A MAXIMIZATION PROBLEM
INVOLVING CRITICAL SOBOLEV
EXONENTS

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Dedicated to Professor L.A. Medeiros on his 60th birthday

ABSTRACT: We consider the functional

\[ J(u) = \int_\Omega u^{p^*} \, dx + \lambda \int_\Omega u^2 \, dx \] (*)

where \( \lambda \) is a given positive constant, \( p = (N + 2)/(N - 2) \), \( \Omega \) is a bounded domain of \( \mathbb{R}^N \), \( N \geq 3 \) and \( u \) varies on the set \( M = \{ u \in H_0^1(\Omega), \int_\Omega \Delta u^2 \, dx = 1 \} \). The problem we discuss in this work answers the question of whether the supremum \( \sup \{ J(u), u \in M \} \) is achieved or not.

KEY WORDS: Critical Sobolev exponent • maximization problem • best Sobolev constant

RESUMO: Consideramos o funcional (*) \( \text{onde} \lambda \text{é uma constante positiva dada, } p = (N + 2)/(N - 2), \Omega \text{é um domínio limitado do } \mathbb{R}^N, \text{ } N \geq 3 \text{ e } u \text{ varia no conjunto } M = \{ u \in H_0^1(\Omega), \int_\Omega \Delta u^2 \, dx = 1 \}. \) O problema discutido neste trabalho responde à pergunta: é o valor \( \sup \{ J(u), u \in M \} \) atingido ou não?

PALAVRAS-CHAVE: Exponente crítico de Sobolev • problema de maximização • melhor constante de Sobolev

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1. INTRODUCTION

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 3$. Let us define the set $M$ and the functional $J(u)$ by

$$M = \{ u \in H^1_0(\Omega) ; \int \sum_{\nu} u^2 = 1 \}$$

(1)

and

$$J(u) = \int |u|^{p+1} + \lambda \int |u|^2$$

(2)

where $\lambda$ is a given positive constant and $p = (N+2)/(N-2)$ so that $p+1 = 2N/(N-2)$ is the critical Sobolev exponent.

Let

$$J = \sup_{u \in M} J(u) .$$

(3)

We would like to know whether the supremum in (3) is achieved or not. Of course the main difficulty comes from the fact that the Sobolev imbedding $H^1_0 \subset L^{p+1}$ is not compact. Note that if $u \geq 0$ is a maximizer in (3), then $u$ satisfies the Euler equation.

$$-m \Delta u = u^p + \frac{2\lambda}{p+1} u$$

for some positive constant $m$ (Lagrange multiplier). For this reason our problem is closely related to the one studied by Brézis-Nirenberg [4], which corresponds to the case $m = 1$.

Our main results are the following:

**Theorem 1.** Assume $N \geq 4$. The supremum in (3) is achieved for every $\lambda > 0$.

**Theorem 2.** Assume $N = 3$. Then, there exists a constant $\lambda^* > 0$ (depending on $\Omega$) such that the supremum in (3) is achieved for $\lambda > \lambda^*$ and is not achieved for $\lambda < \lambda^*$.

**Remark 1.** When $N = 3$ we do not know whether the supremum is achieved or not for $\lambda = \lambda^*$. In view of the results of [4] for the ball we suspect that it is not achieved.

As pointed by Th. Aubin [2] the best Sobolev constant plays an important role in such kind of problems.
Let
\[ \Sigma = \sup_{u \in M} \left( \int |u|^{p+1}/(p+1) \right) \]
be the inverse of the best Sobolev constant $S$. Note that, by definition of $J$, we have
\[ J \geq \Sigma^{(p+1)/2} \]
Adapting a technique of [4] we shall base the proofs of Theorems 1 and 2 on the following two lemmas:

**Lemma 3.** Assume $N \geq 3$ and
\[ J > \Sigma^{(p+1)/2} \]
then the supremum in (3) is achieved.

**Lemma 4.** If $N \geq 4$, then (4) holds for every $\lambda > 0$. If $N = 3$, then (4) holds for $\lambda > \lambda^*$ (large enough).

2. **PROOF OF LEMMA 3**

Let $\{u_n\}$ be a maximizing sequence for (3). By passing to a subsequence we may assume $u_n \rightharpoonup u$ weakly in $H^1_0$. We write
\[ u_n = u + v_n \]
so that $v_n \rightharpoonup 0$ weakly in $H^1_0$ and
\[ 1 = \int |\nabla u_n|^2 = \int |\nabla u|^2 + \int |\nabla v_n|^2 + o(1) \quad (5) \]
On the other hand, using a result of Brezis-Lieb [3] we find that
\[ \int |u_n|^{p+1} = \int |u|^{p+1} + \int |v_n|^{p+1} + o(1) \quad (6) \]
Since $\{u_n\}$ is a maximizing sequence we have
\[ \int |u_n|^{p+1} + \lambda \int |u_n|^2 = J + o(1) \quad (7) \]
Combining (6) and (7) we obtain
\[
\int |u|^{p+1} + \int |v_n|^{p+1} + \lambda \int |u|^2 = J + o(1). \tag{8}
\]

Set \[t = \int |\nabla u|^2\]
so that \(0 \leq t \leq 1\). It suffices to show that \(t = 1\) (since this implies that \(u_n \to u\) strongly in \(L^{p+1}\)).

