\mathbb{R}^3$ with smooth boundary. Consider the problem

$$\inf_{\substack{\phi \in H_0^1 \\ \|\phi\|_6 = 1}} \left(\int |\nabla\phi|^2 - \mu \int \phi^2 \right) . \tag{14}$$

We recall (see [4]) that there exists a positive constant μ^* (depending on Ω) such that

- if $\mu > \mu^*$, then the infimum in (14) is achieved
- if $\mu < \mu^*$, then the infimum in (14) is not achieved.

(When Ω is the unit ball then $\mu^* = \pi^2/4$). We conjecture that λ^* and μ^* are always related by the relation $\lambda^* = 3\Sigma^3 \mu^*$.

5. REFERENCES

[1] AUBIN, Th. - "Problèmes isopérimétriques et espaces de Sobolev", *J. Diff. Geom.*, Vol. 11, pp. 573-598, 1975.

[2] AUBIN, Th. - "Equations différentielles non-linéaires et problème de Yamabe concernant la courbure scalaire", *J. Math. Pures Appl.*, Vol. 55, pp. 269-293, 1976.

[3] BRÉZIS, H. and LIEB, E. - "A relation between pointwise convergence of functions and convergence of integrals", *Proc. Amer. Math. Soc.*, Vol. 88, pp. 486-490, 1983.

[4] BRÉZIS, H. and NIRENBERG, L. - "Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents", *Comm. Pure Appl. Math.*, Vol. 36, pp. 637-677, 1983.

[5] LIEB, E. - "Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities", *Ann. Math.*, Vol. 118, pp. 349-374, 1983.

[6] TALENTI, G. - "Best constant in Sobolev inequality", *Annali di Mat.*, Vol. 110, pp. 353-372, 1976.