We claim that

\[\int |u|^{p+1} + \lambda \int |u|^2 \leq Jt. \tag{9}\]

Indeed \(\tilde{u} = \tilde{u}/\sqrt{t}\) satisfies \(\int |\nabla \tilde{u}|^2 = 1\) and thus

\[\int |\tilde{u}|^{p+1} + \lambda \int |\tilde{u}|^2 \leq J\]

i.e.

\[\frac{1}{t^{(p+1)/2}} \int |u|^{p+1} + \frac{\lambda}{t} \int |u|^2 \leq J\]

which implies (9) since \(0 \leq t \leq 1\).

From (8) and (9) we deduce that

\[J(1-t) \leq \int |v_n|^{p+1} + o(1). \tag{10}\]

On the other hand, by the definition of \(\Sigma\) and by (5) we have

\[\int |v_n|^{p+1} \leq \Sigma^{(p+1)/2} \left(\int |\nabla v_n|^2\right)^{(p+1)/2} = \Sigma^{(p+1)/2} (1-t)(p+1)/2 + o(1). \tag{11}\]

From (10) and (11) we obtain

\[J(1-t) \leq \Sigma^{(p+1)/2} (1-t)(p+1)/2 \tag{12}\]

and since \(J \geq \Sigma^{(p+1)/2}\) we conclude that either \(t = 0\) or \(t = 1\).

Under the assumption (4) \(t = 0\) is excluded by (12).

3. PROOF OF LEMMA 4

where $\phi(x)$ is a smooth function with compact support in $\Omega$ such that $\phi(x) = 1$ near 0 (assuming $0 \in \Omega$). We find that

\[
\left|\nabla u_\varepsilon\right|^2 = K_1 + O(\varepsilon^{N-2})
\]
\[
\left|u_\varepsilon\right|^{p+1} = K_2 + O(\varepsilon^N)
\]

where

\[
K_1 = (N-2)^2 \int_{\mathbb{R}^N} \frac{y^2}{(1+|y|^2)^N} \, dy
\]
\[
K_2 = \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^N} \, dy
\]

and

\[
\int u_\varepsilon^2 = \varepsilon^2 K_3 + O(\varepsilon^{N-2}) \quad \text{with}
\]
\[
K_3 = \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{(N-2)}} \, dy \quad \text{for } N \geq 5
\]

and

\[
\int u_\varepsilon^2 = \varepsilon^2 \log \left(\frac{1}{\varepsilon}\right) K_3 + O(\varepsilon^2)
\]

where $K_3$ equals the area of $S^3$ for $N = 4$. Therefore

\[
\begin{align*}
J(u_\varepsilon) & = K_2 + \lambda \varepsilon^2 K_3 + O(\varepsilon^{N-2}) \\
\left|\nabla u_\varepsilon\right|^2 & = K_1 + O(\varepsilon^{N-2}) \quad \text{for } N \geq 5
\end{align*}
\]

and

\[
\begin{align*}
J(u_\varepsilon) & = K_2 + \lambda \varepsilon^2 \log \left(\frac{1}{\varepsilon}\right) K_3 + O(\varepsilon^2) \\
\left|\nabla u_\varepsilon\right|^2 & = K_1 + O(\varepsilon^2) \quad \text{for } N = 4
\end{align*}
\]
Normalizing $u_\varepsilon$ we introduce a function $v_\varepsilon$ such that $\int|\nabla v_\varepsilon|^2 = 1$ and then we can see that for $\varepsilon$ small enough

$$J(v_\varepsilon) > \frac{(p+1)/2}{K_2^{p+1}}$$

(Note that $\frac{2}{K_2^{p+1}}$ see [1], [5] or [6]).

**Case of Dimension 3.** Taking $u_\varepsilon$ as in (13) we find:

$$\int |\nabla u_\varepsilon|^2 = K_1 + \varepsilon \int \frac{|\nabla \phi|^2}{|x|^2} \, dx + O(\varepsilon^2)$$

$$\int |u_\varepsilon|^6 = K_2 + O(\varepsilon^2)$$

$$\int u_\varepsilon^2 = \varepsilon \int \frac{\phi^2}{|x|^2} \, dx + O(\varepsilon^2)$$

where $K_1 = 3 \int_{\mathbb{R}^3} \frac{dy}{(1+|y|^2)^3}$ and $K_2 = \int_{\mathbb{R}^3} \frac{dy}{(1+|y|^2)^3}$. Hence

$$J(u_\varepsilon) = K_2 + \lambda \varepsilon \int \frac{\phi^2}{|x|^2} \, dx + O(\varepsilon^2).$$

Stretching $u_\varepsilon$ into $v_\varepsilon$ so that $\int|\nabla v_\varepsilon|^2 = 1$, we find that

$$J(v_\varepsilon) = \frac{K_2}{K_1^2} + \varepsilon \left( \frac{\lambda}{K_1} \int \frac{\phi^2}{|x|^2} - \frac{3K_2}{K_1^4} \int \frac{|\nabla \phi|^2}{|x|^2} \right) + O(\varepsilon^2)$$

Since $\Sigma^3 = K_2/K_1^3$ (cf. [1], [5] or [6]), we see that for $\lambda$ large enough, we can find $\varepsilon$ small enough so that $J > \Sigma^{(p+1)/2}$.

**4. PROOF OF THEOREM 2**

**Step 1.** For $\lambda$ small enough the supremum is not achieved.

Let us suppose by contradiction that the supremum in (3) is achieved for arbitrarily small values of $\lambda$. Then we would have a positive solution to the
equation
\[
\begin{aligned}
- \Delta u &= u^p + \frac{2\lambda}{p+1} u \quad \text{on } \Omega \\
0 &= \quad \text{on } \partial \Omega.
\end{aligned}
\]

Multiplying by \(u\) and integrating over \(\Omega\) we obtain
\[
m = m \left( \int |\nabla u|^2 \right) = \int u^{p+1} + \frac{2\lambda}{p+1} \int u^2 \geq \frac{2}{p+1} \int J \geq \frac{2}{p+1} \lambda^{(p+1)/2}.
\]

Hence \(m\) is bounded from below. Letting \(u = \alpha v\) with \(\alpha^{p-1} = m\) we obtain a positive solution of
\[
\begin{aligned}
- \Delta u &= v^p + \lambda v \quad \text{on } \Omega \\
0 &= \quad \text{on } \partial \Omega
\end{aligned}
\]
in dimension 3 and with \(\lambda = \frac{2\lambda}{(p+1)m}\) arbitrarily small. This is a contradiction with a result of Brézis-Nirenberg [4] (Theorem 1.2 and 1.2').

Step 2. The set of all \(\lambda\) for which the supremum in (3) is achieved is an interval.

We shall show that if the supremum is achieved for some \(\lambda_o\), then it is achieved for every \(\lambda\) greater than \(\lambda_o\). By assumption, there exists some \(u_o \in M\) such that
\[
J_{\lambda_o} = \sup J_{\lambda_o} (u) = J_{\lambda_o} (u_o)
\]
where
\[
J_\lambda (u) = \int u^{p+1} + \lambda \int u^2.
\]

On the other hand, we always have
\[
J_\lambda \geq \lambda^{(p+1)/2}
\]
so that
\[
J_{\lambda_o} (u_o) \geq \lambda^{(p+1)/2}.
\]
Therefore
\[ J_{\lambda}(u_o) = J_{\lambda_0}(u_o) + (\lambda - \lambda_0) \int_{B_0} u_o^2 > \sum (p+1)/2 \]

By Lemma 3 we deduce that the supremum in (3) is achieved.

Remark 2. Suppose \( \Omega \) is the unit ball in \( \mathbb{R}^3 \). The expansion given in Section 3 for the case \( N = 3 \) shows that if
\[ \lambda > 3 \sum_3 \inf_{\phi'(0) = 0, \phi(1) = 0} \left( \int_0^1 \phi'(r) dr / \int_0^1 \phi^2(r) dr \right) = 3 \sum_3 \frac{\pi^2}{4} \]
then the supremum in (3) is achieved. This says that \( \lambda^* \leq 3 \sum_3 \frac{\pi^2}{4} \). We conjecture that \( \lambda^* = 3 \sum_3 \frac{\pi^2}{4} \).

More generally, suppose \( \Omega \) is any bounded domain in \( \mathbb{R}^3 \) with smooth boundary. Consider the problem
\[ \inf_{\phi \in H_0^1} \left( \int_B |\nabla \phi|^2 - u \int_B \phi^2 \right) \quad (14) \]

We recall (see [4]) that there exists a positive constant \( \mu^* \) (depending on \( \Omega \)) such that
- if \( \mu > \mu^* \), then the infimum in (14) is achieved
- if \( \mu < \mu^* \), then the infimum in (14) is not achieved.

(When \( \Omega \) is the unit ball then \( \mu^* = \pi^2/4 \). We conjecture that \( \lambda^* \) and \( \mu^* \) are always related by the relation \( \lambda^* = 3 \sum_3 \mu^* \).
5. REFERENCES


